# Review Sheet: Chain Rule, Product Rule, Quotient Rule

## Content Review

### Overview

*Product Rule*:

$$\frac{d}{dx} f(x) \cdot g(x) = f'(x) \cdot g(x) + g'(x) \cdot f(x)$$

Product Rule is sometimes also written as:

$$y = u \cdot v \Longrightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = v'u + u'v \text{ where u,v are both functions in terms of x}$$

**Common Mistake:** Don't try to take the derivative of expressions like $y = 8x$ using product rule. This will be a normal derivative the way you have learned so far. Don't try to make constants (like 8) functions of x, this will only lead to confusion.

*Quotient Rule*:

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$$

Quotient Rule is sometimes also written as:

$$y = \frac{u}{v} \Longrightarrow \frac{dy}{dx} = \frac{u'v - v'u}{v^2} \text{ where u,v are both functions in terms of x}$$

Quotient rule is essentially a special case of product rule where one of the functions has the power (-1), i.e $y = u \cdot v^{-1}$. If correctly applied, product rule can be used to derive quotient rule. However, at this level, it is just easier to remember the quotient rule.

*Chain Rule*: The Chain Rule is probably one of the most important parts of fundamental calculus that you will learn. This idea will continue to appear at different places in engineering, mathematics, biology, chemistry, economics or other coursework.
In simple terms, the chain rule can be stated as:

$$\frac{dy}{dx} = \frac{dy}{dm} \times \frac{dm}{dx}$$

The main idea here is the presence of the same variable (in this case m) on the diagonal. As long as you take the derivative of $y$ with respect to $m$, and take the derivative of $m$ with respect to $x$ separately, you can find the expression for $\frac{dy}{dx}$ without having to explicitly find $y$ in terms of $x$.

This idea can also be extended to multiple levels of 'chains'. For instance:

$$\frac{dy}{dx} = \frac{dy}{dm} \times \frac{dm}{dt} \times \frac{dt}{dw} \times \frac{dw}{dx}$$

Once you learn to recognize this pattern of having the same variables on the 'diagonal', you will find chain rule to be more intuitive.

**For example:**

$$y = 4m^2 \text{ and } m = \sin x\text{, then we can find } \frac{dy}{dm} = 8m \text{ and } \frac{dm}{dx} = \cos x$$

By applying chain rule:

$$\frac{dy}{dx} = \frac{dy}{dm} \times \frac{dm}{dx} = 8m \cdot \cos x = 8(\sin x) \cdot \cos x = 8 \sin x \cos x$$

Skip to Worked Problems

Created for the MSP by Asmi Kawatkar

## Resources

*Chain Rule*

- Video: Chain Rule (Khan Academy, 5 min)
- Video: Chain Rule for Finding Derivatives - with Worked Examples (Organic Chemistry Tutor, 20 min)
- Worksheet: Chain Rule (has solutions)
- Worksheet: Chain Rule (has worked solutions)

*Product Rule*

- Video: Product Rule (Khan Academy, 3 min)
- Video: Product Rule for Derivatives - with Worked Examples
- Practice Problems: Product & Quotient Rule (with Worked Solutions)
- Practice Problems: Product Rule (Worked Solutions)
- Explainer Worksheet (Detailed Concept + Worked Examples)

*Quotient Rule*

- Video: Quotient Rule (Khan Academy, 4 min)
- Video: Quotient Rule for Derivatives - with Worked Examples (Organic Chemistry Tutor, 12 min)
- Practice Problems: Quotient Rule - with Worked Solutions (UC Davis Math)

## Acknowlegement

Questions in the Worked Problems section of this sheet have been taken from external sources that have been linked where appropriate. All solutions have been developed independently.

# Worked Problems

## Chain Rule

Find the derivatives of the following functions (with respect to x):
Source and another Source

1. $y = (5x+8)^4$

$y = (5x+8)^4$ Want to find $\frac{dy}{dx}$

Let $m = 5x+8$
So then $y = (5x+8)^4 = m^4$

Chain Rule states that : $\frac{dy}{dx} = \underbrace{\frac{dy}{dm} \cdot \frac{dm}{dx}}$ Need to find each of these components.

$m = 5x+8 \implies \frac{dm}{dx} = 5$  $y = m^4 \implies \frac{dy}{dm} = 4m^3$

Substituting into formula for chain rule gives:
$\frac{dy}{dx} = \frac{dy}{dm} \times \frac{dm}{dx} = 4m^3 \cdot 5 = 20m^3$

Generally, we want the answer to be in terms of the specified variables (in this case, x and y), without the "intermediate" variables m.
So substituting for m:

$\frac{dy}{dx} = 20m^3 = 20(5x+8)^3 \implies \frac{dy}{dx} = 20(5x+8)^3$

2. $y = 2(9-x^2)^{\frac{1}{4}}$

$y = 2(9-x^2)^{\frac{1}{4}}$ Want to find $\frac{dy}{dx}$

Let $m = 9-x^2$
So then $y = 2(9-x^2)^{\frac{1}{4}} = 2m^{\frac{1}{4}}$

$m = 9-x^2 \implies \frac{dm}{dx} = -2x$  $y = 2m^{\frac{1}{4}} \implies \frac{dy}{dm} = 2 \cdot \frac{1}{4} \cdot m^{\frac{1}{4}-1} = \frac{1}{2} m^{-\frac{3}{4}}$

Substituting into formula for chain rule gives:
$\frac{dy}{dx} = \frac{dy}{dm} \cdot \frac{dm}{dx} = \frac{1}{\cancel{2}} m^{-3/4} \cdot (-\cancel{2}x) = (9-x^2)^{-3/4} \cdot (-x)$

Substituting m

$\implies \frac{dy}{dx} = (9-x^2)^{-3/4}(-x)$

3. $y = \sqrt{x^2 - 4x + 2}$

$y = \sqrt{x^2-4x+2}$ want to find $\frac{dy}{dx}$

Let $m = x^2-4x+2$

So then $y = \sqrt{x^2-4x+2} = \sqrt{m}$

$m = x^2-4x+2 \Longrightarrow \frac{dm}{dx} = 2x-4$ $\quad$ $y = \sqrt{m} \Longrightarrow \frac{dy}{dm} = \frac{1}{2}m^{\frac{1}{2}-1} = \frac{1}{2}m^{-\frac{1}{2}} = \frac{1}{2\sqrt{m}}$

Substituting into formula for chain rule gives:

$$\frac{dy}{dx} = \frac{dy}{dm}\cdot\frac{dm}{dx} = \frac{1}{2\sqrt{m}}\cdot(2x-4) = \frac{\cancel{2}x-\cancel{4}^{2}}{\cancel{2}\sqrt{x^2-4x+2}} = \frac{x-2}{\sqrt{x^2-4x+2}}$$

$$\Longrightarrow \frac{dy}{dx} = \frac{x-2}{\sqrt{x^2-4x+2}}$$

4. $y = \sin^3 x + \cos^3 x$

$y = \sin^3 x + \cos^3 x$ Want to find $\frac{dy}{dx}$

Let's split up this function:

$y = a + b$ where $a = \sin^3 x$ and $b = \cos^3 x$.

We know from derivatives that: $\frac{d}{dx}y = \frac{d}{dx}a + \frac{d}{dx}b$.

Let's compute $\frac{d}{dx}a$ and $\frac{d}{dx}b$ separately

$a = \sin^3 x$ [Want to find $\frac{da}{dx}$]

Let $m = \sin x$

So then $a = m^3$

$m = \sin x \Longrightarrow \frac{dm}{dx} = \cos x$

Also, $a = m^3 \Longrightarrow \frac{da}{dm} = 3m^2$

Applying Chain Rule: $\frac{da}{dx} = \frac{da}{dm}\cdot\frac{dm}{dx}$

$\frac{da}{dx} = 3m^2\cdot\cos x = 3\sin^2 x\cos x$

$b = \cos^3 x$

Let $w = \cos x$

So then $b = w^3$

$w = \cos x \Longrightarrow \frac{dw}{dx} = -\sin x$

$b = w^3 \Longrightarrow \frac{db}{dw} = 3w^2$

Applying Chain Rule: $\frac{db}{dx} = \frac{db}{dw}\cdot\frac{dw}{dx}$

$\frac{db}{dx} = 3w^2\cdot(-\sin x) = -3\cos^2 x\sin x$

Adding them back together, we get:

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx} = 3\sin^2 x\cos x - 3\cos^2 \sin x$$

5. $y = \sin^2(\cos^4 x)$

$y = \sin^2(\cos^4 x)$ [want to find $\frac{dy}{dx}$]

Let $m = \cos x$
Let $\omega = \cos^4 x = m^4$
Let $t = \sin(m^4) = \sin(\omega)$
So $y = t^2 = \sin^2(\omega) = \sin^2(m^4) = \sin^2(\cos^4 x)$

Setting up chain rule:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{d\omega} \cdot \frac{d\omega}{dm} \cdot \frac{dm}{dx}$$

$y = t^2 \Longrightarrow \frac{dy}{dt} = 2t$

$t = \sin(\omega) \Longrightarrow \frac{dt}{d\omega} = \cos(\omega)$

$\omega = m^4 \Longrightarrow \frac{d\omega}{dm} = 4m^3$

$m = \cos x \Longrightarrow \frac{dm}{dx} = -\sin x$

Substituting into formula for chain rule

$$\frac{dy}{dx} = 2t \cdot \cos(\omega) \cdot 4m^3 \cdot -\sin x$$

$$\frac{dy}{dx} = 2\sin\omega \cdot \cos(m^4) \cdot 4\cos^3 x \cdot -\sin x$$

$$\frac{dy}{dx} = 2\sin m^4 \cdot \cos(\cos^4(x)) \cdot 4\cos^3 x \cdot -\sin x$$

$$\frac{dy}{dx} = 2\sin(\cos^4 x) \cdot \cos(\cos^4 x) \cdot 4\cos^3 x \cdot -\sin x$$

$$\frac{dy}{dx} = -8\sin x \cos^3 x \sin(\cos^4 x)\cos(\cos^4 x)$$

6. $y = \sin^3(2x+3)$

$y = \sin^3(2x+3)$ want to find $\frac{dy}{dx}$

Let $m = 2x+3$; So $y = \sin^3(m)$
Let $\omega = \sin(m)$; So $y = \omega^3$

Setting up chain rule: $\frac{dy}{dx} = \frac{dy}{d\omega} \cdot \frac{d\omega}{dm} \cdot \frac{dm}{dx}$

$y = \omega^3 \Longrightarrow \frac{dy}{d\omega} = 3\omega^2$

$\omega = \sin(m) \Longrightarrow \frac{d\omega}{dm} = \cos(m)$

$m = 2x+3 \Longrightarrow \frac{dm}{dx} = 2$

Substituting into formula from chain rule:

$$\frac{dy}{dx} = 3\omega^2 \cdot \cos(m) \cdot 2$$

$$\frac{dy}{dx} = 3\sin^2 m \cdot \cos(2x+3) \cdot 2$$

$$\frac{dy}{dx} = 3\sin^2(2x+3) \cdot \cos(2x+3) \cdot 2$$

$$\frac{dy}{dx} = 6\sin^2(2x+3)\cos(2x+3)$$

## Product Rule

Find the derivatives of the following functions (with respect to x):
Source and another Source

1. $y = (x^8 + 2x - 3)e^x$

$y = \underbrace{(x^8+2x-3)}_{u}\underbrace{e^x}_{v}$

Product Rule: $\frac{d}{dx}(uv) = u'v + v'u$

u, v are functions of x

$u = x^8+2x-3 \Longrightarrow u' = 8x^7+2$

$v = e^x \Longrightarrow v' = e^x$

Substituting into product rule formula:

$\frac{dy}{dx} = (8x^7+2)e^x + e^x(x^8+2x-3)$

Simplifying:

$\frac{dy}{dx} = e^x(8x^7+2+x^8+2x-3)$

$\frac{dy}{dx} = e^x(x^8+8x^7+2x-1)$

2. $y = 6xe^{2x} + 8\tan 3x$

$y = \underbrace{6xe^{2x}}_{\text{product rule + chain rule}} + \underbrace{8\tan(3x)}_{\text{chain rule.}}$

Let $f(x) = \underbrace{6x}_{u}\underbrace{e^{2x}}_{v}$

$u = 6x \Longrightarrow u' = 6$

$v = e^{2x} \Longrightarrow v' = 2e^{2x}$

getting this 2 involves chain rule, but if you solved by applying formula for $\frac{d}{dx}e^{f(x)}$, thats OK too!

Substituting into product rule formula:

$f'(x) = 6 \cdot e^{2x} + 2e^{2x} \cdot 6x$

$f'(x) = e^{2x}(6+12x)$

Let $g(x) = 8\tan(3x) = w$

Let $m = 3x. \Longrightarrow w = 8\tan(m)$

Setting up chain rule: $\frac{dw}{dx} = \frac{dw}{dm} \cdot \frac{dm}{dx}$

$w = 8\tan(m) \Longrightarrow \frac{dw}{dm} = 8\sec^2(m)$

$m = 3x \Longrightarrow \frac{dm}{dx} = 3$

So $\frac{dw}{dx} = 8\sec^2(m) \cdot 3 = 24\sec^2(3x)$

$g'(x) = 24\sec^2(3x)$

$\frac{dy}{dx} = f'(x) + g'(x) = e^{2x}(6+12x) + 24\sec^2(3x)$

3. $y = 3e^x \sin x \cos x$

$y = 3e^x \sin x \cos x$

Recall: $\sin(2x) = 2\sin(x)\cos(x) \Longrightarrow \sin(x)\cos(x) = \frac{\sin(2x)}{2}$

This is a helpful trigonometric identity.

$y = \frac{3e^x \sin(2x)}{2}$

$y = \underbrace{\frac{3e^x}{2}}_{u} \cdot \underbrace{\sin(2x)}_{v}$

$u = \frac{3e^x}{2} \Longrightarrow u' = \frac{3e^x}{2}$

$v = \sin(2x) \Longrightarrow v' = 2\cos(2x)$

This 2 is here because of chain rule.

Substituting into formula for product rule:

$$\frac{dy}{dx} = \frac{3e^x}{2} \cdot \sin(2x) + \frac{3e^x}{2} \cdot 2\cos(2x)$$

$$\frac{dy}{dx} = \frac{3e^x}{2}\left[\sin 2x + 2\cos 2x\right]$$

4. $y = (x + \sqrt{x})(3^x)$

$y = \underbrace{(x+\sqrt{x})}_{u}\underbrace{(3^x)}_{v}$

$u = x + \sqrt{x} \Longrightarrow u' = 1 + \frac{1}{2}x^{\frac{1}{2}-1} = 1 + \frac{1}{2}x^{-\frac{1}{2}} = 1 + \frac{1}{2\sqrt{x}}$

$v = 3^x \Longrightarrow v' = \underbrace{3^x \cdot \ln(3)}_{\text{This is by rule. The derivative of } a^x \text{ is } a^x \cdot \ln(a).}$

Substituting into formula for product rule:

$$\frac{dy}{dx} = \left(1 + \frac{1}{2\sqrt{x}}\right)(3^x) + (3^x \ln(3))(x + \sqrt{x})$$

You could algebraically simplify this further if required by the question. But for the sake of this worksheet, we leave it here.

$$\frac{dy}{dx} = 3^x\left[1 + \frac{1}{2\sqrt{x}} + x\ln(3) + \sqrt{x}\ln(3)\right]$$

## Quotient Rule

Find the derivatives of the following functions (with respect to x):
Source

1. $f(x) = \frac{4\cos(x)-1}{2+3e^x}$

$$f(x) = \frac{[4\cos(x)-1]}{[2+3e^x]} \begin{matrix} \swarrow u \\ \swarrow v \end{matrix}$$

Quotient Rule: $\frac{d}{dx}\frac{u}{v} = \frac{u'v - v'u}{v^2}$

u, v are functions of x

$u = 4\cos(x) - 1 \implies u' = -4\sin(x)$

$v = 2 + 3e^x \implies v' = 3e^x$

Substituting into formula for quotient rule:

$$f'(x) = \frac{(-4\sin x)(2+3e^x) - (3e^x)(4\cos x - 1)}{(2+3e^x)^2}$$

$$f'(x) = \frac{-8\sin x - 12e^x \sin x - 12e^x \cos x + 3e^x}{(2+3e^x)^2}$$

$$f'(x) = \frac{-8\sin x - 3e^x(4\sin x + 4\cos x - 1)}{(2+3e^x)^2}$$

Tip: Generally there is no need to expand the denominator unless explicitly asked to simplify.

2. $y = \frac{3x^2}{5x-7}$

$$y = \frac{3x^2}{5x-7} \begin{matrix} \swarrow u \\ \swarrow v \end{matrix}$$

$u = 3x^2 \implies u' = 6x$

$v = 5x - 7 \implies v' = 5$

Substituting into quotient rule formula:

$$\frac{dy}{dx} = \frac{(6x)(5x-7) - (5)(3x^2)}{(5x-7)^2} = \frac{30x^2 - 42x - 15x^2}{(5x-7)^2} = \frac{15x^2 - 42x}{(5x-7)^2}$$

$$\frac{dy}{dx} = \frac{3(5x^2 - 14)}{(5x-7)^2}$$

3. $y = \frac{e^x}{1+x}$

$$y = \frac{e^x}{1+x} \quad \begin{matrix} \leftarrow u \\ \leftarrow v \end{matrix}$$

$u = e^x \Longrightarrow u' = e^x$

$v = 1+x \Longrightarrow v' = 1$

Substituting into quotient rule formula:

$$\frac{dy}{dx} = \frac{e^x \cdot (1+x) - (1) \cdot (e^x)}{(1+x)^2}$$

$$\frac{dy}{dx} = \frac{e^x(1+x-1)}{(1+x)^2} = \frac{e^x \cdot x}{(1+x)^2} = \boxed{\frac{xe^x}{(1+x)^2}}$$

4. $y = \frac{\sin x}{x^2}$

$$y = \frac{\sin x}{x^2} \quad \begin{matrix} \leftarrow u \\ \leftarrow v \end{matrix}$$

$u = \sin x \Longrightarrow u' = \cos x$

$v = x^2 \Longrightarrow v' = 2x$

Substituting into quotient rule formula:

$$\frac{dy}{dx} = \frac{\cos x \cdot x^2 - 2x \cdot \sin x}{(x^2)^2}$$

$$\frac{dy}{dx} = \frac{x(x\cos x - 2\sin x)}{x^4} = \boxed{\frac{x\cos x - 2\sin x}{x^3}}$$