Review Sheet: Related Rates

Content Review

Overview

Related rates problems involve finding the rate at which one quantity is changing using information about other known quantities.

Most related rates problems you will encounter at the level of Math 31A/B will require roughly the following steps in their solutions:

1. **Relationship:** Establish a relationship between the variables.
   Relationships/Equations to consider: Trigonometric, Equations of area, surface area and volume of common geometrical shapes, mathematical similarity, equation of speed etc.

2. **Derivative**: Take a derivative of that relationship (often with respect to time). This step involves implicit differentiation. Review this concept if you need to before attempting these problems.

3. **Substitution:** Substitute known values (sometimes this step will also involve using the relationship established before to find some value that you need)

Helpful Equations

Geometry

$$\text{Area of a circle} = \pi r^2$$

$$\text{Volume of sphere} = \frac{4}{3}\pi r^2$$

$$\text{Surface area of sphere} = 4\pi r^2$$

$$\text{Volume of cone or pyramid} = \frac{1}{3} \times \text{base area} \times \text{height}$$

$$\text{Area of curved surface of cone} = \pi r \times \text{slant height}$$

$$\text{Arc length of circle} = r\theta \ \ (\theta \text{ in radians})$$

$$\text{Area of sector of circle} = \frac{r^2\theta}{2} \ \ (\theta \text{ in radians})$$

Trigonometry

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\cos^2\theta + \sin^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$\cot^2\theta + 1 = \csc^2\theta$$

Skip to Worked Problems.

Resources

Mathematical Similarity

- Khan Academy: Similarity (Geometric)
- Video: Similarity (Khan Academy, 10 min)
- Practice Problems: Similarity of Triangles (Basic)
- Practice Problems: Similarity of Shapes (Advanced)

Implicit Differentiation

- MSP Review Sheet on Implicit Differentiation
- Khan Academy: Implicit Differentiation
- Paul's Online Notes (With worked problems)

Related Rates

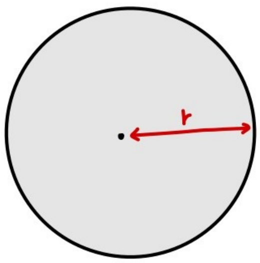
- Video: Introduction to Related Rates (Organic Chemistry Tutor, 10 min) - with worked examples
- Video: Related Rates - Conical Tank, Ladder & Shadow, Calculus (Organic Chemistry Tutor) - with extensive worked examples
- Practice Problems: UC Davis Math

Acknowlegement

Questions in the Worked Problems section of this sheet have been taken from external sources like Khan Academy and Paul's Online Notes. All solutions have been created independently.

Worked Problems

1. The radius $r(t)$ of a circle is increasing at the rate of 3 cm per second. At a certain instant t_0, the radius is 8 cm. What is the rate of change of the area $A(t)$ of the circle at that instant?



Given: $r(t)$ is increasing at 3 cm/sec.

increasing → derivative

sec → derivative w.r.t time.

$$\Rightarrow \frac{dr}{dt} = +3\text{ cm/sec}$$

+ because increasing

Goal: Want to find $\frac{dA}{dt}$ when $r = 8$ cm

$\frac{dA}{dt}$ → Rate of change of area

① Establish relationship between $A(t)$ and $r(t)$.

Area of circle $= \pi r^2$

$\Rightarrow A(t) = \pi \cdot [r(t)]^2$ which is the same as saying $A = \pi r^2$

② Find the derivative of this relationship by differentiating (implicit differentiation)

$$A = \pi r^2$$

$$\frac{dA}{dt} = \pi \cdot 2r \cdot \underbrace{\frac{dr}{dt}}$$

This term exists because of implicit differentiation. (we are taking derivative w.r.t. t)

③ Substitute the knowns

$$\frac{dA}{dt} = \pi \cdot 2\underbrace{r}_{8\text{ cm}} \cdot \underbrace{\frac{dr}{dt}}_{\frac{dr}{dt} = +3} \Rightarrow \frac{dA}{dt} = \pi \times 2 \times 8 \times 3 = 48\pi \text{ cm}^2/\text{sec}$$

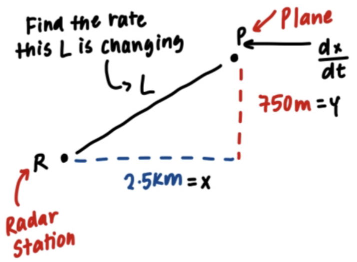
8 cm because we want to find $\frac{dA}{dt}$ when $r = 8$ cm

$\frac{dr}{dt} = +3$

2. A plane is 750 meters in the air flying parallel to the ground at a speed of 100 m/s and is initially 2.5 kilometers away from a radar station. At what rate is the distance between the plane and the radar station changing

   (a) initially

   (b) 30 seconds after it passes over the radar station



Given: Plane flies parallel to the ground (height above ground remains const.)

$\Rightarrow \underbrace{y = 750\,m}_{\text{const.}}$ so $\frac{dy}{dt} = 0$

Plane flies at speed 100 m/s. So horizontal distance covered by plane changes at the rate of 100 m/s.

$\Rightarrow \frac{dx}{dt} = -100\,m/s$ → As a convention, let's denote L.H.S. as negative and R.H.S. as positive

Goal: Find $\frac{dL}{dt}$ when (a) $t = 0$ and (b) 30 seconds after it crosses R.

Part (a):

1. Establish a relationship between x, y, t.

Notice that figure forms a right angled triangle. So use Pythagoras' theorem.

$L^2 = x^2 + y^2$ (y is a constant; $y = 750m$)

$L^2 = x^2 + 750^2$

2. Find the derivative of this relationship (implicit differentiation)

$L^2 = x^2 + 750^2$ (take derivative w.r.t. time)

$2L \cdot \frac{dL}{dt} = 2x \cdot \frac{dx}{dt} + 0 \xrightarrow{\text{Rearranging}} \frac{dL}{dt} = \frac{2x}{2L} \cdot \frac{dx}{dt} = \frac{x}{L} \cdot \frac{dx}{dt}$

these terms exist due to implicit differentiation.

$\Rightarrow \frac{dL}{dt} = \frac{x}{L} \cdot \frac{dx}{dt}$

3. Substitute the knowns

$\frac{dL}{dt} = \frac{x}{L} \cdot \frac{dx}{dt}$

initially, $x = 2.5km = 2500m$ and $y = 750m$

So $L = \sqrt{x^2 + y^2} = 2610.076627$

$\frac{dL}{dt} = \frac{2500}{2610.08} \cdot -100 = -95.78262852 = -95.8\,m/s$

Part (b):

1. Find how far away (in x distance) the plane is 30s after passing over R.

We measure x w.r.t. R.

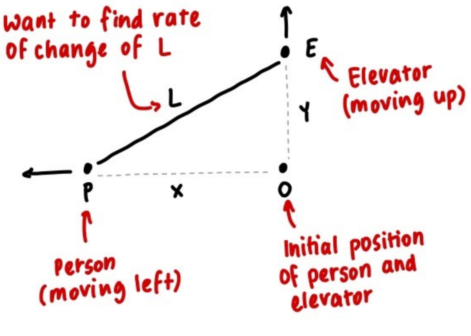
So when plane passes over R, $x = 0$. Plane travels at steady speed, so it travels $x = -100 \cdot 30 = -3000m$

2. Find $\frac{dL}{dt}$ when $x = -3000m$

$L = \sqrt{x^2 + y^2} = \sqrt{3000^2 + 750^2} \approx 3092.329219$

$\frac{dL}{dt} = \frac{x}{L} \cdot \frac{dx}{dt} = \frac{-3000}{3092.3} \times -100 = 97.01425001 = 97.0\,m/s$

3. Two people are at an elevator. At the same time one person starts to walk away from the elevator at a rate of 2 ft/sec and the other person starts going up in the elevator at a rate of 7 ft/sec. What rate is the distance between the two people changing 15 seconds later?



Given: Person walks away from initial position at 2 ft/sec

$\Rightarrow \frac{dx}{dt} = 2 \text{ ft/sec}$

Elevator goes up at 7 ft/sec

$\Rightarrow \frac{dy}{dt} = 7 \text{ ft/sec}$

Goal: Find $\frac{dL}{dt}$ when $t = 15$

① Establish a relationship between x, y, L.

Since the figure above shows a right angled triangle is formed, use Pythagoras' Theorem.

$$x^2 + y^2 = L^2$$

② Take derivative w.r.t. time (t) ⟶ Implicit differentiation

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 2L \cdot \frac{dL}{dt}$$

$$x \cdot \frac{dx}{dt} + y \cdot \frac{dy}{dt} = L \cdot \frac{dL}{dt}$$

$$\frac{dL}{dt} = \frac{x}{L} \cdot \frac{dx}{dt} + \frac{y}{L} \cdot \frac{dy}{dt}$$

③ Substitute values that we know

we know $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 7$.

When $t = 15$ sec. distance travelled by person $(x) = \underbrace{2}_{\text{speed}} \times \underbrace{15}_{\text{time}} = 30$ ft (distance)

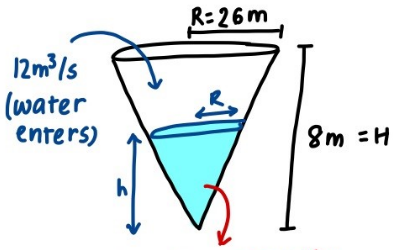
Similarly, distance travelled by elevator $(y) = 7 \times 15 = 105$ ft

So distance between person and elevator at $t = 15$ $(L) = \sqrt{x^2 + y^2} = \sqrt{30^2 + 105^2} = 15\sqrt{53}$ ft

Substituting values into equation:

$$\frac{dL}{dt} = \frac{30}{15\sqrt{53}} \cdot 2 + \frac{105}{15\sqrt{53}} \cdot 7 = \sqrt{53} = 7.280109889 = 7.28 \text{ ft/sec}$$

4. A tank of water in the shape of a cone is being filled with water at a rate of 12 m3/sec. The base radius of the tank is 26 meters and the height of the tank is 8 meters. At what rate is the depth of the water in the tank changing when the radius of the top of the water is 10 meters?



Notice that the water also forms a cone. The two shapes (water and container) are mathematically similar.

Given: water enters tank at the rate of $12 m^3/s$.

$\Longrightarrow \frac{dV}{dt} = 12 m^3/s$ (V denotes volume of water in tank)

Also, radius = 26m, height = 8m — dimensions of tank. Note that these are constants, and won't change.

Goal: Find rate of change of depth of water when the radius of the top of the water is 10m. $\left(\frac{dh}{dt} \text{ when } r=10\right)$

① Establish relationship between R, H, r, h, V

R, H are dimensions of the container. They are constant.

r, h are dimensions of the cone formed by water. They vary with time / are functions of time.

At any given point in time, due to the mathematical notion of "similarity", the following ratio/relationship will always hold:

$$\frac{H}{h} = \frac{R}{r} \Longrightarrow \frac{r}{h} = \frac{R}{H} = \frac{26}{8} = \frac{13}{4} \Longrightarrow \frac{r}{h} = \frac{13}{4} \Longrightarrow \boxed{r = \frac{13h}{4}}$$

Substituting dimensions of container for R, H. — Rearranging

Volume of a cone:

$$V = \frac{\pi r^2 h}{3} \Longrightarrow \text{substitute } r = \frac{13h}{4}$$

$$V = \frac{\pi h}{3} \cdot \frac{169h^2}{16} = \frac{169\pi h^3}{48} \Longrightarrow \boxed{V = \frac{169\pi h^3}{48}}$$

② Take derivative w.r.t. t (Implicit differentiation)

$$\frac{dV}{dt} = \frac{169\pi}{\cancel{48}\,16} \cdot \cancel{3}h^2 \cdot \frac{dh}{dt} \Longrightarrow \text{Substitute } h = \frac{4r}{13}$$

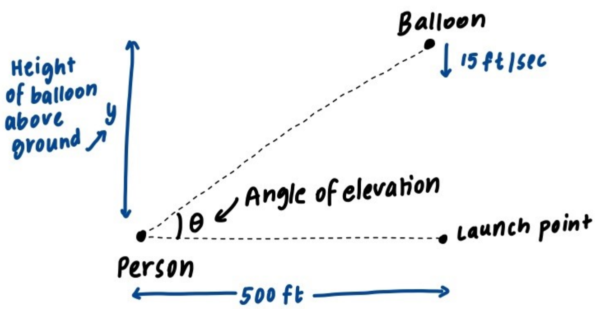
$$\frac{dV}{dt} = \frac{169\pi}{16} \cdot \frac{16r^2}{169} \cdot \frac{dh}{dt} \Longrightarrow \frac{dV}{dt} = \pi r^2 \cdot \frac{dh}{dt} \Longrightarrow \boxed{\frac{dh}{dt} = \frac{1}{\pi r^2} \cdot \frac{dV}{dt}}$$

③ Substitute the knowns

we know from before that $\frac{dV}{dt} = 12$; want to find $\frac{dh}{dt}$ when $r = 10$.

$$\frac{dh}{dt} = \frac{1}{\pi \cdot 10^2} \times 12 = 0.03819718634 = \boxed{0.0381 \text{ m/s}}$$

5. The angle of elevation is the angle formed by a horizontal line and a line joining the observer's eye to an object above the horizontal line. A person is 500 feet way from the launch point of a hot air balloon. The hot air balloon is starting to come back down at a rate of 15 ft/sec. At what rate is the angle of elevation, θ changing when the hot air balloon is 200 feet above the ground?



Given: Balloon is coming down at 15 ft/sec.

$\Longrightarrow \frac{dy}{dt} = -15$ **Note:** we take "down" to be negative by convention.

Also, distance between person & launch point $(x) = 500$ ft.

Goal: Find $\frac{d\theta}{dt}$ when $y = 200$ ft.

① Establish a relationship between x, y, θ.

Using trigonometry, $\tan\theta = \frac{y}{x} = \frac{y}{500} \Longrightarrow \tan\theta = \frac{y}{500}$

② Take derivative w.r.t. t (implicit differentiation)

$\tan\theta = \frac{y}{500}$

$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{500} \cdot \frac{dy}{dt} \Longrightarrow \frac{d\theta}{dt} = \frac{1}{500} \cdot \frac{dy}{dt} \cdot \frac{1}{\sec^2\theta}$

Rearranging

Recall that $\sec\theta = \frac{1}{\cos\theta}$, so $\frac{1}{\sec\theta} = \cos\theta$

And thus, $\frac{1}{\sec^2\theta} = \cos^2\theta$

$$\frac{d\theta}{dt} = \frac{1}{500} \cdot \frac{dy}{dt} \cdot \cos^2\theta$$

③ Substitute known values

when $y = 200$ ft, $\tan\theta = \frac{200}{500} \Longrightarrow \theta = \tan^{-1}\left(\frac{2}{5}\right)$

$$\frac{d\theta}{dt} = \frac{1}{500} \cdot \frac{dy}{dt} \cdot \cos^2\theta = \frac{1}{500} \times -15 \times \cos^2\left(\tan^{-1}\left(\frac{2}{5}\right)\right) = -\frac{3}{116} = -0.0258620689 7$$
$$= -0.0259 \text{ rad/sec}$$

BONUS: NO CALCULATOR VERSION

Want to compute $\cos^2\left(\tan^{-1}\left(\frac{2}{5}\right)\right)$ without a calculator.

Let $\tan^{-1}\left(\frac{2}{5}\right) = \theta$. So we want to compute $\cos^2\theta$. Rearranging, it is equivalent to say $\tan(\theta) = \frac{2}{5}$

Using the trigonometric identity: $\tan^2\theta + 1 = \sec^2\theta$

we know $\tan\theta = \frac{2}{5}$, so $\left(\frac{2}{5}\right)^2 + 1 = \sec^2\theta = \frac{1}{\cos^2\theta} \Longrightarrow \frac{4}{25} + 1 = \frac{1}{\cos^2\theta} \Longrightarrow \frac{29}{25} = \frac{1}{\cos^2\theta}$

$\Longrightarrow \boxed{\cos^2\theta = \frac{25}{29}}$ Plug this back into above equation and simplify manually.