Review Sheet: Logarithmic & Exponential Functions

Content Review

Overview

Introduction to Exponential & Logarithmic Functions

$e \approx 2.718$ is a number, just like π. It is most intuitive to think about e^x in comparison to another function like 2^x since you are used to thinking about 2 as a number.
Common mistake: e is NOT a variable, don't treat it like one.
We know that

$$\begin{aligned} 2^x = 1 &\implies x = 0 \\ 2^x = 2 &\implies x = 1 \\ 2^x = 4 &\implies x = 2 \\ 2^x = 10 &\implies x = ? \end{aligned}$$

To solve the last one, we need to reverse the 'exponentiation' process, using a function called a **logarithm**. This does the opposite of 2^x, and essentially divides x by the 'base' (here 2) multiple times and keeps track of how many times it divides.
Logarithms are written as the following:

$$\log_2(10) = x$$

And are read as 'log base 2 of 10 is equal to x'.
For example:

$$\log_2(8) = 3 \implies 2^3 = 8$$

Similarly for e^x:

$$\begin{aligned} e^x = 1 &\implies x = 0 \\ e^x = e &\implies x = 1 \\ e^x = e^2 &\implies x = 2 \\ e^x = 10 &\implies x = ? \end{aligned}$$

Now we know that to solve the last one, we use $\log_e(10) = x$.
Revisiting the examples above:

$$\begin{aligned} e^x = 1 &\implies x = \ln(1) = 0 \\ e^x = e &\implies x = \ln(e) = 1 \\ e^x = e^2 &\implies x = \ln(e^2) = 2 \\ e^x = 10 &\implies x = \ln(10) \end{aligned}$$

In math, this function $\log_e$ is used so often, that it has a special name: 'natural logarithm' or $\ln(x)$.

Logarithmic Function Rules

Similar to how you have learned rules for exponents, there exist the following rules for logarithmic functions which are used to significantly simplify logarithmic expressions.

$$\begin{aligned} \ln(a) + \ln(b) &= \ln(a \cdot b) \\ \ln(a) - \ln(b) &= \ln(a - b) \\ a \ln(b) &= \ln(b^a) \end{aligned}$$

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Calculus of Exponential & Logarithmic Functions

We will take the following as a rule since it's origins are beyond the scope of MATH 31A/B:

$$\frac{d}{dx}2^x = \ln(2) \cdot 2^x$$

More generally,

$$\frac{d}{dx}a^x = \ln(a) \cdot a^x \text{ for some real constant } a$$

Following this rule, we see that

$$\frac{d}{dx}e^x = \ln(e) \cdot e^x = 1 \cdot e^x \text{ because we know that } \ln(e) = 1$$

The following can similarly be taken as rules. The last 2 formulae originate from applications of chain rule.

$$\begin{aligned}
\int e^x dx &= e^x + c \\
\frac{d}{dx}\ln(x) &= \frac{1}{x} \implies \int \frac{1}{x}dx = \ln x \\
\frac{d}{dx}\ln(f(x)) &= \frac{f'(x)}{f(x)} \\
\frac{d}{dx}e^{f(x)} &= f'(x) \cdot e^{f(x)}
\end{aligned}$$

Important Note: Logarithms will only accept positive values. There can never be an expression like $\ln(0)$ because no constant raised to a power can equal 0 except 0 itself. So $\ln 0$ is undefined.

Resources

Logarithmic & Exponential Functions

- Video: Introduction to Related Rates (Organic Chemistry Tutor, 10 min) - with worked examples
- Worksheet: Logarithmic Function (with Solutions)
- Explainer: Exponentials & Logarithms (with Worked Examples)
- Video: Logarithms & Exponentials Review (Organic Chemistry Tutor, 1h20min) - detailed concept breakdown & Explanations
- Video: Derivatives of Exponentials & Logarithms (Organic Chemistry Tutor, 42min)
- Video:Logarithms Explained Rules & Properties, Graphing etc (Organic Chemistry Tutor, 1h23min)

Acknowlegement

Questions in the Worked Problems section of this sheet have been taken from external sources that have been linked where appropriate. All solutions have been created independently.

Worked Problems

Equations in Logarithms and Exponentials

Find all solutions to the following equations. If there is no solution, explain clearly why.
Source

1. $12 - 4e^{7+3x} = 7$

$$12 - 4e^{7+3x} = 7$$

$\Longrightarrow 12 - 4e^{7+3x} - 7 = 7 - 7$

$\Longrightarrow 12 - 7 \underbrace{- 4e^{7+3x}} = 0$

↳ Adding this term to both sides

$\Longrightarrow 5 = 4e^{7+3x}$ (Dividing by 4 on both sides)

$\Longrightarrow \frac{5}{4} = e^{7+3x}$ OR $e^{7+3x} = \frac{5}{4}$ (Taking natural log on both sides)

$\Longrightarrow \underline{\ln}\left(\underline{e}^{7+3x}\right) = \ln\left(\frac{5}{4}\right)$

since ln and e are inverses of each other, they cancel out

$\Longrightarrow 7 + 3x = \ln\left(\frac{5}{4}\right)$ (Subtracting 7 from both sides)

$\Longrightarrow 3x = \ln\left(\frac{5}{4}\right) - 7$ (Dividing by 3 on both sides)

$\Longrightarrow \boxed{x = \frac{1}{3}\left(\ln\left(\frac{5}{4}\right) - 7\right)}$ ⟶ if you are not allowed to use a calculator, it's sufficient to leave your answer here.

2. $2t - te^{6t-1} = 0$

$2t - te^{6t-1} = 0$ Goal: Isolate t as a variable.

$\Rightarrow t(2 - e^{6t-1}) = 0$ either $t = 0$ or $2 - e^{6t-1} = 0$

$2 - e^{6t-1} = 0$

$\Rightarrow 2 = e^{6t-1}$ (Taking natural log on both sides)

$\Rightarrow \ln(2) = \ln(e^{6t-1})$ (e and ln cancel out as they are inverses)

$\Rightarrow \ln(2) = 6t - 1$ (Add 1 to both sides)

$\Rightarrow \ln(2) + 1 = 6t$ (Divide by 6 on both sides)

$\Rightarrow \frac{\ln(2)+1}{6} = t$

Solution: $t = 0$ OR $t = \frac{\ln(2)+1}{6}$

3. $2\log(x) - \log(7x - 1) = 0$

$2\log(x) - \log(7x-1) = 0$ Goal: Isolate x as a variable.

$\Rightarrow 2\log(x) = \log(7x-1)$ Recall: $a\log(x) = \log(x^a)$

$\Rightarrow \log(x^2) = \log(7x-1)$ (If $\log(a) = \log(b)$, then $a = b$)

$\Rightarrow x^2 = 7x - 1$ (Moving all terms to one side)

$\Rightarrow x^2 - 7x + 1 = 0$ (Solve as a quadratic)

Applying quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{7 \pm \sqrt{(-7)^2 - 4\times1\times1}}{2\times1} = \frac{7 \pm \sqrt{49-4}}{2} = \frac{7 \pm \sqrt{45}}{2}$$

Recall that a logarithm to any base (e.g. $\log_{10}(x)$ OR $\log_5(x)$ OR $\log_{20}(x)$) can only take positive, non-zero values for x.

This means that $x > 0$.

$\frac{7-\sqrt{45}}{2} < 0 \longrightarrow$ invalid solution for x.

$\therefore x = \frac{7+\sqrt{45}}{2}$

4. $\ln(y-1) = 1 + \ln(3y+2)$

$\ln(y-1) = 1 + \ln(3y+2)$ Goal: Isolate y as a variable.

$\implies \ln(y-1) - \ln(3y+2) = 1$ Recall: $\ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right)$

$\implies \ln\left(\frac{y-1}{3y+2}\right) = 1$ (Raising to e on both sides)

$\implies e^{\ln\left(\frac{y-1}{3y+2}\right)} = e^1$ (Recall: $e^{\ln(x)} = x$)

$\implies \frac{y-1}{3y+2} = e$ (multiplying both sides by $3y+2$)

$\implies y-1 = e(3y+2)$ (opening bracket)

$\implies y-1 = 3ey + 2e$ (Grouping terms containing y and constants.)

$\implies y - 3ey = 2e+1$ (Factoring y)

$\implies y(1-3e) = 2e+1$ (Dividing on both sides by $1-3e$)

$\implies \boxed{y = \frac{2e+1}{1-3e}}$

Derivatives of Logarithms and Exponentials

Differentiate the given function.
Source

1. $f(x) = 2e^x - 8^x$

$f(x) = 2e^x - 8^x$

Goal: Find $f'(x)$

$f'(x) = \frac{d}{dx}(2e^x) - \frac{d}{dx}(8^x)$

Note: $\frac{d}{dx} 2^x = 2^x \cdot \ln(2)$

$\frac{d}{dx} e^x = e^x$

$f'(x) = 2e^x - 8^x \ln(8)$

2. $y = \ln(x^2 + 1)$

$y = \ln(x^2+1)$

Goal: Find $\frac{dy}{dx}$

METHOD #1: Exponentiate and then take derivative.

$\Longrightarrow e^y = e^{\ln(x^2+1)}$ (e and ln are inverses so they cancel out)

$\Longrightarrow e^y = x^2+1$ (Differentiate implicitly w.r.t. x)

$\Longrightarrow \frac{d}{dx}(e^y) = \frac{d}{dx}(x^2+1)$

$\Longrightarrow e^y \frac{dy}{dx} = 2x$ (Dividing by e^y on both sides)

$\Longrightarrow \frac{dy}{dx} = \frac{2x}{e^y}$ (Substitute $e^y = x^2+1$)

$\Longrightarrow \frac{dy}{dx} = \frac{2x}{x^2+1}$

METHOD #2: Use Chain Rule

$y = \ln(x^2+1)$

$\Longrightarrow \frac{dy}{dx} = \frac{1}{x^2+1} \cdot \frac{d}{dx}(x^2+1)$

$\Longrightarrow \frac{dy}{dx} = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}$

3. $y = x^5 - e^x \ln x$

$y = x^5 - e^x \ln(x)$

Goal: Find $\frac{dy}{dx}$

Recall: Product Rule

$$\frac{d}{dx}(u \cdot v) = u'v + v'u$$

where u, v are expressions in terms of x.

$$\frac{dy}{dx} = \underbrace{\frac{d}{dx}(x^5)}_{\text{Normal differentiation}} - \underbrace{\frac{d}{dx}\left(e^x \ln(x)\right)}_{\text{Product Rule}}$$

$\frac{d}{dx}(x^5) = 5x^4$

$\frac{d}{dx}\left(e^x \ln(x)\right)$: Let $u = e^x$ and $v = \ln(x)$

Then $u' = e^x$ and $v' = \frac{1}{x}$

Applying product Rule:

$$\frac{d}{dx}(e^x \ln(x)) = e^x \cdot \ln(x) + \frac{1}{x} \cdot e^x = e^x\left(\ln(x) + \frac{1}{x}\right)$$

Substituting back into main expression:

$$\frac{dy}{dx} = 5x^4 + e^x\left(\ln(x) + \frac{1}{x}\right)$$

4. $y = e^{\ln(x)} + \ln(e^x)$

$y = e^{\ln(x)} + \ln(e^x)$

Goal: Find $\frac{dy}{dx}$

Trick: simplify before taking derivative.

Recall: e and $\ln$ are inverses of each other.

So, $e^{\ln(x)} = x$ and $\ln(e^x) = x$

Thus, the above expression simplifies to:

$\Rightarrow y = e^{\ln(x)} + \ln(e^x)$

$\Rightarrow y = x + x = 2x$

Now take derivative normally.

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(2x) = 2$

5. $y = (x^2 + 1)^x$

$y = (x^2+1)^x$ Goal: Find $\frac{dy}{dx}$

Trick: take natural log on both sides before differentiating

$\Rightarrow \ln(y) = \ln\left((x^2+1)^x\right)$ Recall: $\ln(a^b) = b\ln(a)$

$\Rightarrow \ln(y) = x\ln(x^2+1)$

Now differentiating implicitly w.r.t x:

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \underbrace{\frac{d}{dx}\left(x\ln(x^2+1)\right)}$

Apply product rule to take derivative

$u = x \qquad v = \ln(x^2+1)$

$u' = 1 \qquad v' = \frac{1}{x^2+1} \cdot \underbrace{\frac{d}{dx}(x^2+1)} = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}$

↳ Due to chain Rule

Substituting into formula for product rule:

$\frac{d}{dx}\left(x\ln(x^2+1)\right) = 1 \cdot \ln(x^2+1) + x \cdot \frac{2x}{x^2+1}$

$= \ln(x^2+1) + \frac{2x^2}{x^2+1}$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln(x^2+1) + \frac{2x^2}{x^2+1}$ (multiplying by y on both sides)

$\Rightarrow \frac{dy}{dx} = y\left(\ln(x^2+1) + \frac{2x^2}{x^2+1}\right)$ (Rewriting y in terms of x)

$\Rightarrow \boxed{\frac{dy}{dx} = (x^2+1)^x\left(\ln(x^2+1) + \frac{2x^2}{x^2+1}\right)}$ → For the purposes of this worksheet, we leave this answer here. Generally, there is no need to expand a complex expression like this unless explicitly stated.

6. $y = \ln(\sin(x^2))$

$y = \ln(\sin(x^2))$ Goal: Find $\frac{dy}{dx}$

Trick: Apply exponentiation to both sides before taking a derivative.

$\Longrightarrow e^y = e^{\ln(\sin(x^2))}$ (e and ln are inverses of each other)

$\Longrightarrow e^y = \sin(x^2)$

Differentiate implicitly w.r.t x

$$\underbrace{\frac{d}{dx}(e^y)}_{\text{Implicit Differentiation}} = \underbrace{\frac{d}{dx}(\sin(x^2))}_{\text{Chain Rule.}}$$

$$\Longrightarrow e^y \cdot \frac{dy}{dx} = \cos(x^2) \cdot \overbrace{\frac{d}{dx}}^{\text{Because of Chain Rule}}(x^2)$$

$\Longrightarrow e^y \cdot \frac{dy}{dx} = \cos(x^2) \cdot 2x$ (Dividing on both sides by e^y)

$\Longrightarrow \frac{dy}{dx} = \frac{\cos(x^2) \cdot 2x}{e^y}$ (write y in terms of x)

$\Longrightarrow \frac{dy}{dx} = \frac{\cos(x^2) \cdot 2x}{e^{\ln(\sin(x^2))}}$ (e and ln are inverses, they cancel out)

$\Longrightarrow \frac{dy}{dx} = \frac{\cos(x^2) \cdot 2x}{\sin(x^2)} = 2x \cdot \frac{\cos(x^2)}{\sin(x^2)} = \frac{2x}{\tan(x^2)}$

$\boxed{\frac{dy}{dx} = \frac{2x}{\tan(x^2)}}$

can also leave answer at this stage unless explicitly asked to simplify to the maximum.