Review Sheet: Hyperbolic Functions

Content Review

Overview

Basic Definitions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\cosh x = \frac{e^x + e^{-x}}{2}$$

For the remaining hyperbolic functions, apply the same relationships that you already know for basic trigonometry:

$$\begin{aligned}\tanh x &= \frac{\sinh x}{\cosh x}\\ \coth x &= \frac{\cosh x}{\sinh x}\\ \mathrm{sech} x &= \frac{1}{\cosh x}\\ \mathrm{csch} x &= \frac{1}{\sinh x}\end{aligned}$$

Basic Properties
Notice that the hyperbolic functions have effectively the same properties as their trigonometric counterparts

$$\sinh(-x) = -\sinh x$$
$$\cosh(x) = \cosh x$$
$$\tanh(-x) = -\tanh x$$
$$\coth(-x) = -\coth x$$
$$\mathrm{sech}(-x) = \mathrm{sech} x$$
$$\mathrm{csch}(-x) = -\mathrm{csch} x$$

Identities
Note that the signs for these identities are not the same as trigonometric identities. Be very careful of which signs you use.

$$\cosh^2 x - \sinh^2 x = 1$$
$$\tanh^2 x + \mathrm{sech}^2 x = 1$$

Additive identities for hyperbolic functions can also be derived in a similar way to trigonometric functions.

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

Double angle identities for hyperbolic functions are as follows:

$$\sinh 2x = 2 \sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

Inverse Hyperbolic Functions

$$\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$$
$$\cosh^{-1}(x) = \ln(x + \sqrt{x^2-1})$$
$$\tanh^{-1}(x) = \frac{1}{2}\ln\frac{1+x}{1-x}$$

An example of deriving these inverses is shown below:
Note: $y = \sinh x \iff \sinh^{-1} y = x$

Thus,

$$y = \sinh x = \frac{e^x - e^{-x}}{2} \longleftrightarrow 2y = e^x - e^{-x}$$

$$\begin{aligned}
&\implies && 2y = e^x - \frac{1}{e^x} \\
&\implies && 2y = \frac{e^{2x}-1}{e^x} \\
&\implies && 2ye^x = e^{2x} - 1 \\
&\implies && e^{2x} - 2ye^x - 1 = 0 && \text{Solve this by treating } e^x \text{ as a variable and applying the quadratic formula} \\
&\implies && e^x = \frac{2y \pm \sqrt{4y^2+4}}{2} = y \pm \sqrt{y^2+1} && \text{Since log only takes positive numbers, choose the positive sign} \\
&\implies && x = \ln(y + \sqrt{y^2+1}) \\
&\implies && \sinh^{-1}(y) = \ln(y + \sqrt{y^2+1})
\end{aligned}$$

Derivatives of Hyperbolic Functions

$$\begin{aligned}
\frac{d}{dx}\sinh x &= \cosh x \\
\frac{d}{dx}\cosh x &= \sinh x \\
\frac{d}{dx}\tanh x &= \text{sech}^2 x \\
\frac{d}{dx}\coth x &= \text{csch}^2 x \\
\frac{d}{dx}\text{sech} x &= \tanh x \cdot \text{sech} x \\
\frac{d}{dx}\text{csch} x &= -\text{coth} x \cdot \text{csch} x
\end{aligned}$$

Resources

Related Rates

- Content: Hyperbolic Functions (Whitman Edu)
- Content: Derivatives of Hyperbolic Functions (Paul's Online Notes)
- Content: Hyperbolic Functions Explained (MathCenter UK)
- Video: Hyperbolic Trig Functions, Basic Introduction (Organic Chemistry Tutor, 10 min)
- Video: Graphs of Hyperbolic Trig Functions (Organic Chemistry Tutor, 23 min)
- Video: Evaluating Hyperbolic Trig Functions (Organic Chemistry Tutor, 9min)

- Video: Hyperbolic Trig Identities (Organic Chemistry Tutor, 10 min)
- Practice Problems: Hyperbolic Functions
- Practice Problems: Hyperbolic Function (Calculus)

Acknowledgement

Questions in the Worked Problems section of this sheet have been taken from external sources that have been linked where appropriate. All solutions have been created independently.

Worked Problems

Find the derivative of the following functions:
(Source)

1.

$$y = \tanh x$$

$y = \tanh x$ **Goal:** Find $\frac{dy}{dx}$

Note: You can just remember the answer. But treat this problem as an exercise in what to do if you don't know the answer.

$$y = \tanh(x) = \frac{\sinh(x) \leftarrow u}{\cosh(x) \leftarrow v}$$

Apply Quotient Rule:

$u = \sinh(x)$ $v = \cosh(x)$

$u' = \cosh(x)$ $v' = \sinh(x)$

By Quotient Rule:

$$\frac{d}{dx}\frac{\sinh(x)}{\cosh(x)} = \frac{u'v - v'u}{v^2} = \frac{\cosh^2(x) - \sinh^2 x}{\cosh^2 x}$$

Remember hyperbolic identity:

$\cosh^2 x - \sinh^2 x = 1$

Simplifying using identity:

$$\frac{d}{dx}\frac{\sinh(x)}{\cosh(x)} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2(x) \longrightarrow \frac{dy}{dx} = \operatorname{sech}^2 x$$

2.

$$y = e^x \cosh x$$

$y = e^x \cosh x$ **Goal:** Find $\frac{dy}{dx}$

Need to apply Product Rule:

$u = e^x$ $v = \cosh x$

$u' = e^x$ $v' = \sinh x$

Applying product Rule:

$$\frac{dy}{dx} = e^x \cosh x + e^x \sinh x = e^x(\cosh x + \sinh x)$$

3.

$$y = \cosh x^3$$

$y = \cosh(x^3)$ **Goal:** Find $\frac{dy}{dx}$

Need to apply chain Rule.

Let $y = \cosh(a) \longrightarrow \frac{dy}{da} = \sinh(a)$

and $a = x^3 \longrightarrow \frac{da}{dx} = 3x^2$

Applying chain Rule:

$$\frac{dy}{dx} = \frac{dy}{da} \cdot \frac{da}{dx}$$

$$\frac{dy}{dx} = \sinh(a) \cdot 3x^2 = \sinh(x^3) \cdot 3x^2$$

4.

$$y = \cosh \ln x$$

$y = \cosh(\ln(x))$ **Goal:** Find $\frac{dy}{dx}$

Need to apply chain Rule.

Let $y = \cosh(a) \longrightarrow \frac{dy}{da} = \sinh(a)$

and $a = \ln(x) \longrightarrow \frac{da}{dx} = \frac{1}{x}$

Applying chain Rule:

$$\frac{dy}{dx} = \frac{dy}{da} \cdot \frac{da}{dx}$$

$$\frac{dy}{dx} = \sinh(a) \cdot \frac{1}{x} = \sinh(\ln(x)) \cdot \frac{1}{x}$$

5.

$$y = \sin^{-1}(\tanh x^2)$$

$y = \sin^{-1}(\tanh(x^2))$ Goal: Find $\frac{dy}{dx}$.

Note: It might be tempting to directly apply chain rule. But unless you know the derivative of $\sin^{-1}(x)$, this can get messy. We will use a much cleaner method involving implicit differentiation.

$$y = \sin^{-1}(\tanh(x^2)) \implies \sin(y) = \tanh(x^2)$$

Now take derivative w.r.t. x on both sides.

$$\cos(y) \cdot \underbrace{\frac{dy}{dx}}_{\text{Comes from implicit differentiation}} = \operatorname{sech}^2(x^2) \cdot \underbrace{2x}_{\text{comes from chain rule.}}$$

Rearranging in terms of $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\operatorname{sech}^2(x^2) \cdot 2x}{\cos(y)}$$

substituting y in terms of x.

$$\boxed{\frac{dy}{dx} = \frac{\operatorname{sech}^2(x^2) \cdot 2x}{\cos(\sin^{-1}(\tanh(x^2)))}}$$

Find the following antiderivatives:
(Source)

1.

$$\int \sinh^4(x) \cdot \cosh x dx$$

$$\int \sinh^4(x) \cdot \cosh(x)\, dx$$

Strategy: Try to apply u substitution. Notice that cosh(x) is a derivative of sinh(x).
So, if we set sinh(x) = u then something might cancel out.

$$\text{let } u = \sinh(x) \implies \frac{du}{dx} = \cosh(x) \implies dx = \frac{1}{\cosh(x)}\, du$$

Performing substitution:

$$\int \underbrace{\sinh^4(x)}_{u^4} \cdot \cosh(x)\, \underbrace{dx}_{\frac{1}{\cosh x} du} = \int u^4 \cdot \cancel{\cosh(x)} \cdot \frac{1}{\cancel{\cosh(x)}}\, du = \int u^4\, du$$

Integrating:

$$\int u^4\, du = \frac{u^5}{5} + c$$

we want final answer in terms of x not u. Substitute u = sinh(x)

$$= \boxed{\frac{\sinh^5(x)}{5} + c}$$

2.

$$\int e^x \cdot \cosh e^x \cdot \sinh e^x dx$$

$$\boxed{\int e^x \cosh(e^x) \sinh(e^x)\, dx}$$

Strategy: Notice that $\frac{d}{dx} \sinh(x) = \cosh(x)$ and by chain rule, $\frac{d}{dx} \sinh(e^x) = \cosh(e^x) \cdot e^x$

Setting up u-substitution:

$$u = \sinh(e^x) \implies \frac{du}{dx} = \cosh(e^x) \cdot e^x \implies dx = \frac{1}{e^x \cosh(e^x)} du$$

Performing u-substitution:

$$\int e^x \cosh(e^x) \underbrace{\sinh(e^x)}_{u} \underbrace{dx}_{\frac{1}{\cosh(e^x) e^x} du} = \int \cancel{e^x \cosh(x)} \cdot u \cdot \frac{1}{\cancel{e^x \cosh(x)}} du = \int u\, du$$

Integrating:

$$\int u\, du = \frac{u^2}{2} + C$$

substitute u in terms of x $(u = \sinh(e^x))$

$$= \boxed{\frac{\sinh^2(e^x)}{2} + C}$$

3.

$$\int \frac{\sinh x}{1+\cosh x}$$

$$\boxed{\int \frac{\sinh x}{1+\cosh x}}$$

Strategy: Notice that $\frac{d}{dx}(1+\cosh x) = \sinh x$ so if we use $u = 1+\cosh x$ then some terms might cancel out.

Let $u = 1+\cosh x \implies \frac{du}{dx} = \sinh x \implies du = \sinh x \, dx \implies dx = \frac{1}{\sinh x} du$

Applying u-substitution:

$$\int \frac{\sinh x}{\underbrace{1+\cosh x}_{u}} \underbrace{dx}_{\frac{1}{\sinh x} du} = \int \frac{\cancel{\sinh x}}{u} \cdot \frac{1}{\cancel{\sinh x}} du = \int \frac{1}{u} du$$

Integrating:

Resubstitute $u = 1+\cosh x$

$$\int \frac{1}{u} du = \ln|u| + C$$

$$= \boxed{\ln|1+\cosh x| + C}$$

Answer the following questions
(Source and Source)

1. At what point on the curve $y = \cosh x$ does the tangent to the curve have a slope of 1?

Slope of the tangent is given by the derivative.
To rephrase the question: at what point on the curve $y = \cosh x$ is the derivative equal to 1?

- Find the derivative of $y = \cosh x$

$$y = \cosh x$$
$$\frac{dy}{dx} = \sinh x$$

- Solve the equation $\frac{dy}{dx} = 1$

$$\frac{dy}{dx} = \sinh x = 1$$

Recall from definition of hyperbolic functions that $\sinh x = \frac{e^x - e^{-x}}{2}$

$$\Longrightarrow \sinh(x) = \frac{e^x - e^{-x}}{2} = 1$$

$$\Longrightarrow e^x - e^{-x} = 2$$

$$\Longrightarrow e^x - \frac{1}{e^x} = 2$$

$$\Longrightarrow \frac{e^{2x} - 1}{e^x} = 2$$

$$\Longrightarrow e^{2x} - 1 = 2e^x$$

$$\Longrightarrow e^{2x} - 2e^x - 1 = 0$$

$\longrightarrow$ Notice that this is a quadratic in e^x.
i.e. if $m = e^x$, then $e^{2x} - 2e^x - 1 = 0$ becomes
$m^2 - 2m - 1 = 0$

Recall: Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solve using the quadratic formula:

$$e^x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$$

$$e^x = \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$$

Remember that for all real values of x, e^x is always positive, so discard the $(-)$ solution.

$$e^x = 1 + \sqrt{2} \Longrightarrow x = \ln(1 + \sqrt{2})$$

2. Show that

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

.

Want to show that $\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$

Let $x = \sinh(y)$

Thus $\sinh^{-1}(x) = y \implies$ want to find an expression for y.

$\therefore \sinh(y) = \frac{e^y - e^{-y}}{2} = x \longrightarrow$ solve for y

$\implies e^y - e^{-y} = 2x$

$\implies e^y - \frac{1}{e^y} = 2x$

$\implies e^{2y} - 1 = 2xe^y$

$\implies e^{2y} - 2xe^y - 1 = 0 \longrightarrow$ Notice that this is also a quadratic in e^y. Solve using quadratic formula.

$$e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(-1)}}{2(1)}$$

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = \frac{2x \pm \sqrt{4(x^2+1)}}{2} = \frac{\cancel{2}x \pm \cancel{2}\sqrt{x^2+1}}{\cancel{2}} = x \pm \sqrt{x^2+1}$$

Since e^y is always positive for all real values of y, we choose $(+)$ answer.

more intuition on this:

$$x = \sqrt{x^2} \implies x < \sqrt{x^2+1} \implies x - \sqrt{x^2+1} < 0$$

$\therefore e^y = x + \sqrt{x^2+1}$

$y = \ln(x + \sqrt{x^2+1})$

$\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$

3. Find expressions for $\cosh^{-1} x$ and $\tanh^{-1} x$

Want to find expression for $\cosh^{-1}(x)$.

Note: will be using similar method to previous question.

$x = \cosh(y) \implies x = \dfrac{e^y + e^{-y}}{2}$

$\cosh^{-1}(x) = y \longrightarrow$ want to find an expression for y.

$$\therefore \frac{e^y + e^{-y}}{2} = x$$

$$\implies e^y + e^{-y} = 2x$$

$$\implies e^y + \frac{1}{e^y} = 2x$$

$$\implies \frac{e^{2y} + 1}{e^y} = 2x$$

$$\implies e^{2y} + 1 = 2xe^y$$

$\implies e^{2y} - 2xe^y + 1 = 0 \longrightarrow$ Solve as a quadratic in e^y using the quadratic formula.

$$e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(1)}}{2(1)}$$

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = \frac{\cancel{2}x \pm \cancel{2}\sqrt{x^2 - 1}}{\cancel{2}} = x \pm \sqrt{x^2 - 1}$$

We choose the positive sign

$$e^y = x + \sqrt{x^2 - 1}$$

$$y = \ln(x + \sqrt{x^2 - 1})$$

$$\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1})$$

Want to find an expression for $\tanh^{-1}(x)$

$x = \tanh(y) \implies x = \dfrac{\sinh(y)}{\cosh(y)} = \dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$

$\tanh^{-1}(x) = y \longrightarrow$ want to find an expression for y.

$$x = \frac{e^{y}-e^{-y}}{e^{y}+e^{-y}}$$

$$\implies x\left(e^{y}+e^{-y}\right) = e^{y}-e^{-y}$$

$$\implies x\left(e^{y}+\frac{1}{e^{y}}\right) = e^{y}-\frac{1}{e^{y}}$$

$$\implies x\left(\frac{e^{2y}+1}{e^{y}}\right) = \frac{e^{2y}-1}{e^{y}}$$

$$\implies x\left(e^{2y}+1\right) = e^{2y}-1$$

$$\implies xe^{2y}+x = e^{2y}-1$$

$$\implies xe^{2y}-e^{2y}+x+1 = 0$$

$$\implies e^{2y}(x-1)+(x+1) = 0$$

Solve this as a quadratic in e^{y}

$$\implies e^{2y}(x-1) = -(x+1)$$

$$\implies e^{2y} = \frac{-(x+1)}{x-1}$$

$$\implies e^{2y} = \frac{1+x}{1-x}$$

Absolute value because logarithms only take positive values.

$$\implies 2y = \ln\left|\frac{1+x}{1-x}\right|$$

$$\implies y = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| \implies \boxed{\tanh^{-1}(x) = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|}$$

4. Find all values of x that satisfy

$$\sinh 2x - 3\tanh x - \sinh x = 0$$

- Use double angle formula to write expression in terms of $\sinh(x)$ and $\cosh(x)$

<u>Recall:</u> $\sinh(2x) = 2\sinh(x)\cosh(x)$

$$\sinh(2x) - 3\tanh(x) - \sinh(x) = 0$$

$$\Longrightarrow 2\sinh(x)\cosh(x) - \frac{3\sinh(x)}{\cosh(x)} - \sinh(x) = 0$$

$$\Longrightarrow \frac{2\sinh(x)\cosh^2(x)}{\cosh(x)} - \frac{3\sinh(x)}{\cosh(x)} - \frac{\sinh(x)\cosh(x)}{\cosh(x)} = 0$$

$$\Longrightarrow 2\sinh(x)\cosh^2(x) - 3\sinh(x) - \sinh(x)\cosh(x) = 0$$

$$\Longrightarrow \sinh(x)\Big[\underbrace{2\cosh^2(x) - 3 - \cosh(x)}_{\text{Quadratic in } \cosh(x)}\Big] = 0$$

$$\Longrightarrow \sinh(x)\left[2\cosh^2(x) - \cosh(x) - 3\right] = 0$$

$\sinh(x) = 0$ OR $2\cosh^2(x) - \cosh(x) - 3 = 0$

$\sinh(x) = 0$:

$$\frac{e^x - e^{-x}}{2} = 0$$

$$\Longrightarrow e^x - e^{-x} = 0$$

$$\Longrightarrow e^x - \frac{1}{e^x} = 0$$

$$\Longrightarrow \frac{e^{2x} - 1}{e^x} = 0$$

$$\Longrightarrow e^{2x} - 1 = 0$$

$$\Longrightarrow e^{2x} = 1$$

$$\Longrightarrow 2x = \ln(1)$$

$$\Longrightarrow x = \frac{1}{2}\ln(1)$$

$$\Longrightarrow x = 0$$

$2\cosh^2(x) - \cosh(x) - 3 = 0$:

$$\cosh(x) = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-3)}}{2(2)}$$

$$\Longrightarrow \cosh(x) = \frac{1 \pm \sqrt{1+24}}{4} = \frac{1 \pm \sqrt{25}}{4}$$

$$\Longrightarrow \cosh(x) = \frac{1 \pm 5}{4} = \frac{6}{4} \text{ OR } \frac{-4}{4}$$

$$\Longrightarrow \cosh(x) = \frac{3}{2} \text{ OR } -1.$$

From prev. questions in this worksheet, we know that:

$$\cosh^{-1}(x) = \ln\left|x + \sqrt{x^2 - 1}\right|$$

$$\Longrightarrow \cosh^{-1}\left(\frac{3}{2}\right) = \ln\left(\frac{3}{2} + \sqrt{\frac{9}{4} - 1}\right) = \ln\left(\frac{3+\sqrt{5}}{2}\right)$$

$$\Longrightarrow \cosh^{-1}(-1) = \ln\left(-1 + \sqrt{1-1}\right) = \ln(-1)$$

invalid expr.