Review Sheet: U-Substitution

Content Review

Overview

Overall procedure:

1. Choose a substitution $u = g(x)$. Look for a smaller, inner expression whose derivative (or some form of the derivative) is also present in the integrand (the question given to you). Note that the goal of this substitution is to completely remove all terms in x from the integrand either through substitution or cancellation.
2. Compute $\frac{du}{dx} = g'(x) \Longrightarrow dx = \frac{du}{g'(x)}$
3. Substitute in $g(x) = u$ and $dx = \frac{du}{g'(x)}$
4. Integrate with respect to u
5. If computing an indefinite integral, then replace u with $g(x)$ to get the final answer. If computing a definite integral, then change the limits (demonstrated in Worked Problems)

Helpful Equations

Trigonometric/Hyperbolic Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cosh^2 t - \sinh^2 t = 1$$

Integration by Parts

$$\int u dv = uv - \int v du$$

Resources

U-Subsitution

- Video: How to Integrate using U-Substitution (Organic Chemistry Tutor, 20 min) - with worked examples
- U-Substitution Introduction (Khan Academy)
- How to Integrate using U-Substitution (NancyPi, 25 min) - with worked examples
- Worksheet: University of South Carolina (with hints, challenge problems & solutions)
- Worksheet: U-Subsitution (with Answers)
- Worksheet Solutions: U-Subsitution (detailed solutions)
- Worksheet: U-Substitution (Easy, with worked solutions)

Acknowlegement

Questions in the Worked Problems section of this sheet have been taken from external sources that have been linked where appropriate. Content overview for this worksheet has been developed with reference to this resource from Purdue. All solutions have been created independently.

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Worked Problems

Compute the following indefinite integrals.
Source

1.

$$\int 2x\sin(x^2)dx$$

$$\int 2x\ \sin(x^2)\,dx$$

Notice that $2x = \frac{d}{dx}x^2$, so if we substitute $u=x^2$, then the $2x$ might cancel out.

Let $u=x^2 \implies \frac{du}{dx}=2x$

$\implies du = 2x\,dx \implies dx = \frac{du}{2x}$

We want to substitute dx in terms of u as well.

Carry out the substitution:

$$\int 2x\sin(\underbrace{x^2}_{u})\underbrace{dx}_{\frac{du}{2x}} = \int \cancel{2x}\sin(u)\cdot\frac{du}{\cancel{2x}} = \underbrace{\int \sin(u)\ du}$$

This looks like an integral that we know how to solve!

Integrate:

$$\int \sin(u)\ du = -\cos(u)+c$$

But we want final answer in terms of x, so substitute $u=x^2$

$$\int 2x\sin(x^2)\,dx = -\cos(x^2)+c$$

2.

$$\int \frac{1}{x\ln x}dx$$

$$\int \frac{1}{x\,\ln x}dx = \int \frac{1}{x}\cdot\frac{1}{\ln x}\,dx$$

Notice that $\frac{1}{x}$ is a derivative of $\ln x$. So, if we take $u=\ln x$, then $\frac{1}{x}$ might cancel out.

Let $u=\ln(x) \implies \frac{du}{dx}=\frac{1}{x}$

$\implies du = \frac{dx}{x} \implies dx = x\,du.$

This looks like an integral that we know how to solve

Carry out the substitution:

$$\int \frac{1}{x}\cdot\frac{1}{\underbrace{\ln x}_{u}}\underbrace{dx}_{x\,du} = \int \frac{1}{\cancel{x}}\cdot\frac{1}{u}\cdot\cancel{x}\,du = \int \frac{1}{u}\,du$$

Integrate:

$$\int \frac{1}{u}\,du = \ln|u|+c = \ln|\ln(x)|+c$$

substituting $u=\ln(x)$

3.

$$\int 5\sqrt{5x+3}dx$$

$$\int 5\sqrt{5x+3}\, dx$$

Notice that 5 is the derivative of 5x+3. So if we substitute u = 5x+3, then the 5 might cancel out.

Let $5x+3 = u. \implies \frac{du}{dx} = 5 \implies du = 5\,dx \implies dx = \frac{1}{5}du.$

Carry out the substitution:

$$\int 5\sqrt{\underbrace{5x+3}_{u}}\,\underbrace{dx}_{\frac{1}{5}du} = \int \cancel{5}\sqrt{u}\,\frac{du}{\cancel{5}} = \int \sqrt{u}\,du$$

Integrate

Power rule of integration

$$\int \sqrt{u}\,du = \int u^{\frac{1}{2}}\,du = \frac{1}{\left(\frac{1}{2}+1\right)}u^{\frac{1}{2}+1} = \frac{3}{2}u^{\frac{3}{2}} = \boxed{\frac{2}{3}(5x+3)^{\frac{3}{2}} + C}$$

Substituting u = 5x+3

4.

$$\int 14(7x+2)^2dx$$

$$\int 14(7x+2)^2\,dx$$

Notice that 14 = 7·2 where 7 is the derivative of 7x+2

Let $u = 7x+2 \implies \frac{du}{dx} = 7 \implies du = 7\,dx \implies dx = \frac{du}{7}$

Carry out substitution

$$\int 14(\underbrace{7x+2}_{u})^2\,\underbrace{dx}_{\frac{du}{7}} = \int 14\cdot u^2\cdot\frac{du}{7} = \int 2u^2\,du$$

Integrate:

Substituting u = 7x+2

$$\int 2u^2\,du = 2\cdot\frac{1}{3}u^3 = \frac{2}{3}u^3 = \boxed{\frac{2}{3}(7x+2)^3 + C}$$

5.

$$\int \sin^5 x \cos x dx$$

$$\int \sin^5 x \cos x \, dx$$

Notice that $\cos x$ is a derivative of $\sin x$. So if we do $u = \sin x$, then the $\cos x$ might cancel out in substitution.

Let $u = \sin x \implies \frac{du}{dx} = \cos x \implies du = \cos x \, dx \implies dx = \frac{1}{\cos x} du$

Carry out substitution:

$$\int \underbrace{\sin^5 x}_{u^5} \cos x \underbrace{dx}_{\frac{1}{\cos x} du} = \int u^5 \cancel{\cos x} \cdot \frac{1}{\cancel{\cos x}} du = \int u^5 du$$

Integrate:

$$\int u^5 du = \frac{1}{6} u^6 = \boxed{\frac{\sin^6 x}{6} + C}$$

Substituting $u = \sin x$

6.

$$\int \frac{45x^2}{(3x^3+2)^4} dx$$

$$\int \frac{45x^2}{(3x^3+2)^4} dx$$

Notice that $45x^2 = 5 \cdot 9x^2$ and $9x^2$ is a derivative of $3x^3$. So if we do $u = 3x^3+2$, maybe some stuff will cancel out.

Let $u = 3x^3+2 \implies \frac{du}{dx} = 9x^2 \implies du = 9x^2 dx \implies dx = \frac{1}{9x^2} du$

Carry out substitution:

$$\int \frac{45x^2}{\underbrace{(3x^3+2)^4}_{u}} \underbrace{dx}_{\frac{1}{9x^2} du} = \int \frac{\overset{5}{\cancel{45x^2}}}{u^4} \cdot \frac{1}{\cancel{9x^2}} du = \int \frac{5}{u^4} du$$

Substituting $u = 3x^2+2$

Integrate

$$\int \frac{5}{u^4} du = \int 5u^{-4} du = 5 \cdot -\frac{1}{3} u^{-3} = -\frac{5}{3}(3x^2+2)^{-3} + C$$

$$\boxed{\int \frac{45x^2}{(3x^2+2)^4} dx = \frac{-5}{3(3x^2+2)^3} + C}$$

Compute the following definite integrals.
Source and another Source

1.

$$\int_0^2 \frac{x}{(x^2+25)^3}dx$$

$$\boxed{\int_0^2 \frac{x}{(x^2+25)^3}dx}$$

Notice that $x = \frac{2x}{2}$ and $2x$ is a derivative of (x^2+25) so if we set $u = x^2+25$, it might cancel the x.

Let $u = x^2+25 \implies \frac{du}{dx} = 2x \implies du = 2x\,dx \implies dx = \frac{1}{2x}du$

Since this is a definite integral, we want to integrate between $x=0$ and $x=2$. But because of the u-substitution, we need to convert these limits in terms of u as well.

$u = x^2+25$
$\implies$ when $x=0$, $u = 0^2+25 = 25$
$\implies$ when $x=2$, $u^2 = 2^2+25 = 29$

Carry out the substitution:

$$\int_0^2 \frac{x}{(x^2+25)^3}dx = \int_{25}^{29} \frac{\cancel{x}}{u^3}\cdot\frac{1}{2\cancel{x}}du = \int_{25}^{29}\frac{1}{2u^3}du = \int_{25}^{29}\frac{u^{-3}}{2}du$$

$2 \to 29$, $0 \to 25$, $(x^2+25)^3 \to u^3$, $dx \to \frac{1}{2x}du$

Integrate:

$$\int_{25}^{29}\frac{u^{-3}}{2}du = \frac{1}{2}\left[-\frac{1}{2}u^{-2}\right]_{25}^{29} = -\frac{1}{4}\left[\frac{1}{29^2}-\frac{1}{25^2}\right] = 1.027\times10^{-4}$$

2.

$$\int_0^1 \frac{5x}{(4+x^2)^2}dx$$

$$\boxed{\int_0^1 \frac{5x}{(4+x^2)^2}dx}$$

Notice that $x = \frac{2x}{2}$ which is a derivative of x^2+4.

Let $u = 4+x^2 \implies \frac{du}{dx} = 2x \implies du = 2x\,dx \implies dx = \frac{1}{2x}du$

Change the limits:

- when $x=0$, $u = 4+0^2 = 4$
- when $x=1$, $u = 4+1^2 = 5$

Substitute:

$$\int_0^1 \frac{5x}{(4+x^2)^2}dx = \int_4^5 \frac{5\cancel{x}}{u^2}\cdot\frac{1}{2\cancel{x}}du = \int_4^5\frac{5}{2u^2}du$$

$1 \to 5$, $0 \to 4$, $(4+x^2)^2 \to u^2$, $dx \to \frac{1}{2x}du$

Integrate:

$$\int_4^5\frac{5}{2u^2}du = \frac{5}{2}\int_4^5 u^{-2}du = \frac{5}{2}\left[-u^{-1}\right]_4^5 = \frac{5}{2}\left[-\frac{1}{5}+\frac{1}{4}\right] = \frac{1}{8}$$

3.

$$\int_0^{\pi} 3\cos^3 x \sin x dx$$

$$\int_0^{\pi} 3\cos^3 x \sin x \, dx$$

Notice that $\sin x = -\frac{d}{dx}\cos x \Longleftrightarrow \frac{d}{dx}\cos x = -\sin x$

Let $u = \cos x \implies \frac{du}{dx} = -\sin x \implies du = -\sin x \, dx \implies dx = \frac{-1}{\sin x} du$

Change the limits:

- when $x = 0$, $u = \cos(0) = 1$
- when $x = \pi$, $u = \cos(\pi) = -1$

Substitute:

Note that we add a (-) sign here to flip the limits so the smaller number is below.

$$\int_{0 \to 1}^{\pi \to -1} 3\underbrace{\cos^3 x}_{u^3} \sin x \underbrace{dx}_{\frac{-1}{\sin x}dx} = \int_1^{-1} 3u^3 \cancel{\sin x} \cdot \frac{-1}{\cancel{\sin x}} du = -\int_{-1}^{1} -3u^3 \, du$$

Integrate:

$$-\int_{-1}^{1} -3u^3 du = -\left[-u^4\right]_{-1}^{1} = \left[u^4\right]_{-1}^{1} = 1 - 1 = 0$$

4.

$$\int_0^1 \frac{e^{3x}}{4 - e^{3x}} dx$$

$$\int_0^1 \frac{e^{3x}}{4 - e^{3x}} dx$$

Note that $\frac{d}{dx} e^{3x} = 3e^{3x} \Longleftrightarrow \frac{1}{3}\frac{d}{dx} e^{3x} = e^{3x}$

Let $u = 4 - e^{3x} \implies \frac{du}{dx} = 3e^{3x} \implies du = 3e^{3x} dx \implies dx = \frac{1}{3e^{3x}} du$

Change the limits:

- when $x = 0$, $u = 4 - e^{3(0)} = 4 - 1 = 3$
- when $x = 1$, $u = 4 - e^3$

Substitute:

$$\int_{0 \to 3}^{1 \to 4-e^3} \frac{e^{3x}}{\underbrace{4 - e^{3x}}_{u}} \underbrace{dx}_{\frac{1}{3e^{3x}}du} = \int_3^{4-e^3} \frac{\cancel{e^{3x}}}{u} \cdot \frac{1}{3\cancel{e^{3x}}} du = \int_3^{4-e^3} \frac{1}{3u} du$$

Integrate:

$$\int_3^{4-e^3} \frac{1}{3u} du = \frac{1}{3}\int_3^{4-e^3} \frac{1}{u} du = \frac{1}{3}\left[\ln|u|\right]_3^{4-e^3} = \frac{1}{3}\left[\ln(4-e^3) - \ln(3)\right]$$

$$= \frac{1}{3}\ln\left(\frac{4-e^3}{3}\right)$$

If you are not allowed a calculator in exam, it is sufficient to leave your answer here.

See logarithm rules: $\ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right)$