Review Sheet: L'Hopital's Rule

Content Review

Overview

L'Hopital's rule is a technique used to simplify the evaluation of limits. Note that it does not directly evaluate limits, but rather simplifies the process of evaluating limits.

When can we apply L'Hopital's Rule?
L'Hopital's Rule can only be applied in situations where the limit as it currently stands evaluates to one of the two following indeterminate forms:

$$\frac{0}{0} \quad \text{or} \quad \frac{\pm\infty}{\pm\infty}$$

Note that L'Hopitals rule **CANNOT** be applied if the limit evaluates to any of the following forms (or more generally, to any form that is not the above two):

$$\frac{0}{\pm\infty} \quad \text{or} \quad \frac{\pm\infty}{0} \quad \text{or} \quad \frac{1}{\pm\infty} \quad \text{or} \quad \frac{\pm\infty}{1} \quad \text{or} \quad \frac{1}{0}$$

How to apply L'Hopital's Rule?
Suppose we want to evaluate

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

where the limit a could be an ordinary number or $\pm\infty$. Provided that the limit satisfies one of the two indeterminate forms shown above, we can apply L'Hopital's Rule to state that:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Note that this is not the same as applying the quotient rule.

Resources

L'Hopital's Rule

- Video: L'Hopital's Rule (Organic Chemistry Tutor, 13 min) - with worked examples
- Content & Worked Examples (Paul's Online Notes)
- L'Hopital's Rule (Khan Academy Module)
- Practice Problems: L'Hopital's Rule (UC Davis Math) - has worked solutions
- Content & Worked Examples (Textbook)

Acknowlegement

Questions in the Worked Problems section of this sheet have been taken from external sources that have been linked where appropriate. All solutions have been created independently.

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Worked Problems

Source and another Source
Find the following limits. You may use L'Hopital's rule where appropriate. Note that L'Hopital's rule may not apply to every limit, and it may not be helpful even when it does apply.

1.
$$\lim_{x\to-\infty} \frac{3x-1}{x^2+1}$$

$$\lim_{x\to-\infty} \frac{3x-1}{x^2+1}$$

If we directly substitute $x \mapsto -\infty$ then the limit evaluates to $\frac{-\infty}{+\infty}$ which is a valid indeterminate form to apply L'Hôpital's rule.

$\frac{d}{dx}(3x-1) = 3$

$\frac{d}{dx}(x^2+1) = 2x$

By L'Hôpital's rule:

$$\lim_{x\to-\infty} \frac{3x-1}{x^2+1} \overset{H}{=} \lim_{x\to-\infty} \frac{3}{2x} = \frac{3}{-\infty} = 0$$

L'Hôpitals Rule applied here.

Alternate method: (without L'Hôpitals Rule)

$$\lim_{x\to-\infty} \frac{3x-1}{x^2+1}$$

Notice that the 3x is the largest contributor to the numerator.

Notice that the x^2 is the largest contributor to the denominator.

Thus,

$$\lim_{x\to-\infty} \frac{3x-1}{x^2+1} \approx \lim_{x\to-\infty} \frac{3x}{x^2} = \lim_{x\to-\infty} \frac{3}{x} = \frac{3}{-\infty} = 0$$

This approximation works for large (negative or positive) values of x.

2.

$$\lim_{x\to 0}\frac{e^x - x - 1}{\cos x - 1}$$

$$\lim_{x\to -2}\frac{x^3-x^2-10x-8}{5x^3+12x^2-2x-12}$$

Direct evaluation gives: $\dfrac{(-2)^3-(-2)^2-10(-2)-8}{5(-2)^3+12(-2)^2-2(-2)-12} = \dfrac{0}{0}$

Apply L'Hôpital's Rule

$$\frac{d}{dx}(x^3-x^2-10x-8) = 3x^2-2x-10$$

$$\frac{d}{dx}(5x^3+12x^2-2x-12) = 15x^2+24x-2$$

① Applying L'Hôpital's Rule:

$$\lim_{x\to -2}\frac{x^3-x^2-10x-8}{5x^3+12x^2-2x-12} \overset{H}{=} \lim_{x\to -2}\frac{3x^2-2x-10}{15x^2+24x-2} = \frac{3(-2)^2-2(-2)-10}{15(-2)^2+24(-2)-2} = \frac{6}{10} = \frac{3}{5}$$

3.

$$\lim_{x\to -2}\frac{x^3-x^2-10x-8}{5x^3+12x^2-2x-12}$$

$$\lim_{x\to 0}\frac{e^x-x-1}{\cos x - 1}$$

If we directly evaluate the limit it gives

$$\frac{e^0-0-1}{\cos(0)-1} = \frac{1-1}{1-1} = \frac{0}{0}$$ } Indeterminate form that we can apply L'Hopital's Rule to.

$$\frac{d}{dx}(e^x-x-1) = e^x-1$$

$$\frac{d}{dx}(\cos x - 1) = -\sin x$$

By L'Hôpital's Rule:

$$\lim_{x\to 0}\frac{e^x-x-1}{\cos x-1} \overset{H}{=} \lim_{x\to 0}\frac{e^x-1}{-\sin x} = \lim_{x\to 0}\frac{1-e^x}{\sin x}$$

} If we directly evaluate this limit, it gives:

$$\frac{1-e^0}{\sin(0)} = \frac{1-1}{0} = \frac{0}{0}$$

Apply L'Hôpital's Rule again

$$\frac{d}{dx}(1-e^x) = -e^x$$

$$\frac{d}{dx}(\sin x) = \cos x$$

By L'Hôpital's Rule:

$$\lim_{x\to 0}\frac{1-e^x}{\sin x} \overset{H}{=} \lim_{x\to 0}\frac{-e^x}{\cos x} = \frac{-e^0}{\cos(0)} = \frac{-1}{1} = -1$$

4.

$$\lim_{x\to 0}\frac{\sin(6x)}{\sin(3x)}$$

$$\lim_{x\to 0}\frac{\sin(6x)}{\sin(3x)}$$

Direct evaluation gives: $\frac{\sin(0)}{\sin(0)} = \frac{0}{0}$

can apply L'Hôpital's Rule

$$\frac{d}{dx}\sin(6x) = 6\cos(6x)$$

$$\frac{d}{dx}\sin(3x) = 3\cos(3x)$$

Applying L'Hôpital's Rule:

$$\lim_{x\to 0}\frac{\sin(6x)}{\sin(3x)} \overset{H}{=} \lim_{x\to 0}\frac{6\cos(6x)}{3\cos(3x)} = \frac{6\cos(0)}{3\cos(0)} = \frac{6}{3} = 2$$

5.

$$\lim_{x\to 1}\frac{\ln x^2}{9x^2-9}$$

$$\lim_{x\to 1}\frac{\ln(x^2)}{3x^2-9}$$

Direct evaluation gives: $\frac{\ln(1)}{9-9} = \frac{0}{0}$

can apply L'Hôpital's Rule

$$\frac{d}{dx}(\ln(x^2)) = 2x\cdot\frac{1}{x^2} = \frac{2}{x}$$

$$\frac{d}{dx}(3x^2-9) = 6x$$

Applying L'Hôpital's Rule:

$$\lim_{x\to 1}\frac{\ln(x^2)}{3x^2-9} \overset{H}{=} \lim_{x\to 1}\frac{\frac{2}{x}}{6x} = \lim_{x\to 1}\frac{2}{x}\div 6x = \lim_{x\to 1}\frac{2}{x}\cdot\frac{1}{6x}$$

$$= \lim_{x\to 1}\frac{2}{6x^2} = \frac{2}{6(1)^2} = \frac{2}{6} = \frac{1}{3}$$

6.

$$\lim_{x\to 0} \frac{\tan x - x}{x^3}$$

$$\lim_{x\to 0} \frac{\tan x - x}{x^3}$$

Direct evaluation gives: $\frac{\tan(0) - 0}{0^3} = \frac{0}{0}$

Can apply L'Hôpital's Rule

$\frac{d}{dx} \tan x - x = \sec^2 x - 1$

$\frac{d}{dx} x^3 = 2x^2$

① Applying L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} \overset{H}{=} \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2}$$

Direct evaluation gives: $\frac{\sec^2(0) - 1}{3(0)^2} = \frac{1-1}{0} = \frac{0}{0}$

Can apply L'Hôpital's Rule

$$\frac{d}{dx} \sec^2 x - 1 = 2\sec x \cdot \underbrace{\sec x \tan x}_{\text{due to chain Rule}} = 2\sec^2 x \tan x$$

$\frac{d}{dx} 3x^2 = 6x$

② Applying L'Hôpital's Rule (again)

$$\lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} \overset{H}{=} \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x}$$

Direct evaluation gives: $\frac{2\sec^2(0)\tan(0)}{6 \cdot 0} = \frac{1 \cdot 0}{0} = \frac{0}{0}$

can apply L'Hôpital's Rule

$\frac{d}{dx}(2\sec^2 x \tan x)$

$u = 2\sec^2 x$ $\quad v = \tan x$

$u' = 4\sec^2 x \tan x$ $\quad v' = \sec^2 x$

Applying Product Rule:

$$\frac{d}{dx}(2\sec^2 x \tan x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$$

$\frac{d}{dx}(6x) = 6$

③ Applying L'Hôpital's Rule (again)

$$\lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x} \overset{H}{=} \lim_{x\to 0} \frac{4\sec^2 x \tan^2 x + 2\sec^4 x}{6} = \lim_{x\to 0} \frac{2\sec^2 x(2\tan^2 x + \sec^2 x)}{6}$$

$$= \frac{2\sec^2(0)(2\tan^2(0) + \sec^2(0))}{6} = \frac{2(0+1)}{6} = \frac{2}{6} = \boxed{\frac{1}{3}}$$

7.

$$\lim_{x \to 0} \frac{\sin x - x}{x^3}$$

$\lim_{x \to 0} \frac{\sin x - x}{x^3}$

Direct evaluation gives: $\frac{\sin(0) - (0)}{0} = \frac{0}{0}$ can apply L'Hôpital's Rule.

$\frac{d}{dx} \sin x - x = \cos x - 1$

$\frac{d}{dx} x^3 = 3x^2$

① Applying L'Hôpital's Rule:

$\lim_{x \to 0} \frac{\sin x - x}{x^3} = \lim_{x \to 0} \frac{\cos x - 1}{3x^2}$

Direct evaluation gives: $\frac{\cos(0) - 1}{3 \cdot 0^2} = \frac{1-1}{0} = \frac{0}{0}$

can apply L'Hôpital's Rule

$\frac{d}{dx} \cos x - 1 = -\sin x$

$\frac{d}{dx} 3x^2 = 6x$

② Applying L'Hôpital's Rule:

$\lim_{x \to 0} \frac{\cos x - 1}{3x^2} \overset{H}{=} \lim_{x \to 0} \frac{-\sin x}{6x}$

Direct evaluation gives: $\frac{-\sin(0)}{6(0)} = \frac{0}{0}$

can apply L'Hôpitals Rule

$\frac{d}{dx} -\sin x = -\cos x$

$\frac{d}{dx} 6x = 6$

③ Applying L'Hôpital's Rule:

$\lim_{x \to 0} \frac{-\sin x}{6x} \overset{H}{=} \lim_{x \to 0} \frac{-\cos x}{6} = \frac{-\cos(0)}{6} = \frac{-1}{6}$

8.

$$\lim_{x\to 0} x \sin \frac{1}{x}$$

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$$

Note that regardless of the value of x (large, small, zero, negative etc.), the following always holds true:

$-1 \leq \sin x \leq 1$

Even when $x \longmapsto \infty$, $-1 \leq \sin x \leq 1$.

Using this fact:

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = \lim_{x\to 0} x \cdot \underbrace{\lim_{x\to 0} \sin\left(\frac{1}{x}\right)}$$

this value will be some constant c such that $-1 \leq c \leq 1$.

$$= 0 \cdot c = 0$$

Note that this problem did not actually require the use of L'Hôpital's Rule despite being in a strange looking form.

9.

$$\lim_{x\to+\infty}\frac{\ln(x^2+1)}{\ln(x^3+1)}$$

$$\lim_{x\to+\infty}\frac{\ln(x^2+1)}{\ln(x^3+1)}$$

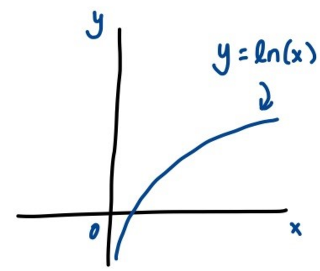
Direct evaluation gives: $\dfrac{\ln(\infty)}{\ln(\infty)} = \dfrac{\infty}{\infty}$

Can apply L'Hopital's Rule

Chain Rule

$$\frac{d}{dx}\ln(x^2+1) = \frac{1}{x^2+1}\cdot 2x = \frac{2x}{x^2+1}$$

$$\frac{d}{dx}\ln(x^3+1) = \frac{1}{x^3+1}\cdot 3x^2 = \frac{3x^2}{x^3+1}$$



Even though it looks "flat", this curve actually keeps increasing as x approaches ∞.

Applying L'Hôpital's Rule:

$$\lim_{x\to\infty}\frac{\ln(x^3+1)}{\ln(x^2+1)} \overset{H}{=} \lim_{x\to+\infty}\frac{\frac{2x}{x^2+1}}{\frac{3x^2}{x^3+1}}$$

$$= \lim_{x\to+\infty}\frac{2x}{x^2+1}\cdot\frac{x^3+1}{3x^2}$$

$$= \lim_{x\to+\infty}\frac{2(x^3+1)}{3x(x^2+1)} \quad = \frac{\infty}{\infty} \text{ by direct evaluation.}$$

Note that applying L'Hopital's Rule here might be long and tedious (you can try) so we use an approximation trick.

Since we want $\lim_{x\to+\infty}$, at very large values of x, $2x^3$ contributes the most to the numerator and $3x^3$ contributes the most to the denominator.

Applying this logic:

$$\lim_{x\to+\infty}\frac{2(x^3+1)}{3x(x^2+1)} = \lim_{x\to+\infty}\frac{2x^3+2}{3x^3+3x} \approx \lim_{x\to+\infty}\frac{2x^3}{3x^3} = \lim_{x\to+\infty}\frac{2}{3} = \frac{2}{3}$$

10.

$$\lim_{x\to 0} \frac{xe^{3x} - x}{1 - \cos(2x)}$$

$$\boxed{\lim_{x\to 0} \frac{xe^{3x} - x}{1 - \cos(2x)}}$$

Direct evaluation gives: $\dfrac{0 \cdot e^0 - 0}{1 - \cos(0)} = \dfrac{0}{1-1} = \dfrac{0}{0}$

can apply L'Hôpital's Rule

$$\frac{d}{dx}(xe^{3x} - x) = \underbrace{\frac{d}{dx} xe^{3x}}_{\text{use product rule}} - \frac{d}{dx} x$$

$u = x \qquad v = e^{3x}$

$u' = 1 \qquad v' = 3e^{3x}$

$$\frac{d}{dx} xe^{3x} = 3xe^{3x} + e^{3x}$$

$$\frac{d}{dx} xe^{3x} - x = 3xe^{3x} + e^{3x} - 1$$

$$\frac{d}{dx} 1 - \cos(2x) = 2\sin 2x$$

Applying L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{xe^{3x} - x}{1 - \cos(2x)} \overset{H}{=} \lim_{x\to 0} \frac{3xe^{3x} + e^{3x} - 1}{2\sin 2x}$$

Direct evaluation gives: $\dfrac{3(0)e^0 + e^0 - 1}{2\sin(0)} = \dfrac{0+1-1}{0} = \dfrac{0}{0}$

can apply L'Hôpital's Rule

$$\frac{d}{dx}(3xe^{3x} + e^{3x} - 1) = 3(3xe^{3x} + e^{3x})$$
$$= 9xe^{3x} + 3e^{3x}$$

$$\frac{d}{dx} 2\sin(2x) = 4\cos(2x)$$

Applying L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{3xe^{3x} + e^{3x} - 1}{2\sin 2x} \overset{H}{=} \lim_{x\to 0} \frac{9xe^{3x} + 3e^{3x}}{4\cos(2x)} = \frac{9(0)e^0 + 3e^0}{4\cos(0)} = \boxed{\frac{3}{4}}$$