Review Sheet: Improper Integrals

Content Review

Overview

Improper integrals are said to be

- convergent if the limit is finite and that limit is the value of the improper integral
- divergent if the limit does not exist

To simplify, if the limit exists, we say the improper integral **converges**. If either the limit fails to exist or is infinite, then the integral **diverges**.

A formal definition is:

Let $f(x)$ be continuous over an interval of the form $[a, +\infty)$. Then (provided this limit exists):

$$\int_a^{+\infty} f(x)dx = \lim_{t \to +\infty} \int_a^t f(x)dx$$

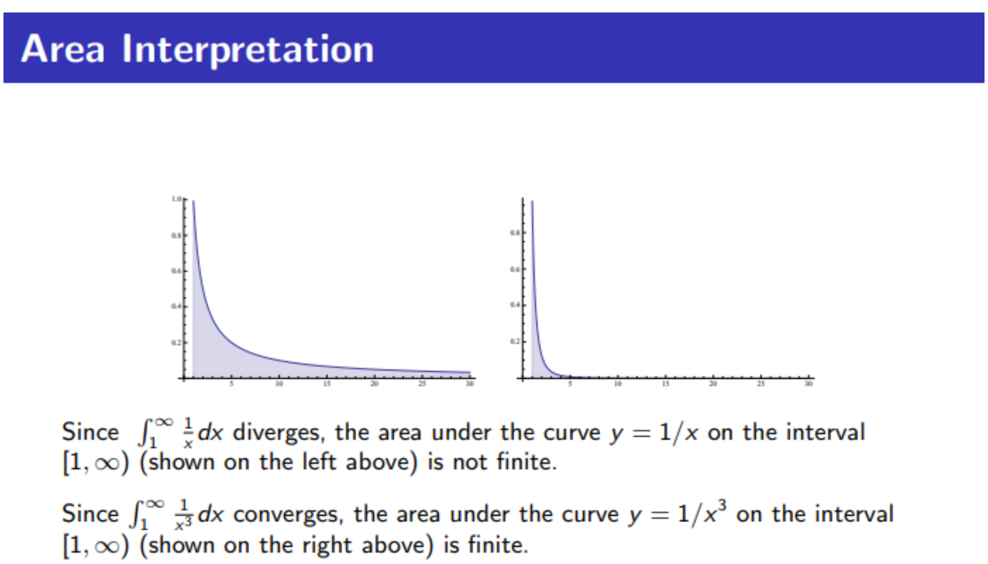
Let $f(x)$ be continuous over an interval of the form $(-\infty, b]$. Then (provided this limit exists):

$$\int_{-\infty}^{b} f(x)dx = \lim_{t \to -\infty} \int_t^b f(x)dx$$

Let $f(x)$ be continuous over $(-\infty, +\infty)$. Then

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_0^{+\infty} f(x)dx$$

provided that both integrals on the R.H.S converge. If either of these two integrals diverge, then their sum diverges.



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Skip to Worked Problems

Resources

Improper Integrals

- Notes: Paul's Online Notes
- Introduction to Improper Integrals (Khan Academy)
- Notes + Practice Questions: Simon Fraser University
- Video: Improper Integrals - Convergence and Divergence (Organic Chemistry Tutor - 14 min)
- Practice Problems with Solutions: Paul's Online Notes

Acknowlegement

Questions in the Worked Problems section of this sheet have been taken from external sources that have been linked where appropriate. All solutions have been created independently.

Worked Problems

Determine if each of the following integrals converge or diverge. If the integral converges, determine its value Source and Source and Source

1.
$$\int_0^\infty (1+2x)e^{-x}dx$$

$$\int_0^\infty (1+2x)e^{-x}\,dx$$

Recall: Integration by parts

$$\int u\,dv = uv - \int v\,du$$

use integration by parts.

$u = 1+2x$ $\quad dv = e^{-x}$

$du = 2$ $\quad v = -e^{-x}$

$$\int_0^\infty (1+2x)e^{-x}\,dx = \Big[(1+2x)(-e^{-x})\Big]_0^\infty - \int_0^\infty -2e^{-x}\,dx$$

$$= \Big[(1+2x)(-e^{-x})\Big]_0^\infty + \int_0^\infty 2e^{-x}\,dx$$

$$= \Big[(1+2x)(-e^{-x})\Big]_0^\infty + \Big[-2e^{-x}\Big]_0^\infty$$

$$= \Big[(1+2x)(-e^{-x}) + (2)(-e^{-x})\Big]_0^\infty$$

$$= \Big[(-e^{-x})(3+2x)\Big]_0^\infty$$

Notice that we cannot substitute ∞ directly. So we take a limit.

$$= \lim_{t\to\infty} \Big[(-e^{-x})(3+2x)\Big]_0^t$$

$$= \lim_{t\to\infty} \Big[(-e^{-t})(3+2t) - (-e^{0})(3+2\cdot 0)\Big]$$

$$= \lim_{t\to\infty} \Big[-e^{-t}(3+2t)\Big] + 3$$

$$= \lim_{t\to\infty} \left[-\left(\frac{3}{e^t} + \frac{2t}{e^t}\right)\right] + 3$$

$$= \underbrace{\lim_{t\to\infty}\left[-\frac{3}{e^t}\right]} + \underbrace{\lim_{t\to\infty}\left[\frac{-2t}{e^t}\right]} + 3 = 0 + 0 + 3 = 3$$

as $t\mapsto\infty$, $e^t\mapsto\infty$

So $\frac{1}{e^t} \mapsto 0$

$\therefore \lim_{t\to\infty}\left[-\frac{3}{e^t}\right] = 0$

as $t\mapsto\infty$, $2t\mapsto\infty$ and $e^t\mapsto\infty \Longrightarrow \frac{\infty}{\infty}$

Apply L'Hôpital's Rule.

$$= \lim_{t\to\infty}\left[\frac{-2}{e^t}\right] = 0$$

2.

$$\int_{-5}^{1} \frac{1}{10+2z} dz$$

$$\int_{-5}^{1} \frac{1}{10+2z} dz$$

Apply u substitution.

Let $u = 10+2z$

$\frac{du}{dz} = 2 \implies du = 2\,dz \implies dz = \frac{1}{2}\,du$

Changing Limits:

- When $x = -5$, $u = 10+2(-5) = 0$
- When $x = 1$, $u = 10+2(1) = 12$

Substituting:

$$\int_{-5 \to 0}^{1 \to 12} \underbrace{\frac{1}{10+2z}}_{u} \underbrace{dz}_{\frac{1}{2}du} = \int_0^{12} \frac{1}{u} \cdot \frac{1}{2}\, du = \frac{1}{2} \int_0^{12} \frac{1}{u}\, du$$

Integrating:

$= \frac{1}{2}\left[\ln|u|\right]_0^{12}$ ← Note that we can't directly have $\ln(0)$, so we take a limit.

$= \lim_{t\to 0} \frac{1}{2}\left[\ln|u|\right]_0^{12}$

$= \lim_{t\to 0} \frac{1}{2}\left[\ln(12) - \ln(t)\right]$

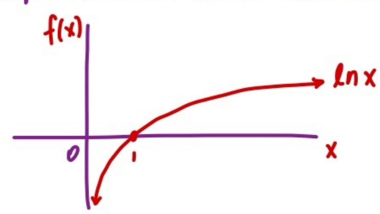
$= \frac{1}{2}\ln(12) - \lim_{t\to 0} \ln(t)$

$= \frac{1}{2}\ln(12) - (-\infty)$

$= \infty$

In other words, the limit does not exist. (Integral diverges)

Graph of $\ln(x)$ looks something like this:



Based on the graph, we see that as $x \mapsto 0$, $\ln(x) \mapsto -\infty$

3.

$$\int_{-\infty}^{\infty} \frac{6w^3}{(w^4+1)^2} dw$$

$$\boxed{\int_{-\infty}^{\infty} \frac{6w^3}{(w^4+1)^2}\, dw}$$

Notice that $\frac{d}{dw}(w^4) = 4w^3 \longleftarrow$ looks similar to numerator term. This is an indication that u-substitution would be a good strategy.

Let $u = w^4+1$

$$\frac{du}{dw} = 4w^3 \implies du = 4w^3\, dw \implies dw = \frac{1}{4w^3}\, du$$

TRICK: Split up the integral since both limits are ∞. Pick a convenient split value (e.g. 0)

$$\int_{-\infty}^{\infty} \frac{6w^3}{(w^4+1)^2}\, dw = \int_{-\infty}^{0} \frac{6w^3}{(w^4+1)^2}\, dw + \int_{0}^{\infty} \frac{6w^3}{(w^4+1)^2}\, dw$$

$$= \lim_{t\to-\infty} \int_{t}^{0} \frac{6w^3}{(w^4+1)^2}\, dw + \lim_{x\to\infty} \int_{0}^{x} \frac{6w^3}{(w^4+1)^2}\, dw \text{ —— (i)}$$

Evaluate the integral using u substitution. (Ignore limits for a sec.)

$$\int \frac{6w^3}{(w^4+1)^2}\, dw = \int \frac{\overset{3}{\cancel{6w^3}}}{u^2} \cdot \frac{1}{\underset{2}{\cancel{4w^3}}}\, du = \int \frac{3}{2u^2}\, du = \frac{3}{2}\int u^{-2}\, du$$

$$= \frac{3}{2}\left[-u^{-1}\right] = \frac{-3}{2u} = \frac{-3}{2(w^4+1)}$$

Going back to expression (i):

$$\int_{-\infty}^{\infty} \frac{6w^3}{(w^4+1)^2}\, dw = \lim_{t\to-\infty}\left[-\frac{3}{2(w^4+1)}\right]_t^0 + \lim_{x\to\infty}\left[\frac{-3}{2(w^4+1)}\right]_0^x$$

$$= \lim_{t\to-\infty}\left[\frac{-3}{2(0+1)} + \frac{3}{2(t^4+1)}\right] + \lim_{x\to\infty}\left[\frac{-3}{2(x^4+1)} + \frac{3}{2(0+1)}\right]$$

$$= \lim_{t\to-\infty}\left[\frac{3}{2(t^4+1)}\right] - \frac{3}{2} + \lim_{x\to\infty}\left[\frac{-3}{2(x^4+1)}\right] + \frac{3}{2}$$

Note: as $t \mapsto -\infty$, $t^4 \mapsto +\infty$ and $\frac{1}{t^4+1} \mapsto 0$

Note: as $x \mapsto \infty$, $x^4 \mapsto \infty$ and $\frac{1}{x^4+1} \mapsto 0$

$$= 0 - \frac{3}{2} + 0 + \frac{3}{2} = 0$$

4.

$$\int_{-2}^{2} \frac{1}{x^2} dx$$

$$\boxed{\int_{-2}^{2} \frac{1}{x^2} dx}$$

Note that this is an improper integral because it is discontinuous at $x=0$

Strategy: split the integral at $x=0$

$$\int_{-2}^{2} \frac{1}{x^2} dx = \int_{-2}^{0} \frac{1}{x^2} dx + \int_{0}^{2} \frac{1}{x^2} dx$$

Integrating:

$$= \left[\frac{-1}{x}\right]_{-2}^{0} + \left[\frac{-1}{x}\right]_{0}^{2}$$

Since it is not defined at $x=0$, take a limit.

$$= \lim_{R \to 0^-} \left[-\frac{1}{x}\right]_{-2}^{R} + \lim_{t \to 0^+} \left[-\frac{1}{x}\right]_{t}^{2}$$

$$= \lim_{R \to 0^-} \left[-\frac{1}{R} - \frac{1}{2}\right] + \lim_{t \to 0^+} \left[-\frac{1}{2} + \frac{1}{t}\right]$$

$$= \underbrace{\lim_{R \to 0^-} \left[-\frac{1}{R}\right]}_{\text{As } R \mapsto 0^-,\ \frac{1}{R} \mapsto \infty} + \underbrace{\lim_{t \to 0^+} \left[\frac{1}{t}\right]}_{\text{As } t \mapsto 0^+,\ \frac{1}{t} \mapsto \infty} - \frac{1}{2} - \frac{1}{2}$$

$\Longrightarrow$ Since both limits diverge, the overall integral also diverges.

5.

$$\int_{-\infty}^{\infty} \cos(\pi t)dt$$

$$\int_{-\infty}^{\infty} \cos(\pi t)\, dt$$

Applying the same strategy as the previous question, split the integral about $t = 0$

$$\int_{-\infty}^{\infty} \cos(\pi t)\, dt = \int_{0}^{\infty} \cos(\pi t)\, dt + \int_{-\infty}^{0} \cos(\pi t)\, dt$$

Integrating:

$$= \left[-\frac{\sin(\pi t)}{\pi} \right]_0^{\infty} + \left[-\frac{\sin(\pi t)}{\pi} \right]_{-\infty}^{0}$$

Since we can't directly substitute ∞, take limits.

$$= \lim_{R \to \infty} \left[-\frac{\sin(\pi t)}{\pi} \right]_0^{R} + \lim_{x \to -\infty} \left[-\frac{\sin(\pi t)}{\pi} \right]_x^{0}$$

$$= \lim_{R \to \infty} \left[-\frac{\sin(\pi R)}{\pi} + \cancelto{0}{\frac{\sin(0)}{\pi}} \right] + \lim_{x \to -\infty} \left[\cancelto{0}{-\frac{\sin(0)}{\pi}} + \frac{\sin(x\pi)}{\pi} \right]$$

$$= \underbrace{\lim_{R \to \infty} \left[-\frac{\sin(\pi R)}{\pi} \right]}_{\substack{\text{as } R \to \infty, \\ -1 \le \sin(\pi R) \le 1 \\ \text{limit diverges.}}} + \underbrace{\lim_{x \to -\infty} \left[\frac{\sin(x\pi)}{\pi} \right]}_{\substack{\text{As } x \to -\infty, \\ -1 \le \sin(x\pi) \le 1 \\ \text{limit diverges.}}}$$

$\Longrightarrow$ since both limits diverge, the integral diverges.

6.

$$\int_0^\infty \frac{e^x}{e^{2x}+3}dx$$

$$\int_0^\infty \frac{e^x}{e^{2x}+3}\,dx$$

<u>Strategy</u>: U-substitution

Let $u = e^x$

$$\frac{du}{dx} = e^x \Longrightarrow du = e^x\,dx \Longrightarrow dx = \frac{1}{e^x}\,du$$

Changing limits:

- when $x=0$, $u = e^0 = 1$
- when $x=\infty$, $u = e^\infty = \infty$

Performing substitution:

$$\int_{\underset{1}{0}}^\infty \frac{e^x}{\underbrace{e^{2x}+3}_{u^2+3}}\,dx \; \hookrightarrow \frac{1}{e^x}du \quad = \int_1^\infty \frac{\cancel{e^x}}{u^2+3}\cdot\frac{1}{\cancel{e^x}}\,du$$

$$= \int_1^\infty \frac{1}{u^2+3}\,du$$

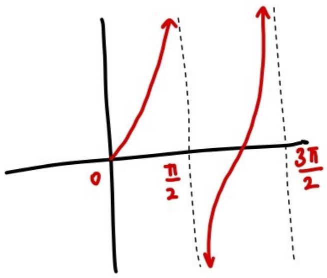
$$= \left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)\right]_1^\infty \longrightarrow \text{Rewrite in limit form}$$

$$= \lim_{R\to\infty}\left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)\right]_1^\infty$$

$$= \underbrace{\lim_{R\to\infty}\left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{R}{\sqrt{3}}\right)\right]} - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}\cdot\frac{\pi}{2} - \frac{1}{\sqrt{3}}\cdot\frac{\pi}{6}$$

$$= \frac{\pi}{3\sqrt{3}}$$

as $R \longmapsto \infty$, $\tan^{-1}(\infty) \longmapsto \frac{\pi}{2}$ because the graph of $\tan x$ looks something like this:



7.

$$\int_1^\infty \frac{\ln(x)}{x^2} dx$$

$$\boxed{\int_1^\infty \frac{\ln(x)}{x^2} dx}$$

Apply integration by parts:

$u = \ln(x)$ $dv = x^{-2}$

$du = \frac{1}{x}$ $v = -x^{-1}$

$$\int_1^\infty \frac{\ln(x)}{x^2} dx = \left[-\frac{\ln(x)}{x}\right]_1^\infty - \int_1^\infty -\frac{1}{x^2} dx$$

$$= \left[-\frac{\ln(x)}{x}\right]_1^\infty + \int_1^\infty \frac{1}{x^2} dx$$

$$= \left[-\frac{\ln(x)}{x}\right]_1^\infty + \left[\frac{-1}{x}\right]_1^\infty \longrightarrow \text{Rewrite as limit}$$

$$= \lim_{R\to\infty}\left[-\frac{\ln(x)}{x}\right]_1^R + \lim_{R\to\infty}\left[\frac{-1}{x}\right]_1^\infty$$

$$= \lim_{R\to\infty}\left[-\frac{\ln(R)}{R} + \frac{\cancel{\ln(1)}}{1}\right] + \lim_{R\to\infty}\left[\frac{-1}{R} + \frac{1}{1}\right]$$

$0 = \ln(1)$

$$= \underbrace{\lim_{R\to\infty}\left[\frac{-\ln(R)}{R}\right]}_{\text{As } R\mapsto\infty,\ \ln(R)\mapsto\infty} + \underbrace{\lim_{R\to\infty}\left[\frac{-1}{R}\right]}_{\text{as } R\mapsto\infty,\ \frac{1}{R}\mapsto 0} + 1$$

$\Rightarrow \frac{\infty}{\infty} \rightarrow$ apply L'Hôpital's Rule

$$\overset{H}{=} \underbrace{\lim_{R\mapsto\infty}\left[-\frac{1/R}{1}\right]}_{\text{as } R\mapsto\infty,\ \frac{1}{R}\mapsto 0} + 0 + 1 = 0 + 0 + 1 = 1$$

8.

$$\int_0^\infty \frac{1}{z^2+3z+2}dz$$

$$\boxed{\int_0^\infty \frac{1}{z^2+3z+2}\,dz}$$

Strategy: complete the square in the denominator

z^2+3z+2

$= \left(z^2+3z+\frac{9}{4}\right)-\frac{1}{4}$

Notice: $\left(z+\frac{3}{2}\right)^2 = z^2+3z+\frac{9}{4}$

$= \left(z+\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2$ } use this as new denominator

$$\int_0^\infty \frac{1}{\left(z+\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}dz \longrightarrow a^2-b^2=(a+b)(a-b)$$

This is algebraically true. You can verify it.

$$= \int_0^\infty \frac{1}{\left(z+\frac{3}{2}+\frac{1}{2}\right)\left(z+\frac{3}{2}-\frac{1}{2}\right)}dz = \int_0^\infty \frac{1}{(z+2)(z+1)}dz$$

Strategy: Use partial fractions to split into simpler fractions.

$$\frac{1}{(z+2)(z+1)} = \frac{A}{(z+2)}+\frac{B}{(z+1)} = \frac{A(z+1)+B(z+2)}{(z+2)(z+1)} = \frac{-1}{z+2}+\frac{1}{z+1}$$

Equating numerators:

$1 = A(z+1)+B(z+2)$

- when $z=-1$:

  $1 = A(-1+1)+B(-1+2)$

  $1 = B$

- when $z=-2$

  $1 = A(-2+1)+B(-2+2)$

  $1 = -A$

  $A=-1$

$$\Longrightarrow \int_0^\infty \frac{1}{(z+1)(z+2)}dz = \int_0^\infty \frac{1}{z+1}-\frac{1}{z+2}\,dz$$

Integrate:

$$= \int_0^\infty \frac{1}{z+1}dz - \int_0^\infty \frac{1}{z+2}dz = \Big[\ln(z+1)\Big]_0^\infty - \Big[\ln(z+2)\Big]_0^\infty$$

$$= \lim_{R\to\infty}\left[\ln\left(\frac{z+1}{z+2}\right)\right]_0^R = \lim_{R\to\infty}\left[\ln\left(\frac{R+1}{R+2}\right)\right]-\ln\left(\frac{1}{2}\right)$$

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$$= \lim_{R \to \infty} \left[\ln\left(\frac{R+1}{R+2}\right) \right] + \ln(2)$$

$$= \ln\left[\lim_{R \to \infty} \left(\frac{R+1}{R+2}\right) \right] + \ln(2) \longrightarrow$$ taking limit inside natural log (see logarithm & limit laws)

To compute $\lim_{R \to \infty} \left(\frac{R+1}{R+2}\right) = \frac{\infty}{\infty} \Longrightarrow$ Apply L'Hôpital's Rule.

$$\overset{\text{H}}{=} \ln\left[\lim_{R \to \infty} \left(\frac{1}{1}\right) \right] + \ln(2)$$

$$= \ln(1) + \ln(2)$$

$$= \ln(2)$$