Review Sheet: Implicit Differentiation

Content Review

Overview

Implicit differentiation involves trying to take the derivative with respect to a variable that you sometimes 'cannot see'. For instance, if we are given that y is a function of x, and we want to find the derivative of y^2 with respect to x, then:

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \cdot \frac{d}{dx}(y) = 2y \cdot \frac{dy}{dx}$$

This is because of the chain rule.
Recall: The Chain Rule states that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Note that the final solution to the problem above contains a term $\frac{dy}{dx}$. This is because we don't explicitly know what y is in terms of x, so we cannot simplify this term further.
However, if it was given that $y = \frac{\sin x}{2}$, for example, then we can simplify

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin x}{2}\right)$$

Frequently Asked Questions

1. Chain Rule vs Implicit Differentiation?

   Ans:Implicit Differentiation is a special application of chain rule, where the $\frac{dy}{dx}$ term appears because y is a function of x, and by chain rule, we must take the derivative of y with respect to x as well.

2. Partial Differentiation vs Implicit Differentiation? *[Ignore this if you haven't come across partial derivatives yet!]*

   Ans: Partial Differentiation involves holding all remaining the variables constant while differentiating with respect to one of the variables. For example:

   $$\frac{\partial}{\partial x}x^2 + y^2 = \frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial x}y^2 = 2x + 0 = 2x$$

   Here, the ***partial derivative*** of y^2 with respect to x is 0, since we pretend that y^2 is a constant with respect to x (i.e. a change in x does not cause a change in y, and y is not a function of x)

   However, in Implicit Differentiation, we assume that y is a function of x, so the same expression above would evaluate to:

   $$\frac{d}{dx}x^2 + y^2 = \frac{d}{dx}x^2 + \frac{d}{dx}y^2 = 2x + 2y\frac{dy}{dx}$$

   Note the use of ∂ for partial differentiation vs the use of d for implicit differentiation.

Skip to Worked Problems

Resources

Implicit Differentiation

- Worksheet (with Solutions): Mr Yang Teacher's Website
- Worksheet (with Solutions): Implicit Differentiation - Extra Practice
- Worksheet (with Solutions): Related Rates & Implicit Differentiation Word Problems

- Worksheet (with Worked Solutions): Simple Implicit Differentiation
- Video: Implicit Differentiation (3Blue1Brown, 15 min) - building intuition
- Implicit Differentiation, explained (Organic Chemistry Tutor, 12 min)
- Implicit Differentiation, Explained (Khan Academy, 8 min)

Acknowlegement

Questions in the Worked Problems section of this sheet have been taken from external sources that have been linked where appropriate. All solutions have been developed independently.

Worked Problems

Basic Implicit Differentiation

For the following section, assume that y is a function of x, and find $\frac{dy}{dx}$
Source

1. $3x^2 + y = 14$

$\boxed{3x^2+y=14}$ Want to find $\frac{dy}{dx}$

$\underbrace{3x^2}_{\text{Take derivative of this normally}} + y = \underbrace{14}$

Take derivative of this "normally", then multiply by $\frac{dy}{dx}$ since we don't know the explicit form of the function y in terms of x.

$\frac{d}{dx}(3x^2) = 3\cdot 2x = 6x$

$\frac{d}{dx}(14) = 0$ (14 is a constant, derivative of a constant w.r.t anything is always 0)

$\frac{d}{dx}(y) = 1\cdot\frac{dy}{dx} = \frac{dy}{dx}$

"Normal" derivative of y w.r.t. y.

$3x^2 + y = 14$

Taking derivative of this expr. w.r.t. x gives:

$6x + \frac{dy}{dx} = 0$

Rearrange to be in terms of $\frac{dy}{dx}$.

$\boxed{\frac{dy}{dx} = -6x}$

2. $x^2y^2 = 1$

$\boxed{x^2y^2=1}$ want to find $\frac{dy}{dx}$

Need to use product rule here, because we are trying to take derivative of 2 expressions in terms of x (x^2 and y^2) multiplied together.

So taking derivative of expression w.r.t. x gives:

$\frac{d}{dx}(\underset{u}{x^2}\underset{v}{y^2}) = \underbrace{\frac{d}{dx}(1)}$ ↳ Derivative of a constant is zero

[Recall: Product rule ⟶ $(uv)' = u'v + v'u$]

$\underbrace{\frac{d}{dx}(x^2)}_{\text{"Normal" derivative}}\cdot y^2 + \underbrace{\frac{d}{dx}(y^2)}_{\text{Implicit derivative}}\cdot x^2 = 0$

$2x\cdot y^2 + 2y\cdot\frac{d}{dx}(y)\cdot x^2 = 0$

$2xy^2 + 2yx^2\frac{dy}{dx} = 0$

Rearranging in terms of $\frac{dy}{dx}$:

$2yx^2\frac{dy}{dx} = -2xy^2$

$\frac{dy}{dx} = \frac{-\cancel{2}x\cancel{y^2}}{\cancel{2y}\cancel{x^2}} = -\frac{y}{x}$

$\boxed{\frac{dy}{dx} = \frac{-y}{x}}$

3. $\sqrt{y} + xy^2 = 15$

$$\underbrace{\sqrt{y}}_{\text{implicit differentiation}} + \underbrace{xy^2}_{\text{product rule \& implicit differentiation}} = 15$$

Want to find $\frac{dy}{dx}$

$$\frac{d}{dx}\left(\sqrt{y} + xy^2\right) = \frac{d}{dx}(15)$$

15 is a constant so its derivative will be 0

$$\frac{d}{dx}(\sqrt{y}) = \frac{d}{dy}(\sqrt{y}) \cdot \frac{dy}{dx}$$

$$= \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx}$$

Applying product rule

$$\frac{d}{dx}(xy^2) = \frac{d}{dx}(x) \cdot y^2 + \frac{d}{dx}(y^2) \cdot x$$

$$= 1 \cdot y^2 + \frac{d}{dy}(y^2) \cdot \frac{dy}{dx} \cdot x$$

Applying chain rule / implicit differentiation

$$= y^2 + 2y \cdot \frac{dy}{dx} \cdot x$$

$$= y^2 + 2xy\frac{dy}{dx}$$

Substituting above results back into equation:

$$\frac{1}{2\sqrt{y}}\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} = 0$$

Now rearrange in terms of $\frac{dy}{dx}$:

$$\frac{dy}{dx}\left(\frac{1}{2\sqrt{y}} + 2xy\right) + y^2 = 0$$

$$\frac{dy}{dx}\left(\frac{1}{2\sqrt{y}} + 2xy\right) = -y^2$$

$$\frac{dy}{dx}\left(\frac{1 + 4xy\sqrt{y}}{2\sqrt{y}}\right) = -y^2$$

$$\frac{dy}{dx} = \frac{-2y^2\sqrt{y}}{1 + 4xy\sqrt{y}}$$

4. $(x - y)^2 = x + y - 1$

$(x-y)^2 = x+y-1$ Want to find $\frac{dy}{dx}$

Taking derivative w.r.t. x on both sides of equation:

$$\underbrace{\frac{d}{dx}(x-y)^2}_{\text{Chain Rule + Implicit Differentiation}} = \underbrace{\frac{d}{dx}(x+y-1)}_{\text{Implicit differentiation.}}$$

$$\frac{d}{dx}(x-y)^2 = 2(x-y)\cdot\frac{d}{dx}(x-y) \quad \text{[Chain Rule]}$$

$$= 2(x-y)\cdot\left[\frac{d}{dx}(x) - \frac{d}{dx}(y)\right] \quad \text{[Implicit Differentiation]}$$

$$= 2(x-y)\cdot\left(1 - \frac{dy}{dx}\right)$$

$$\frac{d}{dx}(x+y-1) = \frac{d}{dx}(x) + \frac{d}{dx}(y) + \frac{d}{dx}(1)$$

$$= 1 + \frac{dy}{dx} \; (+0)$$

↳ since 1 is a constant, its derivative will be 0.

Substituting above results back into equation:

$$2(x-y)\left(1-\frac{dy}{dx}\right) = 1 + \frac{dy}{dx}$$

Rearranging in terms of $\frac{dy}{dx}$

$$2\left(x - x\frac{dy}{dx} - y + y\frac{dy}{dx}\right) = 1 + \frac{dy}{dx}$$

$$2x - 2y - 2x\frac{dy}{dx} + 2y\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$2x - 2y - 1 = \frac{dy}{dx} + 2x\frac{dy}{dx} - 2y\frac{dy}{dx}$$

$$2x - 2y - 1 = \frac{dy}{dx}(1 + 2x - 2y)$$

$$\frac{dy}{dx} = \frac{2x-2y-1}{1+2x-2y}$$

5. $\cos^2 x + \cos^2 y = \cos(2x + 2y)$

$$\underbrace{\cos^2 x}_{\text{Chain Rule}} + \cos^2 y = \cos(2x+2y)$$

Want to find $\frac{dy}{dx}$

Chain Rule + Implicit differentiation

Taking derivative w.r.t x across equation:

$$\frac{d}{dx}\left(\cos^2 x + \cos^2 y\right) = \frac{d}{dx}\left(\cos(2x+2y)\right)$$

$$\frac{d}{dx}\left(\cos^2 x\right) + \frac{d}{dx}\left(\cos^2 y\right) = \frac{d}{dx}\left(\cos(2x+2y)\right)$$

$$\frac{d}{dx}\cos^2 x = 2\cos x \cdot \frac{d}{dx}\cos x \quad \text{[Chain Rule]}$$

$$= 2\cos x \cdot -\sin x$$

$$= -2\sin x \cos x$$

$$\frac{d}{dx}\cos^2 y = 2\cos y \cdot \frac{d}{dx}\cos y \quad \text{[Chain Rule]}$$

$$= 2\cos y \cdot -\sin y \cdot \frac{d}{dx} y \quad \text{[Implicit Differentiation]}$$

$$= -2\sin y \cos y \frac{dy}{dx}$$

$$\frac{d}{dx}\cos(2x+2y) = -\sin(2x+2y) \cdot \frac{d}{dx}(2x+2y) \quad \text{[Chain Rule]}$$

$$= -\sin(2x+2y) \cdot \left(2 + 2\frac{dy}{dx}\right) \quad \text{[Implicit Differentiation]}$$

$$= -2\sin(2x+2y)\left(1 + \frac{dy}{dx}\right)$$

Substituting back into equation:

$$\cancel{-2}\sin x\cos x \cancel{-2}\sin y\cos y\frac{dy}{dx} = \cancel{-2}\sin(2x+2y)\left(1+\frac{dy}{dx}\right)$$

$$\sin x\cos x + \sin y \cos y \frac{dy}{dx} = \sin(2x+2y) + \sin(2x+2y)\frac{dy}{dx}$$

Expressing in terms of $\frac{dy}{dx}$:

$$\sin x\cos x - \sin(2x+2y) = \sin(2x+2y)\frac{dy}{dx} - \sin y\cos y\frac{dy}{dx}$$

$$\sin x\cos x - \sin(2x+2y) = \frac{dy}{dx}\left[\sin(2x+2y) - \sin y\cos y\right]$$

$$\frac{dy}{dx} = \frac{\sin x\cos x - \sin(2x+2y)}{\sin(2x+2y) - \sin y\cos y}$$

There is a way to use trig. identities (e.g. double angle identities) to simplify this further but for the purposes of this worksheet, we stop here.

6. $x = \sqrt{\tan y^2}$

$x = \sqrt{\tan y^2}$ want to find $\frac{dy}{dx}$

Note: Taking derivative of R.H.S. as it currently stands ($\sqrt{\tan y^2}$) would involve multiple applications of Chain Rule. Instead, it might be more useful to rearrange the equation before taking the derivative as follows:

$x = \sqrt{\tan y^2}$
$x^2 = \tan y^2$

Square on both sides.

Now take derivative on both sides w.r.t. x:

$$\underbrace{\frac{d}{dx}(x^2)}_{\text{Normal differentiation}} = \underbrace{\frac{d}{dx}(\tan y^2)}_{\text{Chain Rule + Implicit Differentiation.}}$$

$\frac{d}{dx}(x^2) = 2x$

$\frac{d}{dx}(\tan y^2) = \sec^2(y^2) \cdot \frac{d}{dx}(y^2) = \sec^2(y^2) \cdot 2y \cdot \frac{d}{dx}(y) = \sec^2(y^2) \cdot 2y \cdot \frac{dy}{dx}$

Substituting results into equation above:

$2x = \sec^2(y^2) \cdot 2y \cdot \frac{dy}{dx}$

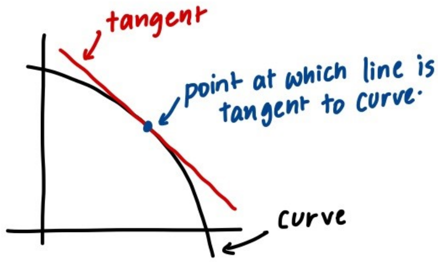
$\frac{dy}{dx} = \frac{2x}{\sec^2(y^2) \cdot 2y} = \frac{x}{y \cdot \sec^2(y^2)} \implies \boxed{\frac{dy}{dx} = \frac{x}{y \sec^2(y^2)}}$

Note: Word Problems continue on the next page.

Word Problems

Source

1. Find the equation of the tangent line to the curve defined by $2x^3 - 5y^2 = 5$ at the point $(2, -1)$.



To find the equation of the tangent
(1) Find slope of line
(2) Find a point on tangent
(3) Substitute into equation of line

$2x^3 - 5y^2 = 5$

(1) Find slope of tangent line at $(2,-1)$

"Slope" ⟶ derivative ⟶ Find $\frac{dy}{dx}$ and evaluate at $(2,-1)$

$2x^3 - 5y^2 = 5$

Taking derivative w.r.t. x:

$$\frac{d}{dx}\left(2x^3 - 5y^2\right) = \frac{d}{dx}(5)$$

$$\underbrace{\frac{d}{dx}(2x^3)}_{\text{Normal differentiation}} - \underbrace{\frac{d}{dx}(5y^2)}_{\text{Implicit differentiation. + Chain Rule}} = \frac{d}{dx}(5)$$

constant, so its derivative will be zero.

$$\frac{d}{dx}(2x^3) = 2 \cdot 3x^2 = 6x^2$$

$$\frac{d}{dx}(5y^2) = 5 \cdot 2y \cdot \frac{d}{dx}(y) = 10y \cdot \frac{dy}{dx}$$

Substituting back into equation gives:

$$6x^2 - 10y\frac{dy}{dx} = 0$$

Rearranging in terms of $\frac{dy}{dx}$:

$$10y\frac{dy}{dx} = 6x^2$$

$$\frac{dy}{dx} = \frac{6x^2}{10y} = \frac{3x^2}{5y}$$

Evaluating $\frac{dy}{dx}$ at $(2,-1)$:

$$\left.\frac{dy}{dx}\right|_{(2,-1)} = \frac{3(2)^2}{5(-1)} = \frac{-12}{5} = -2.4$$

Gradient of tangent line to curve at $(2,-1)$

(2) Point on tangent

Since we know, by definition, the tangent line to the curve at $(2,-1)$ passes through $(2,-1)$, use this point.

(3) Substitute into equation

$y = mx + b$ [Equation of straight line]

$y \to -1$, $m \to \frac{dy}{dx} = -2.4$, $x \to 2$, $b \to$ TBD.

Substituting the values we know:

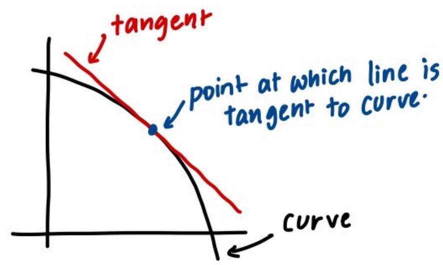
$-1 = -2.4 \times 2 + b$

$-1 = -4.8 + b$

$b = 3.8$

∴ Equation of tangent: $y = -2.4x + 3.8$

2. Find the equation of the tangent line to the curve defined by $y^2 + 2xy + 4 = 0$ at the point $(2, -2)$



To find the equation of the tangent
(1) Find slope of line
(2) Find a point on tangent
(3) Substitute into equation of line

$$y^2 + 2xy + 4 = 0$$

(1) Find slope of tangent line at $(2,-2)$

"slope" $\longrightarrow$ derivative $\longrightarrow$ Find $\frac{dy}{dx}$ and evaluate at $(2,-2)$

$$y^2 + 2xy + 4 = 0$$

Taking derivative w.r.t. x:

$$\frac{d}{dx}(y^2 + 2xy + 4) = 0$$

$$\underbrace{\frac{d}{dx}(y^2)}_{\text{Chain Rule + Implicit Diff.}} + \underbrace{\frac{d}{dx}(2xy)}_{\text{Product Rule + Implicit diff.}} + \underbrace{\frac{d}{dx}(4)}_{\text{Normal differentiation.}} = 0$$

[4 is a constant, so derivative is 0]

$$\frac{d}{dx}(y^2) = 2y \cdot \frac{d}{dx}(y) = 2y\frac{dy}{dx}$$

$$\frac{d}{dx}(2xy) = \frac{d}{dx}(2x)\cdot y + \frac{d}{dx}(y)\cdot 2x = 2y + 2x\frac{dy}{dx}$$

Substituting back into equation gives:

$$2y\frac{dy}{dx} + 2y + 2x\frac{dy}{dx} = 0$$

Rearranging in terms of $\frac{dy}{dx}$:

$$\frac{dy}{dx}(2y + 2x) = -2y$$

$$\frac{dy}{dx} = \frac{-2y}{2y+2x} = \frac{-y}{x+y}$$

Evaluating $\frac{dy}{dx}$ at $(2,-2)$:

$$\left.\frac{dy}{dx}\right|_{(2,-2)} = \frac{-(-2)}{2-2} = \frac{2}{0}$$

∞, meaning the tangent line is vertical

(2) Point on tangent

Since we know, by definition, the tangent line to the curve at $(2,-2)$ passes through $(2,-2)$, use this point.

(3) Substitute into equation

Since we know the tangent line is vertical and passes through $x=2$, the equation of tangent must be $x = 2$