# Gram-Schmidt in 9 Lines of MATLAB

The Gram-Schmidt algorithm starts with $n$ independent vectors $a_1, \ldots, a_n$ (the columns of $A$). It produces $n$ orthonormal vectors $q_1, \ldots, q_n$ (the columns of $Q$). To find $q_j$, start with $a_j$ and subtract off its projections onto the previous $q$'s—and then divide by the length of that vector $v$ to produce a unit vector.

The inner products $q_i^{\mathrm{T}} a_j$ go into a square matrix $R$ that satisfies $A = QR$. This $R$ is upper triangular, because $q_i^{\mathrm{T}} a_j = 0$ when $i$ is larger than $j$ (later $q$'s are orthogonal to earlier $a$'s, that is the point of the algorithm).

Here is a 9-line MATLAB code to build $Q$ and $R$ from $A$. Start with $[m, n] = \texttt{size}(A); Q = \texttt{zeros}(m, n); R = \texttt{zeros}(n, n);$ to get the shapes correct.

```
for j=1:n                       % Gram-Schmidt orthogonalization
  v=A(:,j);                     % v begins as column j of A
  for i=1:j-1
    R(i,j)=Q(:,i)'*A(:,j);      % modify A(:,j) to v for more accuracy
    v=v-R(i,j)*Q(:,i);          % subtract the projection (q_i^T a_j)q_i = (q_i^T v)q_i
  end                           % v is now perpendicular to all of q_1,...,q_{j-1}
  R(j,j)=norm(v);
  Q(:,j)=v/R(j,j);              % normalize v to be the next unit vector q_j
end
```

If you undo the last step and the middle steps, you find column $j$:

$$R(j,j)q_j = (v \textit{ minus its projections}) = (\text{column } j \text{ of } A) - \sum_{i=1}^{j-1} R(i,j)q_i \,.$$

Moving the sum to the far left, this is column $j$ in the multiplication $A = QR$.

That crucial change from $a_j$ to $v$ in line 4 gives "*modified Gram-Schmidt*." In exact arithmetic, the number $R(i,j) = q_i^{\mathrm{T}} a_j$ is the same as $q_i^{\mathrm{T}} v$. (The current $v$ has subtracted from $a_j$ its projections onto earlier $q_1, \ldots, q_{i-1}$. But the new $q_i$ is orthogonal to those directions.) In real arithmetic this orthogonality is not perfect, and computations show a difference in $Q$. Everybody uses $v$ at that step in the code.

EXAMPLE $A$ is 2 by 2. The columns of $Q$, normalized by $\frac{1}{5}$, are $q_1$ and $q_2$:

$$A = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 0 & 2 \end{bmatrix} = QR\,.$$

Starting with the columns $a_1$ and $a_2$ of $A$, Gram-Schmidt normalizes $a_1$ to $q_1$ and subtracts from $a_2$ its projection in the direction of $q_1$. Here are the steps to the $q$'s:

$$a_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad q_1 = \frac{1}{5} \begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad a_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad v = a_2 - (q_1^{\mathrm{T}} a_2) q_1 = \frac{1}{5} \begin{bmatrix} -6 \\ 8 \end{bmatrix} \quad q_2 = \frac{1}{5} \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

Along the way, we divided by $\|a_1\| = 5$ and $\|v\| = 2$. Then 5 and 2 go on the diagonal of $R$, and $q_1^{\mathrm{T}} a_2 = -1$ is $R(1,2)$. This figure shows every vector: