Topics in Algebra Notes

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1 First Isomorphism Theorem

Theorem 1.0.1: First Isomorphism Theorem
If $\varphi : G \to H$ is a homomorphism of groups then $G/\mathrm{Ker}(\varphi) \cong \mathrm{Im}(\varphi)$

Definition 1.0.2
Let G be a group and H_1 and H_2 be subgroups. Then the *join* of H_1 and H_2 is

$$H_1 \vee H_2 = \bigcap_{H_1 H_2 \leq G'} G'.$$

The join of two subgroups is the smallest subgroup containing both.

Lemma 1.0.3
Let H and N be subgroups of a group G, and let N be normal in G. Then $H \vee N = HN$. Moreover, if H is also normal in G, then HN is normal in G.

Proof. We will show that HN is a subgroup. Since $H \vee N$ is the smallest subgroup that contains HN, this will prove the proposition. So let $x = h_1 n_1 \in HN$ and $y = h_2 n_2 \in HN$. We msut show that $xy^{-1} \in HN$. $xy^{-1} = h_1 n_1 n_2^{-1} h_2^{-1}$. Since N is a normal subgroup, this is $h_1 h'(n_1 n_2^{-1})$ for some h', and this is an element of HN.
Now suppose that H is normal in G. Then for any $g \in G$, $g(hn)g^{-1} = ghg^{-1}gng^{-1}$, and since H and N are normal they are closed under conjugation and this is an element of HN. □

Theorem 1.0.4: Second Isomorpism Theorem
Let H and N be subgroups of a group G, and let N be normal in G. Then $H/(H \cap N) \cong NH/N$.

Proof. First we must show that $H/(H \cap N)$ and NH/N are groups at all (done by showing that the subgroups are normal in the appropriate groups). Let $i : H \to HN$ by $h \mapsto he_N$, and let $q : HN \to HN/N$ by $hn \mapsto (hn)N$. Now let $f = q \circ i : H \to HN/N$. Then $\mathrm{Im}(f) = HN/N$ and $\mathrm{Ker}(f) = H \cap N$. Now we apply the first isomorphsim theorem to get the result. □

Theorem 1.0.5: Third Isomorphism Theorem
If H and K are normal subgroups of a group G, and $K \subseteq H$, then $(G/K)/(H/K) \cong G/H$

Theorem 1.0.6: Correspondence Theorem
Let K be a normal subgroup of a group G. There is a one to one correspondence between (normal) subgroups H of G that contain K and (normal) subgroups of G/K. This correspondence is given by $H \mapsto H/K$.

2 Series of Groups

Definition 2.0.1
A group G is *simple* if it has no non-trivial normal subgroups.

Note that this definition is equivalent to saying that G has no non-trivial factor groups.

Definition 2.0.2
Consider a finite series of subgroups of a group G, $\{1_G\} = H_0 \leq H_1 \leq \cdots \leq H_n = G$.

- If $H_i \trianglelefteq H_{i+1}$ for all i the series is a a *subnormal series*.
- If $H_i \trianglelefteq G$ for all i then the series if a *normal series*.
- If a series is normal or subnormal, then the groups H_{i+1}/H_i are called *factor groups* of G.
- If all the factor groups of a subnormal series are simple it is a *composition series*.
- If all the factor groups of a normal series are simple then it is a *principal series*.

Note that if a group is Abelian then all series are normal and principal series.

Example 2.0.3

- In $(\mathbb{Z}, +)$, the series $\{0\} \leq 8\mathbb{Z} \leq 4\mathbb{Z} \leq \mathbb{Z}$ is a subnormal (or normal since $\mathbb{Z}$ Abelian) but not principal series.
- D_8, the dihedral group with 8 elements has the following subnormal series $\{1\} \leq \langle s\rangle \leq \langle r^2, s\rangle \leq D_8$. To check that this series is subnormal note that the index of each subgroup in the following subgroup is 2.
- In $(\mathbb{Z}_{15}, +)$, both of the following series are normal (beacuse the group is Abelian): $\{0\} \leq \angle 5\rangle \leq \mathbb{Z}_{15}$, and $\{0\} \leq \langle 3\rangle \leq \mathbb{Z}_{15}$.

Definition 2.0.4
$\{K_j\}$ is a refinement of a series of subgroups $\{H_i\}$ if $\{H_i\} \subseteq \{K_j\}$.

Definition 2.0.5 (T)
o subnormal series $\{H_i\}$ and $\{K_j\}$ are isomorphic if there exists a bijection between $\{K_{i+1}/K_i\}$ and $\{H_{i+1}/H_i\}$ such that corresponding factor groups are isomorphic.

The third example from above is an example of isomorphic series.

Theorem 2.0.6: Schreier's Theorem
The subnormal series of the same group admit isomorphic refinements.

We do not prove this theorem, but the proof of the theorem is constructive, meaning that it gives the refinements of each series that are isomorphic.

Example 2.0.7
The series $\{0\} \le 8\mathbb{Z} \le 4\mathbb{Z} \le \mathbb{Z}$, and $\{0\} \le 9\mathbb{Z} \le \mathbb{Z}$ can be refined to $\{0\} \le 72\mathbb{Z} \le 8\mathbb{Z} \le 4\mathbb{Z} \le \mathbb{Z}$ and $\{0\} \le 72\mathbb{Z} \le 18\mathbb{Z} \le 9\mathbb{Z} \le \mathbb{Z}$, respectively. These refinements are isomorphic.