Digital Image Processing

Filtering in the Frequency Domain
(Circulant Matrices and Convolution)

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Toeplitz matrices

- Elements with constant value along the main diagonal and sub-diagonals.
- For a NxN matrix, its elements are determined by a $(2N$-$1)$-length sequence $\{t_n \mid -(N-1) \le n \le N-1\}$

$$\mathbf{T}(m,n) = t_{m-n}$$

$$\mathbf{T} = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{-(N-1)} \\ t_1 & t_0 & t_{-1} & \ddots & \vdots \\ t_2 & \ddots & \ddots & \ddots & t_{-2} \\ \vdots & \ddots & \ddots & \ddots & t_{-1} \\ t_{N-1} & \cdots & t_2 & t_1 & t_0 \end{bmatrix}_{N\times N}$$

Toeplitz matrices (cont.)

- Each row (column) is generated by a shift of the previous row (column).
  - The last element disappears.
  - A new element appears.

$$\mathbf{T}(m,n) = t_{m-n}$$

$$\mathbf{T} = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{-(N-1)} \\ t_1 & t_0 & t_{-1} & \ddots & \vdots \\ t_2 & \ddots & \ddots & \ddots & t_{-2} \\ \vdots & \ddots & \ddots & \ddots & t_{-1} \\ t_{N-1} & \cdots & t_2 & t_1 & t_0 \end{bmatrix}_{N\times N}$$

Circulant matrices

- Elements with constant value along the main diagonal and sub-diagonals.
- For a NxN matrix, its elements are determined by a N-length sequence $\{c_n \mid 0 \leq n \leq N-1\}$

$$\mathbf{C}(m,n) = c_{(m-n) \bmod N}$$

$$\mathbf{C} = \begin{bmatrix} c_0 & c_{N-1} & c_{N-2} & \cdots & c_1 \\ c_1 & c_0 & c_{N-1} & \cdots & \vdots \\ c_2 & c_1 & c_0 & \cdots & c_{N-2} \\ \vdots & \vdots & \vdots & \cdots & c_{N-1} \\ c_{N-1} & c_{N-2} & \cdots & c_1 & c_0 \end{bmatrix}_{N \times N}$$

Circulant matrices (cont.)

- Special case of a Toeplitz matrix.
- Each row (column) is generated by a **circular shift** (modulo N) of the previous row (column).

$$\mathbf{C}(m,n) = c_{(m-n)\,\mathrm{mod}\,N}$$

$$\mathbf{C} = \begin{bmatrix} c_0 & c_{N-1} & c_{N-2} & \cdots & c_1 \\ c_1 & c_0 & c_{N-1} & \cdots & \vdots \\ c_2 & c_1 & c_0 & \cdots & c_{N-2} \\ \vdots & \vdots & \vdots & \cdots & c_{N-1} \\ c_{N-1} & c_{N-2} & \cdots & c_1 & c_0 \end{bmatrix}_{N\times N}$$

Convolution by matrix-vector operations

- 1-D **linear convolution** between two discrete signals may be expressed as the product of a **Toeplitz matrix** constructed by the elements of one of the signals and a vector constructed by the elements of the other signal.
- 1-D **circular convolution** between two discrete signals may be expressed as the product of a **circulant matrix** constructed by the elements of one of the signals and a vector constructed by the elements of the other signal.
- Extension to 2D signals.

1D linear convolution using Toeplitz matrices

$$f[n]=\{\underline{1},\ 2,\ 2\},\ \ h[n]=\{\underline{1},\ -1\},\ \ N_1=3, N_2=2$$

- The linear convolution $g[n]=f[n]*h[n]$ will be of length $N=N_1+N_2-1=3+2-1=4$.
- We create a Toeplitz matrix $\mathbf{H}$ from the elements of $h[n]$ (zero-padded if needed) with
  - $N=4$ lines (the length of the result).
  - $N_1=3$ columns (the length of $f[n]$).
  - The two signals may be interchanged.

1D linear convolution using Toeplitz matrices (cont.)

$$f[n]=\{\underline{1},\ 2,\ 2\},\ \ h[n]=\{\underline{1},\ -1\},\ \ N_1=3, N_2=2$$

$$\mathbf{H}=\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}_{4\times 3}$$

Length of the result =4

Length of $f[n]=3$

Zero-padded $h[n]$ in the first column

Notice that H is not circulant (e.g. a -1 appears in the second line which is not present in the first line.

1D linear convolution using Toeplitz matrices (cont.)

$$f[n] = \{\underline{1},\ 2,\ 2\},\ \ h[n] = \{\underline{1},\ -1\},\ \ N_1 = 3, N_2 = 2$$

$$\mathbf{g} = \mathbf{Hf} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix}$$

$$g[n] = \{\underline{1},\ 1,\ 0,\ -2\}$$

1D circular convolution using circulant matrices

$$f[n]=\{\underline{1},\ 2,\ 2\},\ \ h[n]=\{\underline{1},\ -1\},\ \ N_1=3, N_2=2$$

- The circular convolution $g[n]=f[n]✪h[n]$ will be of length $N=\max\{N_1, N_2\}=3$.
- We create a circulant matrix $\mathbf{H}$ from the elements of $h[n]$ (zero-padded if needed) of size $N\mathrm{x}N$.
  - The two signals may be interchanged.

1D circular convolution using circulant matrices (cont.)

$$f[n]=\{\underline{1},\ 2,\ 2\},\ \ h[n]=\{\underline{1},\ -1\},\ \ N_1=3, N_2=2$$

$$\mathbf{H}=\begin{bmatrix} \boxed{\begin{matrix} 1 \\ -1 \\ 0 \end{matrix}} & \begin{matrix} 0 \\ 1 \\ -1 \end{matrix} & \begin{matrix} -1 \\ 0 \\ 1 \end{matrix} \end{bmatrix}_{3\times 3}$$

Zero-padded $h[n]$ in the first column

1D circular convolution using circulant matrices (cont.)

$$f[n]=\{\underline{1},\ 2,\ 2\},\ \ h[n]=\{\underline{1},\ -1\},\ \ N_1=3, N_2=2$$

$$\mathbf{g} = \mathbf{Hf} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$g[n]==\{-\underline{1}, 1, 0\}$$

Block matrices

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ A_{M1} & A_{M2} & \cdots & A_{MN} \end{bmatrix}$$

- A_{ij} are matrices.
- If the structure of A, with respect to its sub-matrices, is Toeplitz (circulant) then matrix A is called **block-Toeplitz** (**block-circulant**).
- If each individual A_{ij} is also a Toeplitz (circulant) matrix then A is called **doubly block-Toeplitz** (**doubly block-circulant**).

2D linear convolution using doubly block Toeplitz matrices

$$
\begin{array}{c|ccc}
m & & & \\
 & 1 & 4 & 1 \\
 & 2 & 5 & 3 \\
\hline
 & & & n
\end{array}
\qquad\qquad
\begin{array}{c|cc}
m & & \\
 & 1 & 1 \\
 & 1 & -1 \\
\hline
 & & n
\end{array}
$$

$$f[m,n] \qquad\qquad h[m,n]$$

$$M_1=2,\ N_1=3 \qquad\qquad M_2=2,\ N_2=2$$

The result will be of size $(M_1+M_2-1) \times (N_1+N_2-1) = 3 \times 4$

2D linear convolution using doubly block Toeplitz matrices (cont.)

$$\begin{array}{l} m \\ \begin{matrix} 1 & 4 & 1 \\ 2 & 5 & 3 \end{matrix} \quad n \\ f\,[m,n] \end{array} \qquad\qquad \begin{array}{l} m \\ \begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix} \quad n \\ h\,[m,n] \end{array}$$

- At first, $h[m,n]$ is zero-padded to $3 \text{ x } 4$ (the size of the result).
- Then, for each row of $h[m,n]$, a Toeplitz matrix with 3 columns (the number of **columns** of $f\,[m,n]$) is constructed.

$$\begin{array}{l} m \\ \begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{matrix} \quad n \\ h[m,n] \end{array}$$

2D linear convolution using doubly block Toeplitz matrices (cont.)

- For each row of $h[m,n]$, a Toeplitz matrix with 3 columns (the number of columns of $f[m,n]$) is constructed.

$$
h[m,n]: \quad
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & -1 & 0 & 0
\end{array}
$$

$$
\mathbf{H}_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}
\quad
\mathbf{H}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\quad
\mathbf{H}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

2D linear convolution using doubly block Toeplitz matrices (cont.)

m

1	4	1
2	5	3

n

$f[m,n]$

m

1	1
1	-1

n

$h[m,n]$

- Using matrices $\mathbf{H}_1$, $\mathbf{H}_2$ and $\mathbf{H}_3$ as elements, a doubly block Toeplitz matrix $\mathbf{H}$ is then constructed with 2 columns (the number of **rows** of $f[m,n]$).

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_3 \\ \mathbf{H}_2 & \mathbf{H}_1 \\ \mathbf{H}_3 & \mathbf{H}_2 \end{bmatrix}_{12\times 6}$$

2D linear convolution using doubly block Toeplitz matrices (cont.)

$$f[m,n] = \begin{matrix} 1 & 4 & 1 \\ 2 & 5 & 3 \end{matrix} \qquad h[m,n] = \begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}$$

- We now construct a vector from the elements of $f[m,n]$.

$$\mathbf{f} = \begin{bmatrix} 2 \\ 5 \\ 3 \\ \hdashline 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} (2 \quad 5 \quad 3)^T \\ \hdashline (1 \quad 4 \quad 1)^T \end{bmatrix}$$

2D linear convolution using doubly block Toeplitz matrices (cont.)

$$\begin{array}{c|ccc} m & & & \\ & 1 & 4 & 1 \\ & 2 & 5 & 3 \\ \hline & & & n \end{array}$$

$f\,[m,n]$

$$\begin{array}{c|cc} m & & \\ & 1 & 1 \\ & 1 & -1 \\ \hline & & n \end{array}$$

$h\,[m,n]$

$$\mathbf{g} = \mathbf{Hf} = \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_3 \\ \mathbf{H}_2 & \mathbf{H}_1 \\ \mathbf{H}_3 & \mathbf{H}_2 \end{bmatrix} \left[\begin{array}{c} 2 \\ 5 \\ 3 \\ \hline 1 \\ 4 \\ 1 \end{array}\right]$$

2D linear convolution using doubly block Toeplitz matrices (cont.)

$$\mathbf{g} = \mathbf{H}\mathbf{f} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} 2 \\ 5 \\ 3 \\ \hdashline 1 \\ 4 \\ 1 \end{array}\right] = \left[\begin{array}{c} 2 \\ 3 \\ -2 \\ 3 \\ \hdashline 3 \\ 10 \\ 5 \\ 2 \\ \hdashline 1 \\ 5 \\ 5 \\ 1 \end{array}\right] = \left[\begin{array}{c} (2 \quad 3 \quad -2 \quad 3)^T \\ \hdashline (3 \quad 10 \quad 5 \quad 2)^T \\ \hdashline (1 \quad 5 \quad 5 \quad 1)^T \end{array}\right]$$

2D linear convolution using doubly block Toeplitz matrices (cont.)

$$\begin{array}{ccc} 1 & 4 & 1 \\ 2 & 5 & 3 \end{array} \quad * \quad \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array}$$

$f\,[m,n]$ * $h\,[m,n]$

$$= \begin{array}{cccc} 1 & 5 & 5 & 1 \\ 3 & 10 & 5 & 2 \\ 2 & 3 & -2 & 3 \end{array}$$

$g\,[m,n]$

(axes: m vertical, n horizontal)

$$\mathbf{g} = \begin{bmatrix} (2 \quad 3 \quad -2 \quad 3)^T \\ \hdashline (3 \quad 10 \quad 5 \quad 2)^T \\ \hdashline (1 \quad 5 \quad 5 \quad 1)^T \end{bmatrix}$$

2D linear convolution using doubly block Toeplitz matrices (cont.)

Another example

$$f[m,n]: \quad \begin{array}{cc} 3 & 4 \\ 1 & 2 \end{array} \qquad h[m,n]: \quad \begin{array}{cc} 1 & -1 \end{array}$$

(m vertical axis, n horizontal axis)

$M_1=2,\ N_1=2$

$M_2=1,\ N_2=2$

The result will be of size (M_1+M_2-1) x $(N_1+N_2-1) = 2$ x 3

2D linear convolution using doubly block Toeplitz matrices (cont.)

$$f[m,n]: \quad \begin{array}{cc} 3 & 4 \\ 1 & 2 \end{array}$$

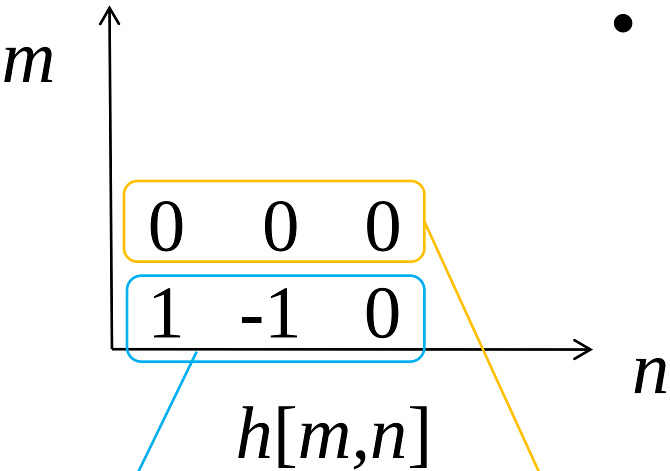
$$h[m,n]: \quad \begin{array}{cc} 1 & -1 \end{array}$$

- At first, $h[m,n]$ is zero-padded to $2 \text{ x } 3$ (the size of the result).
- Then, for each line of $h[m,n]$, a Toeplitz matrix with 2 columns (the number of **columns** of $f[m,n]$) is constructed.

$$h[m,n]: \quad \begin{array}{ccc} 0 & 0 & 0 \\ 1 & -1 & 0 \end{array}$$

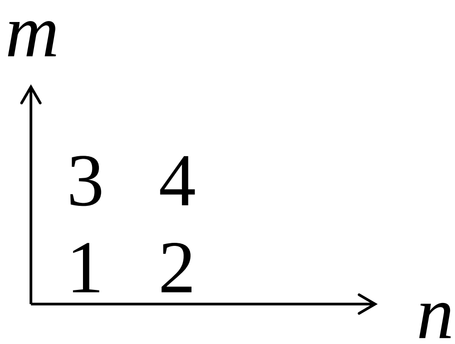
2D linear convolution using doubly block Toeplitz matrices (cont.)

- For each row of $h[m,n]$, a Toeplitz matrix with 2 columns (the number of columns of $f[m,n]$) is constructed.

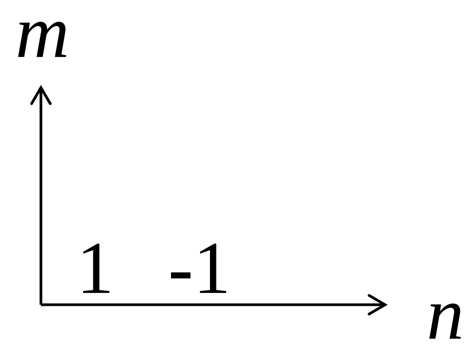


$$\mathbf{H}_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \qquad \mathbf{H}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2D linear convolution using doubly block Toeplitz matrices (cont.)



$f[m,n]$

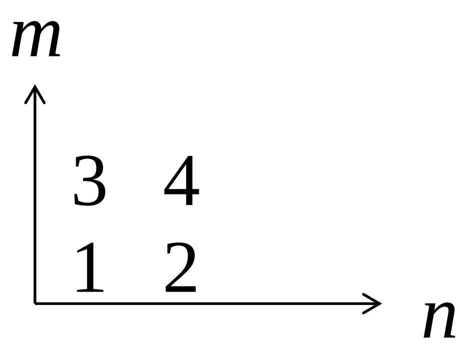


$h[m,n]$

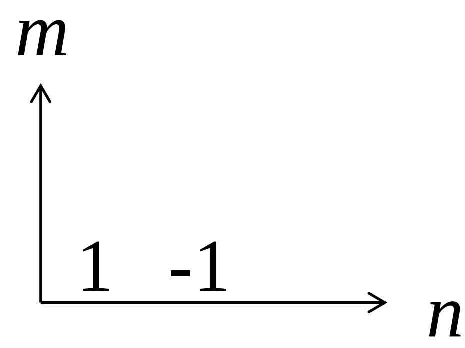
- Using matrices $\mathbf{H}_1$ and $\mathbf{H}_2$ as elements, a doubly block Toeplitz matrix $\mathbf{H}$ is then constructed with 2 columns (the number of **rows** of $f[m,n]$).

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 \\ \mathbf{H}_2 & \mathbf{H}_1 \end{bmatrix}_{6\times 4}$$

2D linear convolution using doubly block Toeplitz matrices (cont.)



$f\,[m,n]$

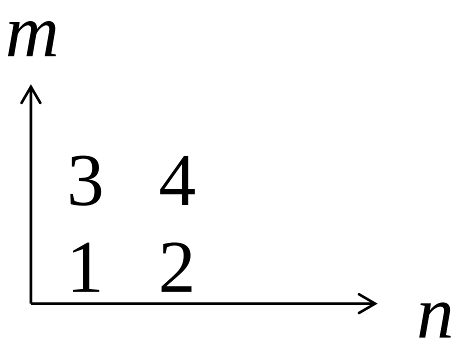


$h\,[m,n]$

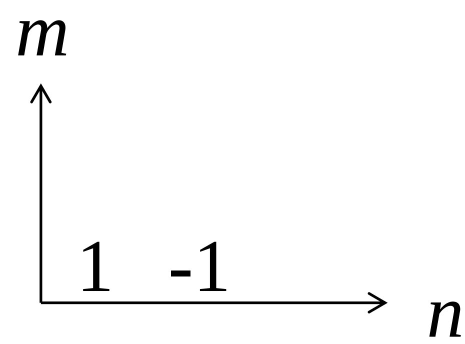
- We now construct a vector from the elements of $f\,[m,n]$.

$$\mathbf{f} = \begin{bmatrix} 1 \\ 2 \\ \hdashline 3 \\ 4 \end{bmatrix} = \begin{bmatrix} (1 \quad 2)^T \\ \hdashline (3 \quad 4)^T \end{bmatrix}$$

2D linear convolution using doubly block Toeplitz matrices (cont.)



$f\,[m,n]$



$h\,[m,n]$

$$\mathbf{g} = \mathbf{H}\mathbf{f} = \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 \\ \mathbf{H}_2 & \mathbf{H}_1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ \hdashline 3 \\ 4 \end{bmatrix}$$

2D linear convolution using doubly block Toeplitz matrices (cont.)

$$\mathbf{g} = \mathbf{H}\mathbf{f} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right] \left[\begin{array}{c} 1 \\ 2 \\ \hdashline 3 \\ 4 \end{array}\right] = \left[\begin{array}{c} 1 \\ 1 \\ -2 \\ \hline 3 \\ 1 \\ -4 \end{array}\right] = \left[\begin{array}{c} (1 \quad 1 \quad -2)^T \\ \hdashline (3 \quad 1 \quad -4)^T \end{array}\right]$$

2D linear convolution using doubly block Toeplitz matrices (cont.)

$$
\begin{array}{c|cc}
m & & \\
 & 3 & 4 \\
 & 1 & 2 \\
\hline
 & & & n
\end{array}
\quad * \quad
\begin{array}{c|cc}
m & & \\
 & 1 & -1 \\
\hline
 & & & n
\end{array}
$$

$f\,[m,n]$ $\qquad$ $h\,[m,n]$

$$
= \quad
\begin{array}{c|ccc}
m & & & \\
 & 3 & 1 & 4 \\
 & 1 & 1 & -2 \\
\hline
 & & & & n
\end{array}
$$

$g\,[m,n]$

$$
\mathbf{g} = \begin{bmatrix} (1 \quad 1 \quad -2)^T \\ \hdashline (3 \quad 1 \quad -4)^T \end{bmatrix}
$$

2D circular convolution using doubly block circulant matrices

The circular convolution $g[m,n]=f\,[m,n]\circledast h\,[m,n]$

with $0 \leq m \leq M-1,\ 0 \leq n \leq N-1,$

may be expressed in matrix-vector form as:

$$\mathbf{g} = \mathbf{Hf}$$

where $\mathbf{H}$ is a doubly block circulant matrix generated by $h\,[m,n]$ and $\mathbf{f}$ is a vectorized form of $f\,[m,n]$.

2D circular convolution using doubly block circulant matrices (cont.)

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_0 & \mathbf{H}_{M-1} & \mathbf{H}_{M-2} & \cdots & \mathbf{H}_1 \\ \mathbf{H}_1 & \mathbf{H}_0 & \mathbf{H}_{M-1} & \cdots & \mathbf{H}_2 \\ \mathbf{H}_2 & \mathbf{H}_1 & \mathbf{H}_0 & \cdots & \mathbf{H}_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \mathbf{H}_{M-1} & \mathbf{H}_{M-2} & \mathbf{H}_{M-3} & \cdots & \mathbf{H}_0 \end{bmatrix}$$

Each $\mathbf{H}_j$, for $j=1,..M$, is a circulant matrix with N columns (the number of **columns** of $f\,[m,n]$) generated from the elements of the j-th **row** of $h\,[m,n]$.

2D circular convolution using doubly block circulant matrices (cont.)

$$\mathbf{H}_j = \begin{bmatrix} h[j,0] & h[j,N-1] & \dots & h[j,1] \\ h[j,1] & h[j,0] & \dots & h[j,2] \\ \vdots & \vdots & \vdots & \vdots \\ h[j,N-1] & h[j,N-2] & \dots & h[j,0] \end{bmatrix}_{N\times N}$$

Each $\mathbf{H}_j$, for $j=1,..M$, is a NxN circulant matrix generated from the elements of the j-th **row** of $h[m,n]$.

2D circular convolution using doubly block circulant matrices (cont.)

$$\underset{f[m,n]}{\begin{matrix}0 & 1 & 0\\ 1 & 3 & -1\\ 1 & 2 & 1\end{matrix}} \quad \circledast \quad \underset{h[m,n]}{\begin{matrix}0 & 0 & 0\\ 1 & 0 & 0\\ 1 & -1 & 0\end{matrix}}$$

(axes: m vertical, n horizontal)

$$\mathbf{H}_0 = \begin{bmatrix} 1 & 0 & -1\\ -1 & 1 & 0\\ 0 & -1 & 1\end{bmatrix} \qquad \mathbf{H}_1 = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad \mathbf{H}_2 = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

2D circular convolution using doubly block circulant matrices (cont.)

$$m \quad \begin{matrix} 0 & 1 & 0 \\ 1 & 3 & -1 \\ 1 & 2 & 1 \end{matrix} \quad n \qquad \circledast \qquad m \quad \begin{matrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{matrix} \quad n$$

$$f[m,n] \qquad\qquad\qquad\qquad h[m,n]$$

$$\mathbf{g} = \mathbf{H}\mathbf{f} = \begin{bmatrix} \mathbf{H}_0 & \mathbf{H}_2 & \mathbf{H}_1 \\ \mathbf{H}_1 & \mathbf{H}_0 & \mathbf{H}_2 \\ \mathbf{H}_2 & \mathbf{H}_1 & \mathbf{H}_0 \end{bmatrix} \left[\begin{array}{c} \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}^T \\ \hdashline \begin{pmatrix} 1 & 3 & -1 \end{pmatrix}^T \\ \hdashline \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T \end{array} \right]$$

2D circular convolution using doubly block circulant matrices (cont.)

$$\mathbf{g} = \mathbf{H}\mathbf{f} = \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{array}\right] \left[\begin{array}{c} 1 \\ 2 \\ 1 \\ \hline 1 \\ 3 \\ -1 \\ \hline 0 \\ 1 \\ 0 \end{array}\right] = \left[\begin{array}{c} 0 \\ 2 \\ -1 \\ \hline 3 \\ 4 \\ -3 \\ \hline 1 \\ 4 \\ -2 \end{array}\right]$$

2D circular convolution using doubly block circulant matrices (cont.)

$$\underbrace{\begin{array}{c|ccc} m & & & \\ & 0 & 1 & 0 \\ & 1 & 3 & -1 \\ & 1 & 2 & 1 \\ \hline & & & & n \end{array}}_{f[m,n]} \quad ✪ \quad \underbrace{\begin{array}{c|ccc} m & & & \\ & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & 1 & -1 & 0 \\ \hline & & & & n \end{array}}_{h[m,n]}$$

$$= \underbrace{\begin{array}{c|ccc} m & & & \\ & 1 & 4 & -2 \\ & 3 & 4 & -3 \\ & 0 & 2 & -1 \\ \hline & & & & n \end{array}}_{g[m,n]} \qquad \mathbf{g} = \begin{bmatrix} \begin{pmatrix} 0 & 2 & -1 \end{pmatrix}^T \\ \hdashline \begin{pmatrix} 3 & 4 & -3 \end{pmatrix}^T \\ \hdashline \begin{pmatrix} 1 & 4 & -2 \end{pmatrix}^T \end{bmatrix}$$

Diagonalization of circulant matrices

Theorem: The columns of the inverse DFT matrix are eigenvectors of any circulant matrix. The corresponding eigenvalues are the DFT values of the signal generating the circulant matrix.

Proof: Let

$$w_N = e^{-j\frac{2\pi}{N}} \Leftrightarrow w_N^{nk} = e^{-j\frac{2\pi n}{N}k}$$

be the DFT basis elements of length N with:

$$0 \leq k \leq N-1,\ 0 \leq n \leq N-1,$$

Diagonalization of circulant matrices (cont.)

We know that the DFT $F[k]$ of a 1D signal $f[n]$ may be expressed in matrix-vector form:

$$\mathbf{F} = \mathbf{Af}$$

where

$$\mathbf{f} = \left[\, f[0], f[1], \ldots, f[N-1] \right]^T, \ \mathbf{F} = \left[F[0], F[1], \ldots, F[N-1] \right]^T$$

$$\mathbf{A} = \begin{bmatrix} \left(w_N^0\right)^0 & \left(w_N^0\right)^1 & \left(w_N^0\right)^2 & \cdots & \left(w_N^0\right)^{N-1} \\ \left(w_N^1\right)^0 & \left(w_N^1\right)^1 & \left(w_N^1\right)^2 & \cdots & \left(w_N^1\right)^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \left(w_N^{N-1}\right)^0 & \left(w_N^{N-1}\right)^1 & \left(w_N^{N-1}\right)^2 & \cdots & \left(w_N^{N-1}\right)^{N-1} \end{bmatrix}$$

Diagonalization of circulant matrices (cont.)

The inverse DFT is then expressed by:

$$\mathbf{f} = \mathbf{A}^{-1}\mathbf{F}$$

where

$$\mathbf{A}^{-1} = \frac{1}{N}\left(\mathbf{A}^{*}\right)^{T} = \frac{1}{N}\left(\begin{bmatrix} \left(w_N^0\right)^0 & \left(w_N^0\right)^1 & \left(w_N^0\right)^2 & \cdots & \left(w_N^0\right)^{N-1} \\ \left(w_N^1\right)^0 & \left(w_N^1\right)^1 & \left(w_N^1\right)^2 & \cdots & \left(w_N^1\right)^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \left(w_N^{N-1}\right)^0 & \left(w_N^{N-1}\right)^1 & \left(w_N^{N-1}\right)^2 & \cdots & \left(w_N^{N-1}\right)^{N-1} \end{bmatrix}^{*}\right)^{T}$$

The theorem implies that any circulant matrix has eigenvectors the columns of $\mathbf{A}^{-1}$.

Diagonalization of circulant matrices (cont.)

Let $\mathbf{H}$ be a NxN circulant matrix generated by the 1D N-length signal $h[n]$, that is:

$$\mathbf{H}(m,n) = h\left[(m-n)_{\mathrm{mod}\,N}\right] \triangleq h[m-n]_N$$

Let also $\boldsymbol{\alpha}_k$ be the k-th column of the inverse DFT matrix $\mathbf{A}^{-1}$.

<u>We will prove that $\boldsymbol{\alpha}_k$, for any k, is an eigenvector of $\mathbf{H}$.</u>

The m-th element of the vector $\mathbf{H}\boldsymbol{\alpha}_\mathrm{k}$, denoted by $\left[\mathbf{H}\boldsymbol{\alpha}_k\right]_m$ is the result of the circular convolution of the signal $h[n]$ with $\boldsymbol{\alpha}_k$.

Diagonalization of circulant matrices (cont.)

$$\left[\mathbf{H}\boldsymbol{\alpha}_k\right]_m = \sum_{n=0}^{N-1} h[m-n]_N\, \alpha_k[n] = \frac{1}{N}\sum_{n=0}^{N-1} h[m-n]_N\, w_N^{-kn}$$

$$\overset{l=m-n}{=} \frac{1}{N} \sum_{l=m}^{m-(N-1)} h[l]_N\, w_N^{-k(m-l)} = \frac{1}{N} w_N^{-km} \sum_{l=m}^{m-(N-1)} h[l]_N\, w_N^{kl}$$

$$= \frac{1}{N} w_N^{-km} \left[\sum_{l=m-(N-1)}^{-1} h[l]_N\, w_N^{kl} + \underbrace{\sum_{l=0}^{m} h[l]_N\, w_N^{kl}}_{} \right]$$

We will break it into two parts

Diagonalization of circulant matrices (cont.)

$$= \frac{1}{N} w_N^{-km} \left[\underbrace{\sum_{l=m-(N-1)}^{-1} h[l]_N \, w_N^{kl}}_{\text{Periodic with respect to } N.} + \overbrace{\sum_{l=0}^{N-1} h[l]_N \, w_N^{kl} - \sum_{l=m+1}^{N-1} h[l]_N \, w_N^{kl}} \right]$$

$$= \frac{1}{N} w_N^{-km} \left[\sum_{l=N+m-(N-1)}^{N-1} h[l]_N \, w_N^{kl} + \sum_{l=0}^{N-1} h[l]_N \, w_N^{kl} - \sum_{l=m+1}^{N-1} h[l]_N \, w_N^{kl} \right]$$

$$= \frac{1}{N} w_N^{-km} \left[\cancel{\sum_{l=m+1}^{N-1} h[l]_N \, w_N^{kl}} + \sum_{l=0}^{N-1} h[l]_N \, w_N^{kl} - \cancel{\sum_{l=m+1}^{N-1} h[l]_N \, w_N^{kl}} \right] \Leftrightarrow$$

Diagonalization of circulant matrices (cont.)

$$\left[\mathbf{H}\boldsymbol{\alpha}_k\right]_m = \frac{1}{N} w_N^{-km} \underbrace{\left[\sum_{l=0}^{N-1} h[l]_N \, w_N^{kl}\right]}_{\text{DFT of } h[n] \text{ at } k.} = H[k]\left[\boldsymbol{\alpha}_k\right]_m$$

This holds for any value of m. Therefore:

$$\mathbf{H}\boldsymbol{\alpha}_k = H[k]\boldsymbol{\alpha}_k$$

which means that $\boldsymbol{\alpha}_k$, for any k, is an eigenvector of $\mathbf{H}$ with corresponding eigenvalue the k-th element of $H[k]$, the DFT of the signal generating $\mathbf{H}$.

Diagonalization of circulant matrices (cont.)

The above expression may be written in terms of the inverse DFT matrix:

$$\mathbf{H}\mathbf{A}^{-1} = \mathbf{A}^{-1}\boldsymbol{\Lambda} \quad \Leftrightarrow \quad \mathbf{H} = \mathbf{A}^{-1}\boldsymbol{\Lambda}\mathbf{A}$$

$$\boldsymbol{\Lambda} = \mathrm{diag}\left\{H[0], H[1], ..., H[N-1]\right\}$$

or equivalently: $\boldsymbol{\Lambda} = \mathbf{A}\mathbf{H}\mathbf{A}^{-1}$

Based on this diagonalization, we can prove the property between circular convolution and DFT.

Diagonalization of circulant matrices (cont.)

$$\mathbf{g} = \mathbf{Hf} \Leftrightarrow \mathbf{g} = \mathbf{HA}^{-1}\mathbf{Af} \quad \Leftrightarrow \mathbf{Ag} = \mathbf{AHA}^{-1}\mathbf{Af} \quad \Leftrightarrow \mathbf{G} = \mathbf{\Lambda F}$$

$$\Leftrightarrow \underbrace{\begin{bmatrix} G[0] \\ G[1] \\ \vdots \\ G[N-1] \end{bmatrix}}_{\text{DFT of } g[n]} = \underbrace{\begin{bmatrix} H[0] & 0 & \dots & 0 \\ 0 & H[1] & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & H[N-1] \end{bmatrix}}_{\text{DFT of } h[n]} \underbrace{\begin{bmatrix} F[0] \\ F[1] \\ \vdots \\ F[N-1] \end{bmatrix}}_{\text{DFT of } f[n]}$$

Diagonalization of doubly block circulant matrices

- These properties may be generalized in 2D.
- We need to define the Kronecker product:

$$\mathbf{A} \in \mathbb{R}^{M \times N}, \mathbf{B} \in \mathbb{R}^{K \times L}$$

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots & a_{1N}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \dots & a_{2N}\mathbf{B} \\ \vdots & \vdots & \vdots & \vdots \\ a_{M1}\mathbf{B} & a_{M2}\mathbf{B} & \dots & a_{MN}\mathbf{B} \end{bmatrix}_{MK \times NL}$$

Diagonalization of doubly block circulant matrices (cont.)

- The 2D signal $f[m,n],\ 0 \leq m \leq M-1,\ 0 \leq n \leq N-1,$

  may be vectorized in lexicographic ordering (stacking one column after the other) to a vector:

$$\mathbf{f} \in \mathbb{R}^{MN \times 1}$$

- The DFT of $f[m,n]$, may be obtained in matrix-vector form:

$$F = \left(\mathbf{A} \otimes \mathbf{A}\right)\mathbf{f}$$

Diagonalization of doubly block circulant matrices (cont.)

Theorem: The columns of the inverse 2D DFT matrix

$$\left(\mathbf{A}\otimes\mathbf{A}\right)^{-1}$$

are eigenvectors of any doubly block circulant matrix. The corresponding eigenvalues are the 2D DFT values of the 2D signal generating the doubly block circulant matrix:

$$\mathbf{\Lambda}=\left(\mathbf{A}\otimes\mathbf{A}\right)\mathbf{H}\left(\mathbf{A}\otimes\mathbf{A}\right)^{-1}$$

Diagonal, containing the 2D DFT of $h[m,n]$ generating $\mathbf{H}$

Doubly block circulant