Mecánica 2014

U02C03: Kepler

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Partiendo de la ec. de la órbita que obtuvimos para un potencial $V = a r^{n+1}$,

$$\Theta = \Theta_0 - \int_{U_0}^{U} \frac{dU}{\sqrt{\frac{2mE}{L^2} - \frac{2ma U^{-n-1}}{L^2} - U^2}} \quad ; \; U = 1/r$$

y para el pot. gravitatorio, $V = K r^{-1} \Rightarrow n = -2$

$$\Theta = \Theta_0 - \int_{U_0}^{U} \frac{dU}{\sqrt{\frac{2mE}{L^2} - \frac{2mKU}{L^2} - U^2}}$$

Qui tieni la formula di

$$\int \frac{dx}{\sqrt{\alpha+\beta u+\gamma u^2}} = \frac{1}{\sqrt{-\gamma}} \arcsin \frac{-\beta+2\gamma x}{\sqrt{\beta^2-4\alpha\gamma}}$$

Con $\alpha = \frac{2mE}{L^2}$; $\beta = \frac{2mK}{L^2}$; $\gamma = -1$

$$\Rightarrow \beta^2-4\alpha\gamma = \frac{4m^2K^2}{L^4} + \frac{8mE}{L^2} = \left(\frac{2mK}{L^2}\right)^2\left(1+\frac{2EL^2}{mK^2}\right)$$

$$\Rightarrow \sqrt{\beta^2-4\alpha\gamma} = \left(\frac{2mK}{L^2}\right)\sqrt{1+\frac{2EL^2}{mK^2}}$$

Para el numerador:

$$\beta + 2\gamma u = \frac{2mk}{L^2} - 2u = \left(\frac{2mk}{L^2}\right)\left(1 - \frac{uL^2}{mk}\right)$$

y con el cambio del signo: $\left(\frac{2mk}{L^2}\right)\left(\frac{uL^2}{mk} - 1\right)$

Luego para el arccos tenemos:

$$-\frac{\beta + 2\gamma u}{\sqrt{\beta^2 - 4\alpha\gamma}} = \frac{\cancel{\left(2mk/L^2\right)} \cdot \left(\frac{uL^2}{mk} - 1\right)}{\cancel{\left(2mk/L^2\right)} \sqrt{\left(1 + \frac{2EL^2}{mk^2}\right)}} = \frac{\frac{uL^2}{mk} - 1}{\sqrt{1 + 2EL^2/mk^2}}$$

y puesto que $\gamma = -1 \Rightarrow 1/\sqrt{-\gamma} = 1 \Rightarrow$

$$\theta = \theta' - \arccos \frac{\overbrace{\frac{L^2}{mk}}^{\gamma} u - 1}{\underbrace{\sqrt{1 + \frac{2EL^2}{mk^2}}}_{\epsilon}}$$

Notar que sólo la u es una variable

$$\theta = \theta' - \arccos \frac{\gamma u - 1}{\epsilon}$$

Resolviendo para u:

$$u = \frac{1}{\gamma}\left(1 + \epsilon \cos\left(\theta - \theta'\right)\right)$$

$\cos -a = \cos a$

Que es la ecuación de una cónica de excentricidad ϵ. $\Rightarrow$ La excentricidad puede definida por

$$\epsilon = \sqrt{1 + \frac{2EL^2}{mk^2}}$$

Notar que m, L^2, k^2 y 2 son positivos $\Rightarrow$ El signo de ϵ y el tipo de órbita queda definido por E:

Si $E > 0 \Rightarrow \epsilon > 1 \longrightarrow$ Hipérbola

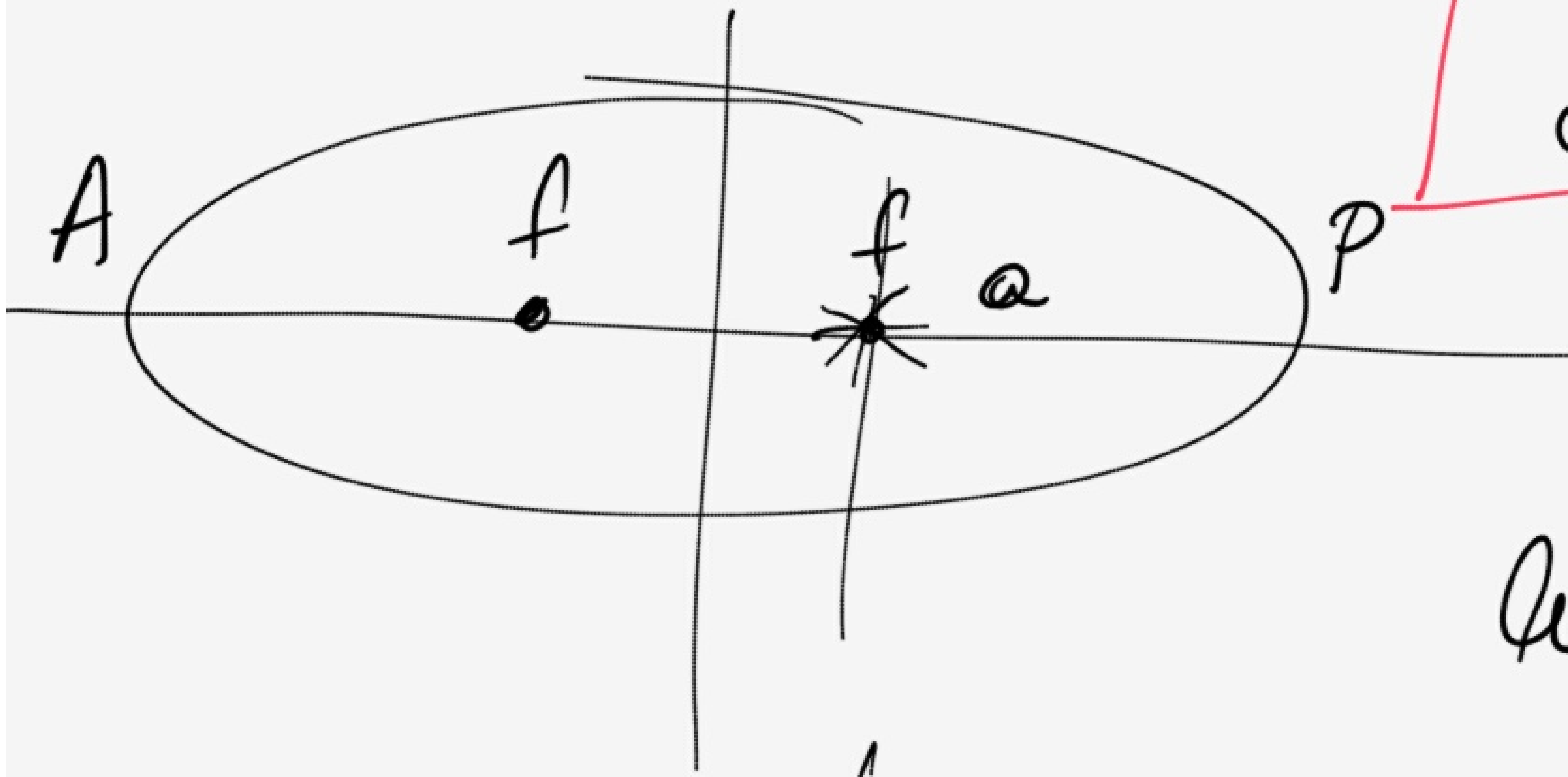
Si $E = 0 \Rightarrow \epsilon = 1 \longrightarrow$ Parábola

Si $-\dfrac{mk^2}{2L^2} < E < 0 \longrightarrow$ Elipse

Si $E = -\dfrac{mk^2}{2L^2} \longrightarrow$ Círculo

Para la órbita circular, $E = V'|_{r_0} \Rightarrow$

$$E = V(r_0) + \frac{L^2}{2mr_0} \quad (\longrightarrow \dot{r} = 0) \Rightarrow$$



$$\boxed{2a = A + P}$$

$$E - \frac{L^2}{2mr^2} + \frac{k}{r} = 0$$

en P y A

mult. por $r^2/E \Rightarrow r^2 - \frac{L^2}{2mE} + \frac{k}{E} r = 0$

$$\Rightarrow r^2 + \left(\frac{k}{E}\right) r - \frac{L^2}{2mE} = 0 \Big|_{A,P}$$

$$\Rightarrow \frac{K}{E} = -(A+P) \Rightarrow -\frac{K}{2E} = \frac{A+P}{2} = a$$

$$\Rightarrow \boxed{a = -\frac{K}{2E}} \qquad \epsilon = \sqrt{1 + \frac{2EL^2}{mk^2}}$$

$$\Rightarrow \epsilon = \sqrt{1 + \frac{2EL^2}{mk^2}} = \sqrt{1 + \frac{L^2}{mk\left(\underbrace{K/2E}_{=-a}\right)}} \Rightarrow$$

$$\boxed{\epsilon = \sqrt{1 - \frac{L^2}{mka}}}$$

obtien

$$\frac{L^2}{mk} = a(1-\epsilon^2)$$

$\Rightarrow$ Recordando la ec. de la órbita:

$$\frac{1}{r} = \frac{1}{\gamma}\left(1+\epsilon\cos(\theta-\theta')\right)$$

$$\Rightarrow r = \underset{\overset{\uparrow}{L^2/mk}}{\gamma}\left(1+\epsilon\cos(\theta-\theta')\right)^{-1} \Rightarrow r = \frac{L^2}{mk}\left(1+\epsilon\cos(\theta-\theta')\right)^{-1}$$

$$\boxed{r(\theta) = a\,\frac{1-\epsilon^2}{1+\epsilon\cos(\theta-\theta')}}$$

$$\text{En } P \Rightarrow (\theta-\theta') = 0 \Rightarrow$$

$$P = a\,\frac{(1-\epsilon^2)}{1+\epsilon} \Rightarrow P = a(1-\epsilon)$$

$$\text{En } A \Rightarrow (\theta-\theta') = \pi \Rightarrow$$

$$A = a\,\frac{(1-\epsilon^2)}{1-\epsilon} \Rightarrow A = a(1+\epsilon)$$

v_{cen} $\theta' = 0 \Rightarrow$

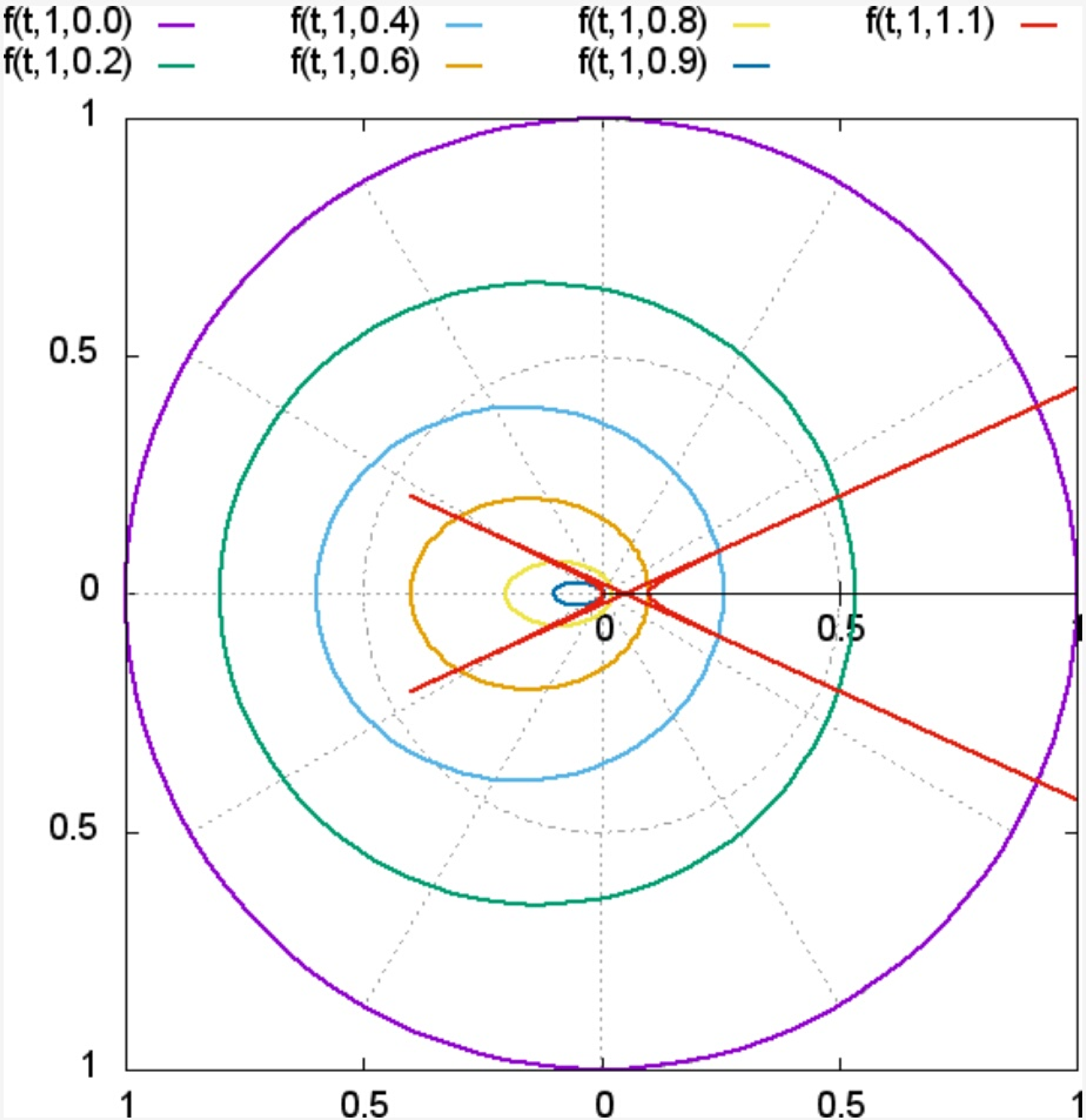
$$r(\theta) = a \frac{(1-\epsilon)^2}{1+\epsilon \cos\theta}$$

Si $\epsilon = 0 \Rightarrow r(\theta) = a$ Mov. Circular en Polares

Si $\epsilon \to 1 \Rightarrow r_{max} < a$

Si $\epsilon = 1 \Rightarrow r \to 0$

Si $\epsilon > 1 \Rightarrow r > a$



$$r(\theta)=a\,\frac{(1-\epsilon)^2}{1+\epsilon\cos\theta}$$

$\epsilon=0, 0.2, 0.4,$
$0.6, 0.8, 0.9$
y 1.1

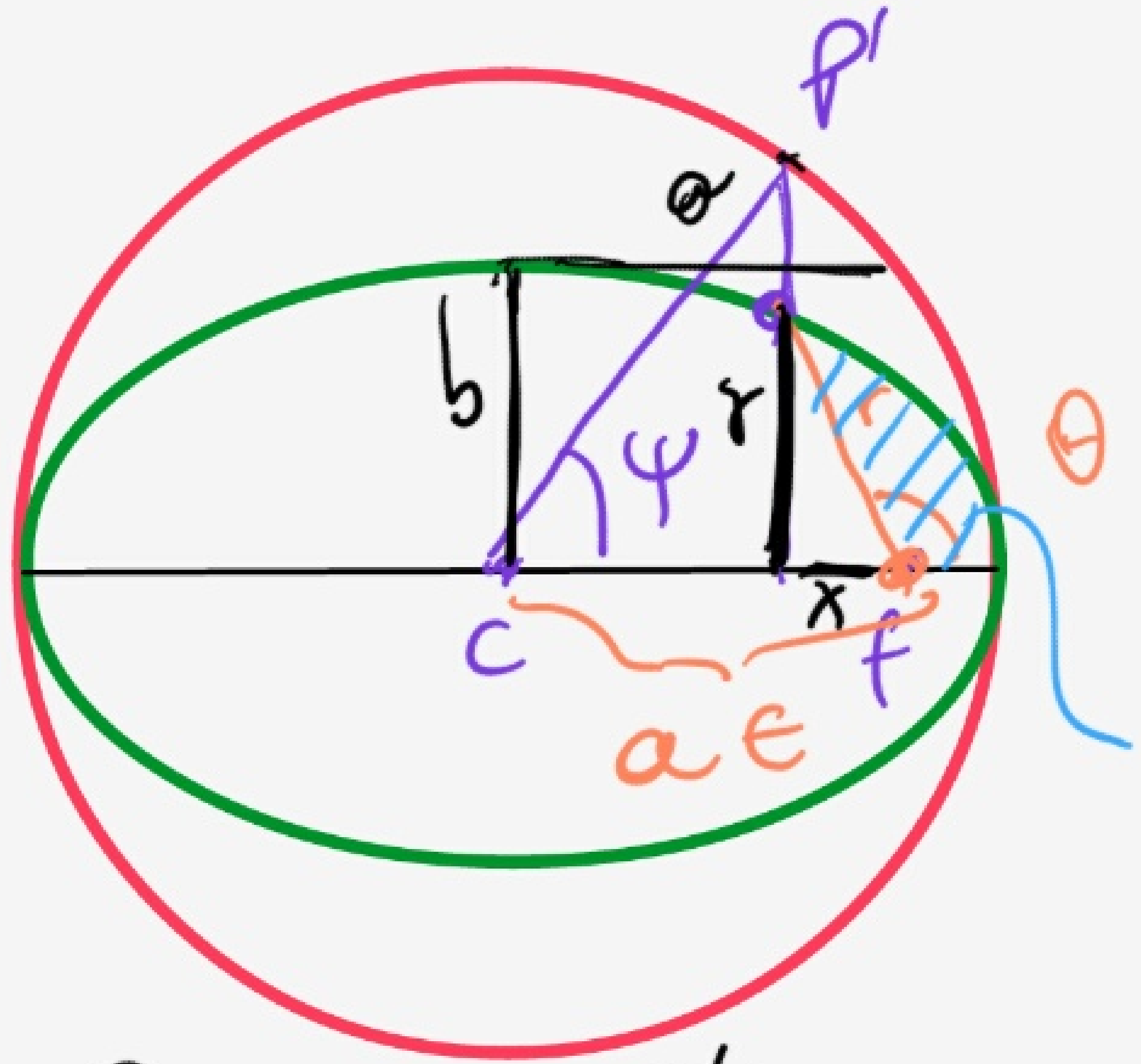
Análisis temporal

Recordamos: $t = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - L^2/2mr^2\right)}}$

$\Rightarrow V = -k/r \Rightarrow$

$$t = \left(\frac{m}{2}\right)^{1/2} \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{k}{r} - \frac{L^2}{2mr^2} + E}}$$

Es conveniente plantearla en términos de la excentricidad anómala:



$x = a\epsilon - a\cos\psi$

$y = b\sin\psi$

ψ es la anomalía Excéntrica

θ es la anomalía verdadera

$M = \sqrt{\frac{G(M+m)}{a^3}}\, t$ es la anomalía Media

y es prop. al área barrida por el radio vector que une el foco con el planeta

Es adimensional y $0 \leq M \leq 2\pi$

$$M = \psi - \epsilon \sin\psi$$