Mecánica 2014

U02C01: Introducción

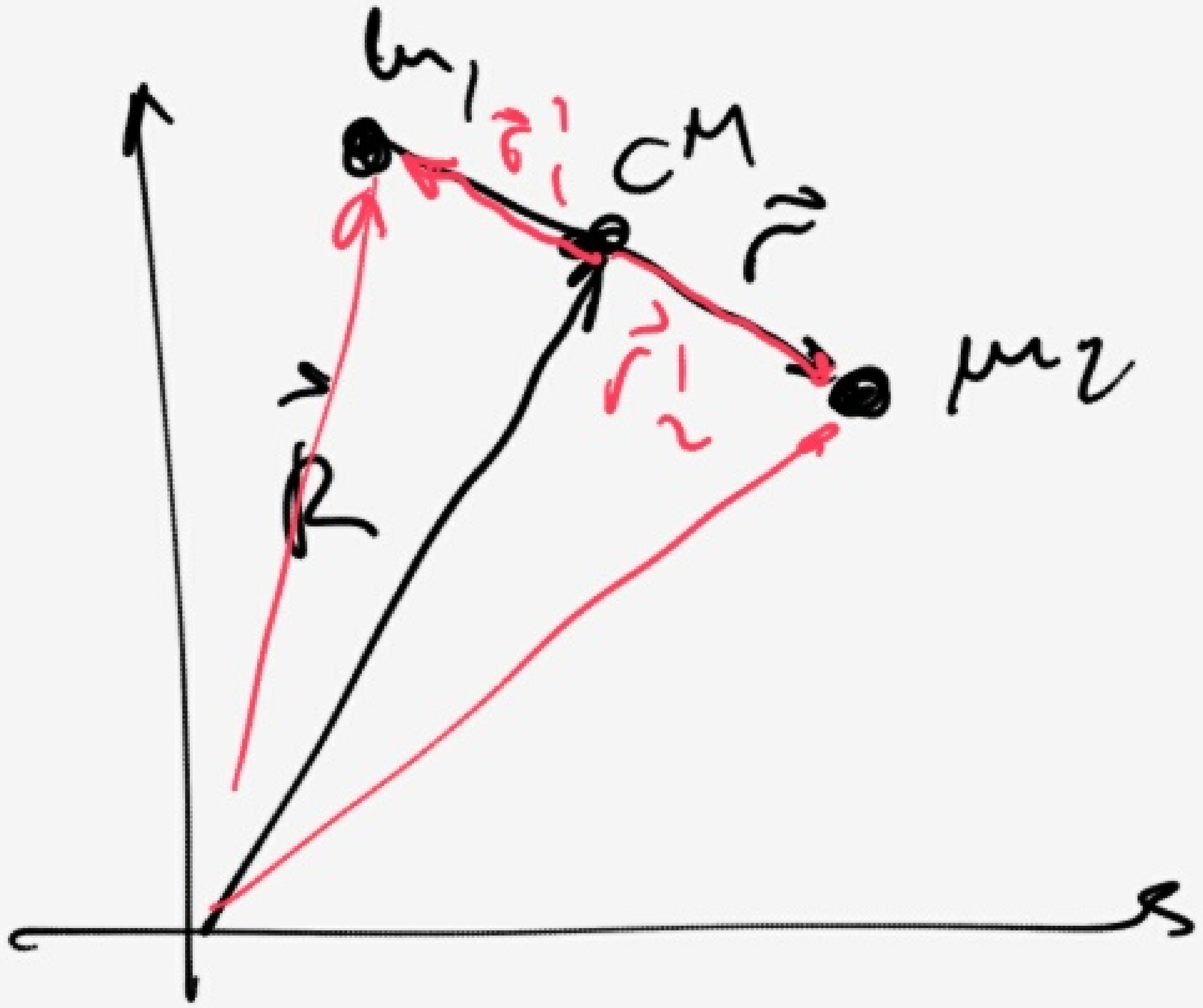
2014/10/07

H. Asorey
hasorey@uis.edu.co
UIS

En un sistema, la energía potencial sólo depende de la posición relativa de los cuerpos interactuantes y de la magnitud de las cargas interactuantes.

$$\Rightarrow U_{ij} \propto f\left(\Delta \bar{r}_{ij}, \Delta \dot{r}_{ij}\right)$$

$\Rightarrow$ Para un sistema con dos cuerpos de masa m_1 y m_2 $\Rightarrow$ 6 grados de libertad.



$$\vec r = \bar r_2 - \bar r_1$$

$$\Rightarrow \mathcal{L} = T - \Omega = T(\dot{\vec R}, \dot{\vec r}) - U(\vec r, \vec r, \ldots)$$

$$T = \frac{1}{2}(m_1 + m_2)\dot R^2 + \frac{1}{2} m_1 \dot{\vec r}_1'^2 + \frac{1}{2} m_2 \dot{\vec r}_2'^2$$

And $r_1' = \dfrac{-m_2}{m_1+m_2}\vec r$ y $\vec r_2' = \dfrac{m_1}{m_1+m_2}\vec r$

$$\Rightarrow T' = \frac{1}{2} m_1 \left(\frac{m_2}{m_1+m_2}\right)^2 \dot r^2 + \frac{1}{2} m_2 \left(\frac{m_1}{m_1+m_2}\right)^2 \dot r^2$$

$$T' = \frac{1}{2}\dot{\vec r}^2 \left(\frac{m_1 m_2^2 + m_2 m_1^2}{(m_1+m_2)^2}\right) = \frac{1}{2}\dot{\vec r}^2 \cdot m_1 m_2 \frac{\cancel{(m_2 + m_1)}}{(m_1+m_2)^2}$$

$$\Rightarrow T' = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\dot{r}^2$$

$$\Rightarrow \mathcal{L} = \underbrace{\frac{1}{2}(m_1+m_2)\dot{R}^2}_{CM} + \underbrace{\frac{1}{2}\overbrace{\frac{m_1 m_2}{m_1+m_2}}^{\mu \,\equiv\, \text{massa reduzida}}\dot{r}^2 - U(r,\dot{r},\dots)}_{\text{mov. relativo.}}$$

$$\Rightarrow \mathcal{L} = \mathcal{L}_{CM} + \mathcal{L}_{rel}$$

$$\mathcal{L}_{rel} = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\dot{r}^2 - U(r,\dot{r},\dots)$$

$$\Rightarrow \mathcal{L}_{rel} = \frac{1}{2}\mu\,\dot{r}^2 - U(r,\dot{r},\dots)$$

Fuerzas conservativas $\longrightarrow$ tiempo homogeneo

$\rightarrow$ Energía total es conservada

$$E = T + V = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + V(r) = cte$$

Si nos centramos en una fuerza central ⇒

El potencial sólo depende de $|\vec{r}|$ ⇒

$\vec{F} \propto \vec{r}$

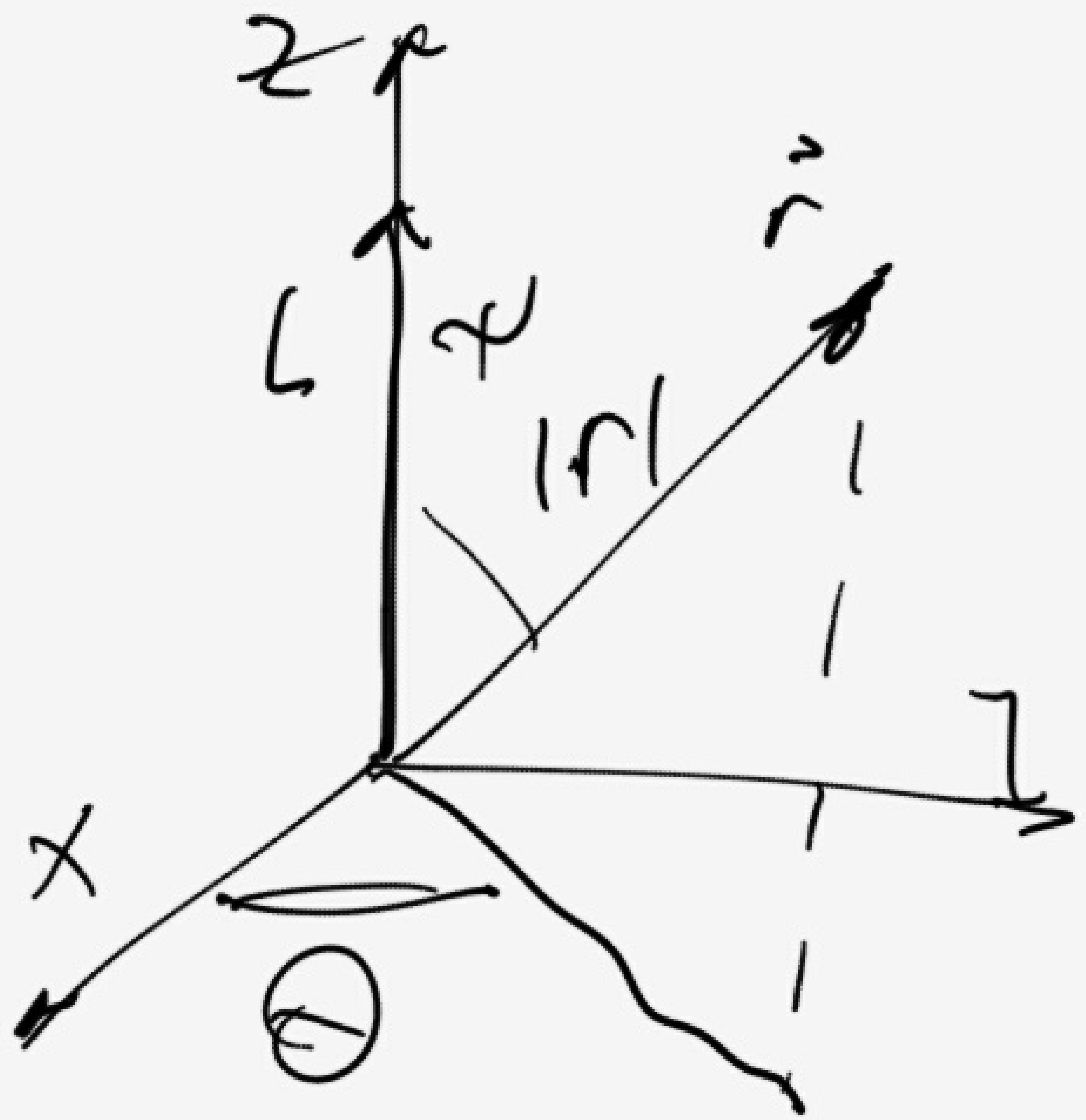
Nöther

Simetrías ⟷ Mag. Conservadas.

Simetría Esférica ⟷ Momento angular

$\Rightarrow \vec{L} = \vec{r} \times \vec{p} \rightarrow \dot{L} = 0$

Esto se cumple si r y $\dot{r}$ están contenidos en un plano $\perp$ a L. Si $\vec{L} = 0 \Rightarrow \vec{r} \parallel \dot{\vec{r}} \to$ MR



Si $\vec{L} = L\hat{k} \Rightarrow \alpha = \pi/2$

$$\Rightarrow \mathcal{L} = T - V$$

$$= \frac{1}{2}\mu \dot{r}^2 - V(r)$$

$$\mathcal{L} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)$$

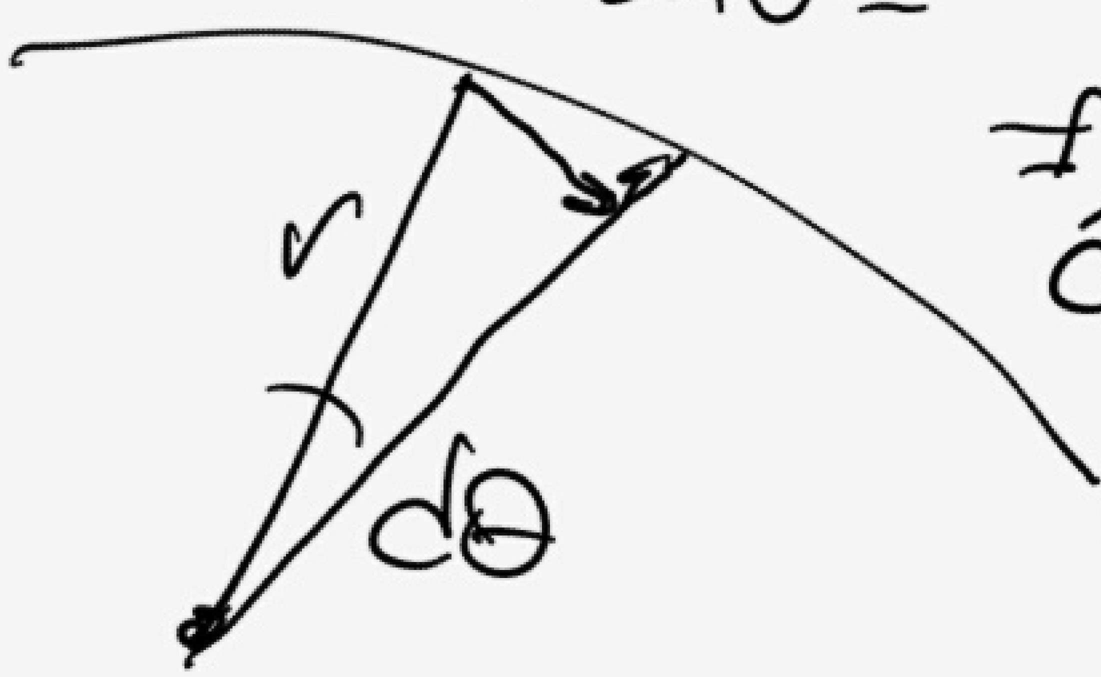
$$\Rightarrow p_\theta = \frac{\partial \mathcal{L}}{\partial \dot\theta} = m r^2 \dot\theta$$

y la conservación de L implica $\dot p_\theta = 0 \rightarrow \frac{d}{dt} m r^2 \dot\theta = 0$

$$\Rightarrow m r^2 \dot\theta = L = cte$$

2° ley de Kepler

$r \sin\theta \simeq r\, d\theta$



$$\Rightarrow dA = \frac{1}{2} r\, r\, d\theta$$

$$dA = \frac{1}{2} r^2 d\theta \Rightarrow \frac{dA}{dt} = \frac{1}{2} r^2 \dot\theta = cte$$

Para r tuem:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}_j} - \partial \mathcal{L}/\partial r_j = 0$$

$$r_j = r$$

$$\Rightarrow \mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} \rightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}_j} = m\ddot{r}$$

$$\frac{\partial \mathcal{L}}{\partial r} = mr\dot{\theta}^2 - \partial r/\partial r$$

$$\frac{d}{dt}m\dot{r} - \left(mr\dot{\theta}^2 - \frac{\partial V}{\partial r}\right) = 0$$

$\Rightarrow$ Sup $m = cte \Rightarrow$

$$m\ddot{r} - mr\dot{\theta}^2 = -\frac{\partial v}{\partial r} = f(r)$$

pero $mr^2\dot{\theta} = L \Rightarrow \underline{m^2 r^4 \dot{\theta}^2 = L^2} \Rightarrow mr\dot{\theta}^2 = \dfrac{L^2}{mr^3}$

$$\Rightarrow m\ddot{r} - \frac{l^2}{mr^3} = f(r)$$

Ahora, Energía $E = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + v(r) = cte$

$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + v(r) = cte$ pero:

$$mr^2\dot{\theta} = L \Rightarrow \frac{1}{2}m^2 r^4 \dot{\theta}^2 = \frac{L^2}{2} \Rightarrow$$

$$\Rightarrow \frac{1}{2} m r^2 \dot{\theta}^2 = \frac{L^2}{2mr^2}$$ y entonces, la energía

$$E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 + V(r) = \text{cte}$$

$$\Rightarrow E = \frac{1}{2} m \dot{r}^2 + \frac{L^2}{2mr^2} + V(r) = \text{cte}$$

$$\Rightarrow \begin{cases} \boxed{L = m r^2 \dot{\theta} = cte} \\ \boxed{E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} \frac{L^2}{m r^2} + V(r) = cte} \end{cases}$$

Resolviendo esta para $\dot{r} \Rightarrow$

$$\Rightarrow \dot{r} = \sqrt{\frac{2}{m}\left(E - \frac{L^2}{2 m r^2} - V(r)\right)} \equiv \alpha$$

$$\Rightarrow dt = \alpha^{-1}\, dr \Rightarrow \boxed{t = \int_{r_0}^{r} \alpha^{-1}\, dr}$$

$\Rightarrow r \equiv t^{-1}(r_0, E, L)$

Para θ, a partir de $r \Rightarrow$

$L = mr^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{d\theta}{dt} = \frac{L}{mr^2} \Rightarrow d\theta = \frac{L}{mr^2}dt$

$$\Rightarrow \boxed{\theta = \int_0^t \frac{L\,dt}{mr^2} + \theta_0}$$

4 constantes de integración: E, L, r_0 y θ_0

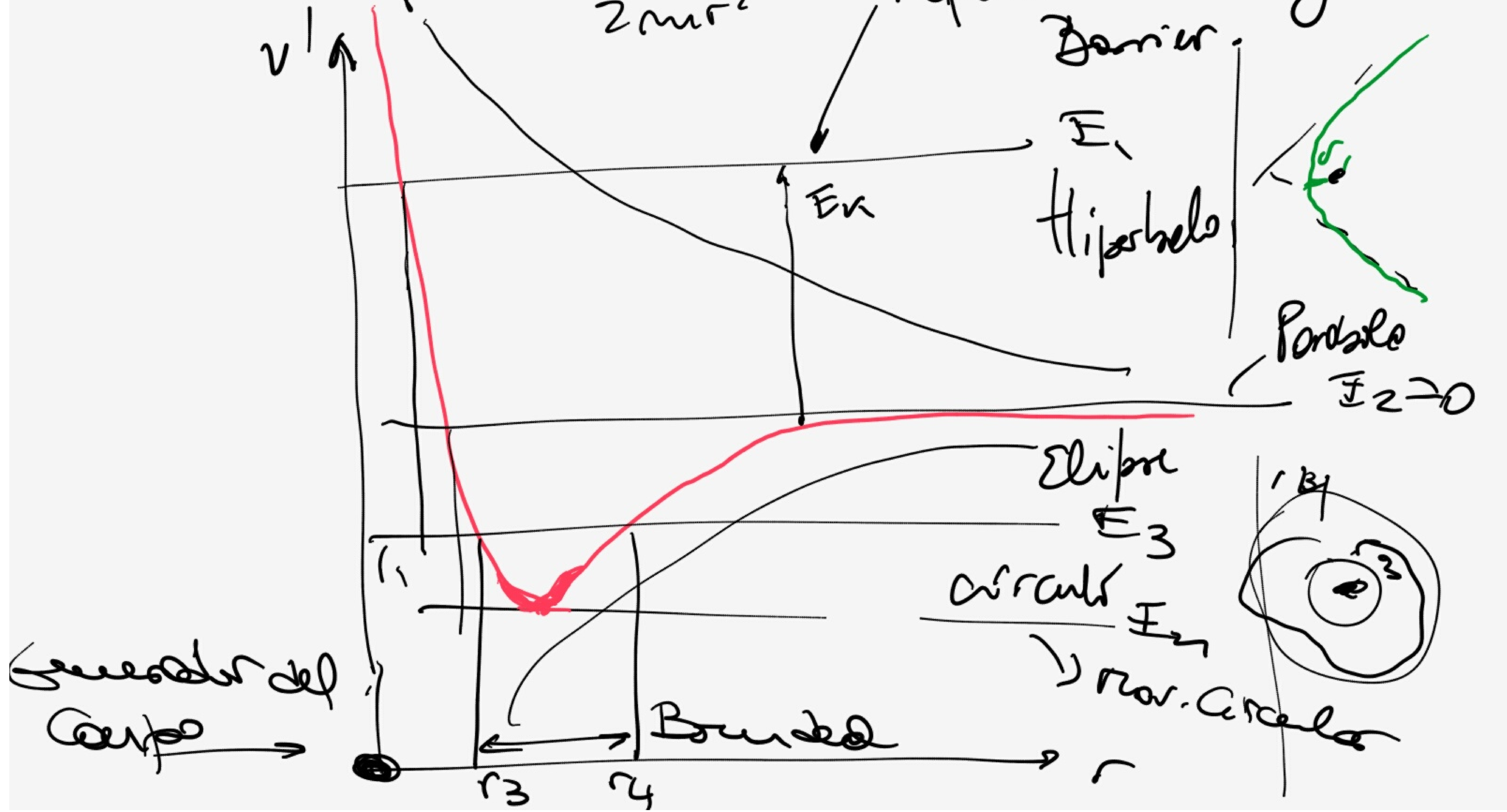
Ahora bien, $m\ddot{r} - \dfrac{l^2}{mr^3} = f(r)$

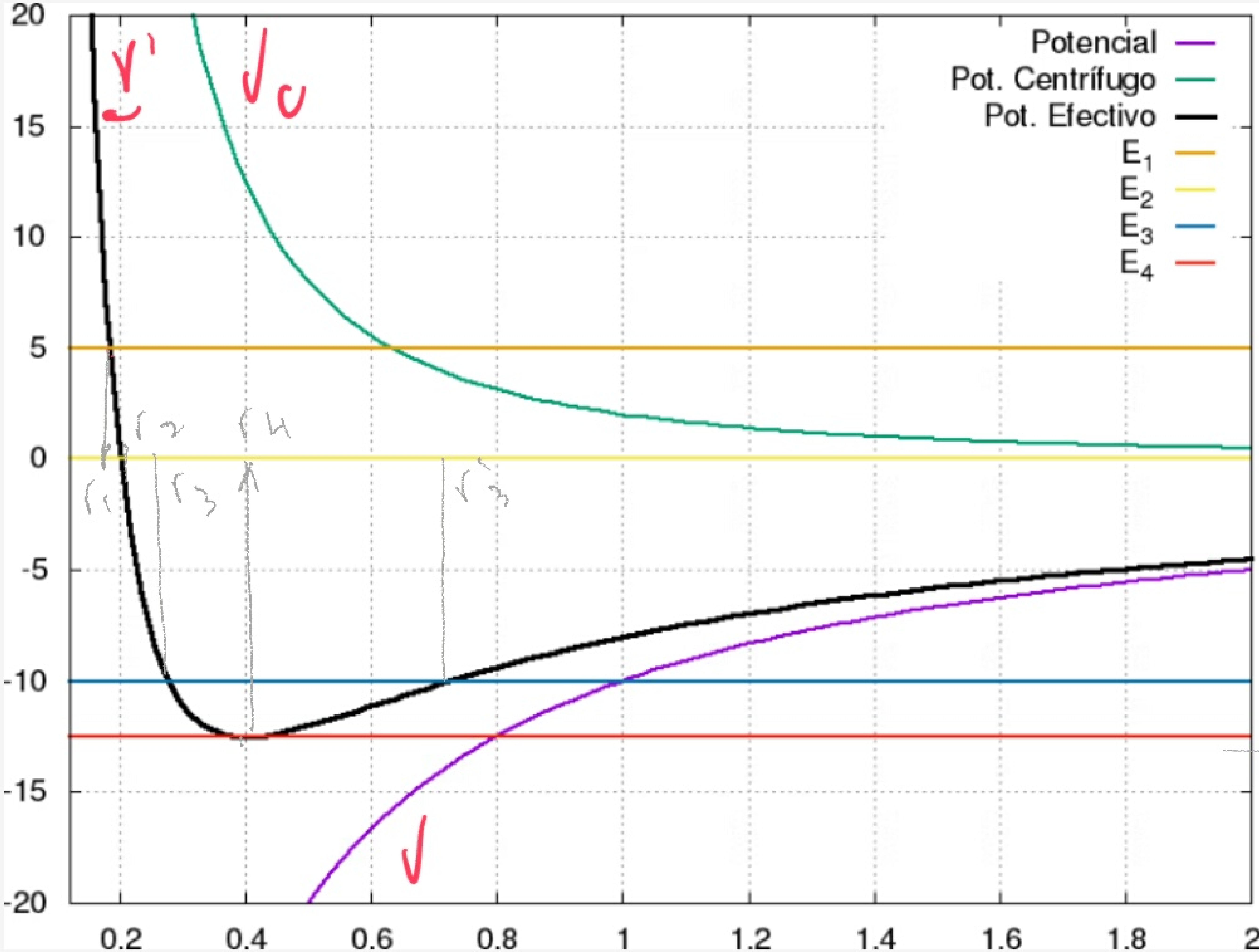
$$\Rightarrow m\ddot{r} = f(r) + \underbrace{l^2/mr^3}_{\text{fuerza centrífuga}} = f'(r)$$

pero $f'(r) = -\partial V'/\partial r \Rightarrow V' = V + \underbrace{\dfrac{l^2}{2mr^2}}_{\text{pot. centrífugo}}$

Fuerza de atracción $\Rightarrow V = -kr^{-1}$

$$\Rightarrow V' = -\frac{K}{r} + \frac{l^2}{2mr^2}$$

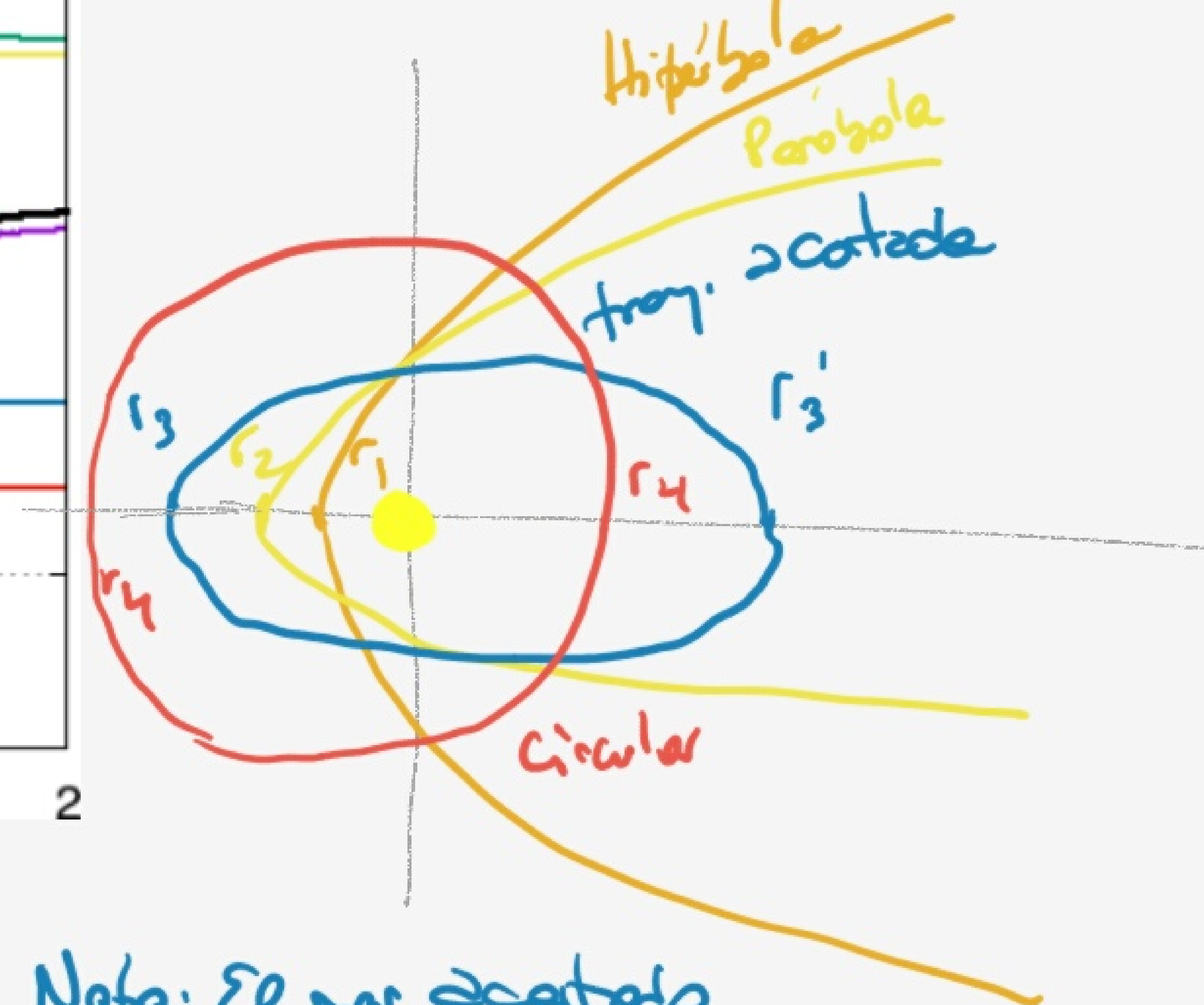




$$V'(r) = V(r) + V_c(r)$$

$$V(r) = -k/r$$

$$V_c(r) = \frac{L^2}{2mr^2}$$



Nota: El mov. acotado podría no ser cerrado