# Mecánica 2014

U02C02: Órbitas

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El teorema del virial.

Relaciona $\langle T \rangle \longleftrightarrow \langle V \rangle$ :

$$\boxed{\langle T \rangle = -\frac{1}{2} \left\langle \sum \vec{F_i} \cdot \vec{r_i} \right\rangle}$$

Si $\vec{F} = -\vec{\nabla} V \Rightarrow \langle T \rangle = \frac{1}{2} \left\langle \sum \partial^i V \, r_i \right\rangle$

Si $V(\vec{r}) \equiv V(r) \Rightarrow \langle T \rangle = \frac{1}{2} \langle \partial_r V \, r \rangle$

Si además $V = a r^{n+1} \Rightarrow \frac{\partial V}{\partial r} = (n+1) a r^n$

$\Rightarrow \partial V / \partial r \cdot r = (n+1) a r^{n+1} = (n+1) V \Rightarrow \langle T \rangle = \frac{n+1}{2} \langle V \rangle$

Finalmente, si $F \propto r^{-2} \Rightarrow V \propto r^{-1} \Rightarrow n = -2 \Rightarrow$

$$\boxed{\langle T \rangle = -\frac{1}{2} \langle V \rangle}$$

## Órbitas

Es posible escribir: $\left( r(t), \theta(t) \right) \rightarrow r(\theta)$

En el caso de F central es simple, ya que el tiempo no es explícito en las ec. de movimiento

Recordando $dt = \sqrt{\frac{2}{m}\left(E - V - \frac{L^2}{2mr^2}\right)}$

y $L\,dt = mr^2\,d\theta$

$$\Rightarrow \cancel{\frac{mr^2}{L}}\,d\theta = \frac{L\,dr}{mr^2\sqrt{\frac{2}{m}\left(E - V - \frac{L^2}{2mr^2}\right)}} \Rightarrow$$

$$\int_{\theta_0}^{\theta} d\theta = \int \frac{dr}{r^2\sqrt{\frac{2m^2}{mL^2}\left(E - V - L^2/2mr^2\right)}} = \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2mE}{L^2} - \frac{2mV}{L^2} - \frac{1}{r^2}}}$$

y ahora $u = 1/r \Rightarrow r = 1/u \Rightarrow dr = -1/u^2 \, du$

$$\Rightarrow \theta - \theta_0 = \int_{u_0}^{u} \frac{-du}{\cancel{u^2} \cdot \frac{1}{\cancel{u^2}} \cdot \sqrt{\frac{2mE}{L^2} - \frac{2mV}{L^2} - u^2}}$$

$$\Rightarrow \theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{L^2} - \frac{2mV}{L^2} - u^2}}$$

$V = a r^{n+1}$

$\Rightarrow V = a\left(\frac{1}{u}\right)^{n+1}$

$V = a u^{-n-1}$

$$\Rightarrow \theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{L^2} - \frac{2ma}{L^2} u^{-n-1} - u^2}}$$

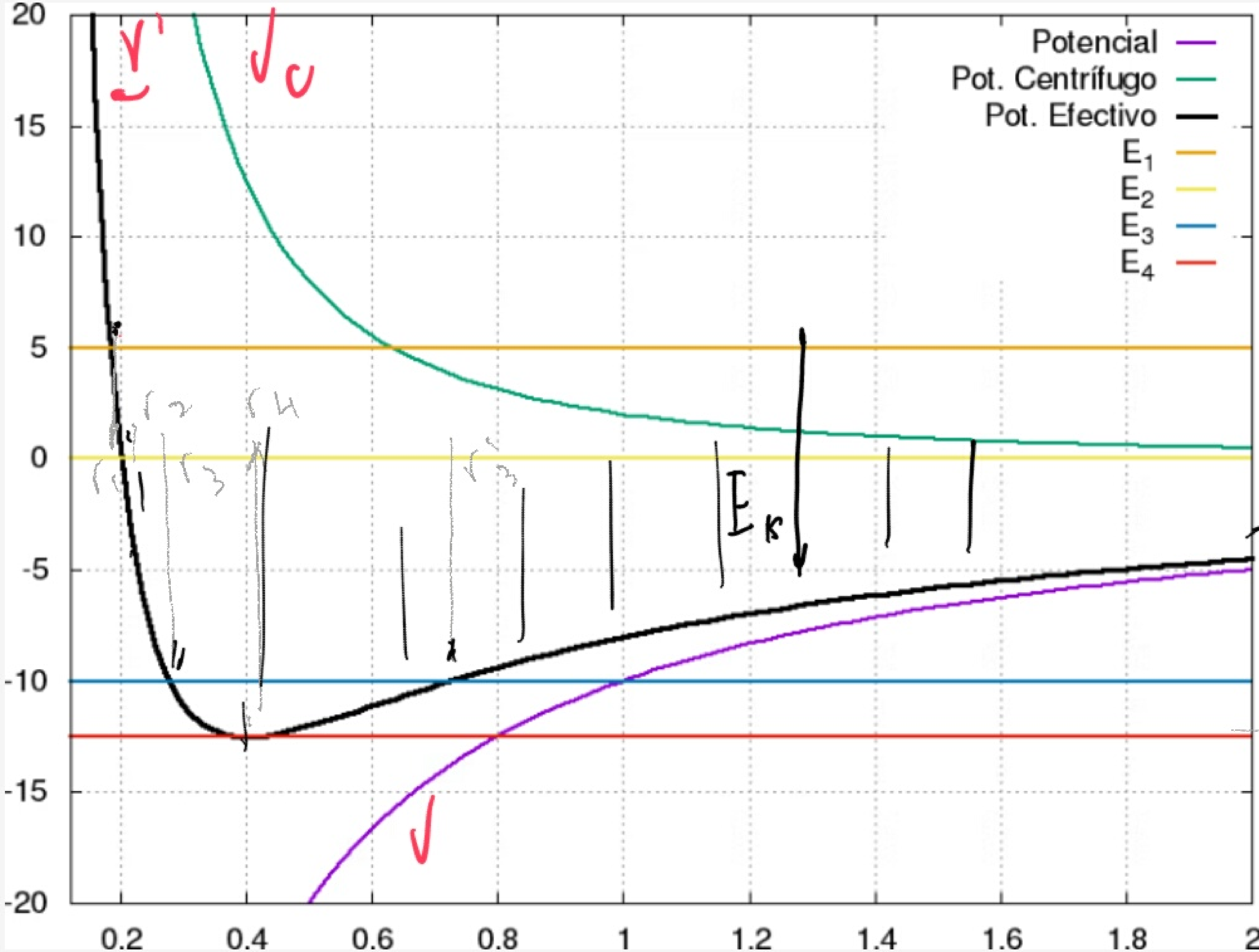
Si $n = 1, -2, -3$ $\left(\Rightarrow V = ar^2; ar^{-1}; ar^{-2}\right)$

$\Leftrightarrow$ es integrable

y n no locos $\rightarrow$ Runge Kutta / Leapfrog...

Teorema de Bertrand. Cond. para órbitas cerradas

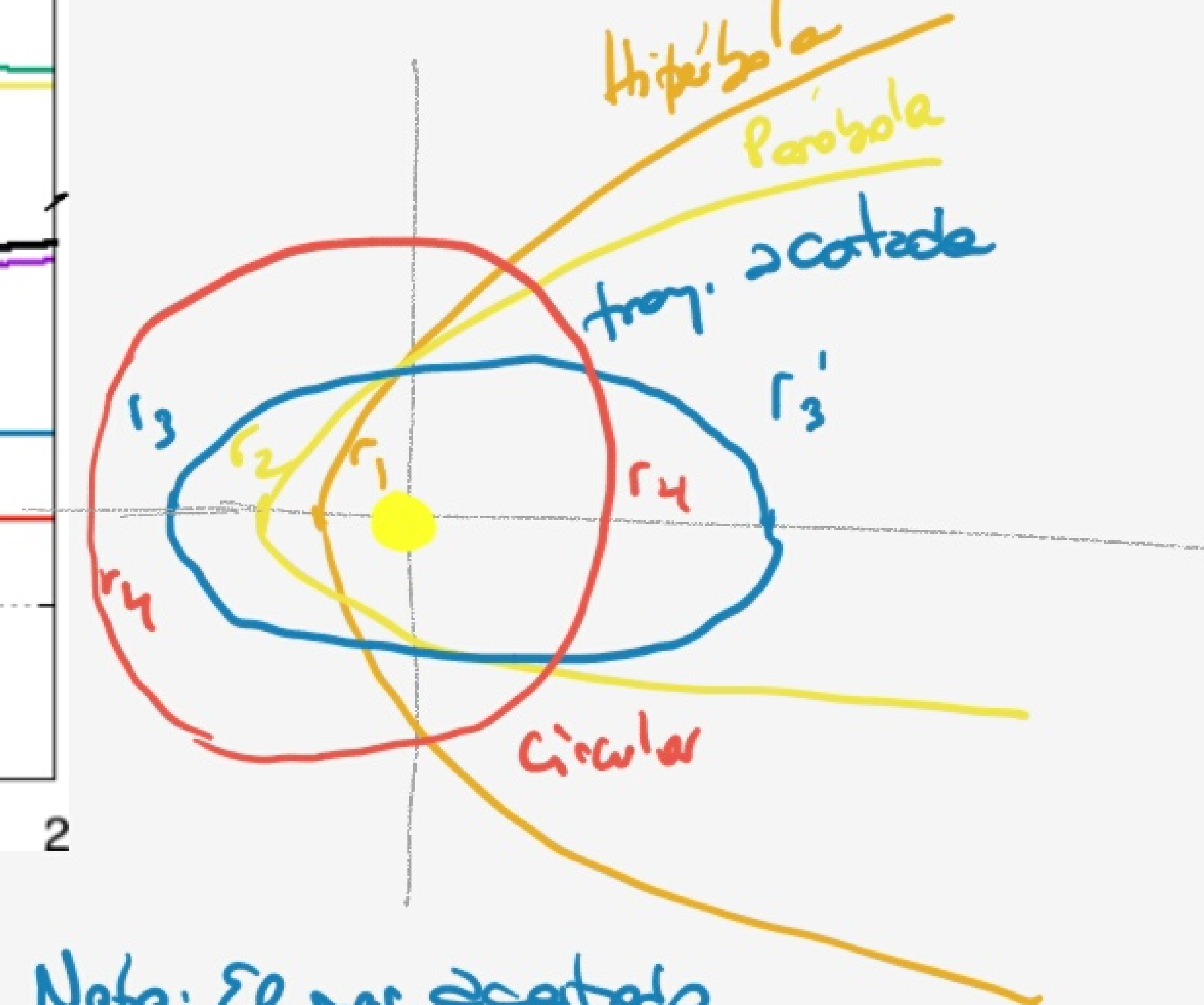
Recordemos:




$$V'(r) = V(r) + V_c(r)$$

$$V(r) = -k/r$$

$$V_c(r) = \frac{L^2}{2mr^2}$$




Nota: El mov. acotado podría no ser cerrado

En general, la existencia de órbitas cerradas requiere que el potencial efectivo,

$$V'(r) = V(r) + \frac{L^2}{2mr^2}$$

presente un extremo. Para la fuerza esto se traduce en

$$f' = -\partial V'/\partial r \Rightarrow f' = -\frac{\partial V}{\partial r} + \left(-\frac{L^2}{mr^3}\right) = f - \frac{L^2}{mr^3}$$

Recordar que $L^2 = m^2 r^4 \dot{\theta}^2 \Rightarrow \frac{L^2}{m r^3} = \frac{m^2 r^4 \dot{\theta}^2}{\cancel{m} r^3}$

$= m r \dot{\theta}^2$ y como $v_\theta = r\dot{\theta} \Rightarrow v_\theta^2 = r^2\dot{\theta}^2 \Rightarrow r\dot{\theta}^2 = v_\theta^2/r$

$$\Rightarrow \frac{L^2}{m r^3} = m \frac{v_\theta^2}{r}$$ Fuerza Centrífuga

$$\Rightarrow f' = f + f_{centrífuga}$$

Luego, $f' = -\partial V/\partial r$. Si $V'$ tiene un extremo $\Rightarrow f' = 0$

$$\Rightarrow f = -L^2/m r_0^3 \quad \text{ó} \quad f = -m v_\theta^2/r$$

En el exterior, la energía total debe ser igual

$$E_0 = V(r_0) + \frac{L^2}{2mr_0^2}$$

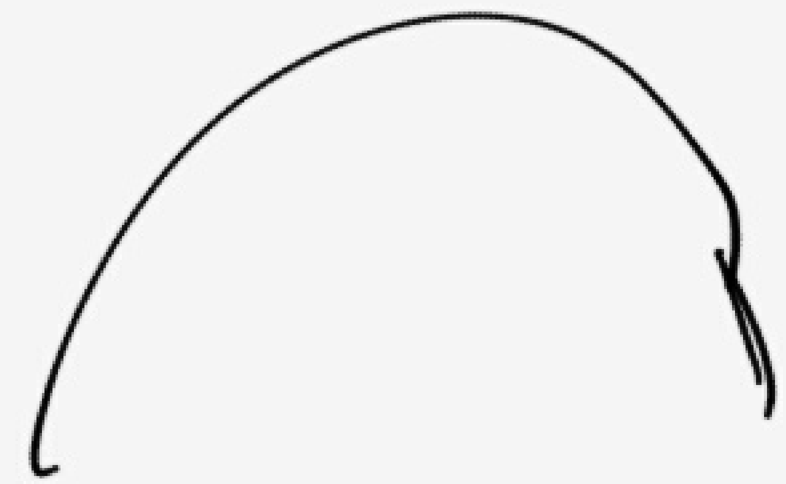
$\Rightarrow E_K = 0 \Rightarrow \dot{r} = 0 \Rightarrow$ Mov. Circular ($r$=cte)

Estas dos condiciones garantizan la existencia de una órbita circular para un potencial dado.

Si $E \gtrsim E_0 \Rightarrow$ Mov. Acotado

Para el caso donde el potencial es:

Estable y acotado



Circular inestable

$$\Rightarrow \text{Estabilidad} \longleftrightarrow \frac{\partial^2 V'}{\partial r^2} > 0$$

$$\Rightarrow \frac{\partial^2 V}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial V}{\partial r}\right) + \frac{\partial^2}{\partial r^2}\left(\frac{L^2}{2mr^2}\right) =$$

$$\Rightarrow \frac{\partial^2 V'}{\partial r^2} = -\frac{\partial f}{\partial r} + \frac{L^2}{\cancel{2}m}\,{}^{3}\cancel{6}\, r^{-4}$$

$$\Rightarrow \frac{\partial^2 V'}{\partial r^2} = -\frac{\partial f}{\partial r}\bigg|_{r_0} + \frac{3L^2}{mr^4}\bigg|_{r_0} > 0$$

es cond. de Estabilidad

Como $f(r_0) = -L^2/mr_0^3 \Rightarrow \dfrac{-3f(r_0)}{r_0} = \dfrac{3L^2}{mr_0^4}$

$$\Rightarrow -\frac{\partial f}{\partial r}\bigg|_{r_0} - \frac{3f(r_0)}{r_0} > 0 \Rightarrow \boxed{\frac{\partial f}{\partial r}\bigg|_{r_0} < -\frac{3f(r_0)}{r_0}}$$

y aplicando la derivada log:

$$\boxed{\left.\frac{d \ln f}{d \ln r}\right|_{r=r_0} > -3}$$

Cond. de Estabilidad y exist. de órbitas cerradas

Si $f = -k r^n \Rightarrow \dfrac{\partial f}{\partial r} = -n k r^{n-1} \Rightarrow$

$$-n k r^{n-2} < +3 \left(+k \frac{r^n}{r}\right)$$

$$\Rightarrow \quad -n k \cancel{r^{n-1}} < 3 k \cancel{r^{n-1}}$$

$$-n < 3 \quad \Rightarrow \quad \boxed{n > -3}$$

Recordando: $\dot{\theta} = L/mr^2 \Rightarrow \frac{d\theta}{dt} = L/mr^2 \Rightarrow$

$dt\, L = d\theta\, mr^2 \Rightarrow \frac{L}{mr^2}\frac{d}{d\theta} = \frac{d}{dt}$

$m\ddot{r} - L^2/mr^3 = f(r)$

$m\frac{d^2r}{dt^2} - L^2/mr^3 = m\frac{d}{dt}\left(\frac{dr}{dt}\right) - L^2/mr^3 =$

$= m\frac{d}{dt}\left(\frac{L}{mr^2}\frac{dr}{d\theta}\right) - L^2/mr^3 = \cancel{m} \cdot \frac{L}{\cancel{m}r^2}\frac{d}{d\theta}\left(\frac{L}{mr^2}\frac{dr}{d\theta}\right) -$

$- L^2/mr^3 \Rightarrow \frac{L}{r^2}\frac{d}{d\theta}\left(\frac{L}{mr^2}\frac{dr}{d\theta}\right) - \frac{L^2}{mr^3} = f(r)$

Multiplicando por $\frac{mr^2}{L^2} \Rightarrow \frac{L}{\cancel{r^2}} \frac{m\cancel{r^2}}{L^{\cancel{2}}} \frac{d}{d\theta}\left(\frac{L}{mr^2}\frac{dr}{d\theta}\right) - \frac{\cancel{L^2}}{\cancel{m}r^3} \frac{\cancel{m}r^2}{\cancel{L^2}} =$

$$- \frac{mr^2}{L^2} \frac{dV}{dr}$$

$$\Rightarrow \frac{m}{L} \frac{d}{d\theta}\left(\frac{L}{mr^2} dr/d\theta\right) + \frac{1}{r} = \frac{mr^2}{L^2} dV/dr$$

Proponho entonces $u = 1/r \Rightarrow du = -1/r^2\, dr$

Se pueden probar que

$$\frac{d^2 u}{d\theta^2} + u = \overbrace{-\frac{m}{L^2}\frac{d}{du} V\left(1/u\right)}^{F(r)}$$

$dr$

Potenciales $\sim r^{-1} \Rightarrow f \sim r^{-2}$ conducen a órbitas estables para todo valor posible de $r_0$.

Una pequeña variación introduce lo siguiente:

$u = 1/r \Rightarrow$

$u = u_0 + a \cos \beta\theta$

Si $\beta$ es racional $\rightarrow$ órbita cerrada.

Puede factorizarse (Ejercicio) por tanto de ela ec. dif. en v que

$$\beta^2 (1-\beta^2)(4-\beta^2) = 0$$

$\Rightarrow \beta = 0$

$\beta = 1 \longrightarrow f \sim r^{-2}$

$\beta = 2 \longrightarrow f \sim r$ — Hooke $\rightarrow$ Armónico