Hidden Markov Models

Theoretical Aspects. Octave Implementation. Applications

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Romanian Asociation for Artificial Intelligence

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PART 1.
Intro

PART 2.
Theory of HMMs

PART 3.
Demo & Discussions

Outline

Intro

Outline

1. ARIA Education Workshops
   - ARIA's Mission
   - ARIA Education
   - Workshop Program

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Outline

1 ARIA Education Workshops
- ARIA's Mission
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ARIA EDU

:)

Outline

1. ARIA Education Workshops
   - ARIA's Mission
   - ARIA Education
   - Workshop Program

Today's Program

9:00	Registration
10:00	Rahaturi despre ARIA
11:00	HMM Theory

Theory of HMMs

Outline

2. Machine Learning Applications for HMM
   - Machine Learning
   - Where do HMMs fit into Machine Learning?
3. Theory of HMMs
   - The 3 things you want from an HMM
   - Mathematical Foundations for HMMs
   - Notation Conventions & Framework Description
4. Implementing HMMs
   - Using the Model for Estimations: the Forward-Backward algorithm
   - Learning from Observations: Baum-Welch algorithm
   - Uncovering Hidden states: Viterbi algorithm

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What is Machine Learning?

Machine Learning

..Trascau is a beautiful horse.

Machine Learning Applications

- Computer Vision: Google Car
- Machine Translation
- Speech Recognition
- Recommender Systems
- Intelligent Advertising

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Machine Learning Classification

Types of Machine Learning Problems

- Regression
- Classification
- Reinforcement Learning

- supervised learning (eg. ..)
- unsupervised

Sequence / Temporal problems (I)

OBJECT TRACKING

Speaker Detection

GPS

Robotics

Surface to air missille

Ship or rocket navigation

SPEECH RECOGNITION

Voice User Interfaces - e.g. SIRI

Speech-to-Text Processing

Direct Voice Input - Aircraft

GESTURE RECOGNITION

Personalized Signature Recognition

Human Activity Recognition

Sign Language Recognition

Sequence / Temporal problems (II)

BIOINFORMATICS

Protein Sequencing

Modeling of a Gene Regulatory Network

ECONOMICS

Stock Price Prediction

Econometrics
- estimate a country's econmic indicators across time -

Probabilistic Reasoning over Time - Models

Consider some of the previously presented problems ...

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How do we model such dynamic situations?

Probabilistic Reasoning over Time - Models

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How do we model such dynamic situations?

States and Observations

- The process of change is viewed as a series of time slices (snapshots)
- Each time slice contains a set of random variables
  - $\mathbf{O}_t$ - set of all *observable* evidence variables at time t
  - $\mathbf{Q}_t$ - set of all *unobservable / hidden* state variables at time t

Probabilistic Reasoning over Time - Assumptions

Consider some of the previously presented problems ...

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What assumptions (if any) do we make?

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Stationary Process

The process of change is governed by laws that do not themselves change over time.
Implication: we need to specify conditional distributions only for the variables within a *representative* timeslice.

Probabilistic Reasoning over Time - Assumptions

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Stationary Process

The process of change is governed by laws that do not themselves change over time.
Implication: we need to specify conditional distributions only for the variables within a *representative* timeslice.

Markov Assumption

The current state in a process of change depends only on a finite history of previous states.
Implication: there is a bounded number of "parents" for the variables in each time slice.
$\mathbf{P}(\mathbf{Q}_t|\mathbf{Q}_{1:t-1}) = \mathbf{P}(\mathbf{Q}_t|\mathbf{Q}_{t-1}) \qquad \mathbf{P}(\mathbf{O}_t|\mathbf{Q}_{1:t}, \mathbf{Q}_{1:t-1}) = \mathbf{P}(\mathbf{O}_t|\mathbf{Q}_t)$

Probabilistic Reasoning over Time - Inference

What are the basic inference tasks that must be solved?

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Filtering (monitoring)

The task of computing the belief state - the posterior distribution over the current state, given all evidence to date.
$\mathbf{P}(\mathbf{Q}_t|\mathbf{o}_{1:t})$

Probabilistic Reasoning over Time - Inference

What are the basic inference tasks that must be solved?

Filtering (monitoring)

The task of computing the belief state - the posterior distribution over the current state, given all evidence to date.
$\mathbf{P}(\mathbf{Q}_t|\mathbf{o}_{1:t})$

Evaluation (likelihood)

The task of computing the likelihood of the evidence up to present.
$\mathbf{P}(\mathbf{o}_{1:t})$

Probabilistic Reasoning over Time - Inference

Prediction

The task of computing the posterior distribution over the future state, given all evidence to date.
$\mathbf{P}(\mathbf{Q}_{t+k}|\mathbf{o}_{1:t})$, for some $k > 0$

Probabilistic Reasoning over Time - Inference

Prediction

The task of computing the posterior distribution over the future state, given all evidence to date.
$\mathbf{P}(\mathbf{Q}_{t+k}|\mathbf{o}_{1:t})$, for some $k > 0$

Smoothing (hindsight)

The task of computing the posterior distribution over a past state, given all evidence to the present.
$\mathbf{P}(\mathbf{Q}_{k}|\mathbf{o}_{1:t})$, for some $1 \leq k < t$
Provides a better estimate of the state than was available at the time.

Probabilistic Reasoning over Time - Inference

Most likely explanation

Given a *sequence of observations*, find the sequence of states that is most likely to have generated those observations. $argmax_{q_{1:t}}\ \mathbf{P}(\mathbf{q}_{t+k}|\mathbf{o}_{1:t})$, for some $k > 0$

Probabilistic Reasoning over Time - Inference

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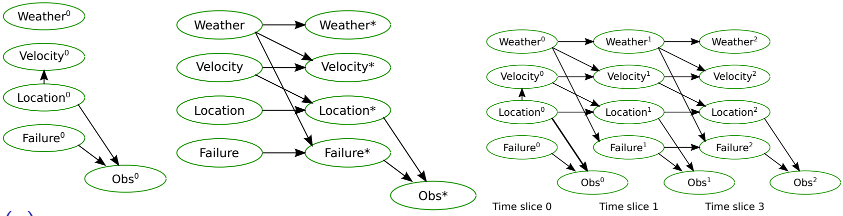
Learning

Given a set of *observation sequences*, find a method to learn the transition (e.g. $\mathbf{P}(\mathbf{q}_{t+1} = s_j|\mathbf{q}_t = s_i)$, $1 \leq i, j < N$) and sensor ($\mathbf{P}(\mathbf{o}_t|\mathbf{q}_t)$) models from the observations.

Probabilistic Reasoning over Time - Known Methods

Dynamic Bayesian Networks (DBN)

A DBN is Bayesian network that represents a temporal probability model.



(a) the 2-time-slice Bayesian Network (b) the time 0 network (c) unrolled DBN over 3 time slices

Figure: A highly simplified DBN for monitoring a vehicle (Koller and Friedman 2009)

Applied in problems like: object tracking, human activity recognition, protein sequencing etc.

Probabilistic Reasoning over Time - Known Methods

Kalman Filters (Linear Dynamical Systems)

A temporal model of one or more real-valued variables that evolve linearly over time, with some Gaussian noise.

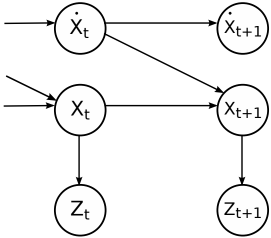


Figure: BN structure for a linear dynamical system with position X_t, velocity $\dot{X}_t$, and position measurement Z_t

- can be viewed as DBNs where all variables are continuous and all dependencies are linear gaussian
- wide application in **object tracking**

Probabilistic Reasoning over Time - Known Methods

Hidden Markov Models (HMM)

An HMM is a temporal probabilistic model in which the state of the process is described by a single discrete random variable. The possible values of the variable are the possible states of the world.

Used successfully in applications like:

- Handwriting Recognition
- Gesture Recognition
- Speech Recognition
- Part-of-Speech Tagging
- DNA Sequencing

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The 3 fundamental problems (Rabiner 1989)

- Particularization of temporal inference problems to the HMM case
- The restricted structure of the HMM allows for elegant implementations of all the basic algorithms

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Evaluation Problem

Given a model and a sequence of observations, how do we compute the probability that the observed sequence was produced by the model?

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Best Explanation of Observations Problem

Given a model and a sequence of observations how do we choose a corresponding sequence of states which *gives meaning* to the observations? How do we *uncover* the hidden part of the model?

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Model Estimation (Training) Problem

Given some observed sequences, how do we adjust the parameters of an HMM model that best tries to explain the observations?

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Elements of an HMM

- N Hidden States : $S_1, S_2, \ldots S_N$
- M Observable Variables : $O_1, O_2, \ldots O_M$

Parameters:

- Transition Function / Matrix between states
- Emission probabilities
- Initial state probabilities

Formalisation of the estimation problem

-

Formalisation of problem # 2

- $P(Q_1) = \sum_{x=\{1,2\}}^{N} P(Q_2)\theta\Pi$
- $P(Q_i | q_i = s_x) = i \times x \cdot i \ldots$

Formalisation of parameters estimation problem

-

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Notation Conventions

Variables in Octave

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Learning from observations - Reminder

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Adjust the model parameters $\lambda = (A, B, \Pi)$ to obtain $\max_\lambda P(O|\lambda)$

Learning from observations - Reminder

Model Estimation (Training) Problem

Given some observed sequences, how do we adjust the parameters of an HMM model that best tries to explain the observations?

Adjust the model parameters $\lambda = (A, B, \Pi)$ to obtain $\max_\lambda P(O|\lambda)$

The observation sequence used to adjust the model parameters is called a training sequence.
Training problem is crucial - allows to create best models for real phenomena.

Learning from observations - Aspects of the approach

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Problem

There is no known way to analytically solve for the model which maximizes the probability of the observation sequence.

Learning from observations - Aspects of the approach

Problem

There is no known way to analytically solve for the model which maximizes the probability of the observation sequence.

Solution

We can choose $\lambda = (A, B, \Pi)$ such that $\max_{\lambda} P(O|\lambda)$ is locally maximized using an iterative procedure such as *Baum-Welch*.
The method is an instance of the *EM algoritm* (Dempster, Laird, and Rubin 1977) for HMMs.

Baum-Welch algorithm (I)

Baum-Welch algorithm (I)

We first define some auxiliary variables:

$\xi_{t,i,j} = \xi_t(i,j) = P(q_t = s_i, q_{t+1} = s_j | O, \lambda)$
The probability of being in state s_i at time t and in state s_j at time $t+1$, given the model and the observation sequence.

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The probability of being in state s_i at time t and in state s_j at time $t+1$, given the model and the observation sequence.

$\gamma_{t,i} = \gamma_t(i) = P(q_t = s_i | O, \lambda)$
The probability of being in state s_i at time t, given the model and the observation sequence.

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The probability of being in state s_i at time t, given the model and the observation sequence.

From the definitions it follows that:

$$\gamma_t(i) = \sum_{j=1}^{N} \xi_t(i,j)$$

Baum-Welch algorithm (II)

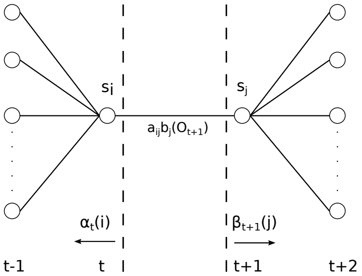

Figure: Sequence of operations required for the computation of the joint event that the system is in state S_i at time t and state S_j at time $t+1$ (Rabiner 1989)

$$\alpha_{t,i} = P(o_1, o_2, \dots, o_t, q_t = S_i | \lambda)$$

$$\beta_{t,i} = P(o_{t+1} o_{t+2} \cdots o_T | q_t = S_i, \lambda)$$

$$\xi_t(i,j) = \frac{\alpha_{t,i} \cdot a_{i,j} \cdot b_j(o_{t+1}) \cdot \beta_{t+1,j}}{P(O|\lambda)} = \frac{\alpha_{t,i} \cdot a_{i,j} \cdot b_j(o_{t+1}) \cdot \beta_{t+1,j}}{\sum_{k=1}^{N} \sum_{l=1}^{N} \alpha_{t,k} \cdot a_{k,l} \cdot b_l(o_{t+1}) \cdot \beta_{t+1,l}}$$

Baum-Welch algorithm (III)

How do these auxiliary variables help?

$$\sum_{t=1}^{T-1} \gamma_t(i) = \text{expected number of transitions from } S_i$$

$$\sum_{t=1}^{T-1} \xi_t(i,j) = \text{expected number of transitions from } S_i \text{ to } S_j$$

Baum-Welch algorithm (IV)

$$\bar{\pi}_i = \text{ expected no. of times in state } S_i \text{ at time } (t = 1) = \gamma_t(i)$$

Baum-Welch algorithm (IV)

$$\bar{\pi}_i = \text{ expected no. of times in state } S_i \text{ at time } (t = 1) = \gamma_t(i)$$

$$\bar{a_{i,j}} = \frac{\text{expected no. of transitions from } S_i \text{ to } S_j}{\text{expcted no. of transition from } S_i}$$

$$= \frac{\sum_{t=1}^{T-1} \xi_t(i,j)}{\sum_{t=1}^{T-1} \gamma_t(i)}$$

Baum-Welch algorithm (IV)

$$\bar{\pi}_i = \text{ expected no. of times in state } S_i \text{ at time } (t = 1) = \gamma_t(i)$$

$$\bar{a_{i,j}} = \frac{\text{expected no. of transitions from } S_i \text{ to } S_j}{\text{expcted no. of transition from } S_i} = \frac{\sum_{t=1}^{T-1} \xi_t(i,j)}{\sum_{t=1}^{T-1} \gamma_t(i)}$$

$$\bar{b_{j,k}} = \frac{\text{expected no. of times in } S_j \text{ observing symbol } v_k}{\text{expcted no. of times in } S_j} = \frac{\sum_{t=1, O_t = v_k}^{T} \gamma_t(j)}{\sum^{T} \cdots}$$

Baum-Welch algorithm (V)

The routine for the general case:

Initialize uniform π_i for $1 \leq i \leq N$
Initialize random (stochastic) $a_{i,j}$
Initialize uniform $b_{j,k}$ for $1 \leq k \leq M$

Repeat until convergence
    E step:
        compute auxiliary variables $\xi_t(i,j)$ and $\gamma_t(i)$
        using current π_i, $a_{i,j}$ and $b_{j,k}$

    M step:
        compute updated parameter models $\bar{\pi}_i$, $\bar{a}_{i,j}$, $\bar{b}_{j,k}$

Baum-Welch Iterative Update

Baum-Welch - Let's write some code

LET'S WRITE SOME CODE :-)

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Viterbi s-a nascut in ...

Demo & Discussions

Outline

5 A Case for HMMs in Symbol Recognition

A simple symbol recognition application

Features:

A simple symbol recognition application

Features:

Define

Define, organize and visualize a dataset of symbols defined with mouse movements.

A simple symbol recognition application

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Train

Train a HMM-based recognition engine on a symbol dataset.

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Recognize new symbols and view classification metrics.

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Train a HMM-based recognition engine on a symbol dataset.

Recognize

Recognize new symbols and view classification metrics.

Default included symbols: **left arrow, right arrow, circle, square, infinity**

A simple symbol recognition application - A View

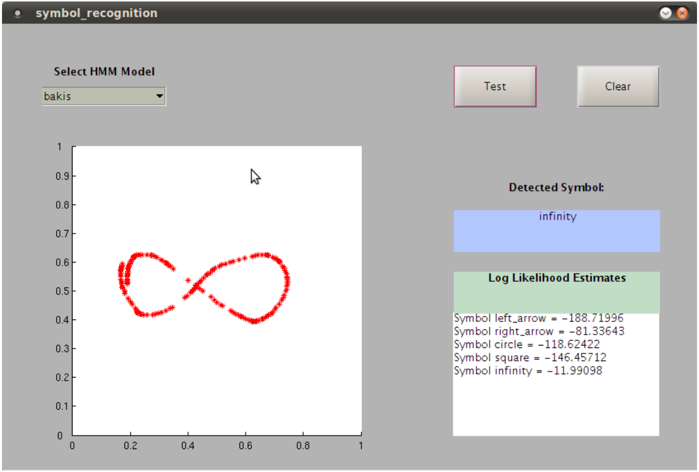
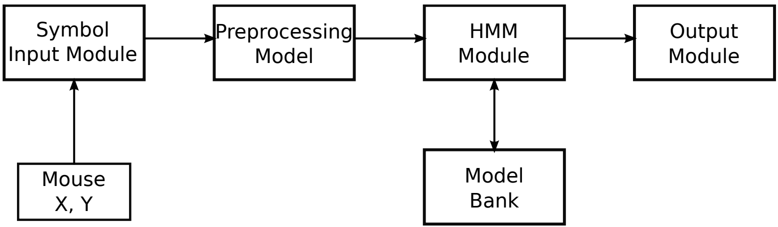


Figure: A view of the symbol recognition application GUI

A simple symbol recognition application - Approach (I)

Adapted from (Yang and Xu 1994).



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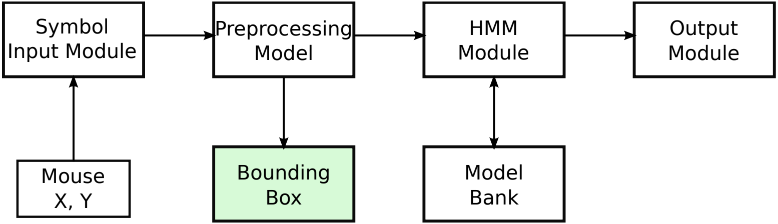

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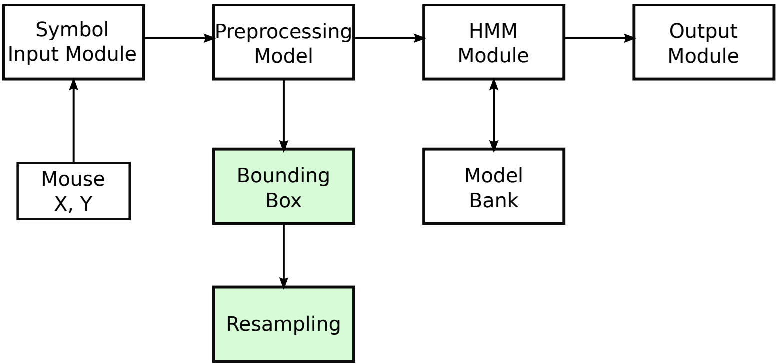

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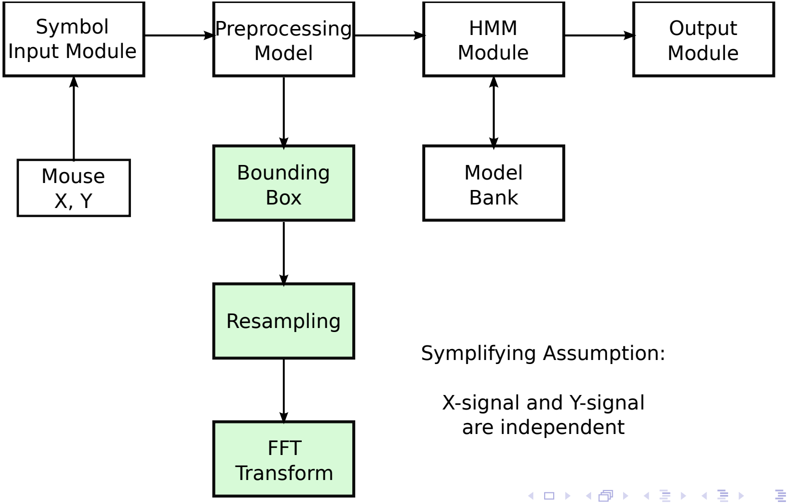

A simple symbol recognition application - Approach (II)

Adapted from (Yang and Xu 1994).

HMM Structure

N(number of states) $= 8$
2 discrete observable variables per state - $coef_{FFT}(x)$, $coef_{FFT}(y)$
M(number of values for each observable variable) $= 256$
Transition model:

- Bakis
- Ergodic

A simple symbol recognition application - Results

Dataset size

5 symbols: **left arrow, right arrow, circle, square, infinity**
100 samples per symbol: **50** training, **10** validation, **40** testing

```
>> symbol_performance_test('ergodic')
========= Testing trained HMM models ========
## Results for the model of symbol "left_arrow":
        Accuracy: 0.97500
        Precision: 1.00000
        Recall: 0.97500
        Confusion matrix line:    39    0    1    0    0    0

## Results for the model of symbol "right_arrow":
        Accuracy: 1.00000
        Precision: 1.00000
        Recall: 1.00000
        Confusion matrix line:     0   40    0    0    0    0

## Results for the model of symbol "circle":
        Accuracy: 0.90244
        Precision: 0.97368
        Recall: 0.92500
        Confusion matrix line:     0    0   37    2    1    0

## Results for the model of symbol "square":
        Accuracy: 0.95238
        Precision: 0.95238
        Recall: 1.00000
        Confusion matrix line:     0    0    0   40    0    0

## Results for the model of symbol "infinity":
        Accuracy: 0.97561
        Precision: 0.97561
        Recall: 1.00000
        Confusion matrix line:     0    0    0    0   40    0
```

```
>> symbol_performance_test('bakis')
========= Testing trained HMM models ========
## Results for the model of symbol "left_arrow":
        Accuracy: 0.90000
        Precision: 1.00000
        Recall: 0.90000
        Confusion matrix line:    36    0    1    0    0    3

## Results for the model of symbol "right_arrow":
        Accuracy: 1.00000
        Precision: 1.00000
        Recall: 1.00000
        Confusion matrix line:     0   40    0    0    0    0

## Results for the model of symbol "circle":
        Accuracy: 0.97561
        Precision: 0.97561
        Recall: 1.00000
        Confusion matrix line:     0    0   40    0    0    0

## Results for the model of symbol "square":
        Accuracy: 0.97500
        Precision: 1.00000
        Recall: 0.97500
        Confusion matrix line:     0    0    0   39    0    1

## Results for the model of symbol "infinity":
        Accuracy: 1.00000
        Precision: 1.00000
        Recall: 1.00000
        Confusion matrix line:     0    0    0    0   40    0
```