13. Small barbershop, operated by single barber, has room for @ most 2 customers. Potential customers arrive @ ~Poi $\lambda = 3$/hr, & sucessive service times are indep ~exp rv w/ $\mu = 0.25$ hr

$\lambda = 3$

$\mu = 4$

$\Rightarrow \frac{\lambda}{\mu} = \frac{3}{4}$

a. What's avg # of customers in shop?

$$P_0 = \frac{1}{1+P_1+P_2} = \frac{1}{1+0.75+(0.75)^2} = \frac{16}{37}$$

$$P_1 = \frac{\lambda_0}{\mu_1}(P_0) = \frac{3}{4}(P_0) = \frac{3}{4}\left(\frac{16}{37}\right) = \frac{12}{37}$$

$$P_2 = \frac{\lambda_1}{\mu_2}(P_1) = \frac{3}{4}\left(\frac{3}{4}\right)P_0 = \frac{9}{16}(P_0) = \frac{9}{16}\left(\frac{16}{37}\right) = \frac{9}{37}$$

$$P_1 + 2P_2 = \frac{12}{37} + 2\left(\frac{9}{37}\right) = \frac{12+18}{37} = \frac{30}{37}$$

The average number of customers in the shop is $\frac{30}{37}$.

b. What's proportion of potential customers that enter shop?

$$\frac{\lambda(1-P_2)}{\lambda} = 1-P_2 = 1 - \frac{9}{37} = \frac{37-9}{37} = \frac{28}{37}$$

The proportion of potential customers that enter the shop is $\frac{28}{37}$.

c. If barber could work twice as fast, how much more business would he do?

$$\lambda = 3 \qquad \lambda = 3$$

$$\mu = 4 \ \underset{2x}{\Longrightarrow} \ \mu = 8$$

$$P_0 = \frac{1}{1+P_1+P_2} = \frac{1}{1+0.375+(0.375)^2} = \frac{64}{97}$$

$$P_1 = \frac{\lambda_0}{\mu_1}(P_0) = \frac{3}{8}(P_0) = \frac{3}{8}\left(\frac{64}{97}\right) = \frac{24}{97}$$

$$P_2 = \frac{\lambda_1}{\mu_2}(P_1) = \frac{3}{8}\left(\frac{3}{8}\right)P_0 = \frac{9}{64}(P_0) = \frac{9}{64}\left(\frac{64}{97}\right) = \frac{9}{97}$$

$$\frac{\lambda(1-P_2)}{\lambda} = 1-P_2 = 1-\frac{9}{97} = \frac{97-9}{97} = \frac{88}{97}$$

$$\lambda\left(\frac{88}{97}-\frac{28}{37}\right) = 3\left(\frac{540}{3589}\right) = \frac{1620}{3589} = 0.4514$$

If the barber could work twice as fast, the business would increase by 45.14%.

17. Each time machine is repaired it remains up for rexp distr time w/ $\lambda = 1$. It then fails, & its failure is either 2 types. If it's TI failure, then time to repair machine is rexp w/ $\lambda = \mu_1$; if it's TII failure, then repair time is rexp w/ $\lambda = \mu_2$. Each failure is, indep of time it took machine to fail, TI failure w/ prob p & TII failure w/ prob $(1-p)$. What proportion of time is machine down due to TI failure? due to TII failure? What proportion of time is it up?

working $= \lambda$ prop T1 vs T2

$T_1 = \mu_1, P(p)$
$T_2 = \mu_2, P(1-p)$ prop of working

$S = \{0, 1, 2\}$

0 = machine ↑ $v_0 = \lambda$
1 = machine ↓ TI $v_1 = \mu_1$
2 = machine ↓ TII $v_2 = \mu_2$

$\lambda_0 P_0 = \mu_1 P_1$

$(\lambda_1 + \mu_1) P_1 = \mu_2 P_2 + \lambda_0 P_0$

$(\lambda_2 + \mu_2) P_2 = \mu_3 P_3 + \lambda_1 P_1$

$(\lambda_n + \mu_n) P_n = \mu_{n+1} P_{n+1} + \lambda_{n-1} P_{n-1}$

$\Rightarrow$

$\mu_1 P_1 = p(\lambda) P_0$

$P_1 = \frac{p\lambda}{\mu_1}(P_0)$

$\mu_2 P_2 = (1-p) P_0 \lambda$

$P_2 = \frac{\lambda(1-p)}{\mu_2} P_0$

$P_0 + P_1 + P_2 = 1 \Rightarrow P_0 + \frac{p\lambda}{\mu_1}(P_0) + \frac{\lambda(1-p)}{\mu_2} P_0 = 1$

$\Rightarrow P_0 \left[1 + \frac{p\lambda}{\mu_1} + \frac{\lambda(1-p)}{\mu_2} \right] = 1$

$\Rightarrow P_0 \left[\frac{\mu_1 \mu_2 + \mu_2 (p) \lambda + \lambda(1-p)\mu_1}{\mu_1 \mu_2} \right] = 1$

$\Rightarrow P_0 = \frac{\mu_1 \mu_2}{\mu_1 \mu_2 + \mu_2 (p) \lambda + \lambda(1-p)\mu_1}$

$$P_1 = \frac{p\lambda}{\mu_1}(P_0) = \frac{p\lambda}{\cancel{\mu_1}}\left[\frac{\cancel{\mu_1}\mu_2}{\mu_1\mu_2 + \mu_2(p)\lambda + \lambda(1-p)\mu_1}\right]$$

$$= \frac{p\lambda\mu_2}{\mu_1\mu_2 + \mu_2(p)\lambda + \mu_1(1-p)\lambda}$$

$$P_2 = \frac{\lambda(1-p)}{\mu_2}P_0 = \frac{\lambda(1-p)}{\cancel{\mu_2}}\left[\frac{\mu_1\cancel{\mu_2}}{\mu_1\mu_2 + \mu_2(p)\lambda + \lambda(1-p)\mu_1}\right]$$

$$= \frac{\mu_1(1-p)\lambda}{\mu_1\mu_2 + \mu_2(p)\lambda + \mu_1(1-p)\lambda}$$

The proportion of time the machine is up is:

$$\frac{\mu_1\mu_2}{\mu_1\mu_2 + \mu_2(p)\lambda + \mu_1(1-p)\lambda}$$

The proportion of time the machine is down due to type I failure is:

$$\frac{p\lambda\mu_2}{\mu_1\mu_2 + \mu_2(p)\lambda + \mu_1(1-p)\lambda}$$

The proportion of time the machine is down due to type II failure is:

$$\frac{\mu_1(1-p)\lambda}{\mu_1\mu_2 + \mu_2(p)\lambda + \mu_1(1-p)\lambda}$$

1. Customers arrive @ full service one-pump gas station @ $\lambda = 20$ cars/hr. However, customers will go to another station if there are @ least 2 cars on the station, ie, one being served & one waiting. Suppose that service time for customers is exp w/ $\mu = 6$ min

a. Formulate MC model for # of cars @ gas station & find its stationary distr

$\lambda = 20$ cars/hr $\quad S = \{0, 1, 2\}$

$\mu = 6$ min

λ of service $= \frac{60}{6} = 10$/hr

$$R = \begin{array}{c|ccc|c} & 0 & 1 & 2 & \\ 0 & -20 & 20 & 0 & = 0 \\ 1 & 10 & -30 & 20 & = 0 \\ 2 & 0 & 10 & -10 & = 0 \end{array}$$

$-20\pi_0 + 10\pi_1 = 0 \Rightarrow -20\pi_0 = -10\pi_1 \Rightarrow \pi_1 = 2\pi_0 = \frac{2}{7}$

$20\pi_1 - 10\pi_2 = 0 \Rightarrow 20\pi_1 = 10\pi_2 \Rightarrow \pi_2 = 2\pi_1 = \frac{4}{7}$

$\pi_0 + \pi_1 + \pi_2 = 1 \Rightarrow \pi_0 + 2\pi_0 + 2\pi_1 = 1 \Rightarrow 3\pi_0 + 4\pi_0 = 1$
$\Rightarrow 7\pi_0 = 1 \Rightarrow \pi_0 = \frac{1}{7}$

b. On avg, how many customers are served/hr

$(1-\pi_0)\lambda = \left(1 - \frac{1}{7}\right)10 = \left(\frac{7-1}{7}\right)10 = \left(\frac{6}{7}\right)10 = \frac{60}{7} = 8.5714$

On average, 9 customers are served per hour.