Ch 6.4

$\{X(t), t \geq 0\}$ = cont time MC

$P[X(t+s) = j \mid X(s) = i] = p_{ij}(t)$

$\mathbb{P}(t) = [p_{ij}(t)]_{i,j \in E}$

$\mathbb{P}(t) = \begin{pmatrix} p_{00}(t) & p_{01}(t) & \cdots \end{pmatrix}$

$T_i \sim Exp(v_i)$

i ———— change

$\mathbb{E}(T_i) = \frac{1}{v_i}$

p_{ij} $\qquad$ $q_{ij} = v_i p_{ij}$

$p_{ii} = 0$

$$R_{ij} = \begin{cases} -v_i, & i = j \\ q_{ij}, & i \neq j \end{cases} \qquad \mathbb{R} = \begin{bmatrix} -v_0 & q_{01} = v_0 (p_{01}) & \cdots & q_{0m} = v_0 p_{0m} \\ q_{10} & -v_1 & & \end{bmatrix}$$

$E = \{0, 1, \ldots, m\}$

$\mathbb{P}'(t) = \mathbb{R} \cdot \mathbb{P}(t)$

$\mathbb{P}'(t) = \mathbb{P}(t)\mathbb{R}$

$P(t) = e^{Rt}$

$P(t) = expm(R \cdot t)$ matlab

Ch 6.5: Lim Probs

$p_j = \lim_{t \to \infty} p_{ij}$

We assume lim exists & is indep of initial state

Note: $p_{ij}(t)$ function

p_{ij} #, element of mat $\mathbb{P}$

p_j lim distr

Forward Kolmogorov Eq

$p_{ij}'(t) = \sum_{k \neq j} p_{ik}(t) q_{kj} - v_j p_{ij}(t)$

$$\lim_{t\to\infty} P_{ij}'(t) = \lim_{t\to\infty} \sum_{k\neq j} P_{ik}(t) q_{kj} - v_j P_{ij}(t)$$

$$\frac{d}{dt} \lim_{t\to\infty} P_{ij}(t) = 0 = \sum_{k\neq j} P_k (q_{kj}) - v_j P_{ij}(t)$$

$P'(t) = P(t)R$

Balance eq

$0 = (P_0, P_1, \ldots, P_m) R$

$$v_j P_j = \sum_{k\neq j} P_k q_{kj}$$

$$\vec{P} = \begin{pmatrix} P_0 \\ P_1 \\ \vdots \\ P_m \end{pmatrix} \quad \vec{P}^T (R) = 0 \quad \sum_{\forall i} P_i = 1$$

$$\sum P_j = 1$$

$P^T \pi = \pi$

$\pi^T P = \pi^T$

$\pi^T (P - I) = 0$

A sufficient condition for P_j exist

1. MC is irreducible

$\forall i,j$, $i \leftrightarrow j$ i,j communicates

$i \to j$ j is accesible from i

$\exists t$ s.t. $P[X(t+s)=i \mid P(s)=i] > 0$

2. MC is (+) recurrent starting in any state, mean time to return to that state is finite

If (1) & (2) holds, P_j is long run prop of time. Process is in state j

Theorem

If cont-time MC, $\{X(t), t \geq 0\}$ is irreducible & has stationary distr. Then

$$p_j = \lim_{t \to \infty} p_{ij}(t)$$

Example

1 = sunny $\quad T_1 \sim Exp\left(\frac{1}{3}\right) \quad T_3 \sim Exp(1)$

2 = smoggy $\quad T_2 \sim Exp\left(\frac{1}{4}\right)$

3 = rainy

$$R = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -\frac{1}{4} & \frac{1}{4} \\ 1 & 0 & -1 \end{pmatrix} \qquad \vec{p}R = 0 \qquad \sum p_j = 1$$

$$(p_1 \ p_2 \ p_3) \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -\frac{1}{4} & \frac{1}{4} \\ 1 & 0 & -1 \end{pmatrix} = (0, 0, 0)$$

$$\begin{cases} -\frac{1}{3}p_1 + p_3 = 0 \\ \frac{1}{3}p_1 - \frac{1}{4}p_2 = 0 \\ p_1 + p_2 + p_3 = 1 \end{cases} \qquad \begin{aligned} p_3 &= \frac{1}{3}p_1 \\ p_2 &= \frac{4}{3}p_1 \\ p_1\left(1 + \frac{4}{3} + \frac{1}{3}\right) &= 1 \end{aligned} \qquad \begin{aligned} p_1 &= \frac{3}{8} \\ p_2 &= \frac{1}{2} \\ p_3 &= \frac{1}{8} \end{aligned}$$

lim prob for birth & death process

$v_0 = \lambda_n \qquad p_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i} \qquad p_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}$

$v_i = \lambda_i + \mu_i$

$$
R \quad
\begin{array}{c|cccccccc}
 & 0 & 1 & 2 & 3 & \cdots & i-1 & i & i+1 \\
\hline
0 & -\lambda_0 & \lambda_0 & 0 & 0 & & 0 & 0 & 0 \\
1 & \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & & 0 & 0 & 0 \\
2 & 0 & \mu_2 & -(\lambda_2+\mu_1) & \lambda_2 & & 0 & & \\
\vdots & & & & & & & & \\
i-1 & & & & & & & & \\
i & 0 & & & & & \mu_i & -(\lambda_i+\mu_i) & \lambda_i \\
i+1 & & & & & & & & \\
\vdots & & & & & & & &
\end{array}
$$

$-P_0\lambda_0 + \mu_1 P_1 = 0$

$\lambda_0 P_0 - (\lambda_1+\mu_1)P_1 + \mu_2 P_2 = 0$

$\lambda_1 P_1 - (\lambda_2+\mu_2)P_2 + \mu_3 P_3 = 0$

$\vdots$

$\lambda_{n-1}P_{n-1} - (\lambda_n+\mu_n)P_n + \mu_{n+1}P_{n+1} = 0$

$\sum P_i = 1$