Brownian Motion

Axioms:
1. $X(0)=0$
2. $\{X(t), t\geq 0\}$ has stationary & indep increments

   $X(t+s)-X(t)$
3. $\forall t>0,\ X(t)\sim N(0,\sigma^2 t)$

$X(t)$
cont $f(t)$

$X(t,w)$
cont $f(t)$ $\forall$ trajectory

$$\lim_{h\to 0}(X(t+h)-X(t))=0$$

$$\lim_{h\to 0}\mathbb{E}(X(t+h)-X(t))=0$$

$$\lim_{h\to 0}\mathbb{V}(X(t+h)-X(t))=\lim_{h\to 0}h=0$$

$$\lim_{h\to 0}\frac{X(t+h)-X(t)}{h}$$

$$\mathbb{E}\left(\frac{X(t+h)-X(t)}{h}\right)=0$$

$$\mathbb{V}\left(\frac{X(t+h)-X(t)}{h}\right)=\lim_{h\to 0}\frac{1}{h^2}(h)=\infty$$

$X(t_1),\dots,X(t_n)$

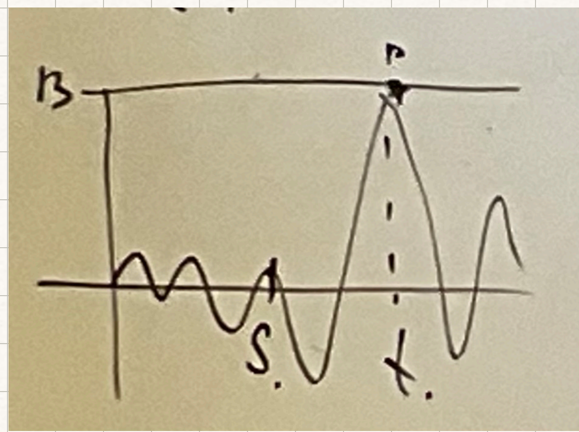
$$f(x)=\frac{1}{\sqrt{2\pi t}}\left(e^{-\frac{x^2}{2t}}\right)$$

$$\begin{matrix}X(t_1)=x_1\\X(t_2)=x_2\\\vdots\\X(t_n)=x_n\end{matrix}\equiv\left\{\begin{matrix}X(t_1)=x_1\sim N(0,t_1)\\X(t_2)-X(t_1)=x_2-x_1\\\sim N(0,t_2-t_1)\\\vdots\\X(t_n)-X(t_{n-1})=x_n-x_{n-1}\end{matrix}\right.$$

$$f(x_1,\dots x_n)=f(x_1)f(x_2-x_1)\cdots f(x_n-x_{n-1})$$

$$f(x_1,\dots,x_n) = \frac{\exp\left\{-\frac{1}{2}\left[\frac{x_1^2}{t_1}+\frac{(x_2-x_1)^2}{t_2-t_1}+\dots+\frac{(x_n-x_{n-1})^2}{t_n-t_{n-1}}\right]\right\}}{(2\pi)^{\frac{n}{2}}\sqrt{t_1(t_2-t_1)\cdots(t_n-t_{n-1})}}$$

$X(t)=B$ $\quad s<t$



$$f_{s|t}(x|B) = \frac{f_s(x)f_{t-s}(B-x)}{f_t(B)}$$

$$= k_1\left\{\exp\left[-\frac{x^2}{2s}-\frac{(B-x)^2}{2(t-s)}\right]\right\}$$

$$= k_1\left[\exp\left(-\frac{x^2}{2s}-\frac{B^2-2Bx-x^2}{2t-2s}\right)\right]$$

$$= k_2\left\{\exp\left[-\frac{x^2}{2s}-\frac{x^2}{2(t-s)}+\frac{x}{t-s}\right]\right\}$$

$$-\frac{x^2}{2}\left(\frac{\frac{1}{t-s}}{s}+\frac{\frac{1}{s}}{t-s}+\frac{Bx}{t-s}\right)=$$

$$= -\frac{1}{2(t-s)}\left[x^2\left(\frac{t}{s}\right)-2Bx\right]$$

$$= -\frac{1}{2(t-s)}\left(\frac{t}{s}x^2-2Bx+\sqrt{\frac{s}{t}}B^2-\sqrt{\frac{s}{t}}B^2\right)$$

$$= -\frac{1}{2(t-s)}\left(x\sqrt{\frac{s}{t}}-B\sqrt{\frac{s}{t}}\right)^2$$

$$= -\frac{1}{2(t-s)}\left(\frac{xt-Bs}{\sqrt{st}}\right)^2 = -\frac{1}{2(t-s)\left(\frac{s}{t}\right)}\left[x-B\left(\frac{t}{s}\right)\right]^2$$

$$= k_3\exp\left\{-\frac{\left[x-B\left(\frac{s}{t}\right)\right]^2}{2(t-s)\frac{s}{t}}\right\}$$

$$k_3 = \frac{1}{\sqrt{2\pi\left(\frac{t}{s}\right)(t-s)}}$$

$$X(s) \mid X(t) \sim \mathcal{N}\left(\frac{s}{t}(B), \frac{s}{t}(t-s)\right)$$

Example

$Y(t)$ = amt of time (in seconds) by which racer * is ahead 100t% of race has been completed

$\{Y(t), 0 \le t \le 1\}$

$\mathbb{E}[Y(t)] = 0$

$\mathbb{V}[Y(t)] = \sigma^2 t$

If inside racer is leading by σ sec @ midpt of race, what's P(she's winner)?

$$P[Y(1) > 0 \mid Y(0.5) = \sigma] = P\left[Y(1) - Y\left(\tfrac{1}{2}\right) > -\sigma \,\middle|\, Y\left(\tfrac{1}{2}\right) = \sigma\right]$$

$$= P\left[Y(1) - Y\left(\tfrac{1}{2}\right) > -\sigma\right]$$

$$Y\left(\tfrac{1}{2}\right) \sim \mathcal{N}\left(0, \frac{\sigma^2}{2}\right)$$

$$= P\left[Y\left(\tfrac{1}{2}\right) > -\sigma\right] = P\left(Z > \frac{-\sigma - 0}{\sigma/\sqrt{2}}\right)$$

$$= P(Z > -\sqrt{2}) = P(Z \le \sqrt{2})$$