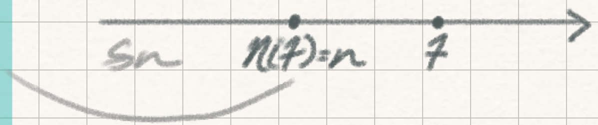
Ch 7.2: Distr of $N(t)$

$N(t) \geq n \iff S_n \leq t$

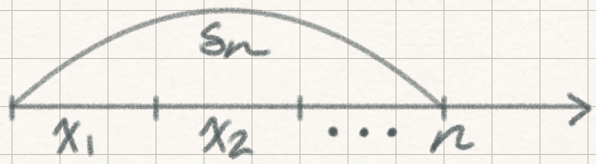


$P(\overset{C}{N(t)=n}) = P(\overset{A}{N(t)\geq n}) - P(\overset{B}{N(t)\geq n+1})$

$A = B \cup C$ $\quad = P(S_n \leq t) - P(S_{n+1} \leq t)$

$P(N(t)\geq n+1) + P(N(t)=n) = P(N(t)\geq t)$

$P(N(t)=n) = F_n(t) - F_{n+1}(t)$



$S_n = \sum_{i=1}^{n} X_i$

$X_i \quad F$

$X_1 + X_2 \sim F_2$

$\vdots$

$S_n \sim F_n$

Example:

$P(X_n = i) = p(1-p)^{i-1}, \; i \geq 1$

$$P(S_n = k) = \begin{cases} \binom{k-1}{n-1} p^n (1-p)^{k-n}, & k \geq n \\ 0, & k < n \end{cases}$$

$P(S_n \leq t) = \sum_{j=n}^{[t]} \binom{j-1}{n-1} p^n (1-p)^{j-n}$

$F_n(t)$

$F_{n+1}(t) = P(S_{n+1} \leq t) = \sum_{j=n+1}^{[t]} \binom{j-1}{n} p^{n+1} (1-p)^{j-n-1}$

$P(N(t)=n) = F_n(t) - F_{n+1}(t) = \binom{[t]}{n} p^n (1-p)^{[t]-n}$

Another expression

$$P(N(t)=n) = \int_0^\infty P(N(t)=n \mid S_n = y) f_{S_n}(y)\,dy$$

$$= \int_0^t P(X_{n+1} > t-y \mid S_n = y) f_{S_n}(y)\,dy$$

right tail

$$= \int_0^t \bar{F}(t-y) f_{S_n}(y)\,dy$$

$F(x) = 1-e^{-\lambda x}$
X_i : exp
$S_n = \text{Gamma}(n,\lambda)$

$$= \int_0^t e^{-\lambda(t-y)} \frac{e^{-\lambda y} (\lambda y)^{n-1}}{(n-1)!}\,dy$$

$$= \int_0^t \frac{e^{-\lambda t}(\cancel{e^{\lambda y}}) \cancel{e^{-\lambda y}} (\lambda y)^{n-1}}{(n-1)!}\,dy$$

$$= \frac{e^{-\lambda t}(\lambda)\lambda^{n-1}}{(n-1)!} \int_0^t y^{n-1}\,dy = \frac{e^{-\lambda t}(\lambda^n) y^n}{(n-1)!\,n} \Bigg|_0^t$$

$$= \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

Mean value renewal $f(x)$

$$m(t) = \mathbb{E}(N(t)) = \sum_{n=1}^\infty P(N(t) \geq n) = \sum_{n=1}^\infty P_n(S_n \leq t)$$

$$= \sum_{n=1}^\infty F_n(t)$$

$m(t) < \infty \;\; \forall\, t < \infty$

$$\mathbb{E}N(t) = \mathbb{E}(\mathbb{E}(N(t) \mid X_1))$$

$$\int_0^\infty \mathbb{E}(N(t) \mid X_1 = x) f_X(x)\,dx = 1 + \mathbb{E}(N(t-(x)))$$ $x < t$

$$m(t) = \int_0^t (1+m(t-x)) f(x)\,dx = F(t) + \int_0 m(t-x)\,dx$$

Renewal Eq

$$m(t) = F(t) + \int_0^t m(t-x)f(x)\,dx$$

$X \sim U(0,1)$

$F(t) = t \quad 0 \le x \le 1$

$$m(t) = t + \int_0^t m(t-x)\,dx \qquad \begin{aligned} y &= t-x \\ dy &= -dx \end{aligned}$$

$$= t - \int_t^0 m(y)\,dy = t + \int_0^t m(y)\,dy$$

$m'(t) = 1 + m(t)$

$h(t) = 1 + m(t)$

$\Rightarrow \frac{dh}{dt} = \frac{dm}{dt}$

$h'(t) = h(t) \Rightarrow \frac{dh}{dt} = h(t)$

$\frac{dh}{h} = dt \Rightarrow \ln h = t + c \Rightarrow h = e^{t+c} \quad c = \ln k$

$\Rightarrow h(t) = ke^t$

$m(t) = ke^t - 1 \qquad \begin{aligned} m(0) &= 0 \\ k &= 1 \end{aligned}$

$m(t) = e^t - 1$
$0 \le t \le 1$

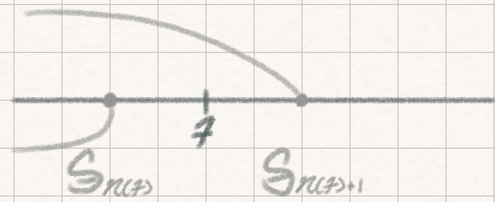
Ch 7.3: Lim Theorems & Their Apps

$\begin{aligned} N(t) &\to \infty \\ t &\to \infty \end{aligned} \Rightarrow$ w/ prob 1

What's the rate?

$S_{N(t)} = \sum_{i=1}^{N(t)} X_i$ time of last renewal prior to t

$S_{N(t)+1} = \sum_{i=1}^{N(t)+1} X_i$



$\mathbb{E}X_i = \mu$

Prop

$$\frac{N(t)}{t} \longrightarrow \frac{1}{\mu} \quad t \to \infty \text{ w/ prob } 1$$

$$\frac{S_{N(t)}}{N(t)} \le \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}$$

$$\frac{S_{N(t)}}{N(t)} = \frac{\sum_{i=1}^{N(t)} X_i}{N(t)} \longrightarrow \mu \quad t \to \infty \text{ w/ prob } 1$$

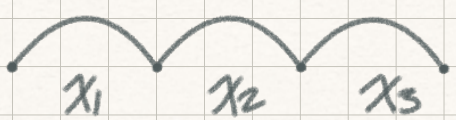
Strong Law of Large #s

$$\frac{S_{N(t)+1}}{N(t)} = \frac{S_{N(t)+1}}{N(t)} \left(\frac{N(t)+1}{N(t)} \right) = \mu(1)$$

$$\frac{S_{N(t)}}{N(t)} \to \mu \le \frac{t}{N(t)} \to \mu < \frac{S_{N(t)+1}}{N(t)} \to \mu$$

$t \to \infty$

Ex 7.4



$X_i \sim U(30, 60)$

$$\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{\mu} = \frac{1}{45}$$