Recitation 1, solutions.

Continuous MC, definition, transition probability
Problem 1:
Consider a birth and death process with birth rates $\lambda_i = (i + 1)\lambda$, $i \geq 0$, and death rates $\mu_i = i\mu$, $i \geq 0$.
(a) Determine the expected time to go from state 0 to state 4.
(b) Determine the expected time to go from state 2 to state 5.
(c) Determine the variances in parts (a) and (b).

$$\lambda_i = (i+1)\lambda \qquad i \geq 0$$

$$\mu_i = i\mu \qquad i \geq 0$$

$$S_4 = T_0 + T_1 + T_2 + T_3$$

$$ES_4 \ -?$$

$$ET_i = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} ET_{i-1}$$

$$ET_0 = \frac{1}{\lambda_0} = \frac{1}{\lambda}$$

$$ET_1 = \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1} \cdot ET_0 = \frac{1}{2\lambda} + \frac{\mu}{2\lambda} \cdot \frac{1}{\lambda} = \frac{1}{2\lambda}\left(1 + \frac{\mu}{\lambda}\right)$$

$$ET_2 = \frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2} ET_1 = \frac{1}{3\lambda} + \frac{\cancel{2}\mu}{3\lambda} \cdot \frac{1}{\cancel{2}\lambda}\left(1 + \frac{\mu}{\lambda}\right) =$$

$$= \frac{1}{3\lambda}\left(1 + \frac{\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2\right)$$

$$ET_3 = \frac{1}{\lambda_3} + \frac{\mu_3}{\lambda_3} ET_2 = \frac{1}{4\lambda} + \frac{\cancel{3}\mu}{4\lambda} \frac{1}{\cancel{3}\lambda} \cdot \left(1 + \frac{\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2\right)$$

$$= \frac{1}{4\lambda}\left(1 + \frac{\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2 + \left(\frac{\mu}{\lambda}\right)^3\right)$$

$$ES_4 = \frac{1}{\lambda} + \frac{1}{2\lambda}\left(1 + \frac{\mu}{\lambda}\right) + \frac{1}{3\lambda}\left(1 + \frac{\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2\right) + \frac{1}{4\lambda}\left(1 + \frac{\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2 + \left(\frac{\mu}{\lambda}\right)^3\right)$$

$$= \frac{1}{\lambda}\left(1 + \frac{1}{2} + \frac{1}{2}\frac{\mu}{\lambda} + \frac{1}{3} + \frac{1}{3}\frac{\mu}{\lambda} + \frac{1}{3}\left(\frac{\mu}{\lambda}\right)^2 + \frac{1}{4} + \right.$$

$$\left. + \frac{1}{4}\frac{\mu}{\lambda} + \frac{1}{4}\left(\frac{\mu}{\lambda}\right)^2 + \frac{1}{4}\cdot\left(\frac{\mu}{\lambda}\right)^3\right) =$$

$$= \frac{1}{\lambda}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)\frac{\mu}{\lambda} + \left(\frac{1}{3} + \frac{1}{4}\right)\left(\frac{\mu}{\lambda}\right)^2 + \frac{1}{4}\left(\frac{\mu}{\lambda}\right)^3\right)$$

$$= \frac{1}{\lambda}\left(\frac{25}{12} + \frac{13}{12}\frac{\mu}{\lambda} + \frac{7}{12}\left(\frac{\mu}{\lambda}\right)^2 + \frac{1}{4}\left(\frac{\mu}{\lambda}\right)^3\right) =$$

$$= \frac{1}{4\lambda}\left(\frac{25}{3} + \frac{13}{3}\frac{\mu}{\lambda} + \frac{7}{3}\left(\frac{\mu}{\lambda}\right)^2 + \left(\frac{\mu}{\lambda}\right)^3\right)$$

b) $$E\,T_4 = \frac{1}{5\lambda}\left(1 + \frac{\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2 + \left(\frac{\mu}{\lambda}\right)^3 + \left(\frac{\mu}{\lambda}\right)^4 + \left(\frac{\mu}{\lambda}\right)^5\right)$$

$$E(T_2 + T_3 + T_4) =$$

$$= \frac{1}{3\lambda}\left(1 + \frac{\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2 + \left(\frac{\mu}{\lambda}\right)^3\right) + \frac{1}{4\lambda}\left(1 + \frac{\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2 + \left(\frac{\mu}{\lambda}\right)^3 + \left(\frac{\mu}{\lambda}\right)^4\right)$$

$$+ \frac{1}{5\lambda}\left(1 + \frac{\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2 + \left(\frac{\mu}{\lambda}\right)^3 + \left(\frac{\mu}{\lambda}\right)^4 + \left(\frac{\mu}{\lambda}\right)^5\right).$$

c) $$\mathrm{Var}\,T_0 = \frac{1}{\lambda}$$

$$\mathrm{Var}\,T_1 = \frac{1}{\lambda_1(\lambda_1 + \mu_1)} + \frac{\mu_1}{\lambda_1}\overbrace{\frac{1}{\lambda}}^{\mathrm{Var}T_0} + \frac{\mu_1}{\mu_1 + \lambda_1}\left(\frac{1}{\lambda} + \frac{1}{2\lambda}\left(1 + \frac{\mu}{\lambda}\right)\right)$$

$$= \frac{1}{2\lambda(2\lambda + \mu)} + \frac{\mu}{2\lambda}\cdot\frac{1}{\lambda} + \frac{\mu}{\mu + 2\lambda}\left(\frac{1}{\lambda} + \frac{1}{2\lambda}\left(1 + \frac{\mu}{\lambda}\right)\right)$$

Similarly you find $V(T_2)$, $V(T_3)$, $V\,T_4$

$$\mathrm{Var}(S_4) = \mathrm{Var}(T_0) + \mathrm{Var}(T_1) + \mathrm{Var}(T_2) + \mathrm{Var}(T_3)$$

Problem 2

In a birth and death process with birth parameter $\lambda_n = \lambda$, n=0,1,..., and death parameters $\mu_n = n\mu$ for n=0,1,... we have $P_{0,j}(t) = \frac{(\lambda p)^j e^{-\lambda p}}{j!}$

Where

$p = \frac{1}{\mu}(1 - e^{-\mu t})$.

Verify that these transition probabilities satisfy the forward equation with i=0.

$$P_{ij}'(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - (\lambda_i + \mu_j) P_{ij}(t)$$

$$P_{ij}'(t) = \lambda_{j-1} P_{i,j-1}(t) + \mu_{j+1} P_{i,j+1}(t) - (\lambda_i + \mu_j) P_{ij}(t)$$

$$P_{0j}'(t) = \lambda P_{0,j-1}(t) + (j+1) P_{0,j+1}(t) - (\lambda + \mu j) P_{0j}$$

$$P_{0j}(t) = \frac{(\lambda p(t))^j e^{-\lambda p(t)}}{j!} \qquad p(t) = \frac{1}{\mu}(1 - e^{-\mu t})$$

$$\frac{dp}{dt} = \frac{1}{\mu} \frac{d}{dt} (1 - e^{-\mu t}) = -\frac{1}{\mu} e^{-\mu t} \cdot (-\mu) = e^{-\mu t}$$

$$= -1 + e^{-\mu t} + 1 = 1 - \frac{\mu(1 - e^{-\mu t})}{\mu} = \boxed{1 - \mu p}$$

$$\boxed{p_t' = 1 - \mu p}$$

$$\frac{dP_{0j}}{dt} = \frac{dP_{0j}}{dp} \cdot \frac{dp}{dt} = \left(\frac{(\lambda p)^j e^{-\lambda p}}{j!} \right)_t' = \frac{j(\lambda p)^{j-1} \cdot \lambda p' e^{-\lambda p} - (\lambda p)^j e^{-\lambda p} \cdot \lambda p'}{j!}$$

$$= \frac{(\lambda p)^{j-1} e^{-\lambda p} \cdot \lambda (j - \lambda p) p'}{j!} =$$

$$= \frac{(\lambda p)^{j-1} e^{-\lambda p} (1 - \mu p) \cdot \lambda (j - \lambda p)}{j!} \quad \checkmark \quad (1)$$

$$P_{0j}'(t) = \lambda P_{0,j-1}(t) + (j+1)\mu P_{0,j+1}(t) - (\lambda + j\mu) P_{0j}$$

$$\lambda \frac{(\lambda p)^{j-1} e^{-\lambda p}}{(j-1)!} + \frac{\cancel{(j+1)}\, \mu\, (\lambda p)^{j+1} e^{-\lambda p}}{\cancel{(j+1)}!\; j!} - \frac{(\lambda + j\mu)(\lambda p)^{j} e^{-\lambda p}}{j!} =$$

$$\frac{\lambda^{j} p^{j-1} e^{-\lambda p}}{(j-1)!}\left[1 + \frac{\mu \cdot \lambda p^2}{j} - \frac{(\lambda + j\mu) p}{j} \right] =$$

$$= \frac{\lambda^{j} p^{j-1} e^{-\lambda p}}{(j-1)!}\left[\frac{j + \mu \lambda p^2 - \lambda p - j\mu p}{j} \right] =$$

$$\frac{\lambda (\lambda p)^{j-1} e^{-\lambda p}}{j!} \quad [j(1-\mu p) - p\lambda(1-\mu p)] =$$

$$= \boxed{\frac{\lambda (\lambda p)^{j-1} e^{-\lambda p}}{j!} (1-\mu p)(j - \lambda p)} \; \checkmark \quad (2)$$

Both sides are equal $((1) = (2))$ ⊠

(Jukes–Cantor Model). In this chain, the states are the four nucleotides A, C, G, T. Jumps, which correspond to nucleotide substitutions, occur according at rate $q_{ij} = \mu$ if $i \neq j$. Find the transition probability matrix $\mathbf{P}(t)$ using forward differential equation.

$R =$

$$R = \begin{bmatrix} -3\mu & \mu & \mu & \mu \\ \mu & -3\mu & \mu & \mu \\ \mu & \mu & -3\mu & \mu \\ \mu & \mu & \mu & -3\mu \end{bmatrix}$$

$$\lambda_1 = -4m \quad \lambda_2 = -4m \quad \lambda_3 = -4m \quad \lambda_4 = 0 .$$

$v_1 = (-1, 0, 0, 1)$ $\quad v_3 = (-1, 1, 0, 0)$

$v_2 = (-1, 0, 1, 0)$ $\quad v_4 = (1, 1, 1, 1)$

$$V^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{2}{4} & -\frac{1}{4} \end{bmatrix} \qquad J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -4\mu & 0 & 0 \\ 0 & 0 & -4\mu & 0 \\ 0 & 0 & 0 & -4\mu \end{pmatrix}$$

$$V = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

use matlab to solve it

$$P(t) = e^{Rt} = V^{-1} e^{Jt} V =$$

$$\begin{pmatrix} \frac{1}{4} & +\frac{1}{4} e^{-4mt} & \frac{1}{4} e^{-4mt} & \frac{1}{4} e^{-4mt} \\ \frac{1}{4} & -\frac{1}{4} e^{-4mt} & -\frac{1}{4} e^{-4mt} & \frac{3}{4} e^{-4mt} \\ \frac{1}{4} & -\frac{1}{4} e^{-4mt} & \frac{3}{4} e^{-4mt} & -\frac{1}{4} e^{-4mt} \\ \frac{1}{4} & \frac{3}{4} e^{-4mt} & -\frac{1}{4} e^{-4mt} & -\frac{1}{4} e^{-4mt} \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

$$\frac{1}{4} + \frac{1}{4} e^{-4mt} + \frac{1}{4} e^{-4mt} + \frac{1}{4} e^{-4\mu t} =$$

$$= \frac{1}{4}\left(1 + \frac{3}{4} e^{-4mt}\right) = e^{-4\mu t} + \frac{1}{4}\left(1 - e^{-4\mu t}\right)$$

$$\frac{1}{4} - \frac{1}{4} e^{-4mt} = \frac{1}{4}\left(1 - e^{-4\mu t}\right)$$

you can also use function expm(Rt)

4) Similar to 3.