Notation

L
avg # cust in system

L_q
# of cust in queue

W
avg amt of time in system

W_q
avg time in queue

λ_a
Δ of arrival

$L = \lambda_a W$

$L_q = L - [0(\pi_0) + 1(1-\pi_0)]$

$= L - 1 + \pi_0$

$\mathbb{E}S = 0(\pi_0) + 1(1-\pi_0)$

$\pi_0 = 1 - \frac{\lambda_a}{\mu}$

Steady-State Prob

let $X(t)$ b # of custs in system @ time t

$\pi_n = \lim_{t \to \infty} P(X(t) = n)$

s-s prob of exactly n custs in system

Notation

a_n
prop of custs that find n in system when they arrive

d_n
prop of custs leavin behind n in system

π_n
prop of time during which there are in system

time of service $= 1$

arrival $\sim U(1,2)$

$a_0 = d_0 = 1$
$\pi_0 \neq 1$

Prop I

In any system in which custs arrive one @ time & finally depart one @ time & leave n in system

$a_n = d_n$

Prop II

Poi arrivals always see time avg

$\pi_n = \frac{\lambda(\pi_n)}{\lambda} a_n$

Result of Poi arrivals see time avg is called PASTA principle

Exercise 2: Bus stop

Ppl arrive accordin to Poi process w/ $\Delta\ \lambda$. Busses arrive according to Poi process w/ $\Delta\ \mu$

$W_Q = \frac{1}{\mu}$

$L_Q = \lambda W_Q = \frac{\lambda}{\mu}$

$\mathbb{E}(X_i | T_i) = \lambda T_i$

$\mathbb{E}(X_i) = \mathbb{E}[\mathbb{E}(X_i | T_i)] = \mathbb{E}(\lambda T_i)$

$= \lambda \mathbb{E}(T_i) = \frac{\lambda}{\mu}$

3.2.3: M/G/1

M

Markovian arrivals

Poi process w/ $\Delta\ \lambda$

G

gen service

$S_i \sim G$

$\mathbb{E}(S_i) = \frac{1}{\mu}$

nth cust enters system

X_n

MC

# of custs in queue

$X_0 = x$

$$a_k = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^k}{k!} g(t) \, dt$$

prob that k custs arrived during service time

$$\sum_{\forall k} k a_k = \lambda \mathbb{E}(S_i) = \frac{\lambda}{\mu}$$

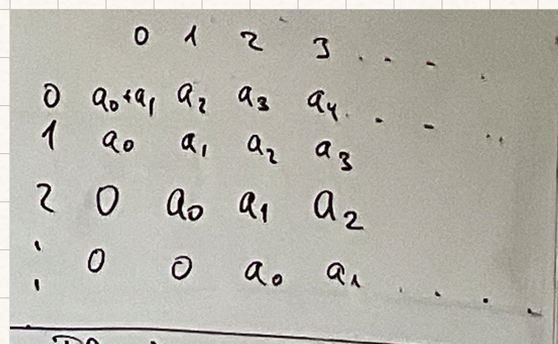
$g(t)t$

$\xi_1, \xi_2 \dots$ be iid rv

$P(\xi_i = k) = a_k$

In $X_n > 0$

$$X_{n+1} = X_n + \xi_n - 1 = (X_n + \xi_n - 1)^+$$



$$\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & \cdots \\ \hline 0 & a_0+a_1 & a_2 & a_3 & a_4 & \cdots \\ 1 & a_0 & a_1 & a_2 & a_3 & \\ 2 & 0 & a_0 & a_1 & a_2 & \\ \vdots & 0 & 0 & a_0 & a_1 & \cdots \end{array}$$

Thm

If $\lambda < \mu$, then MC X_n is (+) recurrent

$$\mathbb{E}_0 T_0 = \frac{\mu}{\mu - \lambda}$$

$\lambda = \mu$ MC is 0 recurrent

$\lambda > \mu$ MC is trans

Busy period

$\lambda < \mu$, M/G/1

$\mathbb{E}(B_n) = \frac{1}{\mu - \lambda}$ $\mathbb{E}(S_i) = \frac{1}{\mu}$

$\lambda = \Delta$ of arrival

$$\pi_0 = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + \mathbb{E}(B_n)} = 1 - \frac{\lambda}{\mu} = \frac{\mu - \lambda}{\mu}$$

$$\mathbb{E}(B_n) + \frac{1}{\lambda} = \frac{1/\lambda}{\pi_0} \Rightarrow \mathbb{E}(B_n) = \frac{1/\lambda}{\pi_0} - \frac{1}{\lambda} = \frac{\mu/\lambda}{\mu - \lambda} - \frac{1}{\lambda}$$

$$= \frac{1}{\lambda}\left(\frac{\mu}{\mu-\lambda} - 1\right) = \frac{1}{\lambda}\left(\frac{\mu - \mu + \lambda}{\mu - \lambda}\right)$$

$$= \frac{1}{\lambda}\left(\frac{\lambda}{\mu-\lambda}\right) = \frac{1}{\mu-\lambda}$$

Thm: Pollaczek-Khintchine Formula

long run avg waitin time in queue

$$W_Q = \frac{\lambda \mathbb{E}\left(\frac{S_i^2}{2}\right)}{1 - \lambda \mathbb{E}(S_i} = \frac{\lambda}{2} \frac{\mathbb{V}(S_i) + (\mathbb{E}S_i)^2}{1 - \mathbb{E}(S_i)}$$

$$\mathbb{E}\left(\frac{S_i^2}{2}\right) = \frac{1}{2}\mathbb{E}(S_i^2) = \frac{1}{2}[\mathbb{V}(S_i) + (\mathbb{E}S_i)^2]$$

$$\mathbb{V}(S_i) = \mathbb{E}(S_i)^2 - (\mathbb{E}S_i)^2$$

Example: Long frac of time service is idle

$\lambda = \frac{1}{6}$ /min $\qquad \pi_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{1}{6}\Big/\frac{1}{5} = 1 - \frac{5}{6} = \frac{1}{6}$

$\mathbb{E}S_i = \frac{1}{\mu} = 5 \qquad W = W_Q + S_i = \frac{\lambda}{2}\frac{\sigma^2 + (1/\mu)^2}{1 - \frac{1}{\mu}} + 5 = \frac{1/6}{2}\frac{49+25}{1 - \frac{1}{6}(5)} + 5$

$\sigma = 7$

$$= \frac{1}{12}\left(\frac{74}{1/6}\right) + 5 = \frac{74}{2} + 5 = 37 + 5 = 42$$

$L = \lambda W$

$L_Q = L - 1 + \pi_0$