Ch 6.4: Trans Prob Mat

Let $P_{ij}(t) = P\{X(t+s)=j \mid X(s)=i\}$

$\mathbb{P}(t)=[P_{ij}(t)]$

mat P w/ elements of $P_{ij}(t)$, change in time

v_i = Δ @ which process makes trans when in state i

$q_{ij} = v_i P_{ij}$

Backward Kolmogorov Eq

$$P_{ij}'(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t) \qquad (1)$$

$\mathbb{P}(t) = (P_{ij}(t)) \qquad \mathbb{R} = (R_{ij})$

$$R_{ij} = \begin{cases} q_{ij}, & i \neq j \\ -v_i, & i = j \end{cases}$$

$\therefore$ (1) $\quad \mathbb{P}'(t) = \mathbb{R}P(t)$

$P(t) = e^{Rt}$

Forward Kolmogorov Eq

$$P_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_i P_{ij}$$

$P'(t) = P(t)R \quad (2)$

Example: Pure Birth Process

$v_i = \lambda_i$ $\quad P_{01} = 1$

$P_{i,i+1} = 1$

$P_{ij} = 0, \; i \neq i+1$

$P'_{ij}(t) = \lambda_i P_{i+1,i}(t) - \lambda_i P_{ij}(t)$ ~ back

$P'_{ij}(t) = \lambda_{j-1} P_{i,j-1} - \lambda_j P_{ij}(t)$ ~ forward (3)

$P_{ij} = 0, \; j < i$

(3) $P'_{ii}(t) = -\lambda_i P_{ii}(t)$

$P'_{ij}(t) = \lambda_{i-1} P_{i,j-1}(t) - \lambda_j P_{ij}(t)$, $j \geq i+1$

Prop

For pure birth process, $P_{ii}(t) = e^{\lambda_i t}$

$$P_{ij}(t) = \lambda_{j-1} e^{-\lambda_j t} \int_0^t e^{\lambda_j s} P_{i,j-1}(s)\, ds$$

$j \geq i+1$

Proof

$P'_{ii}(t) = -\lambda_i P_{ii}(t)$

$$\frac{dP_{ii}(t)}{dt} = -\lambda_i P_{ii}(t)$$

$$\int \frac{dP_{ii}(t)}{P_{ii}(t)} = \int -\lambda_i \, dt$$

$\ln P_{ii}(t) = -\lambda_i t + c$ $\quad c = \ln k$

$\ln P_{ii}(t) = \ln e^{-\lambda_i t + c}$ $\quad \ln e^{-\lambda_i t} \cdot e^{\ln k}$

$P_{ii}(t) = k e^{-\lambda_i t} = e^{-\lambda_i t}$ $\quad P_{ii}(0) = 1 \Rightarrow k = 1$

$$P'_{ij}(t) = \lambda_{j-1} P_{i,j-1}(t) - \lambda_j P_{ij}(t)$$

$$P'_{ij}(t) + \lambda_j P_{ij}(t) = \lambda_{j-1} P_{i,j-1}(t) \qquad \Big| e^{\lambda_j t}$$

$$e^{\lambda_j t}\left[P'_{ij}(t) + \lambda_j P_{ij}(t)\right] = e^{\lambda_j t}(\lambda_{j-1}) P_{i,j-1}(t)$$

$$\frac{d}{dt}\left[e^{\lambda_j t} P_{ij}(t)\right] = e^{\lambda_j t}(\lambda_{j-1}) P_{i,j-1}(t)$$

$$d\left[e^{\lambda_j t} P_{ij}(t)\right] = e^{\lambda_j t}(\lambda_{j-1}) P_{i,j-1}(t)\,dt$$

$$e^{\lambda_j t} P_{ij}(t) = \lambda_{j-1} \int_0^t e^{\lambda_j s} P_{i,j-1}(s)\,ds$$

$$P_{ij}(t) = e^{-\lambda_j t}(\lambda_{j-1}) \int_0^t e^{-\lambda_j s} P_{i,j-1}(s)\,ds + C$$

$P_{ij}(0) = 0 \qquad C = 0$

$$P_{ij}(t) = \lambda_{j-1}\left(e^{-\lambda_j t}\right) \int_0^t e^{-\lambda_j s} P_{ij-1}(s)\,ds$$

$\square$