Chapter 6

$\{X(t), t \geq 0\}$

$\forall s,t \geq 0 \quad i,j \in \mathbb{Z}_+$

$P\{X(t+s)=j \mid X(s)=i, X(u)=x(u)\ 0 \leq u < s\} = P\{X(t+s)=j \mid X(s)=i\}$

1. $\forall i, T_i \sim Exp(v_i)$

$\mathbb{E}(T_i) = \frac{1}{v_i}$

2. P_{ij}

$P_{ii} = 0$ $\qquad R = [r(i,j)]_{ij}$

$\sum P_{ij} = 1 \ \forall i$

Example:

A shoe shine shop has 2 chairs. If customer comes in, chair 1 is cream, chair 2 is shine. Then they leave

$\{N(t), t \geq 0\}, \sim Poi(\lambda)$ $\qquad$ State space

$T_1(\text{chair } 1) \sim Exp(\mu_1) \Rightarrow \mathbb{E}(T_1) = 1/\mu_1$ $\qquad$ 0 = empty (come in)

1 = cust in ch1 (leave)

$T_2 \sim Exp(\mu_2) \Rightarrow \mathbb{E}(T_2) = 1/\mu_2$ $\qquad$ 2 = cust in ch2 (leave)

v_i, v_0, v_1, v_2

if v_0, how long until someone comes?

$T_0 \sim Exp(\lambda), v_0 = \lambda$ (rate)

Define:

$v_0 = \lambda$

$v_1 = \mu_1$

$v_2 = \mu_2$

$$\begin{pmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

6.3 Birth & Death Process

Assume there are n ppl in system

1. New arrivals enter system @ exp $\Delta = \lambda_n$
2. Ppl leave system @ exp $\Delta = \mu_n$

T_a = time until next arrival

$T_a \sim Exp(\lambda_n)$

$\{\lambda_n\}_{n=0}^{\infty}$ = birth rate

$\mathbb{E}(T_a) = 1/\lambda_n$

T_d = time until next departure

$T_d \sim Exp(\mu_n)$

$\{\mu_n\}_{n=0}^{\infty}$ = death rate

$\mathbb{E}(T_d) = 1/\mu_n$

Birth & death process is cont time MC

states: $\{0, 1, 2, \dots\}$

$v_0 = \lambda_0$

$v_i = \lambda_i + \mu_i$

$T \sim \min(T_a, T_d) \sim Exp(\lambda_i + \mu_i)$

P_{ij} $\quad P_{01} = 1$

$$P_{i,j+1} = \frac{\lambda_i}{\lambda_i + \mu_i}$$

$T_a = \min(T_a, T_d)$

$P(T_a < T_d)$

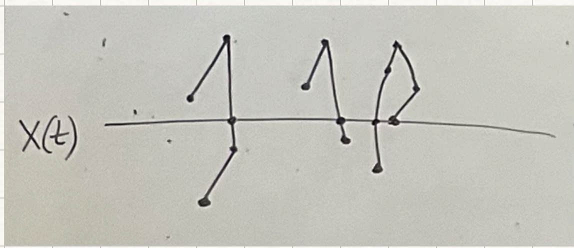
$$P_{i,j-1} = \frac{\mu_i}{\lambda_i + \mu_i}$$

Poisson process

$\mu_n = 0 \quad n \geq 0$

$\lambda_n = \lambda \quad n \geq 0$

Example: birth process w/ linear birth Δ



T /! \! / $T \sim Exp(\lambda)$

let $X(t)$ be pop size @ time t

n = indiv

Yule process: math theory of evo

$T \sim Exp(n\lambda)$

$\lambda_n = n\lambda$

Ex: linear growth model w/ imigration

$\mu_n = n\mu$

$\lambda_n = n\lambda + \theta$

Example: queueing M|M|1

customers arrive according to Poisson w/ $\Delta = \lambda$

$\boxed{\text{server}}$

$S \sim Exp(\mu)$

$E(T_S) = 1/\mu$

Markovian customer arrive accor to $\sim$Poi

Marovian service time is $\sim$exp

$\{X(t), t \geq 0\}$ = birth/death process

$\mu_n = \mu$ $\lambda_n = \lambda$

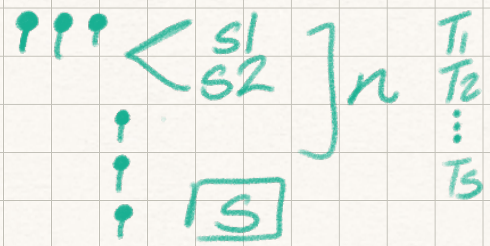
$v_i = \lambda + \mu$

Ex: multiserver exp queueing system M/M/S

now we have 2 servers!

$\lambda_n = \lambda$

$$\mu_n = \begin{cases} n\mu & 1 \le n \le s \\ s\mu & n > s \end{cases}$$



General Case

$\{\lambda_n\}$ = birth Δ

$\{\mu_n\}$ = death Δ

let $\mu_0 = 0$

T_i : $\overset{T_i}{\overline{i \qquad\qquad i+1}}$

time starting from state i it takes for process to enter i+1

$\mathbb{E}(T_i) = ?$

$T_0 \sim Exp(\lambda_0)$

$\mathbb{E}(T_0) = 1/\lambda_0$

$\forall i \ldots \quad i \to i+1$
$\qquad\qquad\;\; \to i-1$

$$I_i = \begin{cases} 1 & \text{if } i \to i+1 \\ 0 & \text{if } i \to i-1 \end{cases}$$

$$\mathbb{E}(T_i) = \mathbb{E}[\mathbb{E}(T_i \mid I_i)] = \mathbb{E}(T_i \mid I_i = 0)\,P(I_i = 0) + \mathbb{E}(T_i \mid I_i = 1)\,P(I_i = 1)$$

$$\mathbb{E}(T_i \mid I_i = 1) = \frac{1}{\lambda_i + \mu_i} \qquad P(I_i = 1) = \frac{\lambda_i}{\lambda_i + \mu_i}$$

$$\mathbb{E}(T_i \mid I_i = 0) = \frac{1}{\lambda_i + \mu_i} + \mathbb{E}(T_{i-1}) + \mathbb{E}(T_i) \qquad P(I_i = 0) = \frac{\mu_i}{\lambda_i + \mu_i}$$