Ex 7.28: Busy Period of $M/G/\infty$ Queue

M - Markovian

arriving according to Poi $\lambda$

G - time of service

Ex 5.18: $\infty$ server queue

# of customers @ time $t$ has

$\sim Poi\left(\lambda\int_0^t (1-G(y))dy = \lambda\int_0^t \overline{G}(y)dy\right)$

System can be: $\begin{cases} \text{Idle}, & \text{no customer} \\ \text{Busy}, & \text{@ least one customer} \end{cases}$

$\mathbb{E}B$ - in one cycle

$\mathbb{E}S = \int_0^\infty (1-G(t))dt = \int_0^\infty \overline{G}(t)dt$

mean of service distr

$\lim_{t\to\infty} P(\text{system off @ } t) = e^{-\lambda(\mathbb{E}S)}$

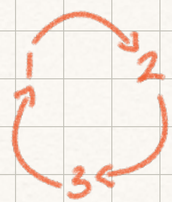
$e^{-\lambda(\mathbb{E}S)} = \dfrac{\mathbb{E}(\text{off time in cycle})}{\mathbb{E}(\text{cycle time})} = \dfrac{\lambda}{\lambda + \mathbb{E}B}$

$\mathbb{E}B = \frac{1}{\lambda}\left(e^{-\lambda(\mathbb{E}S)} - 1\right)$

# Semi-Markov Process

Let $\{X(t), t \geq 0\}$ be stochastic process


$\mathbb{E}T_1 = \mu_1$
$\mathbb{E}T_2 = \mu_2$
$\mathbb{E}T_3 = \mu_3$




Q: Prop of time process spends in state $i$ in long run

$$p_i = \frac{\mu_i}{\mu_1+\mu_2+\mu_3}$$

Similarly, if you have $n$ states

$\mathbb{E}T_i = \mu_i$

$$p_i = \frac{\mu_i}{\sum \mu_i}$$

Generalization

States: $1, 2, \dots, n$

$T_i$ = time spent in state $i$

$\mathbb{E}(T_i) = \mu_i$

$p_{ij}$ = prob that $i \to j$

Such process is called semi-Markov process

If $T_i = 1 \ \forall i$, then process is Markov chain (discrete time)

Q: Find $p_i$ - long run prop of time process spending in state $i$

$\pi_i$ will be lim (stationary) distr

assume embed MC is: aperiodic
pos recur
irreducible

$\sum \pi_i = 1 \quad ; \quad i = 1, \ldots, n$

$\pi_i = \sum_{j=1}^{n} \pi_j p_{ji}$

$$p_i = \frac{\pi_i \mu_i}{\sum_{j=1}^{n} \pi_j \mu_j}$$

Ex: 7.30

Good $\Rightarrow$ 1 $\quad \mathbb{E}T_1 = \mu_1$

Fair $\Rightarrow$ 2 $\quad \mathbb{E}T_2 = \mu_2$

Broken $\Rightarrow$ 3 $\quad \mathbb{E}T_3 = \mu_3$

$p_{12} = \frac{3}{4} \qquad p_{13} = \frac{1}{4}$

$p_{23} = 1$

$p_{31} = \frac{2}{3} \qquad p_{32} = \frac{1}{3}$

$$\begin{vmatrix} 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 1 \\ \frac{2}{3} & \frac{1}{3} & 0 \end{vmatrix}$$

$P^T \pi = \pi$

$(P^T - I)\pi = 0 \qquad \sum \pi_i = 1 \; ,$

$$\begin{vmatrix} -1 & 0 & \frac{2}{3} \\ \frac{3}{4} & -1 & \frac{1}{3} \\ 1 & 1 & 1 \end{vmatrix} \begin{vmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}$$

$\pi_1 = \frac{4}{15}$

$\pi_2 = \frac{5}{15}$

$\pi_3 = \frac{6}{15}$

$$p_1 = \frac{\pi_1 \mu_1}{\sum_{j=1}^{n} \pi_j \mu_j} = \frac{4\mu_1}{4\mu_1 + 5\mu_2 + 6\mu_3}$$

$$p_2 = \frac{5\mu_2}{4\mu_1 + 5\mu_2 + 6\mu_3}$$

$$P_3 = \frac{6\mu_3}{7\mu_1 + 5\mu_2 + 6\mu_3}$$

# Ex 7.18: Avg of Renewal Process

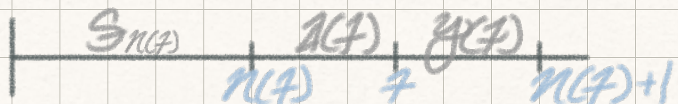
$A(t)$ = time until $t$ since last renewal $= t - S_{N(t)}$

age @ time $t$

$Y(t)$ = time after time $t$ until next renewal

excess/residual @ time $t$

$S_{N(t)}$ = time of last event prior to or @ time




Interested in avg value of age

$$\lim_{S\to\infty} \frac{\int_0^S A(t)\,dt}{S} = \frac{\mathbb{E}(\text{rwd during renew } \circledcirc)}{\mathbb{E}(\text{time of } \circledcirc)}$$

$$= \frac{\mathbb{E}\left(\int_0^X t\,dt\right)}{\mathbb{E}X} = \frac{\mathbb{E}\left(\frac{t^2}{2}\Big|_0^X\right)}{\mathbb{E}X} = \frac{\mathbb{E}(X^2)}{2\mathbb{E}X}$$

$= \text{avg age}$

Avg excess time

$$\lim_{S\to\infty} = \frac{\int_0^S Y(t)\,dt}{S} = \frac{\mathbb{E}(\text{rwd}/\circledcirc)}{\mathbb{E}(\text{time}/\circledcirc)} = \frac{\mathbb{E}(X^2)}{2\mathbb{E}X}$$

$$\text{rwd}/\circledcirc = \int_0^X (X - t)\,dt = \frac{X^2}{2}$$