Time Reversibility

$P_j = \lim_{t\to\infty} P_{ij}(t)$

Embedded chain

$P = (P_{ij})$

$$\begin{pmatrix} 0 & P_{01} & \cdots & P_{0m} \\ P_{10} & \ddots & & \\ \vdots & & \ddots & \\ P_{m} \cdots & & & 0 \end{pmatrix}$$

Assume embedded MC is ergodic (has stationary distr)

$$\begin{cases} \pi_i = \sum_{\forall j} \pi_j P_{ji} \\ \sum \pi_i = 1 \end{cases}$$

$$\begin{cases} P^T \pi = \pi \\ \sum \pi_i = 1 \end{cases}$$

$$\underbrace{(P^T - I)}_{R}\pi = 0$$

$$\pi = \begin{pmatrix} \pi_0 \\ \vdots \\ \pi_m \\ \vdots \end{pmatrix}$$

$$\boxed{P_i = \frac{\pi_i / v_i}{\sum_j \pi_j / v_j}} \quad (1)$$

$R^T \vec{P} = 0$

$\sum P_i = 1$

$(P_0, \dots, P_m, \dots) R = 0$

$$R^T \begin{pmatrix} P_0 \\ \vdots \\ P_m \end{pmatrix} = 0$$

$$v_i P_i = \sum_{j: j \neq i} P_j q_{ji}$$

$$q_{ij} = v_i P_{ij}$$

$$v_i P_i = \sum_{j: j \neq i} P_j v_j P_{ji} \quad \forall i \quad (2)$$

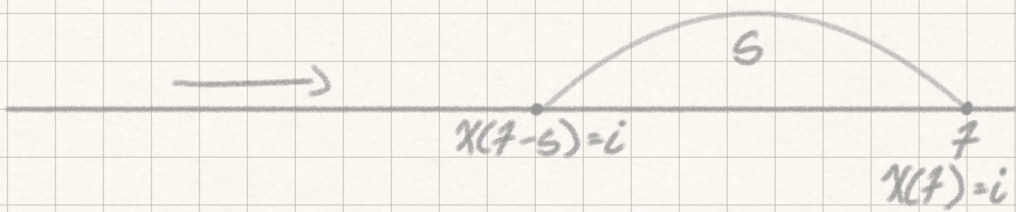
$(1) \rightarrow (2)$

$$\frac{\cancel{v_i}\left(\frac{\pi_i}{\cancel{v_i}}\right)}{\cancel{\sum_k \frac{\pi_k}{v_k}}} = \sum_{j: j \neq i} \frac{\frac{\pi_j}{\cancel{v_j}}(\cancel{v_j})}{\cancel{\sum_k \frac{\pi_k}{v_k}}} P_{ji}$$

$P_{ii} = 0 \quad \forall i$

$$\pi_i = \sum_{j: j \neq i} \pi_j (P_{ji}) \Rightarrow \pi_i = \sum_{\forall j} \pi_j (P_{ji})$$

Let MC have been operating for long time



$$P(X \text{ in state } i \ ((t-s);t) \mid X(t)=i) =$$

$$= \frac{P(\text{process is in state } i \ [t-s,t])}{P(X(t)=i)} = \frac{P(X(t-s)=i)e^{-v_i s}}{P(X(t)=i)}$$

$$= e^{-v_i s}$$

Reverse chain in discrete

$$Q_{ij} = \frac{\pi_j P_{ji}}{\pi_i}$$

Chain is time reversible if

$$Q_{ij} = P_{ij}$$

$$\pi_i P_{ij} = \pi_j P_{ji}$$

$$P_i = \frac{\pi_i / v_i}{\sum_j \frac{\pi_j}{v_j}}$$

$$P_i q_{ij} = P_j q_{ji} \quad \forall i,j$$

$$\frac{\pi_i}{\cancel{v_i}} \cancel{(v_i)} P_{ij} = \frac{\pi_j}{\cancel{v_j}} \cancel{(v_j)} P_{ji}$$

Prop 6.5

An ergodic birth & death process is time reversible

$i \rightarrow i+1$
$i \leftarrow i+1$

Example : Birth & Death Process

$S = \{0, 1, \ldots n\}$

$q_{n,n+1} = \lambda_n$ $\quad\quad$ $p_i = ?$

$q_{n,n-1} = \mu_n$

$$p_i q_{ij} = p_j q_{ji} \Rightarrow \begin{aligned} p_0 q_{01} &= p_1 q_{10} \\ p_1 q_{12} &= p_1 q_{21} \\ &\vdots \\ p_{n-1} q_{n-1,n} &= q_{n,n-1} p_n \end{aligned}$$

$$\begin{aligned} p_0 \lambda_0 &= p_1 \mu_1 \\ p_1 \lambda_1 &= p_2 \mu_2 \\ &\vdots \\ p_{n-1} \lambda_{n-1} &= p_n \mu_n \end{aligned} \Rightarrow \begin{aligned} p_1 &= \frac{\lambda_0}{\mu_1}(p_0) \\ p_2 &= \frac{\lambda_1}{\mu_2}(p_0) \\ p_n &= \frac{\lambda_{n-1} \cdots \lambda_0}{\mu_n \cdots \mu_1}(p_0) \end{aligned}$$

$$p_0 \underbrace{\left(1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1} + \ldots\right)}_{C} = 1 \Rightarrow p_0 C = 1$$

$$\Rightarrow p_0 = \frac{1}{C}$$

Prop 6.5

If for some set $\{P_i\}$

$$\sum P_i = 1$$

$$P_i q_{ij} = P_j q_{ji} \quad \forall i \neq j$$

Then cont MC is time reversible & $P_i = \lim_{t \to \infty} P_{ji}(t)$

Proof

$P_i q_{ij} = P_j q_{ji}$ holds

$P_i = \lim_{t \to \infty} P_{ji}(t)$

$$\sum_{j: j \neq i} P_i q_{ij} = \sum_{j: j \neq i} P_j q_{ji}$$

$$P_i \sum_{j: j \neq i} q_{ij} = \sum_{j: j \neq i} P_j q_{ji}$$

$$\begin{cases} P_i v_i = \sum_{j: j \neq i} P_j q_{ji} \\ \sum P_i = 1 \end{cases}$$

Ex: M/M/1 queue system

$\lambda_i = \lambda$
$\mu_i = \mu$

$$P_j = \frac{(\lambda/\mu)^j}{\sum_{i=0}^{n} \left(\frac{\lambda}{\mu}\right)^i}$$

Prop 6.8

Time reversible chain

$P_j = \lim_{t \to \infty} P_{ij} \qquad j \in S$

$A \subset S$

If remains irreducible & has lim prob p_j^A

$$p_j^A = \frac{p_j}{\sum_{\forall i \in A} p_i}$$

Prop 6.9

If $\{X_i(t), t \geq 0\}$, $i = (1, \dots, n)$ indep time reversible cont-tim MC, then

$$\{(X_1(t) \dots X_n(t))\ t \geq 0\}$$

is also time reversible cont MC