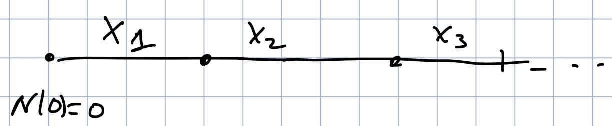
Limit theorems and their applications.

Let $\{N(t),\ t \geq 0\}$ be a renewal process.



$X_1, \ldots X_n, \ldots$ iid

time between renewals.

$$E X_i = \mu$$

$$S_{N(t)} = \sum_{i=1}^{N(t)} X_i$$

Pr. $$\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{\mu}$$ with probability 1 (almost surely).

$$m(t) = E(N(t))$$

Theorem 7.1 (Elementary renewal theorem).

$$\frac{m(t)}{t} \to \frac{1}{\mu} \qquad t \to \infty.$$

Ex. $U \sim \text{Unif}(0,1)$

$$Y_n = \begin{cases} 0 & u > \frac{1}{n} \\ n & u \leq \frac{1}{n} \end{cases}$$

With prob 1 Y_n be 0 for all sufficiently large n, $u > \frac{1}{n}$

$\Rightarrow$ $Y_n \to 0$ $n \to \infty$ (with prob. 1).

$$E Y_n = n\, P\left(U \le \frac{1}{n}\right) = n \cdot \frac{1}{n} = 1 .$$

Therefore even the seq. of r.v.

$Y_n \to 0$, $E\, Y_n \to 1$.
$n \to \infty$.

Def. The nonnegative integer value N is said to be stopping time for a seq. of independent r.v. $X_1, \dots X_n, \dots$ if the event that $\{N = n\}$ is independent of $X_{n+1}, X_{n+2}, \dots$ $\forall\, n = 1, 2, \dots$.

Ex. Let $X_1, X_2 \dots$ be iid seq

$P(X_i = 1) = p = 1 - P(X_i = 0)$ $p > 0$

$X_i \sim Bernoulli(p)$

$N = \min\{ n : X_1 + \dots + X_n = r \}$

N is a stopping time.

Theorem 7.2. (Wald's equation)

If $X_1, X_2 \dots$ iid r.v., $E X < \infty$ and if N is a stopping time for this sequence such that $E(N) < \infty$

then $E\left(\sum_{n=1}^{N} X_n\right) = EN\, EX$

Proof. For $n = 1, 2, \ldots$ let

$$I_n = \begin{cases} 1 & n \le N \\ 0 & n > N \end{cases}$$

$$\sum_{n=1}^{N} X_n = \sum_{n=1}^{\infty} X_n I_n$$

$$E\left(\sum_{n=1}^{N} X_n\right) = E\left(\sum_{n=1}^{\infty} X_n I_n\right) =$$

$$= \sum_{n=1}^{\infty} E(X_n I_n) =$$

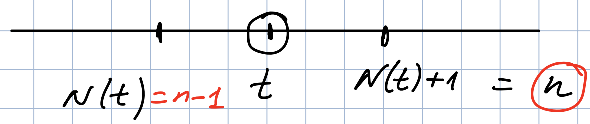
$I_n = 1$ if $n \le N$ which means $I_n = 1$ if we have not stopped yet after having observations $X_1, \ldots X_{n-1}$. But this implies that the value of I_n is determined before X_n has been observed, and thus X_n is independent of I_n

$$= \sum_{n=1}^{\infty} E X_n \cdot E I_n = E X_n \underbrace{\sum_{n=1}^{\infty} E I_n}_{EN}$$

$$= E X_n \cdot E N$$

$N(t)$ be a renewal process

$X_1, X_2, \ldots$ interarrival time.



$N(t)+1 = n$ will be a stopping time, it is the first renewal after time t

$N(t)$ last renewal until time t

$N(t) = n-1$

$$N(t)+1 = n \iff N(t) = n-1 \iff$$

$$X_1 + \cdots + X_{n-1} \leq t$$

$$X_1 + \ldots + X_n > t.$$

Proposition 7.2.

If $X_1, X_2 \ldots$ are interarrival times of a renewal process then

$$E(X_1 + \cdots + X_{N(t)+1}) = E X \cdot E(N(t)+1)$$

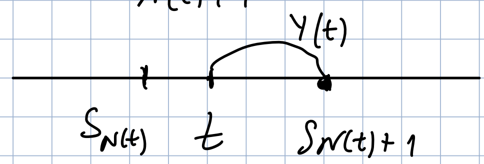
$$E(N(t)+1) = E(N(t)) + 1 = m(t) + 1.$$

$$E(S_{N(t)+1}) = \mu (m(t) + 1)$$

Proof of an elementary renewal theorem.

$$\frac{m(t)}{t} \to \frac{1}{\mu}, \quad t \to \infty.$$

$S_{N(t)+1}$ — be a first renewal after time t



$$S_{N(t)+1} = t + Y(t)$$

$Y(t)$ — excess at time t

$$E(S_{N(t)+1}) = E(t + Y(t))$$

$$\mu(m(t)+1) = t + E(Y(t))$$

$$\frac{m(t)}{t} + \frac{1}{t} = \frac{1}{\mu} + \frac{E(Y(t))}{\mu t}$$

$$\frac{m(t)}{t} = \frac{1}{\mu} + \frac{E(Y(t))}{\mu t} - \frac{1}{t}$$

Since $Y(t) \geq 0$

$$\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu} + \underbrace{\lim_{t\to\infty} \frac{E\,Y(t)}{\mu t}}_{\geq 0} - \underbrace{\frac{1}{t}}_{\to 0} \geq \frac{1}{\mu}$$

Let us show that

$$\lim_{t\to\infty} \frac{m(t)}{t} \leq \frac{1}{\mu}$$

We assume that $\exists\, M < \infty$

$P(X_i \leq M) = 1$ $\qquad E(Y(t)) < M$

$$\frac{m(t)}{t} \leq \frac{1}{\mu} + \frac{M}{t\mu} - \frac{1}{t}$$

$\Downarrow$

$$\lim_{t\to\infty} \frac{m(t)}{t} \leq \frac{1}{\mu} + \underbrace{\lim_{t\to\infty} \frac{1}{t}}_{0} \left(\underbrace{\frac{M}{\mu} - 1}_{} \right)$$

$$\lim_{t\to\infty} \frac{m(t)}{t} \leq \frac{1}{\mu}.$$

Ex. 7.11

$\{N(t),\ t \geq 0\}$ – renewal process

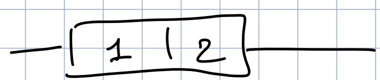
$X_i \sim F = F_1 * F_2$

$F_i(t) = 1 - e^{-\mu_i t}$

$X_i = V_1 + V_2$ $\qquad V_i \sim Exp(\mu_i)$

V_1, V_2 – indep.

$m(t)$ – ?



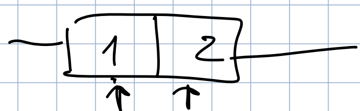
rate to fail for element 1 is μ_1

$$E(V_1) = \frac{1}{\mu_1} \qquad E(V_2) = \frac{1}{\mu_2}$$

1 works until it fails

2 – works until it fails

after that machine is replaces.



1 – element works

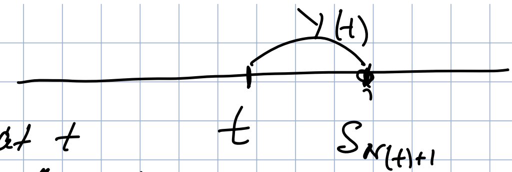
2 – repair time.

$m(t)$ using $E(\underset{\sim}{Y(t)})$

$$\mu(m(t)+1) = t + E(Y(t))$$

$$m(t) = \frac{t}{\mu} + \frac{E(Y(t))}{\mu} - 1$$

$E(Y(t))$



$$I_t = \begin{cases} 1 & \text{if at } t \text{ 1st element works} \\ 2 & \text{if second element works} \end{cases}$$

$$E(Y(t)) = E\left(E(Y(t) \mid I_t)\right)$$

$$E(Y(t) \mid I_t = 1) = \frac{1}{\mu_1} + \frac{1}{\mu_2}$$

$$E(Y(t) \mid I_t = 2) = \frac{1}{\mu_2}$$

$p(t)$ probability that first element works at time t

$1-p(t)$ — second element works at time t.

$$E(Y(t)) = \left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right) p(t) + \frac{1}{\mu_2}\left(1 - p(t)\right)$$

Continuous time MC with 2 states 1, 2.

Ex. 6.11 (continuous MC with 2 states)

$$P_{11}(t) = \frac{\mu_1}{\mu_1+\mu_2} e^{-(\mu_1+\mu_2)t} + \frac{\mu_2}{\mu_1+\mu_2}.$$

$$\parallel$$

$$p(t)$$

$$E(Y(t)) = \left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)\left(\frac{\mu_1}{\mu_1+\mu_2} e^{-(\mu_1+\mu_2)t} + \frac{\mu_2}{\mu_1+\mu_2}\right)$$

$$+ \frac{1}{\mu_2}\Big(\underbrace{1 - \frac{\mu_2}{\mu_1+\mu_2}}_{\frac{\mu_1}{\mu_1+\mu_2}} + \frac{\mu_1}{\mu_1+\mu_2} e^{-(\mu_1+\mu_2)t}$$

$$\frac{1}{\mu_2} + \frac{1}{\mu_1+\mu_2} e^{-(\mu_1+\mu_2)t} + \frac{\mu_2}{\mu_1(\mu_1+\mu_2)}$$

$$m(t) = \frac{t}{\mu} + \frac{E(Y(t))}{\mu} - 1 \qquad \boxed{\mu = \frac{1}{\mu_1} + \frac{1}{\mu_2}}$$

$$m(t) = \frac{\mu_1\mu_2}{\mu_1+\mu_2} t - \frac{\mu_1\mu_2}{(\mu_1+\mu_2)^2}\left[1 - e^{-(\mu_1+\mu_2)t}\right]$$

$$F(t) = p\,\underbrace{F_1(t)}\, + (1-p)\,\underbrace{F_2(t)}$$

$$F(t) = p\,F_1(t) + (1-p)\,F_1 * F_2\,(t)$$

$$F_i(t) = 1 - e^{-\mu_i t}$$

$$I_i = \begin{cases} 1 & \text{if the renewal at time } i \\ 0 & \text{o/w} \end{cases}$$

$$N(n) = \sum_{i=1}^{n} I_i$$

$$m(n) = E\left(N(n)\right) = E \sum_{i=1}^{n} I_i = \sum_{i=1}^{n} E(I_i) =$$

$$= \sum_{i=1}^{n} P(\text{renewal at time } i).$$

$$\frac{\sum_{i=1}^{n} P(\text{renewal at time } i)}{n} \to \frac{1}{E\left(\text{expected time between renewals}\right)}$$

$\forall\ a_1, a_2 \ldots$

$$\lim_{n\to\infty} a_n = a \qquad \lim_{n\to\infty} \frac{\sum_{i=1}^{n} a_i}{n} = a.$$

If $\lim_{n\to\infty} P(\text{renewal at time } n)$ exists $\Rightarrow$

then ~~it~~ is equal $\dfrac{1}{E\left(\text{time between renewals}\right)}$.

The central limit theorem for Renewal processes.

$$\lim_{n\to\infty} P\left(\frac{N(t)-t/\mu}{\sqrt{\frac{t\sigma^2}{\mu^3}}} \le x\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-z^2/2}\,dz.$$

$$N(t) \overset{appr.}{\sim} N\left(\frac{t}{\mu}, \frac{t\sigma^2}{\mu^3}\right)$$

t is large

Heuristic arguments

$$S_n = \sum_{i=1}^{n} X_i \overset{appr.}{\sim} N(n\mu, n\sigma^2)$$

$$N(t) < n \iff S_n > t$$

When n is large

$$P(N(t) < n) = P(S_n > t) =$$

$$= P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} > \frac{t-n\mu}{\sigma\sqrt{n}}\right) \simeq P\left(Z > \frac{t-n\mu}{\sigma\sqrt{n}}\right)$$

$$P\left(\frac{N(t)-t/\mu}{\sqrt{t\sigma^2/\mu^3}} < x\right) = P\left(N(t) < \frac{t}{\mu} + x\sigma\sqrt{\frac{t}{\mu^3}}\right)$$

Treating $\frac{t}{\mu} + x\sigma\sqrt{\frac{t}{\mu^3}}$ as an integer

and letting $n = \frac{t}{\mu} + x\sigma\sqrt{\frac{t}{\mu^3}}$

$$P\left(\frac{N(t) - \frac{t}{\mu}}{\sqrt{t\sigma^2/\mu^3}} < x \right) \approx P\left(Z > \frac{t - t - x\sigma\mu\sqrt{t/\mu^3}}{\sigma\sqrt{\frac{t}{\mu} + x\sigma\sqrt{\frac{t}{\mu^3}}}} \right)$$

$$= P\left(Z > -\frac{x \quad \mu\sqrt{t/\mu^3}}{\sqrt{\frac{t}{\mu} + x\sigma\sqrt{\frac{t}{\mu^3}}}} \right) \approx P(Z > -x)$$

$$= P(Z < x)$$

$$\operatorname{Var}\left(\frac{N(t)}{t} \right) \xrightarrow[t \to \infty]{} \frac{\sigma^2}{\mu^3}.$$