1. Pop of organisms consists of both ♂ & ♀ members
In small colony, any particular ♂ is likely to mate w/ any particular ♀ in any time interval of length h, w/ $P[\lambda h + o(h)]$. Each mating immediately produces offspring, equally likely to be ♂ or ♀.

let $N_1(t)$ & $N_2(t)$ denote # of ♂'s & ♀s in pop @ t

Derive parameters of cont-time MC $\{N_1(t), N_2(t)\}$, ie the v_i, P_{ij} of Sec 6.2

length: h | $\{N_1(t), N_2(t)\} = (m, n)$
probability: $\lambda h + o(h)$ | $T_i \sim Exp(v_i)$
$N_1(t)$: # of ♂ @ t | v_i = rate
$N_2(t)$: # of ♀ @ t | P_{ij} = ?

each mating $\rightarrow$ ♂ / $\rightarrow$ ♀] equally likely

♂ i mates @ rate λ [$\lambda h + o(h)$] w/ any of n ♀s
$\Rightarrow$ mating rate: λn

there are m males $\Rightarrow$ mating rate: $\lambda m n$

$v_{(m,n)} = \lambda n m$

$P_{(m,n),(m,n+1)} = P_{(m,n),(m+1,n)} = 0.5$

Let $\{N_1(t), N_2(t)\}$ be (m, n). There are m males & each one mates at a rate of λ with any of the n females.

Thus, v_i is $v_{(m,n)} = \lambda m n$.

Because each mating immediately produces one offspring, equally likely to be male or female, P_{ij} is:

$P_{(m,n),(m,n+1)} = 0.5$

3. Consider 2 machines that are maintained by repairman. Machine i $f(x) \sim Exp$ w/ $\Delta = \mu_i$ before breaking down, $i = 1, 2$. Repair times for either machine are $\sim Exp(\mu)$.

Can we analyze this as a birth & death process? If so, what are the parameters? If not, how can we analyze it?

1 repairman
Machine $i \sim Exp(\mu_i)$; $i = 1, 2$
Machine j
repair time $\sim Exp(\mu)$

birth & death process:

Suppose that whenever there are n ppl in system, then

i. new arrivals enter system @ $\sim exp(\lambda_n)$

ii. ppl leave system @ $\sim Exp(\mu_n)$

That is, whenever there are n persons in system, then time until next arrival is $\sim$ exp distributed w/ mean $1/\lambda_n$ & is indep of time until departure, which is itself $\sim$ exp distr w/ mean $1/\mu_n$

We can't analyze this as a birth & death process, because there's not enough information. We only have information on one machine, repair time, & that one repairman.

We don't know what machine j functions for, if they break down at the same time or at different times, if the repairman works on one or both when they break down.

We can analyze is it as such:

Because there is only one repairman, let's say they can only fix ~~one~~ one machine at a time. Therefore the states are such:

0 = both working
1 = machine I broken, machine II working
2 = MI working, MII broken
3 = both broken, MI repair
4 = both broken, MII repair

Then the state matrix is:

$$\begin{array}{c} \begin{array}{ccccc} \text{State } 0 & \text{State } 1 & \text{State } 2 & \text{State } 3 & \text{State } 4 \end{array} \\ \begin{bmatrix} -(\mu_1+\mu_2) & \mu_1 & \mu_2 & 0 & 0 \\ \mu & -(\mu_1+\mu_2) & 0 & 0 & \mu_2 \\ \mu & 0 & -(\mu_1+\mu_2) & \mu_1 & 0 \\ 0 & 0 & \mu & -\mu & 0 \\ 0 & \mu & 0 & 0 & -\mu \end{bmatrix} \end{array}$$

5. There are N indivs in pop, some of whom have certain infection that spreads as follows. Contacts bet 2 members of this pop occur in accordance w/ $\sim$ Poi (λ). When contact occurs, it is equally likely to involve any of $\binom{N}{2}$ pairs of indivs in pop. If contact involves infected & noninfected indiv, then w/ P(p), noninfected indiv becomes infected. Once infected, indiv remains infected thruout. Let $X(t)$ denote # of infected members of pop @ time t

a. Is $\{X(t), t \geq 0\}$ a cont MC?

A Markov chain is a stochastic model describing a sequence of possible events in which the probability of each event depends on the state attained in the previous event.

Because the current state of the pair of individuals will determine the future event, $\{X(t), t \geq 0\}$ is a continuous-time Markov chain.

b. Specify its type

It is a birth process, because $X(t)$ denotes the number of infected individuals of the population at time t. As t increases, the number of infected does as well. Because once infected, they remain infected.

Because they don't recover, this is a specifically a pure birth process. Therefore, $\mu_n = 0$.

C. Starting w/ single infected indiv, what's the $\mathbb{E}$ time until all members are infected?

$\{\lambda_n\}_{n=0}^{\infty}$; $\lambda_i \neq \lambda_j$, $i \neq j$

$\sim Poi(\lambda) \Rightarrow \frac{1}{\lambda_n}$; $\binom{N}{2}$

$P(p)$ = infected + uninfected

$\mathbb{E}$ time until full infection = ?

Let T_n denote the time $X(t)$ stays in state n. Then, T_n is exponentially distributed with rate λ_n. Then, let $X(t) = N$.

$\{\lambda_n\}_{n=0}^{\infty} \Rightarrow \{\lambda_n\}_{n=1}^{N}$ is the birth rate.

Let's say there are n infected in the population.

When contact occurs, it's equally likely to involve any of the $\binom{N}{2}$ pairs of individuals in the population. So, the probability that a contact involves an infected and uninfected individuals is:

$$\frac{n(N-n)}{\binom{N}{2}}$$

Therefore $\{\lambda_n\}_{n=1}^{n} = \frac{pn(N-n)\lambda}{\binom{N}{2}}$; $n=1,\ldots,N-1$

$$E\left(\sum_{n=1}^{N-1} T_n\right) = \sum_{n=1}^{N-1} \frac{\binom{N}{2}}{pn(N-n)\lambda} = \frac{\binom{N}{2}}{\lambda p} \sum_{n=1}^{N-1} \frac{1}{n(N-n)}$$

4. Potential customers arrive @ single-server station in accordance w/ ~Poi(λ). However, if arrival finds n customers already in station, then he'll enter system w/ P(α_n).

Assuming ~Exp(μ), set this up as birth & death process & determine birth & death rates

arrival ~ Poi(λ)
n customers $\Rightarrow$ P(α_n)
service ~ Exp(μ)

birth rate = ?
death rate = ?

A birth & death process is a cont-time Markov chain w/ states $\{0,1,\ldots\}$ for which transitions from state n may go only to either state $(n-1)$ or $(n+1)$.

Parameters: $\{\lambda_n\}_{n=0}^{\infty}$

$\{\mu_n\}_{n=1}^{\infty}$

Relationships: $v_0 = \lambda_0$; $v_i = \lambda_i + \mu_i$, $i>0$; $P_{01} = 1$

$P_{i,i+1} = \frac{\lambda_i}{\lambda_i+\mu_i}$, $i>0$; $P_{i,i-1} = \frac{\mu_i}{\lambda_i+\mu_i}$, $i>0$

Let's say we have a state space $\{0,1,2,\ldots,n\}$, where state n is n customers in the system. Because potential customers arrive at a Poisson rate of λ and entering the system with probability α_n, the arrival rate is:

$\lambda_n = \lambda\alpha_n$ for $n \geq 1$

Because there is only one server, departure is μ.

So, we'll need to find state $(n-1)$, because for a custumer to enter the system, there needs to be $(n-1)$ customers already in the system. Thus, the probability of the customer entering the system is α_{n-1}.

Thus, the birth rate for state n is:

$$\lambda_{n-1} = \lambda(1-\alpha_0)(1-\alpha_1)\dots(1-\alpha_{n-2}), \text{ for } n \geq 1$$

The death rate for state n is μ, for $n \geq 1$.