Review

Def. The process $\{X(t),\ t \geq 0\}$ is a continuous time Markov chain if

$\forall\ s, t \geq 0$ and $i, j \in \mathbb{Z}_+$ (non neg integers),

$x(u)$ $0 \leq u < s$

$$P\{X(t+s) = j \mid X(s) = i,\ X(u) = x(u),\ 0 \leq u < s\} =$$
$$= P\{X(t+s) = j \mid X(s) = i\}$$

How can we define MC.

Another approach to define continuous time MC. It is a stochastic process such that each time it enters state i

(i) The amount of time it spends in that state before making a transition into a different state $T_i \sim Exp(\nu_i)$

$ET_i = \frac{1}{\nu_i}$

(ii) P_{ij} - probability that process enters state j after the state i: 1) $P_{ii} = 0$

2) $\sum_{\forall j} P_{ij} = 1 \quad \forall i$

The continuous time MC can be defined by: $P = (P_{ij})_{ij \in S}$ and

ν_i - the rate to leave state i.

Another way is to define a rate matrix R.

$q_{ij} = v_i P_{ij}$ — rate to go from state i to j

$$R = \left(R_{ij} \right)_{i,j \in S} \qquad R_{ij} = \begin{cases} q_{ij} & i \neq j \\ -v_i & i = j \end{cases}$$

$$v_i = \sum_{\forall j} q_{ij}$$

Birth and death process.

Suppose there are n people in the system.

(i) New arrivals enter system at exponential rate λ_n

(ii) People leave the system at exponential rate μ_n

T_a - time until next arrival

$T_a \sim Exp(\lambda_n) \qquad E\, T_a = \frac{1}{\lambda_n}$

T_B - time until next departure

$T_B \sim Exp(\mu_n) \qquad E\, T_B = 1/\mu_n$

T_A and T_B are independent

Parameters: $\begin{cases} \{\lambda_n\}_{n=0}^{\infty} & \text{— birth rate} \\ \{\mu_n\}_{n=0}^{\infty} & \text{— death rate} \end{cases}$

A birth and death process is a continuous-time MC

States : $\{0, 1, 2, \dots\}$

$v_0 = \lambda_0$ $\qquad$ $P_{01} = 1$

$v_i = \lambda_i + \mu_i$ $\qquad$ $P_{i,i+1} = \dfrac{\lambda_i}{\lambda_i + \mu_i}$ $\qquad$ $P_{i+1,i} = \dfrac{\mu_i}{\lambda_i + \mu_i}$

$i > 0$ $\qquad$ $i > 0$.

The rate matrix

$$R_{ij} = \begin{cases} -v_i & i = j \\ q_{ij} & i \neq j \end{cases}$$

$v_0 = \lambda_0$ $\qquad$ $q_{i\,i+1} = \lambda_i$

$v_i = \lambda_i + \mu_i$ $\qquad$ $q_{i,i-1} = \mu_i$

Let $\mu_0 = 0$

T_i – time starting from state i it takes for process to enter state $i+1$.

$$E\,T_i = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} E\left(T_{i-1}\right) \qquad E\,T_0 = \frac{1}{\lambda_0}$$

$$Var\,(T_i) = \frac{1}{(\mu_i + \lambda_i)^2} + \frac{\mu_i}{\mu_i + \lambda_i}\left[Var\,(T_{i-1}) + Var\,(T_i)\right] +$$

$$+ \frac{\mu_i \lambda_i}{(\mu_i + \lambda_i)^2}\left(E\,(T_{i-1}) + E\,T_i\right)^2$$

$$Var(T_i) = \frac{1}{\lambda_i(\lambda_i+\mu_i)} + \frac{\mu_i}{\lambda_i} Var(T_{i-1}) +$$

$$+ \frac{\mu_i}{\mu_i+\lambda_i}\left[E(T_{i-1}) + E\,T_i\right]^2$$

$Var(T_0) = \frac{1}{\lambda_0^2}$ we can get $Var(T_i)$

$$E(\text{time to go from } k \text{ to } j) = \sum_{i=k}^{j-1} E(T_i) =$$

$$= \frac{j-k}{\lambda-\mu} - \frac{\left(\frac{\mu}{\lambda}\right)^{k+1}}{\lambda-\mu}\,\frac{1-(\mu/\lambda)^{j-k}}{1-\mu/\lambda} \qquad \lambda \neq \mu$$

$$Var(\text{time go from } k \text{ to } j) = \sum_{i=k}^{j-1} Var(T_i)$$

6.4. The transition probab. matrix.

Let $\underbrace{P_{ij}}(t) = P\{X(t+s) = j \mid X(s) = i\}$

$$\mathbf{P}(t) = \left(\underbrace{P_{ij}}(t)\right)$$

P_{ij} $\qquad$ $q_{ij} = v_i P_{ij}$

v_i - rate at which process makes a transition when in state i

Backward Kolmogorov equation.

$$P_{ij}'(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t) \qquad (1)$$

$$\mathbf{P}(t) = \left(P_{ij}(t)\right) \qquad \mathbf{R} = \left(R_{ij}\right)$$

$$R_{ij} = \begin{cases} q_{ij} & i \neq j \\ -v_i & i = j \end{cases}$$

$$(1) \quad \mathbf{P}'(t) = \mathbf{R}\,\mathbf{P}(t)$$

$$\mathbf{P}(t) = e^{\mathbf{R}t}$$

Forward Kolmogorov equation.

$$P_{ij}'(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_i P_{ij}$$

$$(2) \quad \mathbf{P}'(t) = \mathbf{P}(t)\mathbf{R}$$

6.5. Limiting Probabilities.

The probability that a continuous MC will be in state j at time t often converges to a limiting value that independent of initial state

$$P_j := \lim_{t \to \infty} P_{ij}(t) \qquad (\text{or } \pi_j = \lim_{t\to\infty} P_{ij}(t).$$

Balance equation

$$v_j P_j = \sum_{k \neq j} q_{kj} P_k$$

$$\sum_{\forall j} P_j = 1$$

Matrix form

$$\vec{P}^T R = 0 \quad (*)$$

$$\vec{P} = \begin{pmatrix} p_0 \\ \vdots \\ p_m \end{pmatrix}.$$

$$R^T \begin{pmatrix} p_0 \\ \vdots \\ p_m \end{pmatrix} = 0 \qquad \sum_{\forall j} P_j = 1$$

Stationary distribution is a solution of

$$\vec{P}^T P(t) = \vec{P}^T$$

Proposition: $\vec{P}$ is a stationary distr. if and only if $\vec{P}^T R = 0$

$\boxtimes$.

Theorem: If a continuous Markov chain $\{X(t), t \geq 0\}$ is irreducible and has a stationary distribution $\vec{P}^T = (P_1, \ldots P_m \ldots)$ then

$$\lim_{t\to\infty} P_{ij}(t) = P_j.$$

Time reversibility.

Consider a continuous-time MC that is ergodic (limiting distr. exist).

Let $P_j = \lim_{t\to\infty} P_{ij}$

Assume embedded chain is ergodic (has a stationary distribution)

$$\begin{cases} \pi_i = \sum_{\forall j} \pi_j P_{ji} & \forall i \\ \sum \pi_i = 1 \end{cases}$$

$$\mathbf{P}^T \pi = \pi$$

$$\pi_i = 1$$

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Then we can show that

$$P_i = \frac{\pi_i / v_i}{\sum_j \pi_j v_j} \qquad (1)$$

Proporsition 6.5 An ergodic birth and death process is time reversible.

Proposition 6.7.

If for some set $\{P_i\}$:

$$\sum_{\forall i} P_i = 1 , \qquad P_i \geq 0$$

$$P_i q_{ij} = P_j q_{ji} \qquad \forall i \neq j \qquad (1)$$

Then the continuous MC is time reversible and

$P_i = \lim_{t \to \infty} P_{ji}$ (limiting probabilities).

Exit distribution.

Let $V_D = \min\{t : X(t) \in D\}$

$T = \min(V_a, V_B)$

Suppose $C = S \cap (A \cup B)'$ is finite.

$\downarrow$

State space

$P_i(T < \infty) > 0$

probability to start at i and come back to i at finite time $\forall i \in C$.

If we have $h(a) = 1 \quad a \in A$

$h(b) = 0 \quad b \in B$

$$h(i) = \sum_{j \neq i} P_{ij}\, h(j) \qquad \forall i \in C$$

Then $h(i) = P_i(V_A < V_B)$ probability to visit set A before B if you start at i.

In terms of rate matrix it is

$$\sum_i q_{ij}\, h(j) - v_i\, h(i) = 0$$

$$\sum_i R_{ij}\, h(j) = 0 \qquad (1)$$

Let $r = (r_{ij})_{i,j \in C}$

$r_{ij} = R_{ij} \quad i, j \in C$

Let $w_i = \sum_{j \in A} R_{ij}$

$R = (R_{ij})$

$$R_{ij} = \begin{cases} q_{ij} & i \neq j \\ -v_i & i = j \end{cases}$$

Let $h(a) = 1 \qquad h(b) = 0$

(1) can be written as $\sum_{j \in C} r_{ij}\, h(j) = -w_i$

$$-\sum r_{ij}\, h(j) = w(i) \qquad -R\,h = w$$

$$h = (-r)^{-1}\, w$$

Exit time

Let $C = S \cap A'$ be finite

$P_i(V_A < \infty) > 0 \qquad i \in C$

probability to start at i and visit A in a finite time.

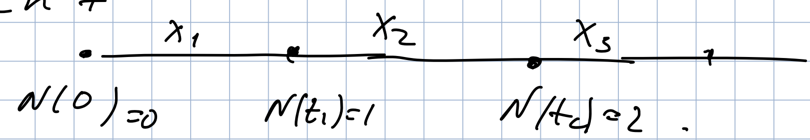
$$g(i) = \frac{1}{V_i} + \sum_{i \neq j} q_{ij}\, g(j)$$

$g(i) = E_i V_A$ (expected time to start at i and visit A)

$$V_i\, g(i) = 1 + \sum_{j \neq i} q_{ij}\, g(j)$$

Renewal process.

Let $\{N(t), t \geq 0\}$ be a counting process.
Let X_n be the time between $(n-1)$ and n^{th} event



Def. If the sequence of nonnegative r.v. $\{X_1, X_2 \ldots\}$ is iid then the counting process $\{N(t), t \geq 0\}$ is said to be a renewal process.

$$S_n = \sum_{k=1}^{n} X_k$$

$$N(t) \geq n \iff S_n \leq t$$

$$\Downarrow$$

$$P(N(t)=n) = P(N(t) \geq n) - P(N(t) \geq n+1)$$
$$= P(S_n \leq t) - P(S_{n+1} \leq t)$$

Therefore

$$P\{N(t)=n\} = F_n(t) - F_{n+1}(t)$$

$$P(N(t)=n) = \int_0^\infty P(N(t)=n / S_n = y) f_{S_n}(y) dy$$

$m(t) = E(N(t))$

$$m(t) = \int_0^t [1 + m(t-x)] f(x) dx$$
$$= F(t) + \int_0^t m(t-x) f(x) dx \tag{7.5}$$

Eq. (7.5) is called the *renewal equation* and can sometimes be solved to obtain the renewal function.

$S_{N(t)}$ – random sum. – time of last renewal prior to or at time t.

$S_{N(t)+1}$ – the time of the first renewal after the time t.

Proposition. 1

With probability 1 $\quad \frac{N(t)}{t} \to \frac{1}{\mu}$ as $t \to \infty$.

$$\lim_{t \to \infty} \frac{m(t)}{t} = \frac{1}{\mu}$$

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