Ch 9.5: Markovian Queue

M/M/1 queue

arrival ~ Poi process ($\Delta = \lambda$)

service

$\mathbb{E}(S_i) = \frac{1}{\mu}$

$S_i \sim Exp(\mu)$

$\{X(t), t \geq 0\}$

$X(t)$ = # of custs in system @ time t

$$X(t) = i \begin{cases} i+1 & \frac{\lambda}{\lambda+\mu} \\ i-1 & \frac{\mu}{\lambda+\mu} \end{cases}$$

cont-time MC

death & birth process

$$R = R_{ij} \begin{cases} q_{ij} & i \neq j \\ -v_i & i = j \end{cases}$$

$v_i = \sum_{\forall j} q_{ij}$

$q_{i,i+1} = \lambda$

$q_{i,i-1} = \mu$

$q_{ij} \neq 0$

To get stationary distr

$\pi_i q_{ij} = q_{ji} \pi_j$

$\pi_n = \frac{\lambda_{n-1} \ldots \lambda_0}{\mu_n \ldots \mu_1} \pi_0 = \left(\frac{\lambda}{\mu}\right)^n \pi_0$

$\pi_0 = 1 - \frac{\lambda}{\mu}$

$\pi_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$ stationary distr $\quad \sum_{n=0}^{\infty} \pi_n$

It's $\propto$ of time when system has n custs

This distr exits if $\lambda < \mu$

Proof

$L = \lambda W = \lambda\left(\frac{1}{\mu-\lambda}\right) = \frac{\lambda}{\mu-\lambda}$

$L = \frac{1}{1-\frac{\lambda}{\mu}} - 1 = \frac{\mu}{\mu-\lambda} - 1 = \frac{\mu-\mu+\lambda}{\mu-\lambda} = \frac{\lambda}{\mu-\lambda}$

$W = \frac{1}{\mu-\lambda}$

Notation

$\frac{\mathbb{E}I}{\mathbb{E}I+\mathbb{E}B} = \pi_0 \Rightarrow \mathbb{E}B+\mathbb{E}I = \frac{\mathbb{E}I}{\pi_0} \Rightarrow \mathbb{E}B = \frac{1/\lambda}{\pi_0} - \frac{1}{\lambda} = \frac{1}{\mu-\lambda}$

M/M/1 w/ finite waiting room

Lemma

Let $X(t)$ be MC w/ stat distr π. Let $Y(t)$ be MC constrained to stay in subset A. That jump which take chain out of A isn't allowed

Notation

$$\overline{q_{ij}} = \begin{cases} q_{ij} & i,j \in A \\ 0 & \text{@ least one} \notin A \end{cases}$$

Δ to go from $i \to j$ in MC $Y(t)$

let $C = \sum_{j \in A} \pi_j$

then $\nu_i = \frac{\pi_i}{C}$ $i \in A$ is stationary distr of $\{Y(t)\}$

$$\pi_i = \frac{\left(\frac{\lambda}{\mu}\right)^i}{C} = \frac{\sum_{i=0}^{n} \left(\frac{\lambda}{\mu}\right)^i}{C} = \frac{C}{C} = 1$$

$\overline{q_{i,i+1}} = \lambda$ $\quad i, i+1 \le n$

$\overline{q_{i,i-1}} = \mu$ $\quad i \le n$
$\quad i-1 \ge 0$

$\nu_i \lambda = \nu_{i+1} \mu \Rightarrow \nu_{i+1} = \frac{\lambda}{\mu}$

$\nu_1 = \frac{\lambda}{\mu} \nu_0$

$\nu_i \overline{q_{i,i+1}} = \nu_{i+1} \overline{q_{i+1,i}}$

$\nu_2 = \left(\frac{\lambda}{\mu}\right)^2 \nu_0$

$\vdots$

$\nu_n = \left(\frac{\lambda}{\mu}\right)^n \nu_0$

$\sum_{i=0}^{n} \nu_i = 1$

$$\sum_{i=0}^{n} \left(\frac{\lambda}{\mu}\right)^i \nu_0 = \nu_0 \sum_{i=0}^{n} \left(\frac{\lambda}{\mu}\right)^i = \nu_0 \left[\frac{1 - \left(\frac{\lambda}{\mu}\right)^{n+1}}{1 - \frac{\lambda}{\mu}}\right]$$

$$= \left[\frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{n+1}}\right] \left[\frac{1 - \left(\frac{\lambda}{\mu}\right)^{n+1}}{1 - \frac{\lambda}{\mu}}\right] = 1$$

$$\nu_i = \begin{cases} \dfrac{\left(\frac{\lambda}{\mu}\right)^i \left(1 - \frac{\lambda}{\mu}\right)}{1 - \left(\frac{\lambda}{\mu}\right)^{n+1}}, & \lambda \neq \mu \\[2ex] \dfrac{1}{n+1} \end{cases}$$

Example: $\mu = 3$
$\lambda = 2$
$n = 3$

Find avg # of ppl in system & avg queue length

$$\pi_0 = \frac{\frac{1}{3}}{1-\left(\frac{2}{3}\right)^4} = \frac{\frac{1}{3}}{1-\frac{16}{81}} = \frac{\frac{1}{3}}{\frac{65}{81}} = \frac{1}{3}\left(\frac{81}{65}\right) = \frac{27}{65}$$

$$\pi_1 = \frac{\lambda}{\mu}(\pi_0) = \frac{2}{3}\left(\frac{27}{65}\right) = \frac{18}{65}$$

$$\pi_2 = \frac{\lambda}{\mu}(\pi_1) = \frac{2}{3}\left(\frac{18}{65}\right) = \frac{12}{65}$$

$$\pi_3 = \frac{2}{3}(\pi_2) = \frac{2}{3}\left(\frac{12}{65}\right) = \frac{8}{65}$$

$$L = 0\left(\frac{27}{65}\right) + 1\left(\frac{18}{65}\right) + 2\left(\frac{12}{65}\right) + 3\left(\frac{8}{65}\right)$$

$$= \frac{18+24+24}{65} = \frac{66}{65}$$

$$L_a = L - 1 + \pi_0 = \frac{66}{65} - 1 + \frac{27}{65} = \frac{1+27}{65} = \frac{28}{65}$$

$$W = \frac{L}{\lambda_a} = \frac{\frac{66}{65}}{2} = \frac{66}{65}\left(\frac{1}{2}\right) = \frac{33}{65}$$

$$W_a = \frac{L_a}{\lambda_a} = \frac{28}{65}\left(\frac{1}{2}\right) = \frac{14}{65}$$