Section 6.6.

Time reversibility.

Consider a continuous-time MC that is ergodic (limiting distr. exist).

Let $P_j = \lim_{t \to \infty} P_{ij}$

Let us ignor time spent in each state during a visit, then this sequence constitutes a discrete-time MC, called embedded chain with $\mathbf{P} = (P_{ij})$ $\begin{pmatrix} 0 & P_{01} & \cdots & P_{0m} \\ P_{10} & \ddots & & \vdots \\ \vdots & & \ddots & \\ & & & 0 \end{pmatrix}$

Assume embedded chain is ergodic (has a stationary distribution)

$$\begin{cases} \pi_i = \sum_{\forall j} \pi_j P_{ji} & \forall i \\ \sum \pi_i = 1 \end{cases} \qquad \mathbf{P}^T \pi = \pi, \quad \sum \pi_i = 1$$

Then we can show that

$$P_i = \frac{\pi_i / v_i}{\sum_j \pi_j v_j} \qquad (1)$$

To check it $\quad v_i P_i = \sum_{j \neq i} P_j q_{ji}$

$q_{ij} = v_i P_{ij} \qquad P_{ii} = 0$

$$v_i P_i = \sum_j P_j v_j P_{ji} \quad \forall i \qquad (2)$$

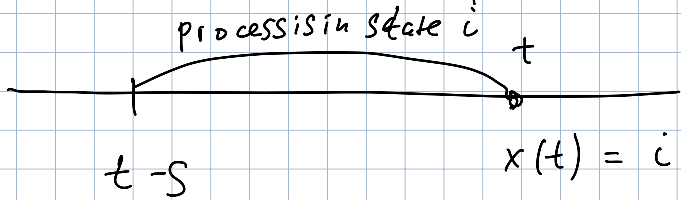
substitute (1) to (2)

$$\frac{\cancel{v_i}\cdot\pi_i}{\cancel{v_i}\sum_k \pi_k v_k} = \sum_j \frac{\frac{\pi_j}{\cancel{v_j}}\cancel{v_j}\, P_{ji}}{\sum_k \pi_k v_k}$$

$$\pi_i = \sum_j \pi_j P_{ji} \quad (*)$$

Let cont. time MC has been operating for a long time and starting at time T we trace process going backward in time

$$P\left(X \text{ in state } i \text{ at } [t-s, t] \,\middle|\, X(t) = i \right) =$$



$$= \frac{P(X(t-s)=i)\cdot e^{-v_i s}}{P(X(t)=i)} = e^{-v_i s}$$

$$P(X(t-s)=i) = P(X(t)=i) = P_i$$

Going back in time the amount of time the process spends in state i is $Exp(v_i)$

Reverse chain in discrete time

$$Q_{ij} = \frac{\pi_j P_{ji}}{\pi_i}$$

if $Q_{ij} = P_{ij}$ then chain is reversible.

$\pi_i P_{ij} = \pi_j P_{ji}$ $\Rightarrow$ detailed balance eq.

$$P_i = \frac{\pi_i}{\nu_i} \Big/ \sum_j \pi_j/\nu_j \qquad \Rightarrow$$

$$P_i q_{ij} = P_j q_{ji} \qquad \forall i, j$$

Since P_i is the proportion of time in state i and q_{ij} is the rate when in state i the process goes to state j directly.

The condition of time reversibility:

The rate at which the process goes directly from state i to state j is equal to the rate at which it goes directly from state j to i.

Proporsition 6.5 An ergodic birth and death process is time reversible.

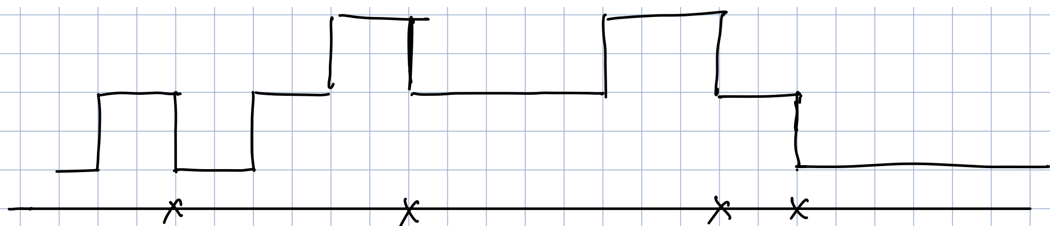
Proof.
We need to show:
The rate $i \to i+1$ is equal to the rate $i+1 \to i$

Proof. We must show that the rate at which a birth and death process goes from state i to state $i+1$ is equal to the rate at which it goes from $i+1$ to i. In any length of time t the number of transitions from i to $i+1$ must equal to within 1 the number from $i+1$ to i (since between each transition from i to $i+1$ the process must return to i, and this can only occur through $i+1$, and vice versa). Hence, as the number of such transitions goes to infinity as $t \to \infty$, it follows that the rate of transitions from i to $i+1$ equals the rate from $i+1$ to i. ■

Corollary 6.6 Consider an $M/M/s$ queue in which customers arrive in accordance with a Poisson process having rate λ and are served by any one of s servers—each having an exponentially distributed service time with rate μ. If $\lambda < s\mu$, then the output process of customers departing is, after the process has been in operation for a long time, a Poisson process with rate λ.

Proof. Let $X(t)$ denote the number of customers in the system at time t. Since the $M/M/s$ process is a birth and death process, it follows from Proposition 6.5 that $\{X(t), t \geqslant 0\}$ is time reversible. Going forward in time, the time points at which $X(t)$ increases by 1 constitute a Poisson process since these are just the arrival times of customers. Hence, by time reversibility the time points at which $X(t)$ increases by 1 when we go backward in time also constitute a Poisson process. But these latter points are exactly the points of time when customers depart (see Figure 6.1). Hence, the departure times constitute a Poisson process with rate λ. ■

Example 6.17 Consider a first come first serve $M/M/1$ queue, with arrival rate λ and service rate μ, where $\lambda < \mu$, that is in steady state. Given that customer C spends a total of t time units in the system, what is the conditional distribution of the number of others that were present when C arrived?



$X(t) \searrow$ $\rightarrow$

$X(t) \nearrow$ $\leftarrow$

Let C arrived at time s and left at time $s+t$.

System is first come first serve.

If it was n people in the system when c arrived then n people should depart during $(s, s+t)$.

In reverse process c arrives at time $(s+t)$ and leave at time t.

The reverse process is M/M/1 queueing system.

# of arrivals at $\underset{s+t \qquad\qquad s}{\overset{t}{\longmapsto}}$ $\sim$ Poisson(λt)

$\boxtimes$

The process is time reversible $\Leftrightarrow$

$$P_i q_{ij} = P_j q_{ij} \quad \forall \ i \neq j$$

Proposition 6.7.

If for some set $\{P_i\}$:

$$\sum_{\forall i} P_i = 1 \quad , \quad P_i \geq 0$$

$$P_i q_{ij} = P_j q_{ji} \quad \forall \ i \neq j \qquad (1)$$

Then the continuous MC is time reversible and

$$P_i = \lim_{t \to \infty} P_{ji} \quad \text{(limiting probabilities)}.$$

Proof. For fixed i

$$P_i q_{ij} = P_j q_{ji}$$

$$\sum_{j:j\neq i} P_i q_{ij} = \sum_{j:j\neq i} P_j q_{ji} \qquad \sum_{j:j\neq i} q_{ij} = v_i$$

$v_i P_i = \sum_{j \neq i} P_j q_{ji} \Rightarrow P_i$ satisfies the

balance eq. $\Rightarrow P_i = \lim_{t \to \infty} P_{ji}(t)$

so is the truncated one.

Proposition 6.8 A time reversible chain with limiting probabilities P_j, $j \in S$ that is truncated to the set $A \subset S$ and remains irreducible is also time reversible and has limiting probabilities P_j^A given by

$$P_j^A = \frac{P_j}{\sum_{i \in A} P_i}, \qquad j \in A$$

Proof. By Proposition 6.7 we need to show that, with P_j^A as given,

$$P_i^A q_{ij} = P_j^A q_{ji} \qquad \text{for } i \in A,\ j \in A$$

or, equivalently,

$$P_i q_{ij} = P_j q_{ji} \qquad \text{for } i \in A,\ j \in A$$

But this follows since the original chain is, by assumption, time reversible. ■

Example 6.19 Consider an $M/M/1$ queue in which arrivals finding N in the system do not enter. This finite capacity system can be regarded as a truncation of the $M/M/1$ queue to the set of states $A = \{0, 1, \ldots, N\}$. Since the number in the system in the $M/M/1$ queue is time reversible and has limiting probabilities $P_j = (\lambda/\mu)^j(1-\lambda/\mu)$ it follows from Proposition 6.8 that the finite capacity model is also time reversible and has limiting probabilities given by

$$P_j = \frac{(\lambda/\mu)^j}{\sum_{i=0}^{N} (\lambda/\mu)^i}, \qquad j = 0, 1, \ldots, N$$ ■

Proposition 6.9 If $\{X_i(t),\ t \geqslant 0\}$ are, for $i = 1, \ldots, n$, independent time reversible continuous-time Markov chains, then the vector process $\{(X_i(t), \ldots, X_n(t)),\ t \geqslant 0\}$ is also a time reversible continuous-time Markov chain.

Ex. Death and birth chain.

$S = \{0, 1, \dots, N\}$ → states. $N \leq \infty$

$q_{n\,n+1} = \lambda_n \qquad n < N$

$q_{n,n-1} = \mu_n \qquad n \geq 0$

Find P_i using detailed balance eq.

$P_i q_{ij} = P_j q_{ji}$

$P_0 q_{01} = P_1 q_{10} \quad \Rightarrow \quad P_0 \lambda_0 = P_1 \mu_1$

$P_1 q_{12} = P_2 q_{21} \qquad P_1 \lambda_1 = P_2 \mu_2$

$P_2 P_{23} = P_3 q_{32} \qquad P_2 \lambda_2 = P_3 \mu_3$

$\vdots \qquad \vdots$

$P_{n-1} q_n = P_n q_{n,n-1} \qquad P_{n-1} \lambda_{n-1} = P_n \mu_n$

P_0

$P_1 = \frac{\lambda_0}{\mu_1} P_0$

$P_2 = \frac{\lambda_1}{\mu_2} \frac{\lambda_0}{\mu_1} P_0$

$P_3 = \frac{\lambda_2}{\mu_3} \cdot \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1} P_0$

$\vdots$

$P_n = \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_1} P_0$

$\sum P_i = 1$

$P_0 \underbrace{\left(1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1} + \cdots + \frac{\lambda_{n-1} \cdots \lambda_0}{\mu_n \cdots \mu_1}\right)}_{c} = 1$

$$p_0 = \frac{1}{c} \qquad p_1 = \frac{\lambda_0}{\mu_0} \cdot \frac{1}{c}$$

$$p_n = \frac{\lambda_{n-1} \cdots \lambda_0}{\mu_n \cdots \mu_1} \cdot \frac{1}{c}.$$

Example 4.23 (M/M/1 Queue). In this system customers arrive to a single server facility at the times of a Poisson process with rate λ, and each requires an independent amount of service that has an exponential distribution with rate μ. From the description it should be clear that the transition rates are

$$q(n, n+1) = \lambda \qquad \text{if } n \geq 0$$
$$q(n, n-1) = \mu \qquad \text{if } n \geq 1$$

so we have a birth and death chain with birth rates $\lambda_n = \lambda$ and death rates $\mu_n = \mu$. Plugging into our formula for the stationary distribution, (4.30), we have

$$\pi(n) = \frac{\lambda_{n-1} \cdots \lambda_0}{\mu_n \cdots \mu_1} \cdot \pi(0) = \left(\frac{\lambda}{\mu}\right)^n \pi(0) \tag{4.34}$$

To have the sum 1, we pick $\pi(0) = 1 - (\lambda/\mu)$, and the resulting stationary distribution is the shifted geometric distribution

$$\pi(n) = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n \quad \text{for } n \geq 0 \tag{4.35}$$

It is comforting to note that this agrees with the idle time formula, (3.5), which says $\pi(0) = 1 - \lambda/\mu$.

Ex. 2. Two state chain.

States $\{1, 2\}$

$q_{12} = \lambda \qquad q_{21} = \mu$ Find p_i using detailed balance eq.

$$\begin{cases} p_1 q_{12} = p_2 q_{21} \\ p_1 + p_2 = 1 \end{cases}$$

$$\begin{cases} \lambda p_1 = \mu p_2 \\ p_1 + p_2 = 1 \end{cases} \Rightarrow \quad p_2 = \frac{\lambda}{\mu} p_1, \quad p_1 + \frac{\lambda}{\mu} p_1 = 1$$

$$p_1\left(1 + \frac{\lambda}{\mu}\right) = 1$$

$$p_1 = \frac{1}{\left(1 + \frac{\lambda}{\mu}\right)} = \frac{\mu}{\mu + \lambda} \qquad p_2 = \frac{\lambda}{\lambda + \mu}$$

Example 4.18 (Two Barbers). Suppose that a shop has two barbers that can each cut hair at rate 3 people per hour customers arrive at times of a rate 2 Poisson process, but will leave if there are two people getting their haircut and two waiting. Find the stationary distribution for the number of customers in the shop.

The transition rate matrix is

	0	**1**	**2**	**3**	**4**
0	−2	2	0	0	0
1	3	−5	2	0	0
2	0	6	−8	2	0
3	0	0	6	−8	2
4	0	0	0	6	−6

The detailed balance conditions say

$$\begin{aligned} 2P_0 &= 3P_1 \\ 2P_1 &= 6P_2 \\ 2P_2 &= 6P_3 \\ 2P_3 &= 6P_4 \end{aligned} \quad \Rightarrow \quad \begin{aligned} P_1 &= \frac{2}{3}P_0 \\ P_2 &= \frac{6}{2}P_1 = \frac{\cancel{6}}{\cancel{2}} \cdot \frac{\cancel{2}}{3} P_0 = 2P_0 \\ P_3 &= \frac{6}{2}P_2 = \frac{6}{2} \cdot 2P_0 = 6P_0 \\ P_4 &= 3P_3 = 18P_0 \end{aligned}$$

$$\left(1 + \frac{2}{3} + 2 + 6 + 18\right) P_0 = 1$$

$$P_0 = \frac{81}{161} \qquad P_1 = \frac{54}{161} \qquad P_2 = \frac{18}{161} \qquad P_3 = \frac{6}{161} \qquad P_4 = \frac{2}{161}.$$