Limit theorems and their applications.

$N(t) \underset{t\to\infty}{\to} \infty$ with probability 1

What is the rate?

$S_{N(t)}$ – random sum. – time of last renewal prior to or at time t.

$S_{N(t)+1}$ – the time of the first renewal after the time t.

Proposition. 1

With probability 1 $\quad \frac{N(t)}{t} \to \frac{1}{\mu}$ as $t\to\infty$.

Proof.

$$S_{N(t)} \le t < S_{N(t)+1}$$

$$\frac{S_{N(t)}}{N(t)} \le \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}$$

$$\frac{S_{N(t)}}{N(t)} = \frac{\sum_{i=1}^{N(t)} X_i}{N(t)} \to \mu \quad \text{as } N(t)\to\infty$$

from a strong law of large numbers

$[\, N(t) \to \infty$ when $t\to\infty$

$$\frac{S_{N(t)+1}}{N(t)} = \underset{\substack{\downarrow\\ \mu}}{\frac{S_{N(t)+1}}{N(t)+1}} \quad \underset{\substack{\downarrow\\ 1}}{\frac{N(t)+1}{N(t)}} \quad \text{as } t\to\infty.$$

Therefore $\quad \underset{\substack{\downarrow\\ \mu}}{\frac{S_{N(t)}}{N(t)}} \le \frac{t}{N(t)} < \underset{\substack{\downarrow\\ \mu}}{\frac{S_{N(t)+1}}{N(t)+1}} \quad t\to\infty$

$\Rightarrow \quad \frac{t}{N(t)} \xrightarrow[t\to\infty]{} \mu \quad \Rightarrow \quad \frac{N(t)}{t} \rightarrow \frac{1}{\mu}$ as $t \to \infty$

Remarks

Remarks. **(i)** The preceding propositions are true even when μ, the mean time between renewals, is infinite. In this case, we interpret $1/\mu$ to be 0.

(ii) The number $1/\mu$ is called the *rate* of the renewal process.

(iii) Because the average time between renewals is μ, it is quite intuitive that the average rate at which renewals occur is 1 per every μ time units. ■

Example 7.4. Beverly has a radio that works on a single battery. As soon as the battery in use fails, Beverly immediately replaces it with a new battery. If the lifetime of a battery (in hours) is distributed uniformly over the interval (30, 60), then at what rate does Beverly have to change batteries?

$N(t)$ denote the # of batteries that have failed by time t, by proposition 1

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu} = \frac{1}{45} \qquad \mu = \frac{30+60}{2} = 45$$

rate $\frac{1}{\mu}$

in a long run, Beverly will have to replace one battery every 45 hours.

Example 7.5. Suppose in Example 7.4 that Beverly does not keep any surplus batteries on hand, and so each time a failure occurs she must go and buy a new battery. If the amount of time it takes for her to get a new battery is uniformly distributed over (0, 1), then what is the average rate that Beverly changes batteries?

Solution: In this case the mean time between renewals is given by

$$\mu = E[U_1] + E[U_2]$$

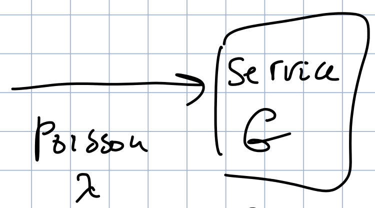
where U_1 is uniform over (30, 60) and U_2 is uniform over (0, 1). Hence,

$$\mu = 45 + \tfrac{1}{2} = 45\tfrac{1}{2}$$

and so in the long run, Beverly will be putting in a new battery at the rate of $\frac{2}{91}$. That is, she will put in two new batteries every 91 hours. ■

Example 7.6. Suppose that potential customers arrive at a single-server bank in accordance with a Poisson process having rate λ. However, suppose that the potential customer will enter the bank only if the server is free when he arrives. That is, if there is already a customer in the bank, then our arriver, rather than entering the bank, will go home. If we assume that the amount of time spent in the bank by an entering customer is a random variable having distribution G, then

(a) what is the rate at which customers enter the bank?
(b) what proportion of potential customers actually enter the bank?



$$EG = \mu_G$$

G – distribution of a service time.

Mean time between entering customers is

$$\mu = \mu_G + \frac{1}{\lambda}$$

rate at which customer enters the bank.

$$\text{rate} = \lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu} = \frac{1}{\mu_G + \frac{1}{\lambda}} = \frac{\lambda}{1 + \mu_G \lambda}$$

Proportion of customers entering bank

λ - rate of arrival

$$\frac{\frac{\lambda}{1+\lambda\mu_G}}{\lambda} = \frac{1}{1+\lambda\mu_G}.$$

If $\lambda = 2$ $\mu_G = 2$

Only one out of 5 customers will enter the bank

Prop. 1: average renewal rate up to time t will converge (a.s) to $\frac{1}{\mu}$ $t \to \infty$. What about renewal rate $\left(\frac{m(t)}{t}\right)$

Theorem 7.1 (Elementary renewal theorem)

$$\frac{m(t)}{t} \to \frac{1}{\mu} \quad t \to \infty$$

$$\frac{1}{\mu} = 0, \quad \text{if} \quad \mu = \infty$$

Remark. At first glance it might seem that the elementary renewal theorem should be a simple consequence of Proposition 7.1. That is, since the average renewal rate will, with probability 1, converge to $1/\mu$, should this not imply that the expected average renewal rate also converges to $1/\mu$? We must, however, be careful; consider the next example.

Ex. Let $U \sim \text{Unif.}(0,1)$

$$Y_n = \begin{cases} 0 & \text{if } U > \frac{1}{n} \\ n & \text{if } U \le \frac{1}{n} \end{cases}$$

With prob. 1 $U > 0$, Y_n will be 0 for all sufficiently large n, $U > \frac{1}{n}$

$\Rightarrow$ with prob. 1.

$$Y_n \to 0 \quad n \to \infty$$

However $E(Y_n) = n P\{U \le \frac{1}{n}\} = n \frac{1}{n} = 1$

Therefore, even though the seq. of r.v $\to 0$

$Y_n \to 0$, $\quad E Y_n \to 1$ $\quad n \to \infty$.

Proof (elementary renewal theorem).

to prove it we need Wald's eq

Definition. The nonnegative integer valued random variable N is said to be a *stopping time* for a sequence of independent random variables $X_1, X_2, \ldots$ if the event that $\{N = n\}$ is independent of $X_{n+1}, X_{n+2}, \ldots$, for all $n = 1, 2, \ldots$.

The idea behind a stopping time is that we imagine that the X_i are observed in sequence, first X_1, then X_2, and so on, and that N denotes the number of them observed before stopping. Because the event that we stop after having observed $X_1, \ldots, X_n$ can only depend on these n values, and not on future unobserved values, it must be independent of these future values.

Ex. Let $X_1 \ldots X_n, \ldots$ be iid sequence

$P(X_i = 1) = p = 1 - P(X_i = 0)$ $\quad p > 0$.

$N = \min\{n, X_1 + \cdots + X_n = r\}$

N is a stopping time

Theorem 7.2 (Wald's equation).

If $X_1, X_2, \ldots$ iid r.v.,

$EX < \infty$, and if N is a stopping time

for this sequence such that $E(N) < \infty$

then $E\left(\sum_{n=1}^{N} X_n\right) = EN \cdot EX$.

Proof. For $n = 1, 2, \ldots$ let

$$I_n = \begin{cases} 1 & n \le N \\ 0 & n > N \end{cases}$$

$$\sum_{n=1}^{N} X_n = \sum_{n=1}^{\infty} X_n I_n$$

$$E\left(\sum_{n=1}^{N} X_n\right) = E\sum_{n=1}^{\infty} X_n I_n = \sum_{n=1}^{\infty} E(X_n I_n)$$

Now $I_n = 1$ if $N \geq n$, which means that $I_n = 1$ if we have not yet stopped after having observed $X_1, \ldots, X_{n-1}$. But this implies that the value of I_n is determined before X_n has been observed, and thus X_n is independent of I_n. Consequently,

$$E[X_n I_n] = E[X_n]E[I_n] = E[X]E[I_n]$$

showing that

$$E\left[\sum_{n=1}^{N} X_n\right] = E[X]\sum_{n=1}^{\infty} E[I_n]$$
$$= E[X]E\left[\sum_{n=1}^{\infty} I_n\right]$$
$$= E[X]E[N] \quad \blacksquare$$

To apply Wald's equation to renewal theory, let $X_1, X_2, \ldots$ be the sequence of interarrival times of a renewal process. If we observe these one at a time and then stop at the first renewal after time t, then we would stop after having observed $X_1, \ldots, X_{N(t)+1}$, showing that $N(t)+1$ is a stopping time for the sequence of interarrival times. For a more formal argument that $N(t)+1$ is a stopping time for the sequence of interarrival times, note that $N(t) = n-1$ if and only if the $(n-1)$st renewal occurs by time t and the nth renewal occurs after time t. That is,

$$N(t)+1 = n \Leftrightarrow N(t) = n-1 \Leftrightarrow X_1 + \cdots + X_{n-1} \leq t,\ X_1 + \cdots + X_n > t$$

showing that the event that $N(t)+1 = n$ depends only on the values of $X_1, \ldots, X_n$. $\blacksquare$

We thus have the following corollary of Wald's equation.

Proposition 7.2.

If $X_1, X_2 \ldots$ are interarrival times of a renewal process then

$$E(X_1 + \cdots + X_{N(t)+1}) = EX\, E(N(t)+1)$$

That is

$$E(S_{N(t)+1}) = \mu(m(t)+1)$$

Proof of Elementary renewal theorem.

time of the first renewal after t $\Rightarrow S_{N(t)+1} = t + Y(t)$

where $Y(t)$, called the *excess* at time t, is defined as the time from t until the next renewal. Taking expectations of the preceding yields, upon applying Proposition 7.2, that

$$\mu(m(t) + 1) = t + E[Y(t)] \tag{7.9}$$

which can be written as

$$\frac{m(t)}{t} = \frac{1}{\mu} + \frac{E[Y(t)]}{t\mu} - \frac{1}{t}$$

Because $Y(t) \geq 0$, the preceding yields that $\frac{m(t)}{t} \geq \frac{1}{\mu} - \frac{1}{t}$, showing that

$$\lim_{t\to\infty} \frac{m(t)}{t} \geq \frac{1}{\mu}$$

To show that $\lim_{t\to\infty} \frac{m(t)}{t} \leq \frac{1}{\mu}$, let us suppose that there is a value $M < \infty$ such that $P(X_i < M) = 1$. Because this implies that $Y(t)$ must also be less than M, we have that $E[Y(t)] < M$, and so

$$\frac{m(t)}{t} \leq \frac{1}{\mu} + \frac{M}{t\mu} - \frac{1}{t}$$

which gives that

$$\lim_{t\to\infty} \frac{m(t)}{t} \leq \frac{1}{\mu}$$

and thus completes the proof of the elementary renewal theorem when the interarrival times are bounded. When the interarrival times $X_1, X_2, \ldots$ are unbounded, fix $M > 0$, and let $N_M(t), t \geq 0$ be the renewal process with interarrival times $\min(X_i, M), i \geq 1$. Because $\min(X_i, M) \leq X_i$ for all i, it follows that $N_M(t) \geq N(t)$ for all t. (That is, because each interarrival time of $N_M(t)$ is smaller than its corresponding interarrival time of $N(t)$, it must have at least as many renewals by time t.) Consequently, $E[N(t)] \leq E[N_M(t)]$, showing that

$$\lim_{t\to\infty} \frac{E[N(t)]}{t} \leq \lim_{t\to\infty} \frac{E[N_M(t)]}{t} = \frac{1}{E[\min(X_i, M)]}$$

where the equality follows because the interarrival times of $N_M(t)$ are bounded. Using that $\lim_{M\to\infty} E[\min(X_i, M)] = E[X_i] = \mu$, we obtain from the preceding upon letting $M \to \infty$ that

$$\lim_{t\to\infty} \frac{m(t)}{t} \leq \frac{1}{\mu}$$

and the proof is completed. ■

Example 7.11. Consider the renewal process whose interarrival distribution is the convolution of two exponentials; that is,

$$F = F_1 * F_2, \quad \text{where } F_i(t) = 1 - e^{-\mu_i t}, \quad i = 1, 2$$

We will determine the renewal function by first determining $E[Y(t)]$. To obtain the mean excess at t, imagine that each renewal corresponds to a new machine being put in use, and suppose that each machine has two components—initially component 1 is employed and this lasts an exponential time with rate μ_1, and then component 2, which functions for an exponential time with rate μ_2, is employed. When component 2 fails, a new machine is put in use (that is, a renewal occurs). Now consider the process $\{X(t),\ t \geq 0\}$ where $X(t)$ is i if a type i component is in use at time t. It is easy to see that $\{X(t),\ t \geq 0\}$ is a two-state continuous-time Markov chain, and so, using the results of Example 6.11, its transition probabilities are

$$P_{11}(t) = \frac{\mu_1}{\mu_1 + \mu_2} e^{-(\mu_1+\mu_2)t} + \frac{\mu_2}{\mu_1 + \mu_2}$$

To compute the expected remaining life of the machine in use at time t, we condition on whether it is using its first or second component: for if it is still using its first component, then its remaining life is $1/\mu_1 + 1/\mu_2$, whereas if it is already using its second component, then its remaining life is $1/\mu_2$. Hence, letting $p(t)$ denote the probability that the machine in use at time t is using its first component, we have

$$\begin{aligned} E[Y(t)] &= \left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right) p(t) + \frac{1 - p(t)}{\mu_2} \\ &= \frac{1}{\mu_2} + \frac{p(t)}{\mu_1} \end{aligned}$$

But, since at time 0 the first machine is utilizing its first component, it follows that $p(t) = P_{11}(t)$, and so, upon using the preceding expression of $P_{11}(t)$, we obtain

$$E[Y(t)] = \frac{1}{\mu_2} + \frac{1}{\mu_1 + \mu_2} e^{-(\mu_1+\mu_2)t} + \frac{\mu_2}{\mu_1(\mu_1 + \mu_2)} \tag{7.10}$$

Now it follows from Eq. (7.9) that

$$m(t) + 1 = \frac{t}{\mu} + \frac{E[Y(t)]}{\mu} \tag{7.11}$$

where μ, the mean interarrival time, is given in this case by

$$\mu = \frac{1}{\mu_1} + \frac{1}{\mu_2} = \frac{\mu_1 + \mu_2}{\mu_1 \mu_2}$$

Substituting Eq. (7.10) and the preceding equation into (7.11) yields, after simplifying,

$$m(t) = \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} t - \frac{\mu_1 \mu_2}{(\mu_1 + \mu_2)^2} [1 - e^{-(\mu_1+\mu_2)t}] \quad \blacksquare$$

Remark. Using the relationship of Eq. (7.11) and results from the two-state continuous-time Markov chain, the renewal function can also be obtained in the same manner as in Example 7.11 for the interarrival distributions

$$F(t) = pF_1(t) + (1 - p)F_2(t)$$

and

$$F(t) = pF_1(t) + (1 - p)(F_1 * F_2)(t)$$

when $F_i(t) = 1 - e^{-\mu_i t}$, $t > 0$, $i = 1, 2$. $\blacksquare$

Suppose the interarrival times of a renewal process are all positive integer valued. Let

$$I_i = \begin{cases} 1, & \text{if there is a renewal at time } i \\ 0, & \text{otherwise} \end{cases}$$

and note that $N(n)$, the number of renewals by time n, can be expressed as

$$N(n) = \sum_{i=1}^{n} I_i$$

Taking expectations of both sides of the preceding shows that

$$m(n) = E[N(n)] = \sum_{i=1}^{n} P(\text{renewal at time } i)$$

Hence, the elementary renewal theorem yields

$$\frac{\sum_{i=1}^{n} P(\text{renewal at time } i)}{n} \to \frac{1}{E[\text{time between renewals}]}$$

Now, for a sequence of numbers $a_1, a_2, \ldots$ it can be shown that

$$\lim_{n\to\infty} a_n = a \quad \Rightarrow \quad \lim_{n\to\infty} \frac{\sum_{i=1}^{n} a_i}{n} = a$$

Hence, if $\lim_{n\to\infty} P(\text{renewal at time } n)$ exists then that limit must equal $\frac{1}{E[\text{time between renewals}]}$.

Theorem 7.3 (Central Limit Theorem for Renewal Processes)**.**

$$\lim_{t\to\infty} P\left\{\frac{N(t)-t/\mu}{\sqrt{t\sigma^2/\mu^3}} < x\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-x^2/2}dx$$

We now give a heuristic argument to show, for t large, that the distribution of $N(t)$ is approximately that of a normal random variable with mean t/μ and variance $t\sigma^2/\mu^3$.

Heuristic Argument for Central Limit Theorem for Renewal Processes. To begin, note that by the central limit theorem it follows when n is large that $S_n = \sum_{i=1}^{n} X_i$ is approximately a normal random variable with mean $n\mu$ and variance $n\sigma^2$. Consequently, using that $N(t) < n \Leftrightarrow S_n > t$, we see that when n is large

$$\begin{aligned} P(N(t) < n) &= P(S_n > t) \\ &= P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} > \frac{t-n\mu}{\sigma\sqrt{n}}\right) \\ &\approx P(Z > \frac{t-n\mu}{\sigma\sqrt{n}}) \end{aligned} \tag{7.12}$$

where Z is a standard normal random variable. Now,

$$P(\frac{N(t)-t/\mu}{\sqrt{t\sigma^2/\mu^3}} < x) = P(N(t) < t/\mu + x\sigma\sqrt{t/\mu^3})$$

Treating $t/\mu + x\sigma\sqrt{t/\mu^3}$ as if it were an integer, we see upon letting $n = t/\mu + x\sigma\sqrt{t/\mu^3}$ in Eq. (7.12) that

$$\begin{aligned} P(\frac{N(t)-t/\mu}{\sqrt{t\sigma^2/\mu^3}} < x) &\approx P\left(Z > \frac{t - t - x\sigma\mu\sqrt{t/\mu^3}}{\sigma\sqrt{t/\mu + x\sigma\sqrt{t/\mu^3}}}\right) \\ &= P\left(Z > \frac{-x\sqrt{t/\mu}}{\sqrt{t/\mu + x\sigma\sqrt{t/\mu^3}}}\right) \\ &\approx P(Z > -x) \qquad \text{when } t \text{ is large} \\ &= P(Z < x) \end{aligned}$$

■

In addition, as might be expected from the central limit theorem for renewal processes, it can be shown that $\text{Var}(N(t))/t$ converges to σ^2/μ^3. That is, it can be shown that

$$\lim_{t\to\infty} \frac{\text{Var}(N(t))}{t} = \sigma^2/\mu^3$$

Example 7.13. Two machines continually process an unending number of jobs. The time that it takes to process a job on machine 1 is a gamma random variable with parameters $n = 4, \lambda = 2$, whereas the time that it takes to process a job on machine 2 is uniformly distributed between 0 and 4. Approximate the probability that together the two machines can process at least 90 jobs by time $t = 100$.

Solution: If we let $N_i(t)$ denote the number of jobs that machine i can process by time t, then $\{N_1(t), t \geq 0\}$ and $\{N_2(t),\ t \geq 0\}$ are independent renewal processes. The interarrival distribution of the first renewal process is gamma with parameters $n = 4, \lambda = 2$, and thus has mean 2 and variance 1. Correspondingly, the interarrival distribution of the second renewal process is uniform between 0 and 4, and thus has mean 2 and variance $16/12$.

Therefore, $N_1(100)$ is approximately normal with mean 50 and variance 100/8; and $N_2(100)$ is approximately normal with mean 50 and variance 100/6. Hence, $N_1(100) + N_2(100)$ is approximately normal with mean 100 and variance $175/6$. Thus, with Φ denoting the standard normal distribution function, we have

$$
\begin{aligned}
P\{N_1(100) + N_2(100) > 89.5\} &= P\left\{\frac{N_1(100) + N_2(100) - 100}{\sqrt{175/6}} > \frac{89.5 - 100}{\sqrt{175/6}}\right\} \\
&\approx 1 - \Phi\left(\frac{-10.5}{\sqrt{175/6}}\right) \\
&\approx \Phi\left(\frac{10.5}{\sqrt{175/6}}\right) \\
&\approx \Phi(1.944) \\
&\approx 0.9741 \qquad \blacksquare
\end{aligned}
$$