M/M/S Queue

Example

$S = 3 \quad \lambda \sim s\mu$

$\lambda = 2 \quad 2 \sim 3 \cdot 1 \Rightarrow (+)$ recurrent

$\mu = 1 \quad 2 < 3$

$$\pi_0 = \frac{1}{1 + \frac{\sum_{n=1}^{s} \left(\frac{\lambda}{\mu}\right)^n}{n!} + \frac{\sum_{n=s+1}^{\infty} \left(\frac{\lambda}{s\mu}\right)^n s^s}{s!}}$$

$$\pi_n = \begin{cases} \frac{\pi_0 \left(\frac{\lambda}{\mu}\right)^n}{n!}, & n \geq s \\ \frac{\pi_0 \left(\frac{\lambda}{s\mu}\right) s^s}{s!}, & n > s \end{cases}$$

$$L = \sum_{n=1}^{\infty} n \pi_n$$

Brownian Motion

Notation

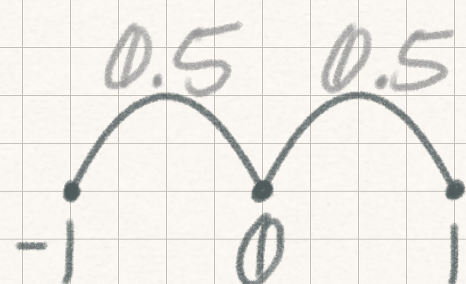
Δx = size of step

Δt = time unit

$X(t) = \Delta x (X_1 + \cdots + X_{\left[\frac{t}{\Delta t}\right]})$

$X(t)$ = position @ time t

$\left[\frac{t}{\Delta t}\right]$ = largest int $\leq \frac{t}{\Delta t}$



$$X_i = \begin{cases} 1 & ; \text{ step } \to \\ -1 & ; \text{ step } \leftarrow \end{cases}$$

$$P(X_i = 1) = \frac{1}{2} = P(X_i = -1)$$

$$\mathbb{E}(X_i) = 0$$

$$\mathbb{E}(X(t)) = \mathbb{E}\left(\Delta x \sum_{i=1}^{\left[\frac{t}{\Delta t}\right]} X_i\right) = 0$$

$$\mathbb{V}(X_i) = \mathbb{E}(X_i^2) - (\mathbb{E}X_i)^2 = (-1)^2 \frac{1}{2} + 1^2\left(\frac{1}{2}\right) - 0 = \frac{1}{2} + \frac{1}{2} = 1$$

$$\mathbb{V}(X(t)) = \mathbb{V}\left(\Delta x \sum_{i=1}^{\left[\frac{t}{\Delta t}\right]} X_i\right) = (\Delta x)^2 \left[\frac{t}{\Delta t}\right]$$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \frac{\Delta x}{\sqrt{\Delta t}} = \sigma \qquad \Delta x = \sigma\sqrt{\Delta t}$$

$$\sigma > 0$$

$\Delta t \to 0$

we obtain lim process

$n \to \infty$

$X_1 + \dots + X_n \Rightarrow \sim N(0,1)$

Properties of

$X_{\Delta t} \xrightarrow{d} X(t)$

1. $\forall t,\ X(t) \sim N(0, \underset{\mathbb{V}}{\sigma^2 t})$

$$X_{\Delta t}(t) = \sigma\sqrt{\Delta t}\left(X_1 + \dots + X_{\left[\frac{t}{\Delta t}\right]}\right)$$

2. $\{X(t), t \geq 0\}$

   $t_1 < t_2 < t_3 < \dots < t_n$

3. $X(t_n) - X(t_{n-1}),\ X(t_{n-1}) - X(t_{n-2}), \dots,$
   $X(t_2) - X(t_1), \dots X(t_1)$

increments are indep & stationary

$X(t+s) - X(t)$

distr of $X(t+s)-X(t)$ depends only from s

Define: Stochastic process $\{X(t), t \geq 0$ is said to be Brownian motion (Wiener process) if

1. $X(0)=0$
2. $\{X(t), t \geq 0\}$ has stationary & indep increments
3. $\forall t>0$, $X(t) \sim N(0, \sigma^2 t)$

Notation

$B(t)$ = standard Brownian motion

$B(t) \sim N(0, t)$

$\sigma = 1$

$\frac{X(t)}{\sigma} = Bt$

$f_t(x) = \frac{1}{\sqrt{2\pi t}}\left(e^{-\frac{x^2}{2t}}\right)$

To obtain joint density function

$t_1 < t_2 < \ldots < t_n$

$X(t_1) = x_1$	$X(t_1) = x_1$
$X(t_2) = x_2$	$X(t_2) - X(t_1) = x_2 - x_1$
$\vdots$	$\vdots$
$X(t_n) = x_n$	$X(t_n) - X(t_{n-1}) = x_n - x_{n-1}$
not indep	indep

$X(t_k) - X(t_{k-1}) \sim N(0, t_k - t_{k-1})$

$(X(t_1), \dots, X(t_n))$

$$f(x) = f_{t_1}(x_1) f_{t_2 - t_1}(x_2 - x_1) \cdots f_{t_n - t_{n-1}}(x_n - x_{n-1})$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}} [t_1(t_2 - t_1) \cdots (t_n - t_{n-1})]^{0.5}} e^{\left(-\frac{1}{2}\left(\frac{x_1^2}{t_1} + \frac{(x_2 - x_1)^2}{t_2 - t_1} + \cdots + \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}}\right)\right)}$$