HW Hint

$i \Rightarrow (n_f, n_m) \to (n_{f+1}, n_m)$

$\qquad p_{ij} \qquad \to (n_f, n_{m+1})$

Ch 6 cont

General Death & Birth Process

birth rate: $\{\lambda_n\}$

death rate $= \{\mu_n\}$

$T_i \sim i \to i+1$

$i \to i-1 \to i \to i+1$

$i \to i-1 \to i \to i-1 \to i \to i+1$

$\mathbb{E}(T_i) = ?$ $\quad T_0 \sim Exp(\lambda_0)$

$i \to i+1$

$\to i-1$

$I_i = \begin{cases} 1 & i \to i+1 \\ 0 & i \to i-1 \end{cases}$

$$\mathbb{E}(T_i | I_i = 1) = \frac{1}{\lambda_i + \mu_i}$$

$$\mathbb{E}(T_i | I_i = 0) = \frac{1}{\lambda_i + \mu_i} + \mathbb{E}(T_{i-1}) + \mathbb{E}(T_i)$$

$$P(I_i = 1) = \frac{\lambda_i}{\lambda_i + \mu_i}$$

$$P(I_i = 0) = \frac{\mu_i}{\lambda_i + \mu_i}$$

$$\mathbb{E}(T_i) = \mathbb{E}[\mathbb{E}(T_i | I_i)] = \mathbb{E}(T_i | I_i = 1) P(I_i = 1) + \mathbb{E}(T_i | I_i = 0) P(I_i = 0)$$

$$= \left[\frac{1}{\lambda_i + \mu_i}\left(\frac{\lambda_i}{\lambda_i + \mu_i}\right) + \frac{1}{\lambda_i + \mu_i}\left(\frac{\mu_i}{\lambda_i + \mu_i}\right)\right] + \mathbb{E}(T_{i-1})\left(\frac{\mu_i}{\lambda_i + \mu_i}\right) + \mathbb{E}(T_i)\left(\frac{\mu_i}{\lambda_i + \mu_i}\right)$$

$$= \frac{1}{\lambda_i + \mu_i} + \frac{\mu_i}{\lambda_i + \mu_i}\left[\mathbb{E}(T_i) + \frac{\mu_i}{\lambda_i + \mu_i}\right] \mathbb{E}(T_i) = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i}(\mathbb{E}T_{i-1})$$

$$\mathbb{E}(T_i)\left(1-\frac{\mu_i}{\lambda_i+\mu_i}\right)=\frac{1}{\lambda_i+\mu_i}\left[1+\mu_i(\mathbb{E}T_{i-1})\right]$$

$$\mathbb{E}(T_0)=\frac{1}{\lambda_0}$$

$$\mathbb{E}(T_1)=\frac{1}{\lambda_1}+\frac{\mu_1}{\lambda_1}\left(\frac{1}{\lambda_0}\right)$$

$$\mathbb{E}(T_2)=\frac{1}{\lambda_2}+\frac{\mu_2}{\lambda_2}\left[\frac{1}{\lambda_1}+\frac{\mu_1}{\lambda_1(\lambda_0)}\right]=\frac{1}{\lambda_2}+\frac{\mu_2}{\lambda_2(\lambda_1)}+\frac{\mu_2(\mu_1)}{\lambda_2(\lambda_1)\lambda_0}$$

$i\to j$, $i<j$ $\qquad \mathbb{E}(T_i)+\mathbb{E}(T_{i+1})+\cdots+\mathbb{E}(T_{j-1})$

Example: Death & Birth Process

$\lambda_i=\lambda$
$\mu_i=\mu$

$$\mathbb{E}(T_i)=\frac{1}{\lambda}+\frac{\mu}{\lambda}(\mathbb{E}T_{i-1})=\frac{1}{\lambda}\left[1+\mu(\mathbb{E}T_i)\right]$$

$$\mathbb{E}(T_1)=\frac{1}{\lambda}+\frac{\ }{\lambda^2}$$

$$\mathbb{E}(T_2)=\frac{1}{\lambda}+\frac{\mu}{\lambda}\left(\frac{1}{\lambda}+\frac{\mu}{\lambda}\right)$$

$$\mathbb{E}(T_3)=\frac{1}{\lambda}+\frac{\mu}{\lambda}+\left\{\frac{1}{\lambda}\left[1+\frac{\mu}{\lambda}+\left(\frac{\mu}{\lambda}\right)^2\right]=\frac{1}{\lambda}\left[1+\frac{\mu}{\lambda}+\left(\frac{\mu}{\lambda}\right)^2+\left(\frac{\mu}{\lambda}\right)^3\right]\right.$$

$$\mathbb{E}(T_i)=\frac{1}{\lambda}\left[1+\frac{\mu}{\lambda}+\left(\frac{\mu}{\lambda}\right)^2+\cdots+\left(\frac{\mu}{\lambda}\right)^i\right]=\frac{1}{\lambda}\left[\frac{1-\left(\frac{\mu}{\lambda}\right)^{i+1}}{1-\frac{\mu}{\lambda}}\right]$$

$$=\frac{1-\left(\frac{\mu}{\lambda}\right)^{i+1}}{\lambda-\mu}\qquad \mu\neq\lambda$$

$\mu=\lambda$, $\mathbb{E}(T_i)=\frac{i+1}{\lambda}$

$$\mathbb{E}(T_{k\to j})=\frac{j(j+1)-k(k+1)}{2\lambda}$$

We can also compute $\mathbb{V}$

$$\mathbb{V}(T_i) = \mathbb{V}[\mathbb{E}(T_i|I_i)] + \mathbb{E}[\mathbb{V}(T_i|I_i)]$$

$$\mathbb{E}(T_i|I_i) = \frac{1}{\lambda_i+\mu_i} + (1-I_i)[\mathbb{E}(T_{i-1}) + \mathbb{E}(T_i)]$$

$$\mathbb{E}(T_i|I_i=1) = \frac{1}{\lambda_i+\mu_i}$$

$$\mathbb{E}(T_i|I_i=\emptyset) = \frac{1}{\lambda_i+\mu_i} + \mathbb{E}(T_{i-1}) + \mathbb{E}(T_i)$$

$$\mathbb{E}(T_i|I_i) = \frac{1}{\lambda_i+\mu_i} + (1-I_i)[\mathbb{E}(T_{i-1}) + \mathbb{E}(T_i)]$$

I_i	$\emptyset$	1
p	$\frac{\mu_i}{\lambda_i+\mu_i}$	$\frac{\lambda_i}{\lambda_i+\mu_i}$

$$\mathbb{V}(I_i) = p(1-p) = \frac{\lambda_i\mu_i}{(\lambda_i+\mu_i)^2}$$

$$\mathbb{V}[\mathbb{E}(T_i|I_i)] = \mathbb{V}\left\{\frac{1}{\lambda_i+\mu_i} + (1-I)[\mathbb{E}(T_{i-1}) + \mathbb{E}(I_i)]\right\}$$

$$= \mathbb{V}\{[\mathbb{E}(T_{i-1}) + \mathbb{E}(T_i)](1-I)\}$$

$$= [\mathbb{E}(T_{i-1}) + \mathbb{E}(T_i)]^2\, \mathbb{V}(1-I)$$

$$= [\mathbb{E}(T_{i-1}) + \mathbb{E}(T_i)]^2 \frac{\lambda_i\mu_i}{(\lambda_i+\mu_i)^2}$$

$$\mathbb{V}(T_i|I=1) = \mathbb{V}(X_i|I=1) = \mathbb{V}(X_i) = \frac{1}{(\lambda_i+\mu_i)^2}$$

$$\mathbb{V}(T_i|I=\emptyset) = \mathbb{V}(X_i) + \mathbb{V}(T_{i-1}) + \mathbb{V}(T_i)$$

$$\mathbb{V}(T_i) = \frac{1}{\lambda_i(\lambda_i+\mu_i)} + \frac{\mu_i}{\lambda_i}[\mathbb{V}(T_{i-1})] + \frac{\mu_i}{\mu_i+\lambda_i}[\mathbb{E}(T_{i-1}) + \mathbb{E}(T_i)]^2$$

$$\mathbb{V}(T_0) = \frac{1}{\lambda_0}$$

$$\mathbb{V}(\text{time to go } k \to j) = \sum_{i=k}^{j-1} \mathbb{V}(T_i)$$

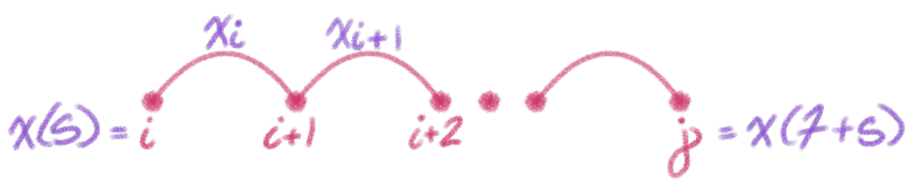
Ch. 6.4: Trans Prob $f(x)$

$P_{ij}(t) = P\{X(t+s) = j \mid X(s) = i\}$

Birth & death process $\Rightarrow$ Pure birth process
distinct birth Δs

$\{\lambda_n\}_{n=0}^{\infty}$; $\lambda_i \neq \lambda_j$, $i \neq j$

X_k = time to go from $k \to k+1$



time to go from $i \to j$: $\sum_{k=i}^{j-1} X_k$

$X(t) < j \iff X_i + X_{i+1} + \cdots + X_{j-1} > t$

$(X_i, X_{i+1}, \ldots, X_{j-1}) \sim$ indep
$\sim$ Exp
$(\lambda_i, \lambda_{i+1}, \ldots, \lambda_{j-1})$

$$P\{X(t) < j \mid X(0) = i\} = \sum_{k=i}^{j-1} e^{-\lambda_k t} \prod_{\substack{r \neq k \\ r=i}}^{j-1} \frac{\lambda_r}{\lambda_r - \lambda_k}$$

5.2.4 Convolution of Exp rv

Let X_i $(i=1,\ldots,n)$ be indep exp rvs w/ Δs λ_i $(i=1,\ldots,n)$ & $\lambda_i \neq \lambda_j$, $i = j$

hyperexp rv $\quad \sum_{i=1}^{n} X_i$

$$X_1 + X_2 \quad \left.\begin{array}{l} \Rightarrow X_1 \sim Exp(\lambda_1) \\ \Rightarrow X_2 \sim Exp(\lambda_2) \end{array}\right] \text{indep}$$

$$f_{X_1+X_2}(t) = \int_{-\infty}^{\infty} f_{X_1}(s) \cdot f_{X_2}(t-s)\, ds$$

$$= \lambda_1 \lambda_2 \int_0^t e^{-\lambda_1 s - \lambda_2 t + \lambda_2 s}\, ds$$

$$= \lambda_1 \lambda_2 \left(e^{-\lambda_2 t}\right) \int_0^t e^{-(\lambda_1 - \lambda_2)}\, ds$$

$$= -\frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} \left(e^{-\lambda_2 t}\right)$$

$$e^{-(\lambda_1-\lambda_2)s} \Big|_0^t = \frac{\lambda_1}{\lambda_1 - \lambda_2} (\lambda_2) \left(e^{-\lambda_2 t}\right) \left[1 - e^{-(\lambda_1 - \lambda_2)t}\right]$$

$$= \frac{\lambda_1}{\lambda_1 - \lambda_2} (\lambda_2) \left(e^{-\lambda_2 t}\right) + \frac{\lambda_2}{\lambda_2 - \lambda_1} (\lambda_1) \left(e^{-\lambda_1 t}\right)$$

Let $C_{in} = \prod_{i \neq j} \frac{\lambda_j}{\lambda_j - \lambda_i}$

$$f_{X_1+X_2+\cdots+X_n}(t) = \sum_{i=1}^{n} C_{in} (\lambda_i) e^{-\lambda_i t}$$

$$f_{X_1+X_2+X_3}(t) = \sum_{i=1}^{3} \lambda_i \left(e^{-\lambda_i t}\right) C_{in}$$

We prove it for $(n+1)$

$S = X_1 + \cdots + X_n$

$$f_{X_1+\cdots+X_{n+1}}(t) = \int_0^t f_{X_1+\cdots+X_n}(s) \cdot f_{X_{n+1}}(t-s)\, ds$$

$$= \sum_{i=1}^{n} C_{in} (\lambda_i) e^{-\lambda_i t} \qquad \text{pdf}$$

$$P(S > t) = \sum_{i=1}^{n} C_{in} (\lambda_i) e^{-\lambda_i t} \qquad \text{cdf}$$

$$P(S \leq t) = 1 - \sum_{i=1}^{n} C_{in} (\lambda_i) e^{-\lambda_i t} \qquad 1-\text{cdf}$$