Exit Dist & Exit Time

$\{X(t), t \geq 0\}$ = cont time MC

$P(t)$ = trans prob mat

embedded MC = disc time

$\mathbb{P} = (P_{ij})_{ij \in S}$ = trans prob mat of embedded chain

v_i = Δ to leave state i

$\mathbb{R} = (R_{ij})_{ij}$

$$R_{ij} = \begin{cases} q_{ij} = v_i P_{ij} & i \neq j \\ -v_i & i = j \end{cases}$$

Let $v_k = \min(t \geq 0 : X(t) = k)$ be time of 1st visit to k

$T_k = \min\{t \geq 0 : X(t) = k, X(s) \neq k \text{ for some } s < t\}$ be time of 1st return

Branching Process

Cont	Discrete
$\forall$ indiv λ = birth Δ μ = death Δ	$X(t) = 0$
$q_{i,i-1} = \mu(i)$	μ = avg # of offsprings
$P_{i,i+1} = \frac{\lambda}{\lambda+\mu}$	P_i = prob to have ith offsprings
$P_{i,i-1} = \frac{\mu}{\lambda+\mu}$	$X_n = \sum_{k=1}^{X_{n-1}} Y_k$ Y_k = # of offspring for kth indiv

0 = absorbing state

$\frac{\lambda}{\mu} \le 1$ pop will die out w/ prob 1

$\lambda > \mu \iff \frac{\mu}{\lambda} < 1$

ρ = prob that pop will die out

$P_1(T_0 = \infty) = 1 - \frac{\mu}{\lambda}$

$\rho = P(T_0 < \infty \mid X(0) = 1)$

If $\frac{\lambda}{\mu} \le 1 \Rightarrow \rho = 1$

If $\frac{\lambda}{\mu} > 1$

let $I = \begin{cases} 1 & \text{death b4 birth} \\ 2 & \text{vice versa} \end{cases}$

$P_1(T_0 < \infty) = P_1(T_0 < \infty \mid I = 1) P(I = 1) + P_1(T_0 < \infty \mid I = 2) P(I = 2)$

$\rho = 1\left(\frac{\mu}{\lambda + \mu}\right) + \rho^2\left(\frac{\lambda}{\lambda + \mu}\right)$

$\rho^2(\lambda) - (\lambda + \mu)\rho = 0$

$(\lambda\rho - \lambda)\left(\rho - \frac{\mu}{\lambda}\right) = 0$

$\lambda(\rho - 1) = 0 \Rightarrow \rho - 1 = 0 \Rightarrow \rho = 1$

$\rho - \frac{\mu}{\lambda} = 0 \Rightarrow \rho = \frac{\mu}{\lambda}$

If we use embedded chain, we can use approach of disc time MC to compute exit distr

$V_D = \min\{t : X(t) \in D\}$

$T = \min(V_A, V_B)$

Suppose $C \setminus (A \cup B) = S \cap (A \cup B)$

C = finite $\quad C \,|\, \frac{A}{B}$

$P_i(T < \infty) = P(T < \infty \mid X(0) = i) > 0 \;\; \forall i \in C$

Goal: Find prob to visit A b4 B if you start @ i

$h(i)$ = prob to exit set $C \to A$ if you start @ i

$P(V_A < V_B) = P(V_A < V_B \mid X(0) = i)$

$h(a) = 1 \;\; \forall a \in A$

$h(b) = 0$

$(v_i) h(i) = \sum_{j \neq i} v_i \cdot P_{ij}\, h(j) = q_{ij}\, h(j)$

$\sum_{j \neq i} q_{ij}\, h(j) - v_i\, h(j) = 0$

$\sum_{\forall j} R_{ij}\, h(j) = 0$

$r_{ij} = R_{ij}$ if $i, j \in C$

$\mathbb{r} = (r_{ij})_{ij}$

$w_i = \sum_{j \in A} R_{ij}$

$\sum_{j \in C} r_{ij}\, h(j) + w_i = 0$

$-\sum_{j \in C} r_{ij}\, h(j) = w_i$

$$-r(h) = w$$

$$h = (-r)^{-1} w$$

Example: Barbershop

$\mu_1 = \mu_2 = 3$

$\lambda = 2$

$$R = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & -2 & 2 & 0 & 0 & 0 \\ 1 & 3 & -5 & 0 & 0 & 0 \\ 2 & 0 & 6 & -8 & 2 & 0 \\ 3 & 0 & 0 & 6 & -8 & 2 \\ 4 & 0 & 0 & 0 & 6 & -6 \end{array}$$

Find $P_i(V_0 < V_4)$

$i = 1, 2, 3$

$$-r = \begin{pmatrix} 5 & -2 & 0 \\ -6 & 8 & -2 \\ 0 & -6 & 8 \end{pmatrix}$$

$$w = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

$$(-r)^{-1} w = \underbrace{\begin{pmatrix} \frac{39}{41} \\ \frac{36}{41} \\ \frac{27}{41} \end{pmatrix}}_{P_i(V_A < V_B)}$$

$P_i(V_B < V_A)$

$\frac{2}{41}$

$\frac{5}{41}$

$\frac{14}{41}$