1. At present, the Economics & Sociology departments each have one typist who can type 25 letters per day. Economics requires an avg of 20 letters/day, while Sociology requires only an avg of 15. Assuming Poisson arrivals & exponentially distributed typing times, find

a. Avg queue length & avg waiting time in each department

$\mu = 25$ stationary distr $= \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$

$\lambda_e = 20$

$\lambda_s = 15$ $L = \frac{\lambda}{\mu - \lambda}$ $W = \frac{L}{\lambda}$

$L_e = \frac{20}{25-20} = \frac{20}{5} = 4$ $W_e = \frac{4}{20} = \frac{1}{5}$

$L_s = \frac{15}{25-15} = \frac{15}{10} = \frac{3}{2}$ $W_s = \frac{1.5}{15} = \frac{3}{2}\left(\frac{1}{15}\right) = \frac{1}{10}$

The Economics department has an average queue length of 4 & an average waiting time of 0.25 day

The Sociology department has an average queue length of $\frac{3}{2}$ & an average waiting time of 0.1 day

b. Avg overall waiting time if they merge their resources to form a typing pool

balance eq

$\lambda \pi_{n-1} = \mu_n \pi_n$

$$\begin{cases} \lambda \pi_{n-1} = \mu(n)\pi_n, & i \le s \\ \lambda_{n-1} = \mu(s)\pi_n, & i \ge s \end{cases} \qquad \pi_k = \begin{cases} \frac{c}{k!}\left(\frac{\lambda}{\mu}\right)^k, & k \le s \\ \frac{c}{s!\,s^{k-s}}\left(\frac{\lambda}{\mu}\right)^k, & k \ge s \end{cases}$$

$$\frac{c}{s!\,s^{-s}}\left(\frac{\lambda}{\mu s}\right)^k$$

$\lambda = 20 + 15 = 35$
$\mu = 25$

$\pi_k = \dfrac{c}{s!s^{-s}}\left(\dfrac{\lambda}{\mu s}\right)^k = \dfrac{c}{2!2^{-2}}\left(\dfrac{35}{2 \cdot 25}\right)^k$

$= \dfrac{4c}{2}\left(\dfrac{7}{10}\right)^k = 2c\left(\dfrac{7}{10}\right)^k$

$\dfrac{1}{c}\sum_{k=0}^{\infty}\pi_k = 1 + \sum_{k=0}^{\infty} 2\left(\dfrac{\lambda}{2\mu}\right)^k = 1 + 2\sum_{k=0}^{\infty}\left(\dfrac{\lambda}{2\mu}\right)^k = 1 + \dfrac{\frac{\lambda}{\mu}}{1 - \frac{\lambda}{2\mu}}$

$= 1 + \dfrac{\frac{7}{5}}{1 - \frac{7}{10}} = 1 + \dfrac{7}{5}\left(\dfrac{10}{3}\right) = 1 + \dfrac{14}{3} = \dfrac{17}{3}$

$\dfrac{1}{c} = \dfrac{17}{3} \Rightarrow c = \dfrac{3}{17}$

$L = c\sum_{k=0}^{\infty} k\pi_k = \dfrac{3}{17}\sum_{k=0}^{\infty} 2k\left(\dfrac{\lambda}{2\mu}\right)^k = \dfrac{3}{17}\left(\dfrac{2}{1 - \frac{\lambda}{2\mu}}\right)\dfrac{\frac{\lambda}{2\mu}}{1 - \frac{\lambda}{2\mu}}$

$= \dfrac{3}{17}(2)\cancel{\dfrac{10}{3}}\left(\dfrac{7}{10}\right)\left(\dfrac{10}{3}\right) = \dfrac{140}{51} \approx 2.745$

$W = \dfrac{L}{\lambda} = \dfrac{2.745}{35} \approx 0.07843$

The average overall waiting time, if they merge their resources together to form a typing pool, is 0.07843 of day

8. Show that W is smaller in an M/M/1 model having arrivals @ rate λ & service @ rate 2μ than it's in a 2-server M/M/2 model w/ arrivals @ rate λ & w/ each server @ rate μ. Can you give an intuitive explanation for this result? Would it also be true for W_Q?

M/M/1	M/M/2
λ = arrival	λ = arrival
2μ = service	μ = service

$L = \frac{\lambda}{\mu - \lambda}$ $\quad W = \frac{L}{\lambda}$

M/M/1

$L_1 = \frac{\lambda}{2\mu - \lambda}$ $\quad W_1 = \frac{\frac{\lambda}{2\mu - \lambda}}{\lambda} = \frac{\lambda}{2\mu - \lambda}\left(\frac{1}{\lambda}\right) = \frac{1}{2\mu - \lambda}$

M/M/2

W_2

$p_0 q_{01} = p_1 q_{10} \Rightarrow \lambda p_0 = \mu p_1$

$p_1 q_{12} = p_2 q_{21} \Rightarrow (\lambda + \mu) p_1 = \lambda p_0 + 2\mu p_2$

$(\lambda + 2\mu) p_n = \lambda p_{n-1} + 2\mu p_{n+1}$

$p_n = \frac{\rho^n}{2^{n-1}} p_0, \quad \rho = \frac{\lambda}{\mu}$

$\sum_{n=0}^{\infty} p_n = 1 \Rightarrow p_0 = \frac{1 - \frac{\rho}{2}}{1 + \frac{\rho}{2}} = \frac{2 - \rho}{2 + \rho}$

$L_2 = \lambda W_2 = \sum_{n=0}^{\infty} n p_n = \rho (p_0) \sum_{n=0}^{\infty} n \left(\frac{\rho}{2}\right)^{n-1} = 2 p_0 \sum_{n=0}^{\infty} n \left(\frac{\rho}{2}\right)^n$

$= 2\left(\frac{\cancel{1 - \frac{\rho}{2}}}{1 + \frac{\rho}{2}}\right) \frac{\frac{\rho}{2}}{\left(1 - \frac{\rho}{2}\right)^{\cancel{2}}} = 2\left(\frac{1}{2+\rho}\right)\frac{2\rho}{2-\rho} = \frac{4\rho}{(2+\rho)(2-\rho)}$

$= \frac{4\left(\frac{\lambda}{\mu}\right)}{\left(2 + \frac{\lambda}{\mu}\right)\left(2 - \frac{\lambda}{\mu}\right)} = \frac{4\mu\lambda}{(2\mu + \lambda)(2\mu - \lambda)}$

$$W_2 = \frac{L}{\lambda} = \frac{4\mu\lambda}{(2\mu+\lambda)(2\mu-\lambda)}\left(\frac{1}{\lambda}\right) = \frac{4\mu}{(2\mu+\lambda)(2\mu-\lambda)}$$

We assume $2\mu > \lambda$ in M/M/1 queue, so that it's stable

Then, $4\mu > 2\mu+\lambda$ $\left(\Rightarrow \frac{4\mu}{2\mu+\lambda} > 1\right)$, Which implies $W_2 > W_1$.

If you find the queue empty in M/M/2, then there's no reason why you should have 2 servers.

W_{a_1} = M/M/1

$$W_{a_1} = W_1 - \frac{1}{\mu} = \frac{1}{2\mu-\lambda} - \frac{1}{2\mu} = \frac{2\mu-(2\mu-\lambda)}{2\mu(2\mu-\lambda)} = \frac{\lambda}{2\mu(2\mu-\lambda)}$$

W_{a_2} = M/M/2

$$W_{a_2} = W_2 - \frac{1}{\mu} = \frac{4\mu}{(2\mu+\lambda)(2\mu-\lambda)} - \frac{1}{\mu}$$

$$= \frac{4\mu^2-(4\mu^2-\lambda^2)}{\mu(2\mu+\lambda)(2\mu-\lambda)} = \frac{\lambda^2}{\mu(4\mu^2-\lambda^2)}$$

$W_{a_1} > W_{a_2}$

$$\frac{\cancel{\lambda}}{2\cancel{\mu}\cancel{(2\mu-\lambda)}} > \frac{\lambda^{\cancel{2}}}{\cancel{\mu}(2\mu+\lambda)\cancel{(2\mu-\lambda)}} \Rightarrow \frac{1}{2} > \frac{\lambda}{2\mu+\lambda}$$

$$\Rightarrow 2\mu+\lambda > 2\lambda$$

$$\Rightarrow 2\mu > \lambda$$

$W_{a_2} < W_{a_1}$ whenever $\lambda < 2\mu$, since we assumed $\lambda < 2\mu$ for stability in M/M/1

9. Consider the M/M/1 queue w/ impatient customers model as presented in Example 8.9. Give your answers in terms of limiting probabilities P_n, $n \geq 0$

$\lambda_n = \lambda$

$\mu_n = \mu + (n-1)\alpha, \ n \geq 1$

$\pi_s = \frac{\lambda_s}{\lambda} = \frac{\mu(1-P_0)}{\lambda}$ service depart $s = 0$

$\lambda_s = \mu(1-P_0)$

a. What's the avg amt of time that a customer spends in queue?

arrive ~ Poi(λ)
service ~ Exp(μ)
W_Q ~ Exp(α) b4 leave

$\sum_n n\left(\frac{P_n}{\lambda}\right)$

The average amount of time that a customer spends in queue is $\sum_n n\left(\frac{P_n}{\lambda}\right)$

b. If e_n denotes the probability that a customer who finds n others in the system upon arrival will be served, find e_n, $n \geq 0$.

$e_n = P$(customer who finds n others in system upon arrival will be served)

$e_0 = 1$

$e_n = \prod_{i=0}^{n-1} \frac{\mu + i\alpha}{\mu + (1+i)\alpha}$

$L = \frac{\lambda}{\mu - \lambda}$ $W = \frac{L}{\lambda}$

c. Find the conditional probability that served customer found n in system upon arrival. That is, find P(arrival finds n | arrival is served)

$P(n | \text{served}) = \frac{e_n P_n}{\sum_n e_n P_n}$

d. Find avg amt of time spent in queue by customer that's served

$$\sum_{n=1}^{\infty} P(n|\text{served}) \sum_{i=0}^{n-1} \frac{1}{\mu+(1+i)\alpha}$$

e. Find avg amt of time spent in queue by customer that departs before entering service

$W_Q(s)$ = avg time spent in queue by those served

$$= \sum_{n=1}^{\infty} P(n|\text{served}) \sum_{i=0}^{n-1} \frac{1}{\mu+(1+i)\alpha}$$

$W_Q(n)$ = avg time spent in queue by those who aren't

$$\sum_{n>0} n-1\left(\frac{\lambda P_n}{\lambda}\right) = W_Q$$

$$= \sum_n e_n P_n W_Q(s) + (1-\sum_n e_n P_n) W_Q(n)$$