6.3. Birth and death process.

Suppose there are n people in the system.

(i) New arrivals enter system at exponential rate λ_n

(ii) People leave the system at exponential rate μ_n

T_a - time until next arrival

$T_a \sim Exp(\lambda_n) \qquad E\, T_a = \frac{1}{\lambda_n}$

T_B - time until next departure

$T_B \sim Exp(\mu_n) \qquad E\, T_B = 1/\mu_n$

T_a and T_B are independent

Parameters: $\{\lambda_n\}_{n=0}^{\infty}$ — birth rate

$\{\mu_n\}_{n=0}^{\infty}$ — death rate

A birth and death process is a continuous-time MC

States: $\{0, 1, 2, \dots\}$

$v_0 = \lambda_0 \qquad P_{01} = 1$

$v_i = \lambda_i + \mu_i \qquad P_{i,i+1} = \dfrac{\lambda_i}{\lambda_i + \mu_i}, \; i > 0 \qquad P_{i+1,i} = \dfrac{\mu_i}{\lambda_i + \mu_i}, \; i > 0.$

Why? discuss.

$T_\lambda \sim Exp(\lambda_1) \qquad T_\mu \sim Exp(\mu)$

$P(T_\lambda < T_\mu) = \dfrac{\lambda}{\lambda + \mu} \qquad \min(T_\lambda, T_\mu) \sim Exp(\lambda + \mu)$

Ex. 6.2. Poisson process.

Consider a birth and death process for which

$\mu_n = 0 \qquad n \geq 0$

$\lambda_n = \lambda \qquad n \geq 0$

Departures never occur

Time between successive arrivals $\sim Exp(\lambda)$

$$E\, T_i = \frac{1}{\lambda}. \quad \Rightarrow \quad \text{Poisson process.}$$

Ex. (A birth process with linear birth rate).

Consider a population whose members can give birth to new members but cant die.

Each member acts independently of others

T - Time until a member gives a birth $\sim Exp(\lambda)$

$$ET = \frac{1}{\lambda}$$

Let $X(t)$ be a population size at time t

Then $\{X(t), t \geq 0\}$ is a pure birth process

$\lambda_n = n\lambda \qquad n \geq 0$

The pure birth process is called Yulle process.

(G. Yule used it in math theory of evolution).

Ex. A linear growth model with immigration.

$$\left.\begin{array}{ll}\mu_n = n\mu & n \geq 1 \\ \lambda_n = n\lambda + \theta & n \geq 0\end{array}\right]$$ is called a linear growth model with immigration

1) Each individual in the population is assumed to give a birth at an exponential rate λ

2) There is an exponential rate of increase θ of the population due to immigration.

3) Deaths are assumed to occure at an exponential rate μ for each member of a population, so $\mu_n = n\mu$.

Let $X(t)$ be a population size at time t.

$X(0) = i$

$M(t) = E\,X(t)$ $\qquad\qquad M(t) - ?$

We will derive and solve a differential equation.

$$M(t+h) = E(X(t+h)) = E(E(X(t+h) \mid X(t)))$$

We ignor events whose probability is $o(h)$

$$X(t+h) = \begin{cases} X(t)+1 & \text{with prob. } [\theta + X(t)\lambda]h + o(h) \\ X(t)-1 & \text{with prob } X(t)\mu h + o(h) \\ X(t) & \text{with prob } 1-[\theta + X(t)\lambda + X(t)\mu]h + o(h) \end{cases}$$

Poisson process $\quad P(N(t+h) - N(t) = 1) = \lambda h + o(h)$

$\qquad P(N(t+h) - N(t) \geq 2) = o(h)$

Therefore

$$E\left(X(t+h) \mid X(t)\right) = X(t) + \left[\theta + X(t)\lambda - X(t)\mu\right]h + o(h)$$

$$M(t+h) = M(t) + (\lambda-\mu)\, M(t)\, h + \theta h + o(h)$$

$$\frac{M(t+h) - M(t)}{h} = (\lambda-\mu)\, M(t) + \theta + \frac{o(h)}{h}$$

$h \to 0$

$$M'(t) = (\lambda-\mu)\, M(t) + \theta$$

define $h(t) = (\lambda-\mu)\, M(t) + \theta$

$$h'(t) = (\lambda-\mu)\, M''(t)$$

$$\frac{h'(t)}{\lambda-\mu} = h(t)$$

$$\frac{h'(t)}{h(t)} = \lambda-\mu$$

$$\frac{dh}{h} = (\lambda-\mu)\, dt$$

$$\ln\left[h(t)\right] = (\lambda-\mu)t + \ln c$$

$$h(t) = K e^{(\lambda-\mu)t}$$

$$(\lambda-\mu)\, M(t) + \theta = K e^{(\lambda-\mu)t}$$

To find K we use an initial conditions

$M(0) = i$

$$\theta + (\lambda-\mu)\, i = K$$

$$M(t) = \frac{K e^{(\lambda-\mu)t} - \theta}{(\lambda-\mu)} = \frac{(\theta + (\lambda-\mu)i)\, e^{(\lambda-\mu)t}}{\lambda-\mu} - \frac{\theta}{\lambda-\mu}$$

$$M(t) = \frac{\theta}{\lambda-\mu}\left(e^{(\lambda-\mu)t} - 1\right) + i e^{(\lambda-\mu)t}. \qquad \lambda \neq \mu.$$

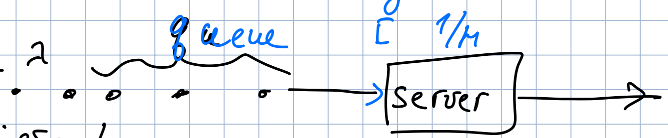
If $\lambda = \mu$

$M'(t) = \theta$

$M(t) = \theta t + c$

Ex. (The queueing system $M/M/1$)

rate λ



arrival

$M/M/1$ queueing system.

customers arrive indep. with rate λ according to poisson process.

service time $T_S \sim Exp(\mu)$

$ET_S = 1/\mu$.

M – Markovian (customers arrive according to a poisson process)

M – Markovian (service time is exponential)

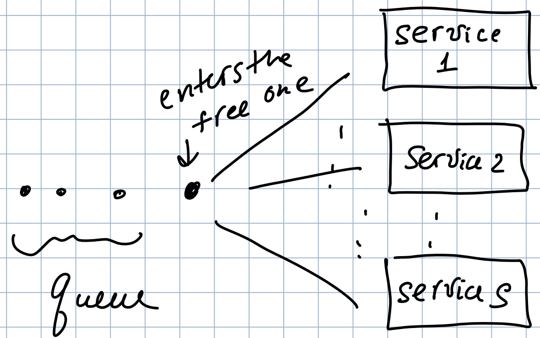
$X(t)$ – # of customers in the system at time t.

$\{X(t), t \geq 0\}$ is a birth and death process.

$\mu_n = \mu \quad n \geq 1$

$\lambda_n = \lambda$

Ex (A Multiserver exponential queueing system).



$$\mu_n = \begin{cases} n\mu & 1 \le n \le s \\ s\mu & n > s \end{cases}$$

$\lambda_n = \lambda \qquad n \ge 0$

Birth and death process.

It is $M/M/s$ queueing model.

General death and birth process.

birth rate $\{\lambda_n\}$

death rate $\{\mu_n\}$

Let $\mu_0 = 0$

T_i – time starting from state i it takes for process to enter state $i+1$.

We will compute $ET_i \quad i \ge 0$ starting from $i=0$.

$T_0 \sim Exp(\lambda_0)$

$ET_0 = \frac{1}{\lambda_0}$

For $i > 0$ $\quad i \to i+1$, $\quad i \to i-1$

Let $I_i = \begin{cases} 1 & \text{if } i \to i+1 \\ 0 & \text{if } i \to i-1 \end{cases}$

$$E\left[T_i \mid I_i = 1\right] = \frac{1}{\lambda_i + \mu_i}$$

$$E\left[T_i \mid I_i = 0\right] = \frac{1}{\lambda_i + \mu_i} + E(T_{i-1}) + E\,T_i$$

$$E\,T_i = E\,E(T_i \mid I_i) = \frac{1}{\lambda_i + \mu_i} P(I_i = 1) +$$

$$+ \left(\frac{1}{\lambda_i + \mu_i} + E(T_{i-1}) + E(T_i)\right) P(I_i = 0) =$$

$$\frac{1}{\lambda_i + \mu_i} + \left[E(T_{i-1}) + E(T_i)\right] \cdot \frac{\mu_i}{\lambda_i + \mu_i}$$

$$E\,T_i - E(T_i) \cdot \frac{\mu_i}{\lambda_i + \mu_i} = \frac{1}{\lambda_i + \mu_i} + \frac{\mu_i}{\lambda_i + \mu_i} E(T_{i-1})$$

$$E\,T_i \left(1 - \frac{\mu_i}{\lambda_i + \mu_i}\right) = \frac{1}{\lambda_i + \mu_i}\left(1 + \mu_i\, E\,T_{i-1}\right)$$

$$\frac{\lambda_i E\,T_i}{\lambda_i + \mu_i} = \frac{1}{\lambda_i + \mu_i}\left(1 + \mu_i \cdot E\,T_{i-1}\right)$$

$$E\,T_i = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} E(T_{i-1}) \qquad E\,T_0 = \frac{1}{\lambda_0}$$

$$E\,T_1 = \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1} \cdot \frac{1}{\lambda_0}$$

$$E\,T_2 = \frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2}\left(\frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1} \cdot \frac{1}{\lambda_0}\right) =$$

$$= \frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2 \cdot \lambda_1} + \frac{\mu_2 \cdot \mu_1}{\lambda_2 \cdot \lambda_1 \cdot \lambda_0} \ldots$$

$i \to j$ $\quad E \cdot$ time - ?

$$E(T_i) + E(T_{i+1}) + \dots + E(T_{j-1})$$

Ex. Death and birth process

$\lambda_i = \lambda$, $\mu_i = \mu$

Find ET_i - ?

$$ET_i = \frac{1}{\lambda} + \frac{\mu}{\lambda} E\, T_{i-1} = \frac{1}{\lambda}\left(1 + \mu E T_{i-1}\right)$$

$$ET_1 = \frac{1}{\lambda}\left(1 + \frac{\mu}{\lambda}\right)$$

$$ET_2 = \frac{1}{\lambda}\left(1 + \mu \cdot \frac{1}{\lambda}\left(1 + \frac{\mu}{\lambda}\right)\right) = \frac{1}{\lambda}\left(1 + \frac{\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2\right)$$

$$E\, T_3 = \frac{1}{\lambda}\left(1 + \mu \cdot \frac{1}{\lambda}\left(1 + \frac{\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2\right)\right) =$$

$$= \frac{1}{\lambda}\left(1 + \frac{\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2 + \left(\frac{\mu}{\lambda}\right)^3\right)$$

$\vdots$

$$E\, T_i = \frac{1}{\lambda}\left(1 + \frac{\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2 + \cdots \left(\frac{\mu}{\lambda}\right)^i\right) =$$

$$= \frac{1}{\lambda} \cdot \frac{1 - \left(\frac{\mu}{\lambda}\right)^{i+1}}{1 - \frac{\mu}{\lambda}} = \frac{1 - \left(\frac{\mu}{\lambda}\right)^{i+1}}{\lambda - \mu}$$

The expected time to reach state j starting at state k

$$E\,(\text{time to go from } k \text{ to } j) = \sum_{i=k}^{j-1} E(T_i) =$$

$$= \frac{j-k}{\lambda - \mu} - \frac{\left(\frac{\mu}{\lambda}\right)^{k+1}}{\lambda - \mu} \quad \frac{1 - (\mu/\lambda)^{j-k}}{1 - \mu/\lambda} \qquad \lambda \neq \mu$$

If $\lambda = \mu$ $\quad E\,T_i = \dfrac{i+1}{\lambda}$

$$E\,(\text{time to go from } k \text{ to } j) = \frac{j(j+1) - k(k+1)}{2\lambda}$$

We can also compute variance

$$Var(T_i) = Var(\underbrace{E(T_i \mid I_i)}) + E\,Var(T_i \mid I_i)$$

$$E\,(T_i \mid I_i = 1) = 1/\lambda_i + \mu_i$$

$$E\,(T_i \mid I_i = 0) = \frac{1}{\lambda_i + \mu_i} + E(T_{i-1}) + E(T_i)$$

$$\underline{E(T_i \mid I_i)} = \frac{1}{\lambda_i + \mu_i} + (1 - I_i)(E(T_{i-1}) + E(T_i))$$

$$Var(E(T_i \mid I_i)) = (E(T_{i-1}) + (E\,T_i))^2\, Var(I_i)$$

I_i	1	0
p	$\dfrac{\lambda_i}{\lambda_i + \mu_i}$	$\dfrac{\mu_i}{\lambda_i + \mu_i}$

— Bernoulli

$$Var(I_i) = p(1-p) = \frac{\lambda_i \mu_i}{(\lambda_i + \mu_i)^2}$$

$$Var(E(T_i \mid I_i)) = (E(T_{i-1}) + E(T_i))^2 \cdot \frac{\lambda_i \mu_i}{\lambda_i + \mu_i}$$

$$Var(T_i \mid I_i = 1) = Var(X_i \mid I_i = 1) = Var(X_i) = \frac{1}{(\lambda_i + \mu_i)^2}$$

X_i — the time until transition from i occurs.

$$Var(T_i \mid I_i = 0) = Var(X_i + \text{time get back to } i + \text{time to reach } i{+}1)$$

$$= Var(X_i) + Var(T_{i-1}) + Var(T_i)$$

$$Var(T_i \mid I_i) = Var(X_i) + (1 - I_i)[Var(T_{i-1}) + Var(T_i)]$$

$$E\left(\mathrm{Var}(T_i \mid I_i)\right) = E\left(\mathrm{Var}(X_i)\right) + E\left[(1-I_i)\left[\mathrm{Var}(T_{i-1}) + \mathrm{Var}\, T_i\right]\right]$$

$$= \frac{1}{(\lambda_i+\mu_i)^2} + \frac{\mu_i}{\lambda_i+\mu_i}\left[\mathrm{Var}(T_{i-1}) + \mathrm{Var}(T_i)\right]$$

$$E\, I_i = \frac{\lambda_i}{\lambda_i+\mu_i} \qquad E(1 - I_i) = 1 - \frac{\lambda_i}{\lambda_i+\mu_i} = \frac{\mu_i}{\lambda_i+\mu_i}$$

$$\mathrm{Var}(T_i) = \frac{1}{(\mu_i+\lambda_i)^2} + \frac{\mu_i}{\mu_i+\lambda_i}\left[\mathrm{Var}(T_{i-1}) + \mathrm{Var}(T_i)\right] +$$

$$+ \frac{\mu_i \lambda_i}{(\mu_i+\lambda_i)^2}\left(E(T_{i-1}) + E\, T_i\right)^2$$

or

$$\mathrm{Var}(T_i) = \frac{1}{\lambda_i(\lambda_i+\mu_i)} + \frac{\mu_i}{\lambda_i}\mathrm{Var}(T_{i-1}) +$$

$$+ \frac{\mu_i}{\mu_i+\lambda_i}\left[E(T_{i-1}) + E\, T_i\right]^2$$

$\mathrm{Var}(T_0) = \frac{1}{\lambda_0^2}$ we can get $\mathrm{Var}(T_i)$

$$\mathrm{Var}(\text{time go from } k \text{ to } j) = \sum_{i=k}^{j-1} \mathrm{Var}(T_i)$$