6.5. Limiting Probabilities.

The probability that a continuous MC will be in state j at time t often converges to a limiting value that independent of initial state

$$P_j := \lim_{t\to\infty} P_{ij}(t) \qquad (\text{or } \pi_j = \lim_{t\to\infty} P_{ij}(t)).$$

where we assume that limit exists and is independent of the initial state.

(forward eq).

$$P_{ij}'(t) = \sum_{k\neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t)$$

$$0 = \lim_{t\to\infty} P_{ij}'(t) = \lim_{t\to\infty}\Big(\sum_{k\neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t)\Big)$$

$$= \sum_{k\neq j} \lim_{t\to\infty}\big(q_{kj} P_{ik}(t) - v_j P_{ij}(t)\big) =$$

$$= \sum_{k\neq j} q_{kj} P_k - v_j P_j$$

$$\lim_{t\to\infty} p'_{ij}(t) = 0$$ (p_{ij} is a bounded function. If p'_{ij} converges then it must be 0)

$$\begin{cases} 0 = \sum_{k\neq j} q_{kj} P_k - v_j P_j \quad \forall j \qquad (1) \\ \sum_j P_j = 1 \end{cases}$$

$$v_j P_j = \sum_{k\neq j} q_{kj} P_k$$

Interpretation.

In any interval $(0,t)$ the # of transitions into state j must equal to within 1 the number of transitions out of state j. Hence, in a long run the rate at which transitions into state j occur must be equal the rate at which transitions out of state j occur.

$v_j \pi_j$ = rate at which process leaves state j

$\sum_{k\neq j} q_{kj} P_k$ = rate at which the process enters state j.

So eq. (1) is just a statement of the equality of the rates at which the process enters and leaves the state j.

(1) is referred to as set of "balance equations".

Matrix form

$$\vec{p}^{\,T} R = 0 \qquad \vec{p} = \begin{pmatrix} p_0 \\ \vdots \\ p_m \end{pmatrix}$$

$$R^T \begin{pmatrix} p_0 \\ \vdots \\ p_m \end{pmatrix} = 0 \qquad \sum_{\forall j} p_j = 1$$

A Sufficien condition for P_j to exist.

1) The Markov Chain is irreducible.

$\forall\ i, j \quad i \longleftrightarrow j$:

$i \longrightarrow j$: starting from state i there is a positive probability of ever being in state j.

2 The Markov Chain is positive recurrent

Starting in any state the mean time

to return to that state is finite

If (1) and (2) holds

p_j is a long run proportion of the time that the process is in State j.

$\vec{p}$, is a stationary distribution

$$\vec{p}^T P(t) = \vec{p}^T$$

Proposition: $\vec{p}$ is a stationary distr. if and only if $\vec{p}^T R = 0$

Proof:

If $\vec{p}^T P(t) = \vec{p}^T$

$$\frac{d}{dt} \vec{p} P(t) = \frac{d}{dt} \vec{p} = 0.$$

$$\frac{d}{dt} \vec{p} P(t) = \frac{d}{dt} \sum_{\forall i} p_i P_{ij}(t) =$$

$$= \sum_{\forall i} p_i \frac{d}{dt} P_{ij}(t) = \sum_{\forall i} p_i \sum_K P_{ik}(t) R_{kj} =$$

$$= \sum_k \underbrace{\sum_i p_i P_{ik}(t)}_{p_k} R_{kj} = \sum_k p_k R_{kj}$$

$$0 = \sum_k p_k R_{kj} \Rightarrow \qquad \vec{p} R = 0.$$

Assume $\vec{p} R = 0$

$$\frac{d}{dt}\left(\sum_i p_i \; P_{ij}(t)\right) = \sum_i p_i \; P'_{ij}(t) =$$

$$= \sum_i p_i \sum_k R_{ik} P_{kj}(t) =$$

$$= \sum_k \underbrace{\sum_i p_i R_{ik}}_{0} P_{kj}(t) = 0$$

$\boxtimes$.

Theorem. If a continuous Markov chain $\{X(t), \; t \geq 0\}$ is irreducible and has a stationary distribution π then

$$\lim_{t \to \infty} P_{ij}(t) = p_j.$$

Ex. 1 (Weather chain).

Example 4.13 (L.A. Weather Chain). There are three states: 1 = sunny, 2 = smoggy, and 3 = rainy. The weather stays sunny for an exponentially distributed number of days with mean 3, then becomes smoggy. It stays smoggy for an exponentially distributed number of days with mean 4, then rain comes. The rain lasts for an exponentially distributed number of days with mean 1, then sunshine returns. Remembering that for an exponential the rate is 1 over the mean, the verbal description translates into the following Q-matrix:

	1	**2**	**3**
1	$-1/3$	$1/3$	0
2	0	$-1/4$	$1/4$
3	1	0	-1

The relation $\pi Q = 0$ leads to three equations:

$$\begin{aligned} -\tfrac{1}{3}\pi_1 \qquad\qquad + \pi_3 &= 0 \\ \tfrac{1}{3}\pi_1 - \tfrac{1}{4}\pi_2 \qquad\quad &= 0 \\ \tfrac{1}{4}\pi_2 - \pi_3 &= 0 \end{aligned}$$

$-\frac{1}{3} p_1 + p_3 = \cdot$

Adding the three equations gives $0 = 0$ so we delete the third equation and add $\pi_1 + \pi_2 + \pi_3 = 1$ to get an equation that can be written in matrix form as

$$\begin{pmatrix}\pi_1 & \pi_2 & \pi_3\end{pmatrix} A = \begin{pmatrix}0 & 0 & 1\end{pmatrix} \quad \text{where} \quad A = \begin{pmatrix} -1/3 & 1/3 & 1 \\ 0 & -1/4 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

here $\pi_i = p_i$

This is similar to our recipe in discrete time. To find the stationary distribution of a k state chain, form A by taking the first $k-1$ columns of Q, add a column of 1's and then

$$\begin{pmatrix}\pi_1 & \pi_2 & \pi_3\end{pmatrix} = \begin{pmatrix}0 & 0 & 1\end{pmatrix} A^{-1}$$

$\pi_i = p_i$

i.e., the last row of A^{-1}. In this case we have

$$\pi(1) = 3/8, \quad \pi(2) = 4/8, \quad \pi(3) = 1/8$$

To check our answer, note that the weather cycles between sunny, smoggy, and rainy spending independent exponentially distributed amounts of time with means 3, 4, and 1, so the limiting fraction of time spent in each state is just the mean time spent in that state over the mean cycle time, 8.

Limiting probabilities for a birth and death process.

$$v_j P_j = \sum q_{kj} P_k \qquad \text{or} \quad \pi R = 0$$

q_{ij} -?

$$v_0 = \lambda_0, \quad v_i = \lambda_i + \mu_i \qquad P_{01} = 0$$

$$P_{i\,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i} \qquad P_{i\,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}$$

$$q_{kj} = v_k P_{kj} \qquad q_{k,k+1} = \lambda_k$$

$$q_{k\,k-1} = \mu_k$$

$$R = \begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & 4 & \dots & i-1 & i & i+1 & \dots \\ \hline 0 & -\lambda_0 & \lambda_0 & 0 & 0 & 0 & & 0 & & & \dots \\ 1 & \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & 0 & & 0 & & & \\ 2 & 0 & \mu_2 & -(\lambda_2+\mu_2) & \mu_2 & 0 & & 0 & & & \\ 3 & 0 & 0 & 0 & \mu_3 & -(\lambda_3+\mu_3) & \dots & 0 & & & \\ 4 & & & & & & & & & & \\ \vdots & & & & & & & & & & \\ i & 0 & 0 & 0 & 0 & & & \mu_i & -(\lambda_i+\mu_i) & \lambda_i & \end{array}$$

$$P^T = (P_0, P_1, P_2 \dots)$$

$$-P_0 \lambda_0 + \mu_1 P_1 = 0 \Rightarrow \qquad \lambda_0 P_0 = \mu_1 P_1$$

$$+ \quad \lambda_0 P_0 - (\lambda_1 + \mu_1) P_1 + \mu_2 P_2 = 0 \qquad (\lambda_1 + \mu_1) P_1 = \mu_2 P_2 + \lambda_0 P_0$$

$+$

$$+\quad \lambda_1 P_1 - (\lambda_2+\mu_2)P_2 + \mu_3 P_3 = 0 \quad \Big| \quad (\lambda_2+\mu_2)P_2 = \lambda_1 P_1 + \mu_3 P_3$$

$$\vdots$$

$$\lambda_{n-1}P_{n-1} - (\lambda_n+\mu_n)P_n + \mu_{n+1}P_{n+1} = 0$$

$$\begin{cases} \mu_1 P_1 - \lambda_0 P_0 = 0 \\ \mu_2 P_2 - \lambda_1 P_1 = 0 \\ \vdots \\ \mu_{n+1}P_{n+1} - \lambda_n P_n = 0 \end{cases} \qquad \begin{cases} p_1 = \dfrac{\lambda_0}{\mu_1} p_0 \\ p_2 = \dfrac{\lambda_1}{\mu_2} p_1 = \dfrac{\lambda_1\lambda_0}{\mu_2\mu_1} p_0 \\ \vdots \\ p_k = \dfrac{\lambda_{k-1}}{\mu_k} p_{k-1} = \dfrac{\lambda_{k-1}\lambda_{k-2}\cdots\lambda_0}{\mu_k\mu_{k-1}\mu_{k-2}\cdots\mu_1} p_0 \\ \vdots \end{cases}$$

$$\sum_{k=0}^{\infty} P_k = 1$$

$$P_0 + \frac{\lambda_0}{\mu_1}P_0 + \frac{\lambda_1\lambda_0}{\mu_2\mu_1}P_0 + \dots + \frac{\lambda_{k-1}\cdots\lambda_0}{\mu_k\cdots\mu_1}P_0 + \dots = 1$$

$$P_0 + P_0 \cdot \underbrace{\sum_{k=1}^{\infty} \frac{\lambda_{k-1}\cdots\lambda_0}{\mu_k\cdots\mu_1}}_{=c} = 1$$

$$P_0(1+c) = 1 \qquad p_0 = \frac{1}{1+c} = \frac{1}{1+\sum_{k=1}^{\infty}\frac{\lambda_{k-1}\cdots\lambda_0}{\mu_k\cdots\mu_1}}$$

$$n = \frac{\lambda_0\cdots\lambda_{n-1}}{\mu_1\cdots\mu_n(1+c)} \qquad n \geq 1$$

$$c = \sum_{k=1}^{\infty} \frac{\lambda_{k-1}\cdots\lambda_0}{\mu_k\cdots\mu_1} < \infty$$

In the multiserver exponential queueing system

$$C = \sum_{n=s+1}^{\infty} \frac{\lambda^n}{(s\mu)^n} < \infty \iff \frac{\lambda}{s\mu} < 1$$

For linear grows model

$$\sum_{n=1}^{\infty} \frac{\theta(\theta+\lambda)\cdots(\theta+(n-1)\lambda)}{n!\,\mu^n} < \infty.$$

Using the ratio test $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1 \Rightarrow$

$$\lim_{n\to\infty} \frac{\theta(\theta+\lambda)\cdots(\theta+n\lambda)}{(n+1)!\,\mu^{n+1}} \cdot \frac{n!\,\mu^n}{\theta(\theta+\lambda)\cdots(\theta+(n-1)\lambda)} =$$

$$= \lim_{n\to\infty} \frac{\theta+n\lambda}{(n+1)\mu} = \frac{\lambda}{\mu} < 1$$

$$\Rightarrow \boxed{\lambda < \mu}$$

Ex. M/M/1

$\lambda_n = \lambda \qquad \mu_n = \mu$

$$P_n = \frac{(\lambda/\mu)^n}{1+\sum_{n=1}^{\infty}\left(\frac{\lambda}{\mu}\right)^n} = \left(\frac{\lambda}{\mu}\right)^n (1-\lambda/\mu)_{n\geq 0}.$$

$\lambda/\mu < 1 \Rightarrow \lambda < \mu$

If customers arrive at a faster rate than they can be served ($\lambda > \mu$) then the queueing size will go to ∞.

$\lambda = \mu$ – behaves like the symmetric random walk which is null recurrent and has no limiting probabilities.