# App to Queuing Theory

## Notation

GI
general input
arrivals to service

G
service time

1
server

$t_i$
arrival
iid $F$ distr
$\mathbb{E}(t_i) = \frac{1}{\lambda}$


$N(t)$
# of arrivals by time $t$

**Thm I**

$$\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{\mathbb{E}(t_i)} = \lambda$$

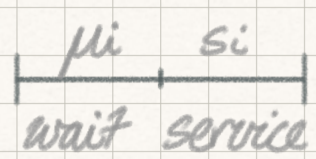
prob 1

Let $G$ be service time

$s_i$
service time

$\mathbb{E}(s_i) = \frac{1}{\mu}$

$(s_1, s_2, \dots) \underset{iid}{\sim} G$ distr



# Thm II

Suppose $\lambda < \mu$. If queue starts w/ some finite # of customers $(k \geq 1)$. Then it'll be empty out w/ prob 1. Furthermore, lim frac of time, service is busy is $\leq \frac{\lambda}{\mu}$

## Proof

$T_n = \tau_1 + \cdots \tau_n$

time of nth arrival

By strong law of large #s,

$$\frac{T_n}{n} \xrightarrow[n\to\infty]{} \frac{1}{\lambda} \text{ w/ prob 1}$$

$S_n = s_1 + \cdots + s_n$

time to serve n customers

$Z_0$

time of service for customers that arrived before service is open

$$Z_0 = s_1 + \cdots + s_k$$

$$\frac{Z_0 + S_n}{n} \to \frac{1}{\mu} \text{ w/ prob 1}$$

$$\frac{Z_0 + S_n}{n+k}\left(\frac{n+k}{n}\right) \to \frac{1}{\mu} \quad (1)$$

$$\frac{\frac{Z_0 + S_n}{n}}{\frac{T_n}{n}} \to \frac{\frac{1}{\mu}}{\frac{1}{\lambda}} = \frac{\lambda}{\mu}$$

Remark

Actual time spend working $[0, T_n]$

$Z_0 + S_n - Z_n$

$Z_n$ = amt of work in system @ time $T_n$

### 3.2.2 Cost Equation

Notation

$L$
avg # of cust in system

$L_Q$
avg # of cust in queue

$W$
avg amt of time cust spend in system

$W_Q$
avg amt of time cust spend in queue

$\lambda_a$
avg $\Delta$ of arriving

Let $X_s$ be # of cust in system @ time s

$$L = \lim_{t\to\infty} \frac{1}{t}\int_0^t X_s \, ds$$

$$W = \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} W_m$$

$W_m$
amt of time mth customer spends in system

$N_a(t)$
# of cust that arrive by time t & enter system

$$\lambda_a = \lim_{t\to\infty} \frac{N_a(t)}{t} \quad \text{w/ prob 1}$$

Thm 3: Little's formula

$L = \lambda_a W$, why is it true?

Example

\$1 each customer pays $\forall$ min in system

$L$ = customer pay \$L

In long run system will earn \$4/min

If cust pays for their entire time when they arrive, system will earn $\lambda_a W$

$4 = \lambda_a W$

$\lambda_a = \Delta$ of arrival

Example: Waiting Time in queue GI/G/1

$W = W_a + \mathbb{E}(S_i)$

$W_a = W - \mathbb{E}(S_i)$

| # | 0 | > 0 |
|---|---|---|
| $p$ | $\pi_0$ | $1-\pi_0$ |

$L_a = \lambda_a W_a = L - 1 + \pi_0$

$\pi_0 = L_a - (L-1) = 1 + \lambda_a(W_a - W) = 1 - \lambda_a \mathbb{E}(S_i)$

$= 1 - \dfrac{\lambda_a}{\mu}$