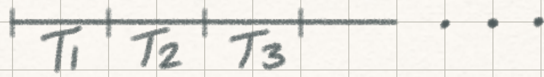
Ch 7.1: Renewal Theory

$\{N(t), t \geq 0\}$, Poi process

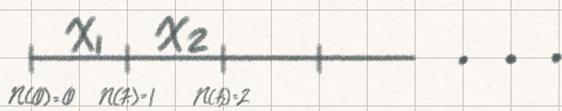


$T_i \sim Exp(\lambda)$

Generalization

Let $\{N(t), t \geq 0\}$ be counting process

X_n be time bet events $(n-1)$ & n



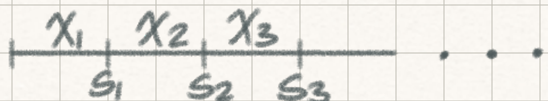
Define: If sequence of non(-) rv $X_1, X_2, \ldots$ is iid, then counting process $\{N(t), t \geq 0\}$ is said to be renewal process

Let F be distr of F

Ex: We have ∞ supply of lightbulbs. We use single bulb @ time. When bulb is failing, we immediately replace it w/ new one

$N(t)$ = # of bubs you replaced by time t

$\{N(t), t \geq 0\}$ = renewal process



Let $X_1, \ldots, X_n$ be interarrival time

$S_0 = 0$

$S_1 = X_1$

$S_2 = X_1 + X_2$

$S_n = \sum_{i=1}^{n} X_i$ = time until n^{th} renewal

$f_X(x)$ = pdf of X

$$f_{X_1+X_2}(s) = \int_{-\infty}^{\infty} f(t)f(s-t)\,dt$$

$M_{S_n}(t) = [M_X(t)]^n$ = mgf

$$M_X(t) = \mathbb{E}e^{ixt} = \int_{-\infty}^{\infty} e^{ixt} f(x)\,dx = \int_{-\infty}^{\infty} e^{-ixt} \varphi(t)\,dt$$

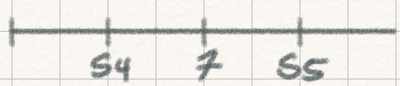
For distr F (cdf of X_i)

$$F(0) = \begin{cases} P(X_n = 0) < 1 \\ P(X_n \le 0) < 1 \end{cases}$$

$\mu = \mathbb{E}(X_n) > 0$

let t be fixed, can $N(t) = \infty$?

$N(t) = \max\{n : S_n \le t\}$



$N(t) = 4$

By strong law of large #s

$\frac{S_n}{n} \to \mu, \; n \to \infty$

$\frac{X_1 + \dots + X_n}{n} \to \mu, \quad \bar{X}_n \to \mu$

w/ prob = 1 (almost surely)

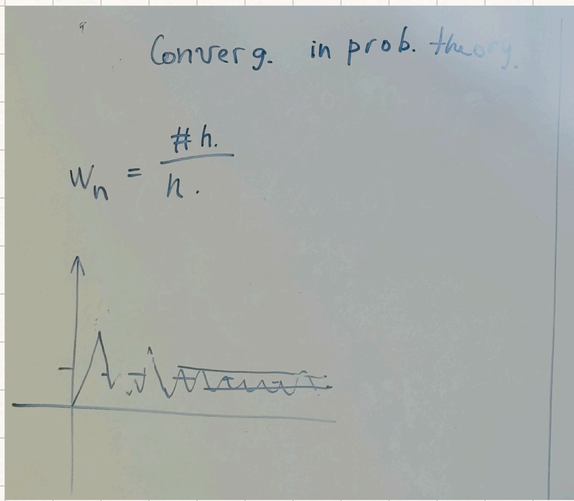
$N(t) = \max\{n : S_n \le t\}$

$N(t) < \infty \; \forall t$ w/ prob 1

$N(\infty) = \lim_{t \to \infty} N(t) = \infty$

$P(N(\infty) < \infty) = P(X_n = \infty \text{ for some } n) = P(\cup_{n=1}^{\infty} X_n = \infty)$

$\le \sum_{n=1}^{\infty} P(X_n=\infty)=0$



$X_n \xrightarrow[n\to\infty]{d} X$

convergence in distr

In distr

$\lim_{n\to\infty} F_{X_n}(x) = F(x)$

$\overline{X_n} \xrightarrow[n\to\infty]{P} \mu$

weak law of large #

$S_n = \sum_{i=1}^{n} X_i$

$\frac{S_n - n\mu}{\sqrt{n}\,\sigma} \xrightarrow{d} Z$

$F_{\frac{S_n-n\mu}{\sqrt{n}\,\sigma}}(x) \to F_Z(x)$

$\overrightarrow{X_n} \xrightarrow{as} \mu, \mu\to\infty$

(almost surely w/ prob 1)

strong law of large #s

Consider seq of rv $(X_1,\dots,X_n,\dots)$

S = sample space

$s \in S$

$X_1(s)\dots X_n(s)\dots$

Seq of rv's is said to converge to rv X w/ prob 1 (almost surely) if

$P(s: \lim_{n\to\infty} X_n(s)=X(s))=1$

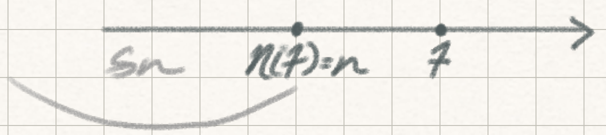
$\lim_{m\to\infty} P\{|X_n-X|<\varepsilon \ \forall n\ge m\}=1$

Convergence in Prob

Seq of rv $(X_1, X_2, \dots)$ converges to rv X in prob if $\forall \varepsilon > 0$ $\lim_{n \to \infty} P(|X_n - X| < \varepsilon) = 1$

Ch 7.2: Distr of $N(t)$

$N(t) \ge n \iff S_n \le t$



$P(\overset{C}{N(t) = n}) = P(\overset{A}{N(t) \ge n}) - P(\overset{B}{N(t) \ge n+1})$

$A = B \cup C$ $\quad = P(S_n \le t) - P(S_{n+1} \le t)$

$P(N(t) \ge n+1) + P(N(t) = n) = P(N(t) \ge t)$

$(X_1, \dots, X_n, \dots)$ are iid

$S_n = \sum_{i=1}^{n} X_n$ $\quad F_n$

$P(N(t) = n) = F_n(t) - F_{n+1}(t)$