6.4. The transition probab. matrix.

Let $\underbrace{P_{ij}}(t) = P\{X(t+s) = j \mid X(s) = i\}$

$$\mathbf{P}(t) = \left(\underbrace{P_{ij}(t)}\right)$$

P_{ij} $\quad v_i$ - rate at which process makes a transition when in state i $\quad \boxed{q_{ij} = v_i P_{ij}}$

Backward Kolmogorov equation.

$$P_{ij}'(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t) \qquad (1)$$

$$\mathbf{P}(t) = \left(P_{ij}(t)\right) \qquad \mathbf{R} = \left(R_{ij}\right)$$

$$R_{ij} = \begin{cases} q_{ij} & i \neq j \\ -v_i & i = j \end{cases}$$

$$(1) \qquad \mathbf{P}'(t) = \mathbf{R}\,\mathbf{P}(t)$$

$$\mathbf{P}(t) = e^{Rt}$$

Forward Kolmogorov equation.

$$P_{ij}'(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_i P_{ij}$$

$$(2) \qquad \mathbf{P}'(t) = \mathbf{P}(t)\mathbf{R}$$

Ex. Pure birth process.

$v_i = \lambda_i$

$P_{01} = 1 \qquad P_{i,i+1} = 1 \qquad P_{ij} = 0 \quad i \neq i+1.$

$$P_{ij}'(t) = \lambda_i P_{i+1,i}(t) - \lambda_i P_{ij}(t)$$

Backward Kolmogorov eq.

Forward Kolmogorov eq.

$$P_{ij}'(t) = \lambda_{j-1} P_{i,j-1} - \lambda_j P_{ij}(t) \qquad (3)$$

$P_{ij} = 0 \qquad j < i$

(3) $$\begin{cases} P_{ii}'(t) = -\lambda_i P_{ii}(t) \\ P_{ij}'(t) = \lambda_{i-1} P_{i,j-1}(t) - \lambda_j P_{ij}(t) \quad j \geq i+1 \end{cases}$$

Proposition. For a pure birth process

$$P_{ii}(t) = e^{\lambda_i t}$$

$$P_{ij}(t) = \lambda_{j-1} e^{-\lambda_j t} \int_0^t e^{\lambda_j s} P_{i,j-1}(s)\, ds \quad j \geq i+1$$

Proof. $P_{ii}'(t) = -\lambda_i P_{ii}(t)$

$$\frac{d P_{ii}(t)}{dt} = -\lambda_i P_{ii}(t)$$

$$\int \frac{d P_{ii}(t)}{P_{ii}(t)} = \int -\lambda_i \, dt$$

$$\ln P_{ii}(t) = -\lambda_i t + c \qquad c = \underbrace{\ln k}$$

$$\ln P_{ii}(t) = \ln e^{-\lambda_i t + c} = \ln e^{-\lambda_i t} \cdot \underbrace{e^{\ln k}}$$

$$P_{ii}(t) = k e^{-\lambda_i t}.$$

$$P_{ii}(0) = 1 \qquad k = 1$$

$$P_{ii}(t) = e^{-\lambda_i t}$$

$$P_{ij}'(t) = \lambda_{j-1} P_{ij-1}(t) - \lambda_j P_{ij}(t)$$

$$P_{ij}'(t) + \lambda_j P_{ij}(t) = \lambda_{j-1} P_{ij-1}(t) \quad \Big| \times e^{\lambda_j t}$$

$$\underbrace{e^{\lambda_j t}\left[P_{ij}'(t) + \lambda_j P_{ij}(t)\right]} = e^{\lambda_j t} \lambda_{j-1} \cdot P_{ij-1}(t)$$

$$\frac{d}{dt}\left[e^{\lambda_j t} P_{ij}(t)\right] = e^{\lambda_j t} \lambda_{j-1} P_{ij-1}(t)$$

$$d\left(e^{\lambda_j t} P_{ij}(t)\right) = e^{\lambda_j t} \underbrace{\lambda_{j-1}} P_{ij-1}(t)\, dt$$

$$\underbrace{e^{\lambda_j t}} P_{ij}(t) = \lambda_{j-1} \int_0^t e^{\lambda_j s} P_{ij-1}(s)\, ds.$$

$$P_{ij}(t) = \lambda_{j-1} e^{-\lambda_j t} \int_0^t e^{-\lambda_j s} P_{ij-1}(s)\, ds. + C$$

$$P_{ij}(0) = 0 \qquad C = 0$$

$$P_{ij}(t) = \lambda_{j-1} e^{-\lambda_j t} \int_0^t e^{-\lambda_j s} P_{ij-1}(s)\, ds$$

Ex. The backward equation for the birth and death process.

$v_0 = \lambda_0$

$v_i = \lambda_i + \mu_i$

$$P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i} \qquad P_{ij} = 0 \quad i \neq i+1 \text{ or } i \neq i-1.$$

$$P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}$$

$$P_{0j}'(t) = \lambda_0 P_{1j}(t) - \lambda_0 P_{0j}(t)$$

$$P_{ij}(t) = (\lambda_i + \mu_i)\left(\frac{\lambda_i}{\lambda_i + \mu_i} P_{i+1,j}(t) + \frac{\mu_i}{\lambda_i + \mu_i} P_{i-1,j}(t)\right] - (\lambda_i + \mu_i) P_{ij}(t)$$

$$P_{0j}'(t) = \lambda_0 \left(P_{1j}(t) - P_{0j}(t)\right)$$

$$P_{ij}'(t) = \lambda_i P_{i+1,j}(t) - \mu_i P_{i-1,j}(t) + (\lambda_i + \mu_i) P_{ij}(t).$$

Forward equation for Birth and Death process.

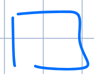
$$P_{i0}'(t) = \sum_{k \neq 0} q_{k0} P_{ik}(t) - \lambda_0 P_{i0}(t) =$$

$$= \mu_1 P_{i1}(t) - \lambda_0 P_{i0}(t)$$

$$P_{ij}'(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - (\lambda_j + \mu_j) P_{ij}(t) =$$

$$= \lambda_{j-1} P_{i,j-1}(t) + \mu_{j+1} P_{i,j+1}(t) - (\lambda_j + \mu_j) P_{ij}.$$

Ex. 6.11. A continuous MC consisting of two states.

 works $\quad T_w \sim \mathrm{Exp}(\lambda)$

$T_r \sim \mathrm{Exp}(\mu)$

States: w, r.

$P(X(t)=w \mid X(0)=w)$ – ?

State 0 – w

1 – r

$\lambda_0 = \lambda$

$\lambda_i = 0 \quad i \neq 0$

$\mu_1 = \mu$

$\mu_i = 0 \quad i \neq 1$

$P_{00}(10)$ – ?

$$\mathbf{P}(t) = \begin{pmatrix} p_{00}(t) & p_{01}(t) \\ p_{10}(t) & p_{11}(t) \end{pmatrix} \qquad \mathbf{R} = \begin{pmatrix} -\nu_0 & q_{01} \\ q_{10} & -\nu_1 \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

$$\mathbf{P}'(t) = \mathbf{R}\,\mathbf{P}(t)$$

$$\begin{pmatrix} p_{00}'(t) & p_{01}'(t) \\ p_{10}'(t) & p_{11}'(t) \end{pmatrix} = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix} \begin{pmatrix} p_{00}(t) & p_{01}(t) \\ p_{10}(t) & p_{11}(t) \end{pmatrix}$$

$$p_{00}'(t) = -\lambda p_{00}(t) + \lambda p_{10}(t) \quad \Big| \times \mu$$
$$p_{10}'(t) = \mu p_{00}(t) - \mu p_{10}(t) \quad \Big| \times \lambda \quad +$$

$$\mu P_{00}'(t) + \lambda P_{10}'(t) = 0$$

$$\forall \quad \mu P_{00}(t) + \lambda P_{10}(t) = c$$

$$P_{00}(0) = 1 \qquad P_{10}(0) = 0$$

$$\mu \cdot 1 + \lambda \cdot 0 = c \quad \Rightarrow \quad \mu = c$$

$$\mu P_{00}(t) + \lambda P_{10}(t) = \mu$$

$$\lambda P_{10}(t) = \mu(1 - P_{00}(t))$$

$$P_{00}'(t) = \mu(1 - P_{00}(t)) - \lambda P_{00}(t)$$

$$P_{00}'(t) = \mu - (\mu + \lambda) P_{00}(t)$$

Let $h(t) = P_{00}(t) - \dfrac{\mu}{\mu+\lambda}$

$$h'(t) = P_{00}'(t)$$

$$h'(t) = \mu - (\mu+\lambda)\cdot\left(h(t) + \frac{\mu}{\mu+\lambda}\right)$$

$$h'(t) = -(\mu+\lambda) h(t)$$

$$\frac{dh}{dt} = -(\mu+\lambda) h(t)$$

$$\frac{dh}{h} = -(\mu+\lambda)\,dt$$

$$\ln h = -(\mu+\lambda)t + c \qquad c = \ln k$$

$$h(t) = k\,e^{-(\mu+\lambda)t}$$

$$P_{00}(t) - \frac{\mu}{\mu+\lambda} = k\,e^{-(\mu+\lambda)t}$$

$$P_{00}(t) = k\,e^{-(\mu+\lambda)t} + \frac{\mu}{\mu+\lambda}$$

$$t = 0$$

$$P_{00}(0) = 1$$

$$1 = k + \frac{\mu}{\mu+\lambda} \qquad k = 1 - \frac{\mu}{\mu+\lambda} = \frac{\lambda}{\mu+\lambda}$$

$$P_{00}(t) = \frac{\lambda}{\lambda+\mu}\,e^{-(\mu+\lambda)t} + \frac{\mu}{\mu+\lambda}$$

$$P_{10}(t) = \frac{\mu}{\mu+\lambda} - \frac{\mu}{\mu+\lambda}\,e^{-(\mu+\lambda)t}$$

$$P_{00}(10) = \frac{\lambda}{\mu+\lambda}\,e^{-10(\mu+\lambda)} + \frac{\mu}{\lambda+\mu}.$$