1. There are 2 tennis courts. Pairs of players arrive @ rate 3/hr & play for exp distr amt of time w/ $\mu = 1$hr. If there are already 2 pairs of players waiting, new arrivals will leave. Find the stationary distr for # of courts occupied (Use detailed balance equation)

$\lambda = 3/hr$
$\mu = 1hr$

$\{0, 1, 2\}$

0 = no court
1 = 1 court
2 = 2 court

$\pi_0 + \pi_1 + \pi_2 = 1$

$P_i q_{ij} = P_j q_{ji}$

$P_0 q_{01} = P_1 q_{10} \Rightarrow P_0 \lambda_0 = P_1 \mu_1$

$\Rightarrow P_1 = \frac{\lambda_0}{\mu_1} P_0 = 3P_0$

$P_0 + P_1 + P_2 = 1$

$P_0 + 3P_0 + 9P_0 = 1$

$P_1 q_{12} = P_2 q_{21} \Rightarrow P_1 \lambda_1 = P_2 \mu_2$

$P_0(1+3+9) = 1$

$\Rightarrow P_2 = \frac{\lambda_1}{\mu_2} P_1 = 3(3P_0)$

$P_0(13) = 1$

$= 9P_0$

$P_0 = \frac{1}{13}$ $P_1 = \frac{3}{13}$ $P_2 = \frac{9}{13}$

X_t = # of pairs

$\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$

2 tennis courts $\Rightarrow$ 4 pairs $\{0, 1, 2, 3, 4\}$

$P_i q_{ij} = P_j q_{ji}$

$\mu_1 = 1$
$\mu_2 = \mu_3 = \mu_4 = 2$

$P_0 q_{01} = P_1 q_{10} \Rightarrow P_0 \lambda_0 = P_1 \mu_1 \Rightarrow P_1 = \frac{\lambda_0}{\mu_1} P_0 = \frac{3}{1} P_0 = \frac{24}{203}$

$P_1 \lambda_1 = P_2 \mu_2 \Rightarrow P_2 = \frac{\lambda_1}{\mu_2} P_1 = \frac{3}{2}(3P_0)$

$= \frac{9}{2} P_0 = \frac{36}{203} P_0$

$P_2 \lambda_2 = P_3 \mu_3 \Rightarrow P_3 = \frac{\lambda_2}{\mu_3} P_0 = \frac{3}{2}\left(\frac{9}{2} P_0\right)$

$$= \frac{27}{4} P_0 = \frac{54}{203}$$

$$P_3 \lambda_3 = P_4 \mu_4 \Rightarrow P_4 = \frac{\lambda_3}{\mu_4} P_3 = \frac{3}{2}\left(\frac{27}{4} P_0\right)$$

$$= \frac{81}{8} P_0 = \frac{81}{203}$$

$$P_0 + P_1 + P_2 + P_3 + P_4 = 1$$

$$\Rightarrow P_0 + 3P_0 + \frac{9}{2} P_0 + \frac{27}{4} P_0 + \frac{81}{8} P_0 = 1$$

$$\Rightarrow 8P_0 + 24P_0 + 36P_0 + 54P_0 + 81P_0 = 8$$

$$\Rightarrow 32P_0 + 90P_0 + 81P_0 = 8$$

$$\Rightarrow 122P_0 + 81P_0 = 8 \Rightarrow 203P_0 = 8$$

$$\Rightarrow P_0 = \frac{8}{203}$$

2. Computer lab has 3 laser printers & 5 toner cartridges. Each machine requires one toner cartridge, which lasts for exp distr amt of time w/ $\mu = 6$ days. When toner cartridge is empty, it's sent to repairman, who takes exp amt of time w/ $\mu = 1$ day to refill it.

a. Compute stationary distr (Use detailed balance equation)

$\boxed{\begin{array}{l} 0 = \text{no printer} \\ 1 = 1 \text{ printer} \\ 2 = 2 \text{ printers} \\ 3 = 3 \text{ printers} \end{array}}$

$\mu = 6$ days work

$\mu = 1$ day refill

exp distr

5 carts, 3 prints

$P_i q_{ij} = P_j q_{ji}$

$$P_0 q_{01} = P_1 q_{10} \Rightarrow P_0 \lambda_0 = P_1 \mu_1 \Rightarrow P_0\left(\frac{5}{6}\right) = P_1\left(\frac{1}{6}\right)$$

0p → 1p, 1 toner; 1p → 0p, no toner

$$\Rightarrow P_1 = 5P_0 = \frac{5}{86}$$

$$P_1 q_{12} = P_2 q_{21} \Rightarrow P_1 \lambda_1 = P_2 \mu_2 \Rightarrow P_1\left(\frac{4}{6}\right) = P_2\left(\frac{1}{6}\right)$$

1p → 2p, 2 toner; 2p → 1p, 1 toner

$$\Rightarrow P_2 = 4P_1 = 20P_0 = \frac{10}{43}$$

$$P_2 q_{23} = P_3 q_{32} \Rightarrow P_2 \lambda_2 = P_3 \mu_3 \Rightarrow P_2\left(\frac{3}{6}\right) = P_3\left(\frac{1}{6}\right)$$

2p → 3p, 3 toner; 3p → 2p, 2 toner

$$\Rightarrow P_3 = 3P_2 = 60P_0 = \frac{30}{43}$$

$$P_0 + P_1 + P_2 + P_3 = 1 \Rightarrow P_0 + 5P_0 + 20P_0 + 60P_0 = 1$$

$$\Rightarrow 6P_0 + 80P_0 = 1 \Rightarrow 86P_0 = 1$$

$$\Rightarrow P_0 = \frac{1}{86}$$

$$\begin{array}{c|cccc|}
 & 0 & 1 & 2 & 3 \\
\hline
0 & -1 & \frac{1}{6} & \frac{2}{6} & \frac{3}{6} \\
1 & -\frac{3}{6} & 0 & \frac{1}{6} & \frac{2}{6} \\
2 & -\frac{2}{6} & -\frac{1}{6} & 0 & \frac{3}{6} \\
3 & -\frac{3}{6} & -\frac{2}{6} & -\frac{1}{6} & 1
\end{array}
\begin{array}{c} \\ = 0 = \\ = 0 = \\ = 0 = \\ = 0 = \end{array}
\begin{array}{|cccc|c}
3 & 2 & 1 & 0 & \\
\hline
\frac{1}{6} & -1 & \frac{5}{6} & 0 & 0 \\
\frac{1}{6} & \frac{4}{6} & 0 & -\frac{5}{6} & 1 \\
\frac{3}{6} & 0 & \frac{1}{6} & -\frac{4}{6} & 2 \\
0 & \frac{1}{6} & \frac{2}{6} & -\frac{3}{6} & 3
\end{array}$$

W/ balance equations in mind

???

0 = no cart
1 = 1 cart
2 = 2 cart
3 = 3 cart
4 = 4 cart
5 = 5 cart

$\lambda_i = 1,\ i < 5$ cartridges

$\mu_i = \frac{i}{6},\ i \leq 3$

$\mu_i = \frac{3}{6},\ i \geq 3$

$$p_0 q_{01} = p_1 q_{10} \Rightarrow p_0 \lambda_0 = p_1 \mu_1$$

$$\Rightarrow p_1 = \frac{\lambda_0}{\mu_1} p_0 = \frac{1}{1/6} p_0 = 6p_0 = \frac{6}{277}$$

$$p_1 q_{12} = p_2 q_{21} \Rightarrow p_1 \lambda_1 = p_2 \mu_2$$

$$\Rightarrow p_2 = \frac{\lambda_1}{\mu_2} p_1 = \frac{6}{2}(6p_0) = 18p_0 = \frac{18}{277}$$

$$p_2 q_{23} = p_3 q_{32} \Rightarrow p_2 \lambda_2 = p_3 \mu_3$$

$$\Rightarrow p_3 = \frac{\lambda_2}{\mu_3} p_2 = \frac{6}{3}(18p_0) = 36p_0 = \frac{36}{277}$$

$$p_3 q_{34} = p_4 q_{43} \Rightarrow p_3 \lambda_3 = p_4 \mu_4$$

$$\Rightarrow p_4 = \frac{\lambda_3}{\mu_4} p_3 = \frac{1}{1/2}(36p_0) = 72p_0 = \frac{72}{277}$$

$$p_4 q_{45} = p_5 q_{54} \Rightarrow p_4 \lambda_4 = p_5 \mu_5$$

$$\Rightarrow p_5 = \frac{\lambda_4}{\mu_5} p_4 = 2(72p_0) = 144p_0 = \frac{144}{277}$$

$$p_0 + p_1 + p_2 + p_3 + p_4 + p_5 = 1$$

$$\Rightarrow p_0 + 6p_0 + 18p_0 + 36p_0 + 72p_0 + 144p_0 = 1$$

$$\Rightarrow 7p_0 + 54p_0 + 216p_0 = 1$$

$$\Rightarrow 61p_0 + 216p_0 = 1 \Rightarrow 277p_0 = 1$$

$$\Rightarrow p_0 = \frac{1}{277}$$

$$
\begin{array}{c|cccccc|l}
 & 0 & 1 & 2 & 3 & 4 & 5 & \\
0 & -\frac{}{6} & \frac{1}{6} & \frac{}{6} & \frac{}{6} & \frac{}{6} & \frac{}{6} & = 0 \\
1 & \frac{1}{6} & -\frac{}{6} & \frac{2}{6} & \frac{}{6} & \frac{}{6} & \frac{}{6} & = 0 \\
2 & \frac{}{6} & \frac{2}{6} & -\frac{}{6} & \frac{3}{6} & \frac{}{6} & \frac{}{6} & = 0 \\
3 & \frac{}{6} & \frac{}{6} & \frac{3}{6} & -\frac{}{6} & \frac{3}{6} & \frac{}{6} & = 0 \\
4 & \frac{}{6} & \frac{}{6} & \frac{}{6} & \frac{}{6} & -\frac{}{6} & \frac{3}{6} & = 0 \\
5 & \frac{}{6} & \frac{}{6} & \frac{}{6} & \frac{}{6} & \frac{3}{6} & -\frac{}{6} & = 0
\end{array}
$$

b. How often are all 3 printers working?

All 3 printers are working when there's
3 or more cartridges is:

$$1 - \left(\frac{1+6+18}{277}\right) = 1 - \frac{25}{277} = \frac{252}{277} = 0.9097 \cdot 100$$

$$= 90.97\%$$

32. Customers arrive @ 2-server station in
accordance w/ poi process having $\lambda = 1$.
Upon arriving, they join single queue.
Whenever server completes service, person 1st
in line enters service. Service times of server i
are exp w/ $\lambda = \mu_i$ $(i = 1, 2)$, where $\mu_1 + \mu_2 > 1$. Arrival
finding both servers free is (=)ly likely to go
to either one. Define appropriate cont-time
MC for this model, show it's time reversible, &
find lim probs

Poisson: λ
μ_i for $i = 1, 2$
$\mu_1 + \mu_2 > 1$

0 = no customer
1 = customer w/ S1
1' = customer w/ S2
$n = n$ customers, $n \geq 2$

let $\mu = \mu_1 + \mu_2$ $\quad p_i q_{ij} = p_j q_{ij} \ \forall\, i \neq j$ $\quad p_i = \lim_{j \to \infty} p_{ji}$

$$p_j = \frac{(\lambda/\mu)^j}{\sum_{i=0}^{n} (\lambda/\mu)^i}$$

$$p_0 q_{01} = p_1 q_{10} \Rightarrow p_0 \lambda_0 = p_1 \mu_1$$

$$p_0\left(\frac{\lambda}{2}\right) = p_1 \mu_1 \Rightarrow p_1 = \frac{0.5\lambda}{\mu_1} p_0$$

$$p_0\left(\frac{\lambda}{2}\right) = p_{1'} \mu_2 \Rightarrow p_{1'} = \frac{0.5\lambda}{\mu_2} p_0$$

$$p_1 q_{12} = p_2 q_{21} \Rightarrow p_1 \lambda_1 = p_2 \mu_2$$

$$p_1 \lambda = p_2 \mu_2 \Rightarrow p_2 = \frac{\lambda}{\mu_2} p_1 = \frac{0.5\lambda^2}{\mu_1 \mu_2} p_0$$

$$p_{1'} \lambda = p_2 \mu_1 \Rightarrow p_2 = \frac{\lambda}{\mu_1} p_{1'} = \frac{0.5\lambda^2}{\mu_1 \mu_2} p_0$$

$$\lambda p_n = \mu p_{n+1} \ (n \geq 2)$$

Solving in terms of p_2 ...

$$p_{n+1} = \left(\frac{\lambda}{\mu}\right) p_n = \left(\frac{\lambda}{\mu}\right)^2 p_{n-1} = \cdots = \left(\frac{\lambda}{\mu}\right)^{n-1} p_2$$

$$\Rightarrow p_{n+2} = \left(\frac{\lambda}{\mu}\right)^n p_2$$

$$p_1 \lambda = p_2 \mu_2 \Rightarrow p_1 = \frac{\mu_2}{\lambda} p_2$$

$$p_{1'} \lambda = p_2 \mu_1 \Rightarrow p_{1'} = \frac{\mu_1}{\lambda} p_2$$

$$p_0\left(\frac{\lambda}{2}\right) = p_1 \mu_1 \Rightarrow p_0 = \frac{2\mu_1}{\lambda} p_1 = \frac{2\mu_1}{\lambda}\left(\frac{\mu_2}{\lambda}\right) p_2 = \frac{2\mu_1\mu_2}{\lambda^2} p_2$$

$$p_0\left(\frac{\lambda}{2}\right) = p_{1'} \mu_2 \Rightarrow p_0 = \frac{2\mu_2}{\lambda} p_{1'} = \frac{2\mu_2}{\lambda}\left(\frac{\mu_1}{\lambda}\right) p_2 = \frac{2\mu_1\mu_2}{\lambda^2} p_2$$

The time reversible equations are:

$$P_0\left(\frac{\lambda}{2}\right) = P_1\mu_1 \qquad P_1\lambda = P_2\mu_2 \qquad \lambda P_n = \mu P_{n+1} \ (n \geq 2)$$

$$P_0\left(\frac{\lambda}{2}\right) = P_{1'}\mu_2 \qquad P_{1'}\lambda = P_2\mu_1$$

And they all hold as shown above when the probabilities are given in terms of P_2. Therefore, the continuous-time Markov Chain is time reversible

The limiting probabilities are then:

$$\sum_j P_j = 1 \qquad v_j P_j = \sum_{k \neq j} q_{kj} P_k$$

$$P_0 + P_1 + P_{1'} + P_2 = 1$$

$$\Rightarrow \frac{2\mu_1\mu_2}{\lambda^2} P_2 + \frac{\mu_2}{\lambda} P_2 + \frac{\mu_1}{\lambda} P_2 + P_2 = 1$$

$$\Rightarrow P_2\left(\frac{2\mu_1\mu_2 + \lambda\mu_2 + \lambda\mu_1 + \lambda^2}{\lambda^2}\right) = 1$$

$$\Rightarrow P_2 = \frac{\lambda^2}{2\mu_1\mu_2 + \lambda\mu_2 + \lambda\mu_1 + \lambda^2}$$

$$P_1 = \frac{\mu_2}{\lambda} P_2 = \frac{\mu_2}{\lambda}\left(\frac{\lambda^2}{2\mu_1\mu_2 + \lambda\mu_2 + \lambda\mu_1 + \lambda^2}\right)$$

$$= \frac{\mu_2\lambda}{2\mu_1\mu_2 + \lambda\mu_2 + \lambda\mu_1 + \lambda^2}$$

$$P_{1'} = \frac{\mu_1}{\lambda} P_2 = \frac{\mu_1}{\lambda}\left(\frac{\lambda^2}{2\mu_1\mu_2 + \lambda\mu_2 + \lambda\mu_1 + \lambda^2}\right)$$

$$= \frac{\lambda\mu_1}{2\mu_1\mu_2 + \lambda\mu_2 + \lambda\mu_1 + \lambda^2}$$

$$P_0 = \frac{2\mu_1\mu_2}{\lambda^2} P_2 = \frac{2\mu_1\mu_2}{\lambda^2}\left(\frac{\lambda^2}{2\mu_1\mu_2 + \lambda\mu_2 + \lambda\mu_1 + \lambda^2}\right)$$

$$= \frac{2\mu_1\mu_2}{2\mu_1\mu_2 + \lambda\mu_2 + \lambda\mu_1 + \lambda^2}$$