7.4. Renewal reward process.

A large number of probability models are special case of the renewal reward process.

Let $\{N(t), t \geq 0\}$ be a renewal process

Let X_n, $n \geq 1$ be an interarrival time

Denote R_n $n \geq 1$ the reward earned during the nth renewal

R_n are iid, but R_n can depend on X_n, the length of n^{th} renewal interval

$$R(t) = \sum_{n=1}^{N(t)} R_n \quad - \text{ total reward earned by time } t$$

Let

$$ER = ER_n \qquad EX = EX_n$$

Proposition 7.3. If $ER < \infty$ and $EX < \infty$, then

(i) $$\lim_{t \to \infty} \frac{R(t)}{t} = \frac{ER}{EX} \quad \text{with probability } 1.$$

(ii) $$\lim_{t \to \infty} \frac{ER(t)}{t} = \frac{ER}{EX}$$

Proof. (i) $$\frac{R(t)}{t} = \frac{\sum_{n=1}^{N(t)} R_n}{t} = \frac{\sum_{n=1}^{N(t)} R_n}{N(t)} \cdot \frac{N(t)}{t}$$

By strong Law of large numbers

$$\lim_{t \to \infty} \frac{\sum_{n=1}^{N(t)} R_n}{N(t)} = ER \quad (\text{with prob. } 1)$$

by Proposition 7.1

$$\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{EX} \quad (\text{with prob} 1)$$

$$\Rightarrow \quad \frac{\sum_{n=1}^{N(t)} R_n}{N(t)} \cdot \frac{N(t)}{t} \xrightarrow[t \to \infty]{} \frac{ER}{EX}$$

Remark. If we say that a cycle is completed every time a renewal occurs then Proposition 7.3 states that the long-run average reward per unit time is equal to the expected reward earned during the cycle divided to the expected length a cycle.

$$\text{limiting reward/time} = \frac{\text{expected reward/cycle}}{\text{expected time/cycle}}$$

Example 7.6. Suppose that potential customers arrive at a single-server bank in accordance with a Poisson process having rate λ. However, suppose that the potential customer will enter the bank only if the server is free when he arrives. That is, if there is already a customer in the bank, then our arriver, rather than entering the bank, will go home. If we assume that the amount of time spent in the bank by an entering customer is a random variable having distribution G, then

We suppose that the amounts that the successive customers deposit in the bank are i.i.d random variables having common distribution H, then the rate at which deposits accumulate —

$$\lim_{t \to \infty} \frac{\text{total deposit by time } t}{t} =$$

$$= \frac{E[\text{deposit during the cycle}]}{E(\text{time of cycle})} = \frac{\mu_H}{\mu_G + \frac{1}{\lambda}}$$

and μ_H is mean of the distribution H

(ii) Although we have suppose that the reward is earned at the time of renewal, the result remains valid when the reward is earned gradually through the renewal cycle.

Example 7.14 (A Car Buying Model)**.** The lifetime of a car is a continuous random variable having a distribution H and probability density h. Mr. Brown has a policy that he buys a new car as soon as his old one either breaks down or reaches the age of T years. Suppose that a new car costs C_1 dollars and also that an additional cost of C_2 dollars is incurred whenever Mr. Brown's car breaks down. Under the assumption that a used car has no resale value, what is Mr. Brown's long-run average cost?

If we say that a cycle is complete every time Mr. Brown gets a new car, then it follows from Proposition 7.3 (with costs replacing rewards) that his long-run average cost equals

$$\frac{E[\text{cost incurred during a cycle}]}{E[\text{length of a cycle}]}$$

Now letting X be the lifetime of Mr. Brown's car during an arbitrary cycle, then the cost incurred during that cycle will be given by

$$\begin{aligned} &C_1, && \text{if } X > T \\ &C_1 + C_2, && \text{if } X \le T \end{aligned}$$

so the expected cost incurred over a cycle is

$$C_1 P\{X > T\} + (C_1 + C_2) P\{X \le T\} = C_1 + C_2 H(T)$$

Also, the length of the cycle is

$$\begin{aligned} &X, && \text{if } X \le T \\ &T, && \text{if } X > T \end{aligned}$$

and so the expected length of a cycle is

$$\int_0^T x h(x)dx + \int_T^\infty T h(x)dx = \int_0^T x h(x)dx + T[1 - H(T)]$$

Therefore, Mr. Brown's long-run average cost will be

$$\frac{C_1 + C_2 H(T)}{\int_0^T x h(x)dx + T[1 - H(T)]} \tag{7.13}$$

Now, suppose that the lifetime of a car (in years) is uniformly distributed over $(0, 10)$, and suppose that C_1 is 3 (thousand) dollars and C_2 is $\frac{1}{2}$ (thousand) dollars. What value of T minimizes Mr. Brown's long-run average cost?

If Mr. Brown uses the value T, $T \le 10$, then from Eq. (7.13) his long-run average cost equals

$$\begin{aligned} \frac{3 + \frac{1}{2}(T/10)}{\int_0^T (x/10)dx + T(1 - T/10)} &= \frac{3 + T/20}{T^2/20 + (10T - T^2)/10} \\ &= \frac{60 + T}{20T - T^2} \end{aligned}$$

We can now minimize this by using the calculus. Toward this end, let

$$g(T) = \frac{60 + T}{20T - T^2}$$

then

$$g'(T) = \frac{(20T - T^2) - (60 + T)(20 - 2T)}{(20T - T^2)^2}$$

Equating to 0 yields

$$20T - T^2 = (60 + T)(20 - 2T)$$

or, equivalently,

$$T^2 + 120T - 1200 = 0$$

which yields the solutions

$$T \approx 9.25 \quad \text{and} \quad T \approx -129.25$$

Since $T \le 10$, it follows that the optimal policy for Mr. Brown would be to purchase a new car whenever his old car reaches the age of 9.25 years. ■

Example 7.18 (The Average Age of a Renewal Process)**.** Consider a renewal process having interarrival distribution F and define $A(t)$ to be the time at t since the last renewal. If renewals represent old items failing and being replaced by new ones, then $A(t)$ represents the age of the item in use at time t. Since $S_{N(t)}$ represents the time of the last event prior to or at time t, we have

$$A(t) = t - S_{N(t)}$$

We are interested in the average value of the age—that is, in

$$\lim_{s\to\infty} \frac{\int_0^s A(t)dt}{s}$$

To determine this quantity, we use renewal reward theory in the following way: Let us assume that at any time we are being paid money at a rate equal to the age of the renewal process at that time. That is, at time t, we are being paid at rate $A(t)$, and so $\int_0^s A(t)dt$ represents our total earnings by time s. As everything starts over again when a renewal occurs, it follows that

$$\frac{\int_0^s A(t)dt}{s} \to \frac{E[\text{reward during a renewal cycle}]}{E[\text{time of a renewal cycle}]}$$

Now, since the age of the renewal process a time t into a renewal cycle is just t, we have

$$\begin{aligned}\text{reward during a renewal cycle} &= \int_0^X t\,dt \\ &= \frac{X^2}{2}\end{aligned}$$

where X is the time of the renewal cycle. Hence, we have that

$$\begin{aligned}\text{average value of age} &\equiv \lim_{s\to\infty} \frac{\int_0^s A(t)dt}{s} \\ &= \frac{E[X^2]}{2E[X]} \end{aligned} \tag{7.14}$$

where X is an interarrival time having distribution function F. ■

Example 7.20. Suppose that passengers arrive at a bus stop according to a Poisson process with rate λ. Suppose also that buses arrive according to a renewal process with distribution function F, and that buses pick up all waiting passengers. Assuming that the Poisson process of people arriving and the renewal process of buses arriving are independent, find

(a) the average number of people who are waiting for a bus, averaged over all time; and

(b) the average amount of time that a passenger waits, averaged over all passengers.

Solution: We will solve this by using renewal reward processes. Say that a new cycle begins each time a bus arrives. Let T be the time of a cycle, and note that T has distribution function F. If we suppose that each passenger pays us money at a rate of 1 per unit time while they wait for a bus, then the reward rate at any time is the number waiting at that time, and so the average reward per unit time is the average number of people that are waiting for a bus. Letting R be the reward earned during a cycle, the renewal reward theorem gives

$$\text{Average Number Waiting} = \frac{E[R]}{E[T]}$$

Let N be the number of arrivals during a cycle. To determine $E[R]$, we will condition on the values of both T and N. Now,

$$E[R|T = t, N = n] = nt/2$$

which follows because given there are n arrivals by time t their set of arrival times are distributed as n independent uniform $(0, t)$ random variables, and so the average amount received per passenger is $t/2$. Hence,

$$E[R|T, N] = NT/2$$

Taking expectations yields

$$E[R] = \frac{1}{2}E[NT]$$

To compute $E[NT]$, condition on T to obtain

$$E[NT|T] = T\,E[N|T] = \lambda T^2$$

where the preceding follows because, given the time T until the bus arrives, the number of people waiting is Poisson distributed with mean λT. Hence, upon taking expectations of the preceding, we obtain

$$E[R] = \frac{1}{2}E[NT] = \lambda E[T^2]/2$$

which gives that

$$\text{Average Number Waiting} = \frac{\lambda E[T^2]}{2E[T]}$$

where T has the interarrival distribution F.

To determine the average amount of time that a passenger waits note that, because each passenger pays 1 per unit time while waiting for a bus, the total amount paid by a passenger is the amount of time the passenger waits. Because R is the total reward earned in a cycle, it thus follows that, with W_i being the waiting time of passenger i,

$$R = W_1 + \cdots + W_N$$

Now, if we consider the rewards earned from successive passengers, namely $W_1, W_2, \ldots$, and imagine that the reward W_i is earned at time i, then this sequence of rewards constitutes a discrete time renewal reward process in which a new cycle begins at time $N + 1$. Consequently, from renewal reward process theory and the preceding identity, we see that

$$\lim_{n\to\infty} \frac{W_1 + \cdots + W_n}{n} = \frac{E[W_1 + \cdots + W_N]}{E[N]} = \frac{E[R]}{E[N]}$$

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Using that

$$E[N] = E[E[N|T]] = E[\lambda T] = \lambda E[T]$$

along with the previously derived $E[R] = \lambda E[T^2]/2$ we obtain the result

$$\lim_{n\to\infty} \frac{W_1 + \cdots + W_n}{n} = \frac{E[T^2]}{2E[T]}$$

Because $\frac{E[T^2]}{2E[T]}$ is the average value of the excess for the renewal process of arriving buses, the preceding equation yields the interesting result that the average waiting time of a passenger is equal to the average time until the next bus arrives when we average over all time. Because passengers are arriving according to a Poisson process, this result is a special case of a general result, known as the PASTA principle, to be presented in Chapter 8. The PASTA principle says that a system as seen by Poisson arrivals is the same as the system as averaged over all time. (In our example, the system refers to the time until the next bus.) ■