1. Suppose that potential customers arrive at a single-server bank in accordance with a Poisson process having rate λ. However, suppose that the potential customer will enter the bank only if the server is free when he arrives. That is, if there is already a customer in the bank, then our arriver, rather than entering the bank, will go home. If we assume that the amount of time spent in the bank by an entering customer is a random variable having distribution G, then

(a) what is the rate at which customers enter the bank?

(b) what proportion of potential customers actually enter the bank?

Solution: In answering these questions, let us suppose that at time 0 a customer has just entered the bank. (That is, we define the process to start when the first customer enters the bank.) If we let μ_G denote the mean service time, then, by the memoryless property of the Poisson process, it follows that the mean time between entering customers is

$$\mu = \mu_G + \frac{1}{\lambda}$$

Hence, the rate at which customers enter the bank will be given by

$$\frac{1}{\mu} = \frac{\lambda}{1 + \lambda\mu_G}$$

On the other hand, since potential customers will be arriving at a rate λ, it follows that the proportion of them entering the bank will be given by

$$\frac{\lambda/(1+\lambda\mu_G)}{\lambda} = \frac{1}{1+\lambda\mu_G}$$

In particular if $\lambda = 2$ and $\mu_G = 2$, then only one customer out of five will actually enter the system. ■

2.

Let $U_1, U_2, \ldots$ be independent uniform (0, 1) random variables, and define N by

$$N = \min\{n : U_1 + U_2 + \cdots + U_n > 1\}$$

What is $E[N]$?

Let $U_1, \ldots, U_n, \ldots$ be independent uniform (0, 1) random variables. Let

$$N = \min\{n : U_n > .8\}$$

and let $S = \sum_{i=1}^{N} U_i$.

(a) Find $E[S]$ by conditioning on the value of U_1.

(b) Find $E[S]$ by conditioning on N.

12. (a)

$$E[S] = \int_0^1 E[S|U_1 = x]dx = \int_0^{.8} (x + E[S])dx + \int_{.8}^1 x dx$$

Hence, $E[S] = 5/2$.

(b) Using that given N, the first $N - 1$ values are uniform on (0, .8) and the last is uniform on (.8, 1), we see that

$$E[S|N = n] = (n - 1)(.4) + .9$$

Hence,

$$E[S] = E[.4(N - 1) + .9] = .5 + .4E[N] = .5 + .4(5) = 2.5$$

where preceding used that N is geometric with parameter $1/5$.

(c) $E[S] = E[X]E[N] = (.5)(5) = 2.5$

A worker sequentially works on jobs. Each time a job is completed, a new one is begun. Each job, independently, takes a random amount of time, having distribution F to complete. However, independently of this, shocks occur according to a Poisson process with rate λ. Whenever a shock occurs, the worker

9. A job completion constitutes a reneval. Let T denote the time between renewals. To compute $E[T]$ start by conditioning on W, the time it takes to finish the next job:

$$E[T] = E[E[T|W]]$$

Now, to determine $E[T|W = w]$ condition on S, the time of the next shock. This gives

$$E[T|W = w] = \int_0^\infty E[T|W = w, S = x]\lambda e^{-\lambda x}dx$$

Now, if the time to finish is less than the time of the shock then the job is completed at the finish time; otherwise everything starts over when the shock occurs. This gives

$$E[T|W = w, S = x] = \begin{cases} x + E[T], & \text{if} \quad x < w \\ w, & \text{if} \quad x \geq w \end{cases}$$

Hence,

$$\begin{aligned} E[T|W = w] &= \int_0^w (x + E[T])\lambda e^{-\lambda x}dx + w\int_w^\infty \lambda e^{-\lambda x}dx \\ &= E[T][1 - e^{-\lambda w}] + 1/\lambda - we^{-\lambda w} - \frac{1}{\lambda}e^{-\lambda w} - we^{-\lambda w} \end{aligned}$$

Thus,

$$E[T|W] = (E[T] + 1/\lambda)(1 - e^{-\lambda W})$$

Taking expectations gives

$$E[T] = (E[T] + 1/\lambda)(1 - E[e^{-\lambda W}])$$

and so

$$E[T] = \frac{1 - E[e^{-\lambda W}]}{\lambda E[e^{-\lambda W}]}$$

In the above, W is a random variable having distribution F and so

$$E[e^{-\lambda W}] = \int_0^\infty e^{-\lambda w} f(w)dw$$