1. Brad's relationship w/ his girlfriend Angelina changes between amorous, bickering, confusion, & depression. According to the following transition rates when t is the time in months:

$$\begin{array}{c|cccc|} & A & B & C & D \\ \hline A & -4 & 3 & 1 & 0 \\ B & 4 & -6 & 2 & 0 \\ C & 2 & 3 & -6 & 1 \\ D & 0 & 0 & 2 & -2 \end{array}$$

a. Find long fraction of time he spends in these 4 states.

$$\begin{vmatrix} -4 & 3 & 1 & 0 \\ 4 & -6 & 2 & 0 \\ 2 & 3 & -6 & 1 \\ 0 & 0 & 2 & -2 \end{vmatrix} \Rightarrow \begin{vmatrix} -4 & 3 & 1 & 1 \\ 4 & -6 & 2 & 1 \\ 2 & 3 & -6 & 1 \\ 0 & 0 & 2 & 1 \end{vmatrix}^{-1} =$$

$$= \begin{vmatrix} -\frac{2}{5} & -\frac{7}{40} & \frac{1}{20} & \frac{21}{40} \\ -\frac{4}{15} & -\frac{17}{60} & \frac{1}{30} & \frac{31}{60} \\ -\frac{1}{5} & -\frac{3}{20} & -\frac{1}{10} & \frac{9}{20} \\ \frac{2}{5} & \frac{3}{10} & \frac{1}{5} & \frac{1}{10} \end{vmatrix}$$

Amorous $= \frac{2}{5}$ Bickering $= \frac{3}{10}$

Confusion $= \frac{1}{5}$ Depression $= \frac{1}{10}$

b. Does the chain satisfy the detailed balance condition?

$p_i q_{ij} = p_j q_{ji}$ $p_0 \lambda_0 = p_1 \mu_1$ $p_1 \lambda_1 = p_2 \mu_2$

$p_2 \lambda_2 = p_3 \mu_3$ $p_3 \lambda_3 = p_4 \mu_4$

Yes, it satisfies the detailed balance equations.

c. They are amorous. What is the $\mathbb{E}$ amount of time until depression sets in?

$A \to D \Rightarrow g(i) = \mathbb{E}_i(V_D)$

$$\begin{vmatrix} -4 & 3 & 1 & 0 \\ 4 & -6 & 2 & 0 \\ 2 & 3 & -6 & 1 \\ 0 & 0 & 2 & -2 \end{vmatrix} \Rightarrow \begin{vmatrix} -4 & 3 & 1 \\ 4 & -6 & 2 \\ 2 & 3 & -6 \end{vmatrix}$$

$$\begin{pmatrix} -4 & 3 & 1 \\ 4 & -6 & 2 \\ 2 & 3 & -6 \end{pmatrix} \begin{pmatrix} g(A) \\ g(B) \\ g(C) \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -4 & 3 & 1 \\ 4 & -6 & 2 \\ 2 & 3 & -6 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{5}{2} & -\frac{7}{4} & -1 \\ -\frac{7}{3} & -\frac{11}{6} & -1 \\ -2 & -\frac{3}{2} & -1 \end{pmatrix}$$

The expected amount of time from amorous to depression is: $\frac{5}{2} + \frac{7}{4} + 1 = \frac{10+7+4}{4} = \frac{21}{4} = 5.25$

2. Al, Betty, Charlie, Diane are working @ math club table during freshmen move-in week. Their attention spans for doing the job are independent & have exp distr w/ μs $= \left(\frac{1}{2}, \frac{2}{3}, 1, 2\right)$ in hours & λs $= \left(2, \frac{3}{2}, 1, \frac{1}{2}\right)$. What is the probability the last 2 left have the same sex, i.e., AC or BD?

Al $\sim \exp\left(\mu = \frac{1}{2}, \lambda = 2\right)$

Charlie $\sim \exp(\mu, \lambda = 1)$

Betty $\sim \exp\left(\mu = \frac{2}{3}, \lambda = \frac{3}{2}\right)$

Diane $\sim \exp\left(\mu = 2, \lambda = \frac{1}{2}\right)$

$P(AC \text{ left}) = ?$

$P(BD \text{ left}) = ?$

let

X_a = attention span of Al $\Rightarrow$ $X_a \sim exp(2)$

X_b = " " Betty $\Rightarrow$ $X_b \sim exp(1.5)$

X_c = " " Charlie $\Rightarrow$ $X_c \sim exp(1)$

X_d = " " Diane $\Rightarrow$ $X_d \sim exp(0.5)$

$X_1 = \min(X_a, X_b, X_c, X_d) = 2 + \frac{3}{2} + 1 + \frac{1}{2} = 3 + 2 = 5$

$X_1 \sim exp(5)$

X_2 = time until only 2 ppl are left

$$
\begin{aligned}
P(X_2 > x) = \; & P(X_2 > x \mid X_a = X_1) P(X_a = X_1) + \\
& P(X_2 > x \mid X_b = X_1) P(X_b = X_1) + \\
& P(X_2 > x \mid X_c = X_1) P(X_c = X_1) + \\
& P(X_2 > x \mid X_d = X_1) P(X_d = X_1) \\
= \; & P(X_2 > x \mid X_a = X_1) \frac{2}{5} + \\
& P(X_2 > x \mid X_b = X_1) \frac{3}{2}\left(\frac{1}{5}\right) + \\
& P(X_2 > x \mid X_c = X_1) \frac{1}{5} + \\
& P(X_2 > x \mid X_d = X_1) \frac{1}{2}\left(\frac{1}{5}\right) \\
= \; & P(X_b > X_1, X_c > x, X_d > x, X_a < x) \frac{2}{5} + \\
& P(X_a > X_1, X_c > x, X_d > x, X_b < x) \frac{3}{10} +
\end{aligned}
$$

$$P(X_a > x_1, X_b > x, X_d > x, X_c < x)\frac{1}{5} +$$

$$P(X_a > x_1, X_b > x, X_c > x, X_d < x)\frac{1}{10}$$

$$P(\male+\male \text{ or } \female+\female) = P(X_1 = x_a \mid X_2 = x_c) + P(X_1 = x_b \mid X_2 = x_d) +$$

$$P(X_1 = x_c \mid X_2 = x_a) + P(X_1 = x_d \mid X_2 = x_b)$$

$$= \frac{2}{5}\left(\frac{1}{1.5+1+0.5}\right) + \frac{3}{10}\left(\frac{0.5}{2+1+0.5}\right) +$$

$$\frac{1}{5}\left(\frac{2}{2+1.5+0.5}\right) + \frac{1}{10}\left(\frac{1.5}{2+1.5+1}\right)$$

$$= \frac{2}{5}\left(\frac{1}{3}\right) + \frac{3}{10}\left(\frac{0.5}{3.5}\right) + \frac{1}{5}\left(\frac{2}{4}\right) + \frac{1}{10}\left(\frac{1.5}{4.5}\right)$$

$$= \frac{2}{15} + \frac{3}{70} + \frac{1}{10} + \frac{1}{30} = \frac{21+28+9+7}{210} = \frac{65}{210}$$

7.1 Is it true that

a. $N(t) < n$ if & only if $S_n > t$?

Yes

$S_n > t > S_{n+1}$

$n > N(t) > n+1$

The inequality still holds when $N(t) < n$

b. $N(t) \le n$ if & only if $S_n \ge t$?

No

$S_n \le t \le S_{n+1}$

$n \le N(t) \le n+1$

If $N(t)=n$, it would contradict the right-hand side

c. $N(t)>n$ if & only if $S_n<t$?

No; if we follow from a

$S_n<t<S_{n+1}$

$n<N(t)<n+1$

We see like b, there's a contradiction on the right-hand side

7.2 Suppose that interarrival distr for renewal process is Poi distributed w/ mean μ. That is, suppose

$$P\{X_n=k\}=e^{-\mu}\left(\frac{\mu^k}{k!}\right), \quad k=0,1,\ldots$$

a. Find distr of S_n.

S_n is a Poisson distribution with mean $n\mu$.

$X\sim Poi(\lambda) \Rightarrow PDF=e^{-\lambda}\left(\frac{\lambda^x}{x!}\right)$

b. Calculate $P\{N(t)=n\}$

Let $[t]$ is the largest intenger not exceeding t

$$P\{N(t)=n\}=P\{N(t)\geq n\}-P\{N(t)\geq n+1\}$$

$$=P\{S_n\leq t\}-P\{S_{n+1}\leq t\}$$

$$=\sum_{k=0}^{[t]} e^{-n\mu}\frac{(n\mu)^k}{k!}-\sum_{k=0}^{[t]} e^{-(n+1)\mu}\frac{[(n+1)\mu]^k}{k!}$$