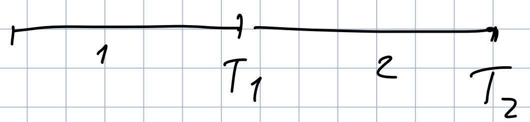
Regenerative process.

Consider a stochastic process $\{X(t), t \geq 0\}$ with state space $\{1, 2, \ldots\}$.

$\exists$ point at which process (probabilistically) restarts itself.

Suppose that with probability 1 $\exists$ time T_1



2 is a replica of 1 ....

$T_2, T_3, T_4 \ldots$ have the same properties as T_1

Such stochastic process is known as a regenerative process.

Then $T_1, T_2, \ldots$ constitute the arrival times of a renewal process and we shall say that a cycle is completed every time a renewal occurs.

Examples. **(1)** A renewal process is regenerative, and T_1 represents the time of the first renewal.

(2) A recurrent Markov chain is regenerative, and T_1 represents the time of the first transition into the initial state.

Q: What is a long-run proportion of the time a regenerative process spends in state j.

Process is in j $\rightarrow$ reward $= 1$

o/w $\rightarrow$ reward $= 0$.

$$\underline{I}(s) = \begin{cases} 1 & X(s) = j \\ 0 & X(s) \neq j \end{cases}$$

Total reward earned by time $t = \int_0^t \underline{I}(s)\,ds$

Cycle time T_1, process restarts

As the preceding is clearly a renewal reward process that starts over again at the cycle time T_1, we see from Proposition 7.3 that

$$\text{average reward per unit time} = \frac{E[\text{reward by time } T_1]}{E[T_1]}$$

However, the average reward per unit is just equal to the proportion of time that the process is in state j. That is, we have the following.

Proposition 7.4. *For a regenerative process, the long-run*

$$\textit{proportion of time in state } j = \frac{E[\textit{amount of time in } j \textit{ during a cycle}]}{E[\textit{time of a cycle}]}$$

Remark. If the cycle time T_1 is a continuous random variable, then it can be shown by using an advanced theorem called the "key renewal theorem" that the preceding is equal also to the limiting probability that the system is in state j at time t. That is, if T_1 is continuous, then

$$\lim_{t\to\infty} P\{X(t) = j\} = \frac{E[\text{amount of time in } j \text{ during a cycle}]}{E[\text{time of a cycle}]}$$

Example 7.21. Consider a positive recurrent continuous-time Markov chain that is initially in state i. By the Markovian property, each time the process reenters state i it starts over again. Thus returns to state i are renewals and constitute the beginnings of new cycles. By Proposition 7.4, it follows that the long-run

$$\text{proportion of time in state } j = \frac{E[\text{amount of time in } j \text{ during an } i-i \text{ cycle}]}{\mu_{ii}}$$

where μ_{ii} represents the mean time to return to state i. If we take j to equal i, then we obtain

$$\text{proportion of time in state } i = \frac{1/v_i}{\mu_{ii}} \qquad \blacksquare$$

Example 7.23. Although a system needs only a single machine to function, it maintains an additional machine as a backup. A machine in use functions for a random time with density function f and then fails. If a machine fails while the other one is in working condition, then the latter is put in use and, simultaneously, repair begins on the one that just failed. If a machine fails while the other machine is in repair, then the newly failed machine waits until the repair is completed; at that time the repaired machine is put in use and, simultaneously, repair begins on the recently failed one. All repair times have density function g. Find P_0, P_1, P_2, where P_i is the long-run proportion of time that exactly i of the machines are in working condition.

Alternating renewal Process.

System can be in two states on and off

on	off	on	off
Z_1	Y_1	Z_2	Y_2

(Z_n, Y_n) – be a random vector iid

$\{Z_n\}$ – iid $\qquad Z_n, Y_n$ can be dependent.

$\{Y_n\}$ – iid

Let $EZ = EZ_n, \quad EY = EY_n$

P_{on} – the long run proportion of time that the system is on.

$X_n = Y_n + Z_n \qquad n \geq 1$

$$P_{on} = \frac{EZ}{EY + EZ} = \frac{E(on)}{E(on) + E(off)}$$

$$P_{off} = 1 - P_{on} = \frac{EY}{EY + EZ}$$

Example 7.24 (A Production Process)**.** One example of an alternating renewal process is a production process (or a machine) that works for a time Z_1, then breaks down and has to be repaired (which takes a time Y_1), then works for a time Z_2, then is down for a time Y_2, and so on. If we suppose that the process is as good as new after each repair, then this constitutes an alternating renewal process. It is worthwhile to note that in this example it makes sense to suppose that the repair time will depend on the amount of time the process had been working before breaking down. ■

Example 7.25. The rate a certain insurance company charges its policyholders alternates between r_1 and r_0. A new policyholder is initially charged at a rate of r_1 per unit time. When a policyholder paying at rate r_1 has made no claims for the most recent s time units, then the rate charged becomes r_0 per unit time. The rate charged remains at r_0 until a claim is made, at which time it reverts to r_1. Suppose that a given policyholder lives forever and makes claims at times chosen according to a Poisson process with rate λ, and find

(a) P_i, the proportion of time that the policyholder pays at rate r_i, $i = 0, 1$;
(b) the long-run average amount paid per unit time.

Solution: If we say that the system is "on" when the policyholder pays at rate r_1 and "off" when she pays at rate r_0, then this on–off system is an alternating renewal process with a new cycle starting each time a claim is made. If X is the time between successive claims, then the on time in the cycle is the smaller of s and X. (Note that if $X < s$, then the off time in the cycle is 0.) Since X is exponential with rate λ, the preceding yields

$$\begin{aligned} E[\text{on time in cycle}] &= E[\min(X, s)] \\ &= \int_0^s x\lambda e^{-\lambda x}\,dx + se^{-\lambda s} \\ &= \frac{1}{\lambda}(1 - e^{-\lambda s}) \end{aligned}$$

Since $E[X] = 1/\lambda$, we see that

$$P_1 = \frac{E[\text{on time in cycle}]}{E[X]} = 1 - e^{-\lambda s}$$

and

$$P_0 = 1 - P_1 = e^{-\lambda s}$$

The long-run average amount paid per unit time is

$$r_0 P_0 + r_1 P_1 = r_1 - (r_1 - r_0)e^{-\lambda s}$$ ■

Example 7.27 (The Excess of a Renewal Process)**.** Let us now consider the long-run proportion of time that the excess of a renewal process is less than c. To determine this quantity, let a cycle correspond to a renewal interval and say that the system is on whenever the excess of the renewal process is greater than or equal to c and that it

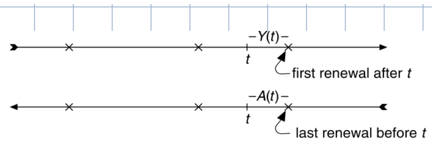


Figure 7.3 Arrowheads indicate direction of time.

is off otherwise. In other words, whenever a renewal occurs the process goes on and stays on until the last c time units of the renewal interval when it goes off. Clearly this is an alternating renewal process, and so we obtain from Eq. (7.16) that

$$\text{long-run proportion of time the excess is less than } c = \frac{E[\text{off time in cycle}]}{E[\text{cycle time}]}$$

If X is the length of a renewal interval, then since the system is off the last c time units of this interval, it follows that the off time in the cycle will equal $\min(X, c)$. Thus,

$$\begin{aligned}\text{long-run proportion of time the excess is less than } c &= \frac{E[\min(X, c)]}{E[X]}\\ &= \frac{\int_0^c (1 - F(x))dx}{E[X]}\end{aligned}$$

where the final equality follows from Eq. (7.17). Thus, we see from the result of Example 7.26 that the long-run proportion of time that the excess is less than c and the long-run proportion of time that the age is less than c are equal. One way to understand this equivalence is to consider a renewal process that has been in operation for a long time and then observe it going backwards in time. In doing so, we observe a counting process where the times between successive events are independent random variables having distribution F. That is, when we observe a renewal process going backwards in time we again observe a renewal process having the same probability structure as the original. Since the excess (age) at any time for the backwards process corresponds to the age (excess) at that time for the original renewal process (see Fig. 7.3), it follows that all long-run properties of the age and the excess must be equal. ■

If μ is the mean interarrival time, then the distribution function F_e, defined by

$$F_e(x) = \int_0^x \frac{1 - F(y)}{\mu} dy$$

is called the *equilibrium distribution* of F. From the preceding, it follows that $F_e(x)$ represents the long-run proportion of time that the age, and the excess, of the renewal process is less than or equal to x.

Example 7.28 (The Busy Period of the $M/G/\infty$ Queue)**.** The infinite server queueing system in which customers arrive according to a Poisson process with rate λ, and have a general service distribution G, was analyzed in Section 5.3, where it was shown that the number of customers in the system at time t is Poisson distributed with mean $\lambda \int_0^t \bar{G}(y)dy$. If we say that the system is busy when there is at least one customer in the system and is idle when the system is empty, find $E[B]$, the expected length of a busy period.

Solution: If we say that the system is on when there is at least one customer in the system, and off when the system is empty, then we have an alternating renewal process. Because $\int_0^\infty \bar{G}(t)dt = E[S]$, where $E[S]$ is the mean of the service distribution G, it follows from the result of Section 5.3 that

$$\lim_{t\to\infty} P\{\text{system off at } t\} = e^{-\lambda E[S]}$$

Consequently, from alternating renewal process theory we obtain

$$e^{-\lambda E[S]} = \frac{E[\text{off time in cycle}]}{E[\text{cycle time}]}$$

But when the system goes off, it remains off only up to the time of the next arrival, giving that

$$E[\text{off time in cycle}] = 1/\lambda$$

Because

$$E[\text{on time in cycle}] = E[B]$$

we obtain

$$e^{-\lambda E[S]} = \frac{1/\lambda}{1/\lambda + E[B]}$$

or

$$E[B] = \frac{1}{\lambda}\left(e^{\lambda E[S]} - 1\right) \qquad \blacksquare$$