7.2. Distribution of $N(t)$.

The distribution of $N(t)$ can be obtained, at least in theory, by first noting the important relationship that *the number of renewals by time t is greater than or equal to n if and only if the nth renewal occurs before or at time t*. That is,

$$N(t) \geq n \iff S_n \leq t \qquad (1)$$

From (1) $\Rightarrow$

$$P(N(t)=n) = P(N(t) \geq n) - P(N(t) \geq n+1)$$

$$= P(S_n \leq t) - P(S_{n+1} \leq t)$$

$X_1, X_2 \ldots$ are iid F – common distribution.

$S_n = \sum_{i=1}^{n} X_i$ is distr. as F_n – the n-fold convolution of F with itself.

Therefore

$$P\{N(t)=n\} = F_n(t) - F_{n+1}(t)$$

Example 7.1. Suppose that $P\{X_n = i\} = p(1-p)^{i-1}$, $i \geq 1$. That is, suppose that the interarrival distribution is geometric. Now $S_1 = X_1$ may be interpreted as the number of trials necessary to get a single success when each trial is independent and has a probability p of being a success. Similarly, S_n may be interpreted as the number of trials necessary to attain n successes, and hence has the negative binomial distribution

$$P\{S_n = k\} = \begin{cases} \binom{k-1}{n-1} p^n (1-p)^{k-n} & k \geq n \\ 0 & k < n \end{cases}$$

$$P(N(t)=n) = \sum_{k=n}^{[t]} \binom{k-1}{n-1} p^n (1-p)^{k-n} -$$

$$- \sum_{k=n+1}^{[t]} \binom{k-1}{n} p^{n+1} (1-p)^{k-n-1}$$

$$\Rightarrow P(N(t)=n) = \binom{[t]}{n} p^n (1-p)^{[t]-n}$$

$\boxtimes$

Another expression for $P(N(t)=n)$ can be obtained by conditioning on S_n.

This yields

$$P(N(t)=n) = \int_0^\infty P(N(t)=n \mid S_n = y)\, f_{S_n}(y)\,dy$$

Now if the n^{th} event occured at time $y > t$ then there would have been less then n events by time t.

On the other hand, if it occured at a time $y \le t$ there would be exactly n events by time t if the next interarrival exceeds $t-y$.

Consequently

$$P(N(t)=n) = \int_0^t P(X_{n+1} > t-y \mid S_n = y)\, f_{S_n}(y)\,dy =$$

$$= \int_0^t \bar{F}(t-y)\, f_{S_n}(y)\,dy \quad , \text{ where } \bar{F} = 1-F$$

Example 7.2. If $F(x) = 1 - e^{\lambda x}$ then S_n, being the sum of n independent exponentials with rate λ, will have a gamma (n, λ) distribution. Consequently, the preceding identity gives

$$\begin{aligned} P\,(N(t) = n) &= \int_0^t e^{-\lambda(t-y)} \frac{\lambda e^{-\lambda y}\,(\lambda y)^{n-1}}{(n-1)!} dy \\ &= \frac{\lambda^n e^{-\lambda t}}{(n-1)!} \int_0^t y^{n-1} dy \\ &= e^{-\lambda t} \frac{(\lambda t)^n}{n!} \end{aligned}$$ ■

By using Eq. (7.2) we can calculate $m(t)$, the mean value of $N(t)$, as

$$\begin{aligned} m(t) &= E[N(t)] \\ &= \sum_{n=1}^{\infty} P\{N(t) \geq n\} \\ &= \sum_{n=1}^{\infty} P\{S_n \leq t\} \\ &= \sum_{n=1}^{\infty} F_n(t) \end{aligned}$$

where we have used the fact that if X is nonnegative and integer valued, then

$$\begin{aligned} E[X] &= \sum_{k=1}^{\infty} k P\{X = k\} = \sum_{k=1}^{\infty} \sum_{n=1}^{k} P\{X = k\} \\ &= \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} P\{X = k\} = \sum_{n=1}^{\infty} P\{X \geq n\} \end{aligned}$$

The function $m(t)$ is known as the *mean-value* or the *renewal function*.

It can be shown that the mean-value function $m(t)$ uniquely determines the renewal process. Specifically, there is a one-to-one correspondence between the interarrival distributions F and the mean-value functions $m(t)$.

Another interesting result that we state without proof is that

$$m(t) < \infty \quad \text{for all } t < \infty$$

Remarks. **(i)** Since $m(t)$ uniquely determines the interarrival distribution, it follows that the Poisson process is the only renewal process having a linear mean-value function.

(ii) Some readers might think that the finiteness of $m(t)$ should follow directly from the fact that, with probability 1, $N(t)$ is finite. However, such reasoning is not valid; consider the following: Let Y be a random variable having the following probability distribution:

$$Y = 2^n \text{ with probability } \left(\tfrac{1}{2}\right)^n, \quad n \geq 1$$

Now,

$$P\{Y < \infty\} = \sum_{n=1}^{\infty} P\{Y = 2^n\} = \sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^n = 1$$

But

$$E[Y] = \sum_{n=1}^{\infty} 2^n P\{Y = 2^n\} = \sum_{n=1}^{\infty} 2^n \left(\tfrac{1}{2}\right)^n = \infty$$

Hence, even when Y is finite, it can still be true that $E[Y] = \infty$.

An integral equation satisfied by the renewal function can be obtained by conditioning on the time of the first renewal. Assuming that the interarrival distribution F is continuous with density function f this yields

$$m(t) = E[N(t)] = \int_0^{\infty} E[N(t) | X_1 = x] f(x) dx \tag{7.4}$$

Now suppose that the first renewal occurs at a time x that is less than t. Then, using the fact that a renewal process probabilistically starts over when a renewal occurs, it follows that the number of renewals by time t would have the same distribution as 1 plus the number of renewals in the first $t - x$ time units. Therefore,

$$E[N(t)|X_1 = x] = 1 + E[N(t - x)] \quad \text{if } x < t$$

Since, clearly

$$E[N(t)|X_1 = x] = 0 \quad \text{when } x > t$$

we obtain from Eq. (7.4) that

$$\begin{aligned} m(t) &= \int_0^t [1 + m(t - x)] f(x) dx \\ &= F(t) + \int_0^t m(t - x) f(x) dx \end{aligned} \tag{7.5}$$

Eq. (7.5) is called the *renewal equation* and can sometimes be solved to obtain the renewal function.

Example 7.3. One instance in which the renewal equation can be solved is when the interarrival distribution is uniform—say, uniform on $(0, 1)$. We will now present a solution in this case when $t \leq 1$. For such values of t, the renewal function becomes

$$\begin{aligned} m(t) &= t + \int_0^t m(t - x) dx \\ &= t + \int_0^t m(y) dy \quad \text{by the substitution } y = t - x \end{aligned}$$

Differentiating the preceding equation yields

$$m'(t) = 1 + m(t)$$

Letting $h(t) = 1 + m(t)$, we obtain

$$h'(t) = h(t)$$

or

$$\log h(t) = t + C$$

or

$$h(t) = Ke^t$$

or

$$m(t) = Ke^t - 1$$

Since $m(0) = 0$, we see that $K = 1$, and so we obtain

$$m(t) = e^t - 1, \quad 0 \leq t \leq 1$$ ■