Recitation 2, solutions.

1 A machine is subject to failures of types i =1; 2; 3 at rates λ_1=1/24, λ_2=1/30, λ_3=1/84. A failure of type i takes an exponential amount of time with rate μ_1=1/3, μ_2=1/5, and μ_3=1/7. Formulate a Markov chain model with state space {0; 1; 2; 3} and find its stationary distribution.

$$R = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & -(\lambda_1+\lambda_2+\lambda_3) & \lambda_1 & \lambda_2 & \lambda_3 \\ 1 & \mu_1 & -\mu_1 & 0 & 0 \\ 2 & \mu_2 & 0 & -\mu_2 & 0 \\ 3 & \mu_3 & 0 & 0 & -\mu_3 \end{array}$$

$v_0 = \lambda_1+\lambda_2+\lambda_3$

$v_1 = \mu_1$

$v_2 = \mu_2$

$v_3 = \mu_3$

$P_{01} = \dfrac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3}$ $\quad P_{02} = \dfrac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3}$ $\quad P_{03} = \dfrac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3}$

$P_{10} = 1 \quad P_{20} = 1 \quad P_{30} = 1$

rest $P_{ij} = 0$.

$$\vec{P} = \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

$\vec{P}^T R = 0$ or $R^T \vec{P} = 0$.

$\sum P_i = 1$

$$(P_0 \ P_1 \ P_2 \ P_3) \cdot \begin{pmatrix} -(\lambda_1+\lambda_2+\lambda_3) & \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & -\mu_1 & 0 & 0 \\ \mu_2 & 0 & -\mu_2 & 0 \\ \mu_3 & 0 & 0 & -\mu_3 \end{pmatrix} = (0,0,0,0)$$

$\sum P_i = 1$

You can also write as

$$\underbrace{\begin{pmatrix} -(\lambda_1+\lambda_2+\lambda_3) & \mu_1 & \mu_2 & \mu_3 \\ \lambda_1 & -\mu_1 & 0 & 0 \\ \lambda_2 & 0 & -\mu_2 & 0 \\ \lambda_3 & 0 & 0 & -\mu_3 \end{pmatrix}}_{R^T} \cdot \underbrace{\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}}_{\vec{p}} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{0}$$

$$\sum_i p_i = 1$$

$$\begin{pmatrix} -(\lambda_1+\lambda_2+\lambda_3) & \mu_1 & \mu_2 & \mu_3 \\ \lambda_1 & -\mu_1 & 0 & 0 \\ \lambda_2 & 0 & -\mu_2 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$A \cdot \vec{p} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{p} = A^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 0.7273 \\ 0.0909 \\ 0.1212 \\ 0.0606 \end{bmatrix} \begin{matrix} \to p_0 \\ \to p_1 \\ \to p_2 \\ \to p_3 \end{matrix}$$

2.A small computer store has room to display up to three computers for sale. Customers come at times of a Poisson process with rate 2 per week to buy a computer and will buy one if at least 1 is available. When the store has only one computer left it places an order for two more computers. The order takes an exponentially distributed amount of time with mean 1 week to arrive. Of course, while the store is waiting for delivery, sales may reduce the inventory to 1 and then to 0. (a) Write down the matrix of transition rates Rij and solve $\vec{P}R = 0$ to find the stationary distribution. (b) At what rate does the store make sales?

States : # computers in the store .

$\{0, 1, 2, 3\}$ you order 2

If you have $0 \rightarrow 2$ with rate 1 per week ,

If you have $1 \nearrow 0$ with rate 2/week (rate of arrival)

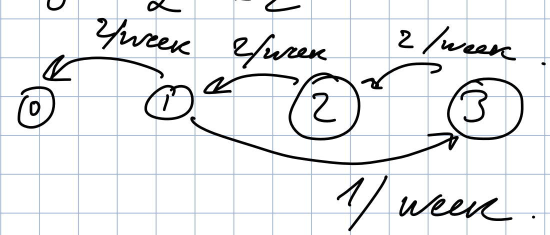
$\searrow 3 \rightarrow$ ordered $\rightarrow$ 1 per week
(delivery per week

$2 \rightarrow 1$ - rate 2/week .

$3 \rightarrow 1$ per week.

$$R = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & -1 & 0 & 1 & 0 \\ 1 & 2 & -3 & 0 & 1 \\ 2 & 0 & 2 & -2 & 0 \\ 3 & 0 & 0 & 2 & -2 \end{array}$$

$\text{rate} = \dfrac{1}{ET}$ ← time until event.



$$A = \begin{pmatrix} -1 & 2 & 0 & 0 \\ 0 & -3 & 2 & 0 \\ 1 & 0 & -2 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad \vec{P} = \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

$$A\vec{P} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \vec{P} = \begin{pmatrix} 0.4 \\ 0.2 \\ 0.3 \\ 0.1 \end{pmatrix}$$

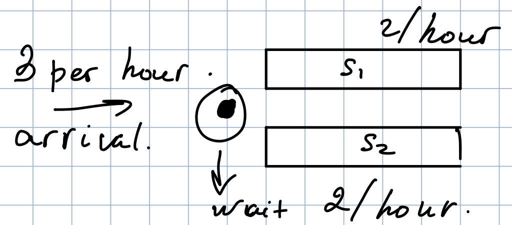
b) Rate $= \lambda p$ $\quad p \to$ probability to sell at least one computer.

$\lambda = 2$

$p = 1 - \underbrace{P_0}_{\text{no sale}} = 1 - 0.4 = 0.6$

Rate $= 2 \cdot 0.6 = 1.2$.

. *3.* A service center consists of two servers, each working at an exponential rate of two services per hour. If customers arrive at a Poisson rate of three per hour, then, assuming a system capacity of at most three customers,
(a) what fraction of potential customers enter the system?
(b) what would the value of part (a) be if there was only a single server, and his rate was twice as fast (that is, μ = 4)?



What fraction of the customers enter system.

$E\, S_1 = \frac{1}{2}$ $\quad E\, S_2 = \frac{1}{2}$ $\leftarrow$ service time

$E\, T_A = \frac{1}{3}$ $\leftarrow$ time until arrival.

It is death and birth process.

$\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = 3 \qquad \lambda_i = 0 \quad i \geq 4.$

$\mu_1 = 2 \quad \mu_2 = \mu_3 = 4.$

$P_1 = \frac{3}{2} P_0 \qquad P_3 = \frac{3}{4} P_4 = \frac{27}{32} P_0$

$P_2 = \frac{3}{2} P_1 = \frac{9}{8} P_0$

$$p_0 + \frac{3}{2}p_0 + \frac{9}{8}p_0 + \frac{27}{32}p_0 = 1$$

$$p_0\left(1 + \frac{3}{2} + \frac{9}{8} + \frac{27}{32}\right) = 1$$

$$p_0 = \frac{32}{143}$$

a) The fraction of customers that enters system.

$$1 - \frac{\text{rate when 3 customers are there}}{\text{total rate}} =$$

$$= 1 - \frac{\lambda P_3}{\lambda} = 1 - P_3 = 1 - \frac{27}{32} \cdot \frac{32}{143} = \frac{116}{143}.$$

b) $\mu_1 = 4 \qquad \mu_2 = \mu_3 = 8$

$\lambda_1 = \lambda_2 = \lambda_3 = 3$

$$P_1 = \frac{3}{4}p_0$$

$$P_2 = \frac{3}{4}P_1 = \frac{9}{16}p_0$$

$$P_3 = \frac{3}{4}P_2 = \frac{27}{64}p_0$$

$$p_0 + \frac{3}{4}p_0 + \frac{9}{16}p_0 + \frac{27}{64}p_0 = 1$$

$$p_0 = \frac{64}{175}$$

$$P_3 = \frac{27}{\cancel{64}} \cdot \frac{\cancel{64}}{175} = \frac{27}{175}$$

$$1 - P_3 = 1 - \frac{27}{175} = \frac{148}{175}.$$

#4 similar problem is in HW. Will skip.

5. Two people are working in a small office selling shares in a mutual fund. Each is either on the phone or not. Suppose that calls come into the two brokers at rate $\lambda_1=\lambda_2=1$ per hour, while the calls are serviced at rate $\mu_1=\mu_2=3$.

(a) Formulate a Markov chain model for this system with state space (0; 1; 2; 12) where the state indicates who is on the phone. (b) Find the stationary distribution.
(c) Suppose they upgrade their telephone system so that a call to one line that is busy is forwarded to the other phone and lost if that phone is busy. Find the new stationary probabilities. (d) Compare the rate at which calls are lost in the two systems.

$$v_0 = \lambda_1 + \lambda_2$$

$$P_{01} = \frac{\lambda_1}{\lambda_1+\lambda_2} \qquad q_{01} = \lambda_1$$

$$P_{02} = \frac{\lambda_2}{\lambda_1+\lambda_2} \qquad q_{02} = \lambda_2.$$

$$v_1 = \mu_1 + \lambda_2$$

$$P_{10} = \frac{\mu_1}{\mu_1+\lambda_2} \qquad q_{10} = \mu_1$$

$$P_{1\,12} = \frac{\lambda_2}{\mu_1+\lambda_2} \qquad q_{1\,12} = \lambda_2.$$

$$v_2 = \lambda_1 + \mu_2$$

$$P_{20} = \frac{\mu_2}{\mu_2+\lambda_1} \qquad q_{20} = \mu_2$$

$$P_{2\,12} = \frac{\lambda_1}{\mu_2+\lambda_1}$$

$$R = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 12 \end{array} \\ \begin{pmatrix} -(\lambda_1+\lambda_2) & \lambda_1 & \lambda_2 & 0 \\ \mu_1 & -(\mu_1+\lambda_2) & 0 & \lambda_2 \\ \mu_2 & 0 & -(\mu_2+\lambda_1) & \lambda_1 \\ 0 & \mu_1 & \mu_2 & -(\mu_1+\mu_2) \end{pmatrix} \end{array}$$

$$A = \begin{pmatrix} -(\lambda_1+\lambda_2) & \mu_1 & \mu_2 & 0 \\ \lambda_1 & -(\mu_1+\lambda_2) & 0 & \mu_1 \\ \lambda_2 & 0 & -(\mu_2+\lambda_1) & \mu_2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$A\vec{p} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$3 \rightarrow$ T2

$$\vec{p} = A^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_1 = \lambda_2 \qquad \mu_1 = \mu_2$$

$$p = \begin{bmatrix} \mu^2/(\lambda+\mu)^2 \\ \lambda\mu/(\lambda+\mu)^2 \\ \lambda\mu/(\lambda+\mu)^2 \\ \lambda^2/(\lambda+\mu)^2 \end{bmatrix} = \begin{bmatrix} 9/16 \\ 3/16 \\ 3/16 \\ 1/16 \end{bmatrix}$$

$\mu = 3$

$\lambda = 1$