# Review

## Ch 7.2

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu}$$

$$m(t) = \mathbb{E}(N(t))$$

$$\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}$$

$\{X(t), t \geq 0\}$

$v_i$ = rate to exit state $i$

$P_{ij}$ = prob to go from $i \to j$

$P_{ii} = 0$

$R$ = rate mat

$$R_{ij} = \begin{cases} q_{ij} = v_i P_{ij}, & i \neq j \\ -v_i & , i = j \end{cases} \Rightarrow P_{ij} = \frac{q_{ij}}{v_i}$$

$P(t)$

$P'(t) = R \cdot P(t)$
backward

$P'(t) = P(t) R$
forward

$P(t) = e^{Rt}$
expm

## Birth & Death

$\lambda_i$ = birth $\Delta$ $\quad i \to i+1$
$\mu_i$ = death $\Delta$ $\quad i \to i-1$

$$v_0 = \lambda_0, \quad v_i = \lambda_i + \mu_i \Rightarrow P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}, \quad P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}$$

$q_{i,i+1} = \lambda_i$ , $q_{i,i-1} = \mu_i$

$T_i$ , $i \to i+1$

$$\mathbb{E}(T_i) = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i}\mathbb{E}(T_{i-1})$$

Example : Time to go from $0 \to 2$

$\lambda_i = (i+1)\lambda$ $\lambda_0 = \lambda$ $\mu_0 = 0$

$\lambda_1 = 2\lambda$ $\mu_1 = \mu$

$\mu_i = i\mu$ $\lambda_2 = 3\lambda$ $\mu_2 = 2\mu$

$0 \to 1$ $\mathbb{E}(T_0) = \frac{1}{\lambda_0}$

$1 \to 2$

$\mathbb{E}(T_1) = \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1}\left(\frac{1}{\lambda_0}\right)$ $\Sigma$

$\mathbb{E}(T_2) = \frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2}\mathbb{E}(T_1)$

Balance Equations

$p_j = \lim_{t \to \infty} p_{ij}(t)$ recurrent

$$\begin{cases} v_i p_j = \sum_{k \neq j} q_{kj} p_k \\ \sum p_i = 1 \end{cases}$$

$\vec{p} = \begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix}$ $(p_1 \dots p_m)R = 0$

$R^T \begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix} = 0$

$p_i q_{ij} = p_j q_{ji}$

Example: Machine has transient states

$i = 1, 2, 3$ $\lambda_1 = 1/5$ $\mu_1 = 1/3$ $(-r)^{-1} w =$ exit distr

$\lambda_2 = 1/6$ $\mu_2 = 1/5$ $(-r)^{-1}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} =$ exit time

$\lambda_3 = 1/7$ $\mu_3 = 1/6$

$$\begin{pmatrix} -(\lambda_1+\lambda_2+\lambda_3) & \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & -\mu_1 & 0 & 0 \\ \mu_2 & 0 & -\mu_2 & 0 \\ \mu_3 & 0 & 0 & -\mu_3 \end{pmatrix}$$

Example: Spring cleaning

J = Jill rake, K bath — J kitch, $\mathbb{E}T = 30$, 2

K = Kelly rake, J kitchen — K bath, $\mathbb{E}T = 1$ hr, $\frac{3}{2}$

(J) = J done, K work

(K) = K done, J work

0 = done

$$\begin{array}{c|ccccccc} & I & J & K & JK & (J) & (K) & \emptyset \\ \hline I & -\frac{7}{2} & 2 & \frac{3}{2} & 0 & 0 & 0 & 0 \\ & 0 & -\frac{5}{2} & 0 & \frac{3}{2} & 0 & 1 & 0 \\ & 0 & 0 & -3 & 2 & 1 & 0 & 0 \\ & 0 & 0 & 0 & -2 & 0 & 0 & 2 \\ & 0 & 0 & 0 & 0 & -2 & 0 & 2 \\ & 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} & \frac{3}{2} \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$