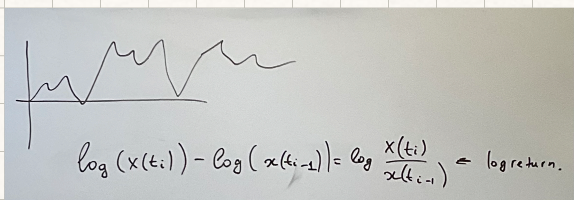
10.3 Vtions of Brownian Motion

Geometric Brownian motion

Let $\{Y(t), t \geq 0\}$ be Brownian motion process w/ drift coeff μ & V σ^2

$\{X(t), t \geq 0\}$ geo Brown motion

$$X(t) = e^{Y(t)}$$



Financial side

$$\mathbb{E}[X(t) \mid X(u), 0 \leq u \leq s] = \mathbb{E}[e^{Y(t)} \mid Y(u), 0 \leq u \leq s]$$

$s < t$

$$= \mathbb{E}[e^{Y(s)+Y(t)-Y(s)} \mid Y(u), 0 \leq u \leq s]$$

$$= \mathbb{E}[e^{Y(s)} e^{Y(t)-Y(s)} \mid Y(u), 0 \leq u \leq s]$$

$$= e^{Y(s)}\, \mathbb{E}[e^{Y(t)-Y(s)}]$$

$$Y(t) - Y(s) \sim N(\mu(t-s), (t-s)\sigma^2)$$

$$W \sim N(\mu, \sigma^2)$$

$$\mu(a) = \mathbb{E}(e^{aW}) = e^{a\mu + a^2(\frac{\sigma^2}{2})}$$

$$= e^{Y(s)}\left[e^{\mu(t-s)+(t-s)\frac{\sigma^2}{2}}\right] = X(s)\left[e^{\mu(t-s)+(t-s)\frac{\sigma^2}{2}}\right]$$

Notation

$\frac{X_n}{X_{n-1}}$ $\quad n \geq 1 \quad$ $Y_n = \frac{X_n}{X_{n-1}} \Rightarrow X_n = Y_n X_{n-1}$

$$X_n = Y_n Y_{n-1} \ldots Y_1 Y_0$$

$\log X_n = \sum_{i=1}^{n} \log(Y_i) + \log(X_0)$

$\log(x_i) - \log(x_{i-1}) = \log \text{return}$

10.6 White Noise

Let $\{X(t), t \geq 0\}$ be stand Brown motion

$f(\cdot)$ = function

$f \in C^1[a,b]$

$$\int_a^b f(t)\,dX(t) \equiv \lim_{n\to\infty} \sum_{i=1}^{n} f(t_{i-1})(X(t_i) - X(t_{i-1}))$$

$(t_i - t_{i-1}) \to 0$

$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$

$$\equiv f(b)X(b) - f(a)X(a) - \sum_{i=1}^{n} X(t_i)(f(t_i) - f(t_{i-1}))$$

$$\equiv f(b)X(b) - f(a)X(a) \int_a^b X(t)\,df(t)$$

$$\mathbb{E}\left[\int_a^b f(t)\,dX(t)\right] = 0$$

$$\mathbb{V}\left\{\sum_{i=1}^{n} f(t_{i-1})[X(t_i) - X(t_{i-1})]\right\} = \sum_{i=1}^{n} \mathbb{V}\{f(t_{i-1})[X(t_i) - X(t_{i-1})]\}$$

$$= \sum_{i=1}^{n} f^2(t_{i-1})\,\mathbb{V}[X(t_i) - X(t_{i-1})]$$

$$= \sum_{i=1}^{n} f^2(t_{i-1})(t_i - t_{i-1})$$

$$\xrightarrow[n\to\infty]{} \int_a^b f^2(t)\,dt$$

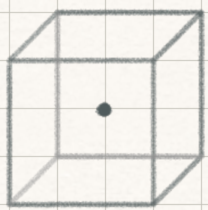
$\{dX(t), 0 \leq t < \infty\}$ white noise transform

$$f(t) \to \int_a^b f(t)\,dX(t)$$

Remark: It's an operator that carries funct f

info val $\int_a^b f(t)\,dX(t)$

Ex 10.4



$V'(t) = -\beta V(t) + \alpha X'(t)$

$\{X(t), t \geq 0\}$
stand Brown motion

$e^{\beta t}(V'(t) + \beta V(t)) = e^{\beta t} \alpha X'(t)$

$\frac{d}{dt}\left(e^{\beta t} V(t)\right) = \alpha e^{\beta t} X'(t)$

$\int_0^t d e^{\beta t} V(t) = \int_0^t \alpha e^{\beta s} X'(s)\,ds$

$e^{\beta t} V(t) = V(0) + \alpha \int_0^t e^{\beta s} X'(s)\,ds$

$V(t) = e^{-\beta t} V(0) + \alpha \int_0^t e^{-\beta t} e^{\beta s} X'(s)\,ds$

$= e^{-\beta t} V(0) + \alpha \int_0^t e^{-\beta(t-s)}\,dX(s)$

$= e^{-\beta t} V(0) + \alpha\left(e^{-\beta(t-t)} X(t)\right) -$

$\int_0^t e^{-\beta(t-s)}\,dX(s)$

$= e^{-\beta t} V(0) + \alpha X(t) - \int_0^t X(s) e^{-\beta(t-s)} \beta\,ds$