Brownian Motion

Example

$y(t)$ = amt of time (in seconds) by which racer * is ahead 100t% of race has been completed

$\{y(t), 0 \le t \le 1\}$

$\mathbb{E}[y(t)] = 0$

$\mathbb{V}[y(t)] = \sigma^2 t$

If inside racer is leading by σ sec @ midpt of race, what's P(she's winner)?

$y(s) \mid y(t) = c \quad s < t$

$\mathbb{E}(y(t)) = \sigma \, \mathbb{E}(X(t)) = \sigma$

$\mathbb{V}(y(t)) = \sigma^2 \, \mathbb{V}(X(t))$

$X(t) = \frac{y(t)}{\sigma}$

$\mathbb{E}(X(s) \mid X(t) = B) = \frac{s}{t}(B) = \mu$

$\mathbb{V}(X(s) \mid X(t) = B) = \frac{s}{t}(t-s) = \sigma^2$

$X(s) \mid X(t) = B \sim N(\mu, \sigma^2)$

$\sim N\left(\frac{s}{t}B, \frac{s}{t}(t-s)\right)$

$$P\left[\frac{y\left(\frac{1}{2}\right)}{\sigma} > 0 \,\middle|\, \frac{y(1)}{\sigma} = \sigma\right] = P\left[X\left(\frac{1}{2}\right) > 0 \,\middle|\, X(1) = \sigma\right]$$

$W = X(0.5) | X(1) = 1 \sim N(0.5, 0.25)$

$$P(W > 0) = P\left(Z > \frac{0 - 0.5}{0.5}\right) = P(Z > -1) = P(Z < 1)$$

$$= \varphi = 0.8413$$

10.2 Hitting Time Max Var & Gambler Ruin Problem

Let T_a be 1st time BM hits a

$P(T_a \le t)$

$$P(X(t) \ge a) = P(X(t) \ge a | T_a \le t) P(T_a \le t) +$$

$$P(X(t) \ge a | T_a > t) P(T_a > t)$$

$$= P(X(t) \ge a | T_a \le t) P(T_a \le t)$$

$$= \frac{1}{2} P(T_a \le t)$$

$$\Rightarrow \quad 2P(X(t) \ge a) = P(T_a \le t)$$

$$P(T_a \le t) = 2P(X(t) \ge a) = 2\int_a^{\infty} \frac{1}{\sqrt{2\pi t}} \left(e^{-x^2/2t}\right) dt$$

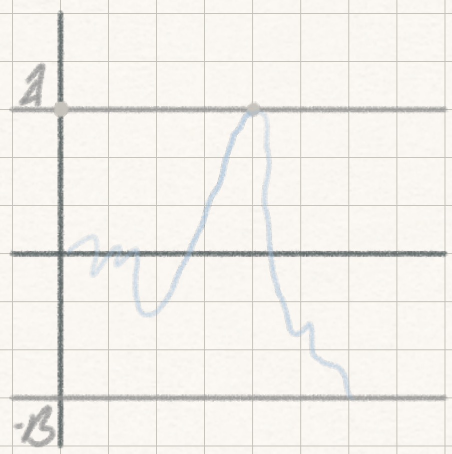
$X(t) \sim N(0, t)$

$$= 2P\left(Z > \frac{a}{\sqrt{t}}\right) = 2P\left(Z \le -\frac{a}{\sqrt{t}}\right)$$

$$= 2\varphi\left(-\frac{a}{\sqrt{t}}\right)$$

$$P\left(\max_{0 \le s \le t} (X(s)) \ge a\right) = P(T_a \le t) = 2\varphi\left(-\frac{a}{\sqrt{t}}\right)$$

$A, B > 0$



Δ - step

$$n = \frac{A+B}{\Delta x}$$

$$i = \frac{B}{\Delta x}$$

$$\frac{B\,\Delta x}{(A+B)\,\Delta x} = \frac{B}{A+B}$$

$$P(\uparrow A \text{ by } B \downarrow) = \frac{B}{A+B}$$

10.3 Variations of Brownian Motion

Define: We say that $\{X(t), t \geq 0\}$ is Brownian motion w/ drift coeff μ & variance σ^2 if

i. $X(0) = 0$

ii. $\{X(t), t \geq 0\}$ has stationary & indep increments

iii. $X(t) \sim N(\mu t, t\sigma^2)$

$X(t) = \sigma B(t) + \mu t$

$$= P\left[y(1) - y\left(\tfrac{1}{2}\right) > -\sigma \,\middle|\, y\left(\tfrac{1}{2}\right) = \sigma\right]$$

$$y\left(\tfrac{1}{2}\right) \sim \mathcal{N}\left(0, \frac{\sigma^2}{2}\right)$$

$$= P\left[y(1) - y\left(\tfrac{1}{2}\right) > -\sigma\right]$$

$$= P\left[y\left(\tfrac{1}{2}\right) > -\sigma\right] = P\left(Z > \frac{-\sigma - 0}{\sigma/\sqrt{2}}\right)$$

$$= P(Z > -\sqrt{2}) = P(Z \le \sqrt{2})$$