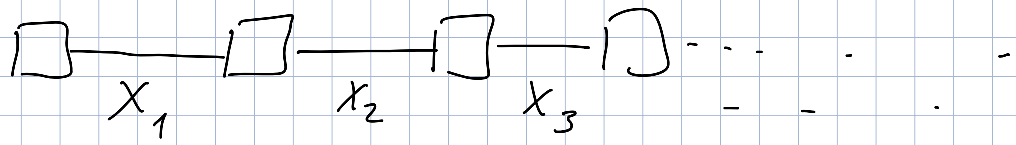
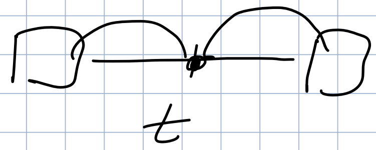
The inspection paradox.



$\{N(t), t \geq 0\}$ - the renewal process.

$X_i \sim F$ which is unknown.

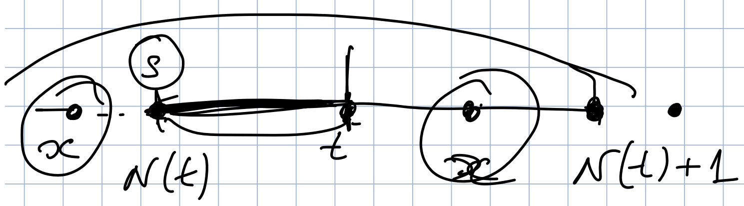
We would to estimate F by the following ispection scheme.



We fix some time t and observe the total life time for the battery that in in use by time t.

It turns out that the battery in use at time t tends to hate a larage life time than an ordinary battery.

inspection paradox.



$$X_{N(t)+1} = S_{N(t)+1} - S_{N(t)}$$

$$P\{X_{N(t)+1} > x\} = E\{P(X_{N(t)+1} > x \mid S_{N(t)} = t-s\}$$

$S_{N(t)}$ – last renewal prior to t

$S_{N(t)+1}$ – first renewal after time t.

Since there are no renewals between $t-s$ and t it follows that $X_{N(t)+1}$ must be larger then x if $s > x$

$$P(X_{N(t)+1} > x \mid S_{N(t)} = t-s) = 1 \quad \text{if } s > x$$

On other hand suppose that $s \le x$

Renewal occurs at time $t-s$ and no additional renewal between $t-s$ and t.

and we ask for the probability that no renewals occur for additional time $x-s$.

That is we are asking probability that an interarrival time will be greater than x given it is greater than s

Therefore for $s \le x$

$$P(X_{N(t)+1} > x \mid S_{N(t)} > t-s) =$$

$$= P(\text{interarrival time} > x \mid \text{interarrival time} > s)$$

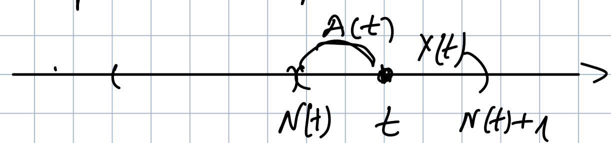
$$= \frac{P(X_{N(t)+1} > x,\ S_{N(t)} > t-s)}{P(S_{N(t)} > t-s)} =$$

$$= \frac{1 - F(x)}{\underbrace{1 - F(s)}} \geq 1 - F(x)$$

$$P\left(X_{N(t)+1} > x \mid S_{N(t)} = t - s \right) \geq \underbrace{1 - F(x)}$$

$$P\left(X_{N(t)+1} \geq x \right) \geq 1 - F(x)$$

Remark. To obtain an intuition for an inspection paradox.



Assume the renewal process is a Poisson process.

$$X_{N(t)+1} = A(t) + Y(t)$$

$A(t)$ – time since last renewal – age

$Y(t)$ — excess time, time until next renewal after t.

$A(t) = t - S_{N(t)}$

$Y(t) = S_{N(t)+1} - t$.

$A(t)$ $Y(t)$ are indep.

$$P(Y(t) \leq x) = 1 - e^{-\lambda x}$$

$$P(A(t) > x) = \begin{cases} P(0 \text{ renewal } [t-x, t]) & x \le t \\ 0 & x > t \end{cases}$$

$$P(A(t) \le x) = \begin{cases} 1 - e^{-\lambda t} & \underbrace{x \le t} \\ 1 & x > t. \end{cases}$$

$$A(t) + Y(t) \sim \text{Gamma}(2, \lambda)$$

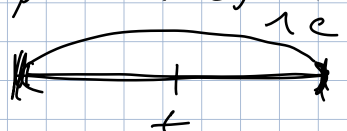
$$f_A(t) * f_Y(t) \qquad E\left(A(t) + Y(t)\right) = \frac{2}{\lambda}$$

$$X_{N(t)+1} = A(t) + Y(t)$$

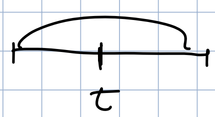
$$\lim_{S \to \infty} \frac{\int_0^\infty X_{N(t)+1} \, dt}{S} = \frac{E X^2}{E X}$$

$$E(X^2) > (E(X))^2$$

We can use an alternating renewal process argument to determine the long-run proportion of time that $X_{N(t)+1} > c$



System on.



$< c \implies$ system is off

Thus if X is the cycle time

$$\text{On time in cycle} = \begin{cases} X & X > c \\ 0 & X \le c \end{cases}$$

long run proportion of time $X_{N(t)+1} > c = \frac{E[\text{on time}]}{E[\text{length cycle}]}$

$$= \frac{\int_c^\infty x f(x)\,dx}{\mu}$$