7.15 Consider a miner trapped in a room that contains 3 doors. Door I leads him to freedom after 2 days of travel; door II returns him to his room after 4-day journey; & door III returns him to his room after 6-day journey. Suppose @ all times he is equally likely to choose any of the 3 doors, & let T denote the time it takes the miner to become free.

a. Define sequence of iid rvs $(X_1, X_2, \dots)$ & a stopping time N s.t. $T = \sum_{i=1}^{n} X_i$

Note: You may have to imagine that the miner cont to randomly choose doors even after he reaches safety

d1 = 2 days → free

d2 = 4 days → room

d3 = 6 days → room

$$\Rightarrow X = \begin{cases} 2, & d1 \ (p = \frac{1}{3}) \\ 4, & d2 \ (p = \frac{1}{3}) \\ 6, & d3 \ (p = \frac{1}{3}) \end{cases}$$

$N = \min\{n : X_n = 2\}$

stopping time as event $N = n$ is determined w/ first n obs of X

b. Use Wald's equation to find $\mathbb{E}T$

$\mathbb{E}(\sum_{n=1}^{n} X_n) = \mathbb{E}N(\mathbb{E}X) \Rightarrow \mathbb{E}T = 3(4) = 12$

$\mathbb{E}N = 3$

$N \sim Geo(p = \frac{1}{3})$

$\mathbb{E}X = \frac{x_1 + x_2 + x_3}{n} = \frac{2+4+6}{3} = \frac{12}{3}$

c. Compute $\mathbb{E}(\sum_{i=1}^{n} X_i \mid N=n)$ & note that it's not equal to $\mathbb{E}(\sum_{i=1}^{n} X_i)$

$$\mathbb{E}(\textstyle\sum_{i=1}^{n} X_i \mid N=n) = \mathbb{E}(\sum_{i=1}^{n} X_i \mid X_1 \neq 2, \dots X_{n-1} \neq 2, X_n = 2)$$

$$= 2 + (n-1)\mathbb{E}(\textstyle\sum_{i=1}^{n} X_i \mid X_i \neq 2) = 2 + (n-1)4$$

$$= 2 + 4n - 4 = 4n - 2$$

$$\mathbb{E}(\textstyle\sum_{i=1}^{n} X_i) = 4n$$

d. Use pt (c) for 2nd derivation of $\mathbb{E}T$

$$\mathbb{E}(T \mid N) = 2 + 4\mathbb{E}(N-1) = 2 + 4(3-1) = 2 + 4(2) = 10$$

$$\mathbb{E}T = 4\mathbb{E}(N) - 2 = 4 \cdot 3 - 2 = 10$$

2. A policeman cruises (on avg) approx 10 min before stopping a car for speeding. 90% of cars stopped are given speeding tickets w/ \$80 fine. It takes the policeman an avg of 5 min to write such a ticket. The other 10% of stops are for more serious offenses, leading to an avg fine of \$300. These more serious charges take an avg of 30 min to process. In the long run, @ what rate (\$/min) does he assign fines?

cruis ~ 10 min $\longrightarrow$ ticket

speed ticket - \$80, 5 min 90%

serious charge - \$300, 30 min 10%

long run = $\dfrac{\text{amt collected by police}}{\text{avg time bet stops}} = \dfrac{\$102}{17.5 \text{ min}} = \dfrac{\$5.8286}{\text{min}}$

stops = cruise + speed fine % (time to write) + serious % (time to write)

$= 10 \text{ min} + 0.9(5 \text{ min}) + 0.1(30 \text{ min})$

$= 10 + 4.5 + 3 = 17.5 \text{ min}$

amt = speed time% (cost) + serious% (cost)

$= 0.9(80) + 0.10(300) = 72 + 30 = \$102$

In the long run, the policeman assigns fines at \$5.83/min

3. A machine tool wears over time & may fail. Failure time measured in months has density $f_T(t) = \frac{2t}{900}$ for $0 \le t \le 30$ & 0 otherwise. If the tool fails, it must be replaced immediately @ cost of \$1200. If it's replaced prior to failure, the cost is only \$300. Consider a replacement policy in which the tool's replaced after c months or when it fails. What's the value of c that minimizes cost/unit of time?

long-run: $\frac{E(r_i)}{E(t_i)} = (c^2+300) \Big/ \left(c - \frac{c^3}{2700}\right)$

$c = \text{months} \Rightarrow r_i \begin{cases} T_i \le c & \$1200 \\ T_i > c & \$300 \end{cases}$

probability of failure

$\int_0^c f_T(t)\,dt = \int_0^c \frac{2t}{900}\,dt = \frac{t^2}{900}\Big|_0^c = \frac{c^2}{900}$

numerator

$E(r_i) = 1200[P(T_i \le c)] + 300[P(T_i > c)]$

$= 1200\left(\frac{c^2}{900}\right) + 300\left(1 - \frac{c^2}{900}\right) = \frac{4}{3}c^2 + 300 - \frac{c^2}{3}$

$= c^2 + 300$

denom

$$\mathbb{E}(T_i) = \int_0^c t\left(\frac{2t}{900}\right)dt + c\left(1-\frac{c^2}{900}\right) = \int_0^c \frac{2t^2}{900}\,dt + c - \frac{c^3}{900}$$

$$= \frac{2t^3}{2700}\Big|_0^c + c - \frac{c^3}{900} = \frac{2c^3}{2700} + c - \frac{3c^3}{2700}$$

$$= c - \frac{c^3}{2700}$$

$$\frac{d}{dc}\left(\frac{c^2+300}{c-\frac{c^3}{2700}}\right) = \frac{2c\left(c-\frac{c^3}{2700}\right) - \left(1-\frac{c^2}{900}\right)(c^2+300)}{\left(c-\frac{c^3}{2700}\right)^2}$$

$$2c\left(c-\frac{c^3}{2700}\right) - \left(1-\frac{c^2}{900}\right)(c^2+300) = 0$$

$$\Rightarrow 2c^2 - \frac{c^4}{1350} - \left(c^2 + 300 - \frac{c^4}{900} - \frac{c^2}{3}\right) = 0$$

$$\Rightarrow 2c^2 - \frac{c^4}{1350} - \frac{2}{3}c^2 - 300 + \frac{c^4}{900} = 0$$

$$\Rightarrow \frac{c^4}{2700} + \frac{4}{3}c^2 - 300 = 0 \quad \Rightarrow \quad c^4 + 3600c^2 - 810000 = 0$$

$$\Rightarrow \quad c^2 = \frac{-3600 \pm \sqrt{(3600)^2 - 4(1)810000}}{2}$$

$$= \frac{-3600 \pm 1800\sqrt{3}}{2} = -1800 \pm 900\sqrt{3}$$

$$c^2 = 900\sqrt{5} \pm 1800 \Rightarrow \quad c = \sqrt{900\sqrt{5} - 1800} = 19.5760$$

$$\Rightarrow \quad c = \sqrt{900\sqrt{5} + 1800} = 61.7451$$

The value of c that minimizes cost/unit time is 19.567 months

7.46 For an interarrival distr F having mean μ, we defined equilibrium distr of F, denoted F_e, by

$$F_e(x) = \frac{1}{\mu}\int_0^x 1 - F(y)\,dy$$

a. Show that if $F \sim \exp$, then $F = F_e$

Assume $X \sim Exp\left(\frac{1}{\mu}\right)$

$$F_e(x) = \frac{1}{\mu}\int_0^x 1-F(y)\,dy = \frac{1}{\mu}\int_0^x e^{-\frac{y}{\mu}}\,dy = \frac{1}{\mu}\left(-\mu e^{-\frac{y}{\mu}}\Big|_0^x\right)$$

$$= \frac{1}{\mu}\left(\mu - \mu e^{-\frac{x}{\mu}}\right) = 1 - e^{-\frac{x}{\mu}}$$

b. If for some constant c,

$$F(x) = \begin{cases} 0 & ; \quad x < c \\ 1 & ; \quad x \geq c \end{cases}$$

show that $F_e \sim U(0, c)$. That is, if interarrival times are identically equal to c, then the equilibrium $\sim U(0, c)$

$$F_e(x) = \frac{1}{\mu}\int_0^x 1 - F(y)\,dy = \frac{1}{c}\int_0^x 1\,dy = \frac{1}{c}\left(y\Big|_0^x\right) = \frac{x}{c} \qquad 0 \leq x \leq c$$

c. The city of Berkeley, CA allows for 2 hrs parking @ all non-metered locations w/in one mile of UC. Parking officials regularly tour around, passing same pt every 2 hrs. When official encounters car, they mark it w/ chalk. If the same car is there on official's return 2 hrs later, then parking ticket is written. If you park your car in Berkeley & return after 3 hrs, what's the probability you'll have received a ticket?

3 hrs → ticket

parking official = renewal ⇒ 2 hrs

$F_e(x) \sim U(0,2)$

Assume 2 hrs have passed, probability of parking official comes in next hr

$$F_e(x) = \frac{1}{\mu}\int_0^x 1-F(y)\,dy = \frac{1}{c}\int_0^1 1\,dy = \frac{1}{2}\left(y\Big|_0^1\right) = \frac{1}{2}$$

If you park your car in Berkeley & return after 3 hrs, the probability you'll have received a ticket is 0.5.