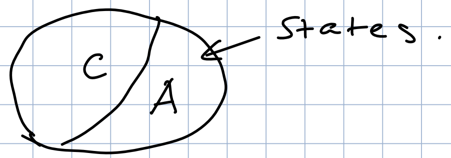
Exit time.

Let $C = S \setminus A = S \cap A'$ be finite.



$$\underbrace{P_i\left(V_A < \infty\right)}\; > 0 \qquad P\left(V_A < \infty \mid X(0) = i\right) > 0.$$

probability to reach A in finite time if you started at state i

Expected Time to exit states C if you started at $i \in C$ $\quad g(i)$

$$g(i) = E_i V_A$$

$$v_i \quad g(i) = v_i\left(\underbrace{\frac{1}{v_i}} + \sum_{j: j\neq i} \underbrace{P_{ij}\, g(j)}\right) \quad \Big|\ \text{condition to the first step.}$$

$$v_i\, g(i) = 1 + \sum_{j: i\neq j} \underbrace{v_i P_{ij}}_{q_{ij}}\, g(j)$$

$$\underbrace{\sum_{j: j\neq i} q_{ij}\, g(j) - v_i\, g(i)} = -1 \qquad R_{ij} = \begin{cases} q_{ij} & j\neq i \\ -v_i & i = j \end{cases}$$

$$\sum_j R_{ij}\, g(j) = -1.$$

$$-\sum_j R_{ij}\, \underbrace{g(j)} = 1 \qquad g = \begin{pmatrix} g(0) \\ \vdots \\ g(m) \\ \vdots \end{pmatrix}$$

$j \in A \quad g(j) = 0 \qquad \mathbb{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

$$r = \begin{cases} R_{ij} & i, j \in C \\ 0 & \text{otherwise.} \end{cases}$$

$$-\sum_{j \in C} r_{ij} g(j) = 1$$

$$-rg = \mathbb{1} \quad \Rightarrow \quad \boxed{g = (-r)^{-1} \mathbb{1}}$$

Ex. $\mu_1 = \mu_2 = 3$, $\lambda_1 = \lambda_2 = 2$. If there are 4 people in the salon customer will leave.

Compute $E_i T_0$ - expected time to start at i and get to 0.

$$R = \begin{array}{c|c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \hline \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} -2 & 2 & 0 & 0 & 0 \\ 3 & -5 & 2 & 0 & 0 \\ 0 & 6 & -8 & 2 & 0 \\ 0 & 0 & 6 & -8 & 2 \\ 0 & 0 & 0 & 6 & -6 \end{bmatrix} \end{array}$$

$$r = \begin{bmatrix} -5 & 2 & 0 & 0 \\ 6 & -8 & 2 & 0 \\ 0 & 6 & -8 & 2 \\ 0 & 0 & 6 & -6 \end{bmatrix} \qquad g = (-r)^{-1} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$(-r)^{-1}\mathbb{1} = \begin{bmatrix} 1/3 & 1/9 & 1/27 & 1/81 \\ 1/3 & 5/18 & 5/54 & 5/16 \\ 1/6 & 5/18 & 7/27 & 7/81 \\ 1/3 & 5/18 & 7/27 & 41/162 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 40/81 \\ 119/162 \\ 155/162 \\ 91/81 \end{bmatrix}$$

Reversed chain.

Let $\{X(t)\ t \geq 0\}$ be continuous time MC.

S – state space.

Assume that the chain is ergodic.

There is a limiting distribution which is a stationary distribution

$$P_i v_i = \sum_{i \neq j} P_j q_{ji}$$

Assume this chain operates for a long time.

The process of states going backwards in time is also a continuous time MC.

$$\boxed{P_i q^*_{ij} = P_j q_{ji}} \quad i \neq j$$

$$R = \begin{cases} q_{ij} & i \neq j \\ -v_i \end{cases}$$

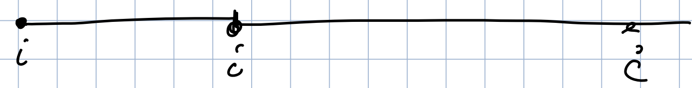
rate matrix for a forward process.

$$q^*_{ij} = \frac{P_j q_{ji}}{P_i}$$

$$R^* = \begin{cases} q^*_{ij} & i \neq j \\ -v^*_i \end{cases}$$

rate matrix for a backward process.

$$v_i = \sum_j q_{ij} \qquad v_i^* = \sum_j q_{ij}^*$$



$$v_i = v_i^*$$

Proposition: Let a continuous-time Markov chain has instanteneous transition rates q_{ij} and limiting probabilities P_i, $i \in S$ and let q_{ij}^* be the instanteneous rates of the reversed chain.

Then with $v_i^* = \sum_{j \neq i} q_{ij}^*$ and

$$v_i = \sum_{j \neq i} q_{ij}$$

$$v_i^* = v_i$$

Moreover P_i, $i \in S$ are also limiting probabilities of the reversed chain.

Proof. $P_i q_{ij}^* = P_j q_{ji}$

$$\underset{\substack{\| \\ v_i^*}}{\sum_{i;\, i \neq j} q_{ij}^*} = \sum_{j:\, i \neq j} \frac{P_j q_{ji}}{P_i} = \frac{1}{P_i} \sum_{j \neq i} P_j q_{ji} = \frac{1}{\cancel{P_i}} \cdot \cancel{P_i} v_i$$

$$v_i^* = v_i \qquad q_{ji} = \frac{q_{ij}^* P_j}{P_i}$$

Ex. 6.21.

$\{X(t),\ t \geq 0\}$ $\quad 0, 1, 2, 3 \ldots$

$0 \xrightarrow{\alpha_i} i$

$P_{0i} = \alpha_i$

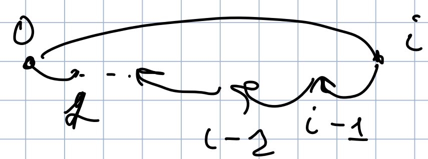
$\sum \alpha_i = 1.$

$P_{i,i-1} = 1$

$q_{0i} = v_0 \alpha_i$

$q_{i,i-1} = v_i$

Let N be a random variable having the distribution of the next state from 0 $\quad P(N=i) = \alpha_i,\ i > 0.$



cycle begins each time the chain goes to state 0.

Forward chain $\quad 0 \to i$

Reversed chain. $\quad 0 \to 1 \to 2 \to 3 \to \cdots \to N$

Now, chain is in state i then the value N for that cycle must be at least i

$$P(N=i \mid N \geq i) = \frac{P(N=i,\ N \geq i)}{P(N \geq i)} = \frac{\alpha_i}{P(N \geq i)}$$

and will be $i+1$ with prob.

$$1 - P(N=i \mid N \geq i) = P(N \geq i+1 \mid N \geq i) =$$

$$= \frac{P(N \geq i+1)}{P(N \geq i)}$$

$$q^*_{i0} = v_i \frac{\alpha_i}{P(N \geq i)} \qquad i > 0.$$

$$q^*_{i,i+1} = v_i \frac{P(N \geq i+1)}{P(N \geq i)}$$

$$P_0 q_{0i} = P_i q^*_{i0} \qquad i \geq 1 \qquad q_{0i} = v_0 \alpha_i$$

$$P_i q_{i,i-1} = P_{i-1} q^*_{i-1,i} \qquad i \geq 1$$

$$P_0 v_0 \alpha_i = P_i v_i \frac{\alpha_i}{P(N \geq i)}$$

$$P_i v_i = P_{i-1} v_{i-1} \frac{P(N \geq i)}{P(N \geq i-1)}$$

$$P_i = P_0 \underline{\underline{v_0}} \frac{P(N \geq i)}{v_i} \qquad i \geq 1$$

$$\sum_{\forall i} P_i = 1$$

$$P_0 + P_1 + \cdot \cdot + \quad =$$

$$= P_0 \left(1 + v_0 \frac{P(N \geq 1)}{v_1} + v_0 \frac{P(N \geq 2)}{v_2} + \ldots \right)$$

$$P_0 v_0 \cdot \sum_{i=0}^{\infty} P(N \geq i) / v_i = 1$$

$$P_0 = \frac{1}{v_0 \left(\sum_{i=0}^{\infty} P(N \geq i) / v_i \right)}$$

$$P_i = \frac{P(N \geq i) / v_i}{\sum_{i=0}^{\infty} P(N \geq i) / v_i} \qquad i \geq 0.$$

Recitation.

Problem 1.

1 repairman.

$T_i \sim Exp(1/60)$ $\qquad P_1, P_2 \quad P_3$

$T_r \sim Exp(1/4)$.

states $(0, 1, 2, 3)$

$$R = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} -1/4 & 1/4 & 0 & 0 \\ 1/60 & -\left(1/60 + \frac{1}{4}\right) & 1/4 & 0 \\ 0 & \frac{1}{60} & -\left(\frac{1}{60} + \frac{1}{4}\right) & \frac{1}{4} \\ 0 & 0 & 1/60 & -1/60 \end{bmatrix} \end{array}$$

W R

W W

$$\boxed{p_i \, q_{ij} = p_j \, q_{ji}} \qquad P \cdot R = R \, P$$

$$p_0 \, q_{01} = p_1 \, q_{10}$$

$$p_1 \, q_{12} = p_2 \, q_{21}$$

$$p_1 = \frac{q_{10}}{q_{01}} p_0 = \frac{1/60}{1/4} p_0 = \frac{2}{30} p_0$$

$$p_2 = \frac{q_{12}}{q_{21}} p_1 = \frac{1/4}{1/60} p_1 = \frac{30}{2} \cdot \frac{2}{30} \cdot p_0$$

$$p_0 + \frac{\cancel{1}}{15} p_0 + p_0 = 1$$