detailed balance
$P_i q_{ij} = P_j q_{ji}$
$\sum P_i = 1$

Recitation 3.
Time-reversible Markov chains.
Detailed balance equation. Exit distribution, exit time.

Problem 1. A factory has three machines in use and one repairman. Suppose each machine works for an exponential amount of time with mean 60 days between breakdowns, but each breakdown requires an exponential repair time with a mean 4 days. What is the long-run fraction of time all three machines are working? (Use a detailed balance equation).

Problem 2. Consider a chain with state space {1, 2, 3} in which $q_{ij} > 0$ if $i \neq j$ and suppose that there is a stationary distribution that satisfies the detailed balance condition. (a) Let $P_1 = c$. Use the detailed balance condition between 1 and 2 to find P2 and between 2 and 3 to find P3. (b) What conditions on the rates must be satisfied for there to be a detailed balance between 1 and 3?

Problem 3
Consider a continuous-time Markov chain with states 1, . . . , n, which spends an exponential time with rate v_i in state i during each visit to that state and is then equally likely to go to any of the other n−1 states.
(a) Is this chain time reversible? (b) Find the long-run proportions of time it spends in each state.

Problem 4
A submarine has three navigational devices but can remain at sea if at least two are working. Suppose that the failure times are exponential with means 1 year, 1.5 years, and 3 years. Formulate a Markov chain with states 0 = all parts working, 1,2,3 = one part failed, and 4 = two failures. Compute E0T4 to determine the average length of time the boat can remain at sea.

Problem 5
Excited by the recent warm weather, Jill and Kelly are doing spring cleaning at their apartment. Jill takes an exponentially distributed amount of time with mean 30 minutes to clean the kitchen. Kelly takes an exponentially distributed amount of time with mean 40 minutes to clean the bath room. The first one to complete their task will go outside and start raking leaves, a task that takes an exponentially distributed amount of time with a mean of one hour. When the second person is done inside, they will help the other and raking will be done at rate 2. (Of course, the other person may already be done raking in which case the chores are done.) What is the expected time until the chores are all done?

2. $\{1,2,3\}$ $q_{ij} > 0$ if $i \neq j$

(a) $P_1 = c$

1–2 : P_2

2–3 : P_3

$P_i q_{ij} = P_j q_{ji} \Rightarrow P_0 q_{01} = P_1 q_{10}$

$P_1 q_{12} = P_2 q_{21} \Rightarrow c q_{12} = P_2 q_{21} \Rightarrow P_2 = \frac{c q_{12}}{q_{21}}$

$P_2 q_{23} = P_3 q_{32} \Rightarrow \frac{c q_{12}}{q_{21}}(q_{23}) = \frac{c q_{13}}{q_{31}}(q_{32})$

$P_1 q_{13} = P_3 q_{31} \Rightarrow c q_{13} = P_3 q_{31} \Rightarrow P_3 = \frac{c q_{13}}{q_{31}}$

$P_1 q_{12} = P_2 q_{21} \Rightarrow P_2 = \frac{q_{12}}{q_{21}} P_1$

$P_2 q_{23} = P_3 q_{32} \Rightarrow P_3 = \frac{q_{23}}{q_{32}} P_2 = \frac{q_{23}(q_{12})}{q_{32}(q_{21})} P_1$

$P_1 q_{13} = P_3 q_{31}$

$\Rightarrow P_1 q_{13} = \frac{q_{31}(q_{23})q_{12}}{q_{32}(q_{21})} P_1 \Rightarrow q_{13} = \frac{q_{31}(q_{23})q_{12}}{q_{32}(q_{21})}$

(b)

$\Rightarrow q_{13}(q_{32})q_{21} = q_{31}(q_{23})q_{12}$

3. $\{1,\dots,n\}$ (a) time reversible?

exp $\Delta = v_i$ $\quad q_{ij} = \frac{v_i}{n-1}$ $\quad P_i q_{ij} = P_j q_{ji}$

$(n-1)$ states

$P_1\left(\frac{v_1}{n-1}\right) = P_2\left(\frac{v_2}{n-1}\right) \Rightarrow P_1 v_1 = P_2 v_2 \Rightarrow P_2 = \frac{v_1}{v_2}(P_1)$

$\sum P_i = 1$

$P_2 v_2 = P_3 v_3 \Rightarrow P_3 = \frac{v_2}{v_3}(P_2) \Rightarrow P_3 = \frac{v_2(v_1)}{v_3(v_2)} P_1$

$\vdots$

$P_n v_n = P_{n+1} v_{n+1} \Rightarrow P_{n+1} = \frac{v_{n+1}}{v_n} P_n$

$P_1 + P_2 + P_3 = 1 \Rightarrow P_1 + \frac{v_1}{v_2}(P_1) + \frac{v_1}{v_3} P_1 = 1$

$\Rightarrow P_1\left(1 + \frac{v_1}{v_2} + \frac{v_1}{v_3}\right) = 1$

$\Rightarrow P_1 = \frac{1}{v_1 \sum_{i=1}^{n} \frac{1}{v_i}} \Rightarrow \boxed{P_i = \frac{1}{v_i \sum_{j=1}^{n} \frac{1}{v_j}}}$

4.
$$\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & -2 & 1 & 2/3 & 1/3 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \\ 2 & 0 & 0 & -4/3 & 0 & 4/3 \\ 3 & 0 & 0 & 0 & -\frac{5}{3} & \frac{5}{3} \\ 4 & (-1 & 0 & 0 & 0 & 1) \end{array}$$

$(-r)^{-1}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$