Problem 1. A factory has three machines in use and one repairman. Suppose each machine works for an exponential amount of time with mean 60 days between breakdowns, but each breakdown requires an exponential repair time with a mean 4 days. What is the long-run fraction of time all three machines are working? (Use a detailed balance equation).

Let X_t be the number of working machines. Since there is one repairman we have $q(i, i+1) = 1/4$ for $i = 0, 1, 2$. On the other hand, the failure rate is proportional to the number of machines working, so $q(i, i-1) = i/60$ for $i = 1, 2, 3$. Setting $\pi(0) = c$ and plugging into the recursion (4.29) gives

$$\pi(1) = \frac{\lambda_0}{\mu_1} \cdot \pi(0) = \frac{1/4}{1/60} \cdot c = 15c$$
$$\pi(2) = \frac{\lambda_1}{\mu_2} \cdot \pi(1) = \frac{1/4}{2/60} \cdot 15c = \frac{225c}{2}$$
$$\pi(3) = \frac{\lambda_2}{\mu_3} \cdot \pi(2) = \frac{1/4}{3/60} \cdot \frac{225c}{2} = \frac{1125c}{2}$$

Adding up the π's gives $(1125 + 225 + 30 + 2)c/2 = 1382c/2$ so $c = 2/1480$ and we have

$$\pi(3) = \frac{1125}{1382} \quad \pi(2) = \frac{225}{1382} \quad \pi(1) = \frac{30}{1382} \quad \pi(0) = \frac{2}{1382}$$

Thus in the long run all three machines are working $1125/1382 = 0.8140$ of the time.

Problem 2

Problem 2. Consider a chain with state space {1, 2, 3} in which $q_{ij} > 0$ if $i \neq j$ and suppose that there is a stationary distribution that satisfies the detailed balance condition. (a) Let $P_1 = c$. Use the detailed balance condition between 1 and 2 to find P2 and between 2 and 3 to find P3. (b) What conditions on the rates must be satisfied for there to be a detailed balance between 1 and 3?

4.23. *Detailed balance for three state chains.* Consider a chain with state space $\{1, 2, 3\}$ in which $q(i, j) > 0$ if $i \neq j$ and suppose that there is a stationary distribution that satisfies the detailed balance condition. (a) Let $\pi(1) = c$. Use the detailed balance condition between 1 and 2 to find $\pi(2)$ and between 2 and 3 to find $\pi(3)$. (b) What conditions on the rates must be satisfied for there to be detailed balance between 1 and 3?

Ans. (a) $\pi(2) = cq(1,2)/q(2,1)$, $\pi(3) = cq(1,2)q(2,3)/q(3,2)q(2,1)$. (b) in order for $\pi(1)q(1,3) = \pi(3)q(3,1)$ we must have

$$c = c \cdot \frac{q(1,2)q(2,3)q(3,1)}{q(1,3)q(3,2)q(2,1)}$$

That is, $q(1,2)q(2,3)q(3,1) = q(1,3)q(3,2)q(2,1)$.

Problem 3

Consider a continuous-time Markov chain with states 1, . . . , n, which spends an exponential time with rate v_i in state i during each visit to that state and is then equally likely to go to any of the other n−1 states.

(a) Is this chain time reversible? (b) Find the long-run proportions of time it spends in each state.

40. The time reversible equations are

$$P(i)\frac{v_i}{n-1} = P(j)\frac{v_j}{n-1}$$

yielding the solution

$$P(j) = \frac{1/v_j}{\sum_{i=1}^{n} 1/v_i}$$

Problem 4

A submarine has three navigational devices but can remain at sea if at least two are working. Suppose that the failure times are exponential with means 1 year, 1.5 years, and 3 years. Formulate a Markov chain with states 0 = all parts working, 1,2,3 = one part failed, and 4 = two failures. Compute E0T4 to determine the average length of time the boat can remain at sea.

Ans. The transition rate matrix is:

	0	**1**	**2**	**3**	**4**
0	−2	1	2/3	1/3	0
1	0	−1	0	0	1
2	0	0	−4/3	0	4/3
3	0	0	0	−5/3	5/3

Let $g(i) = E_i T_4$. By (4.21)

$$g = -\begin{pmatrix} -2 & 1 & 2/3 & 1/3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -4/3 & 0 \\ 0 & 0 & 0 & -5/3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

-1 times =the first row of the inverse is $(1/2, 1/2, 1/4, 1/10)$ which the average amount of time spent in each state before reaching stae 4.

Problem 5
Excited by the recent warm weather Jill and Kelly are doing spring cleaning at their apartment. Jill takes an exponentially distributed amount of time with mean 30 minutes to clean the kitchen. Kelly takes an exponentially distributed amount of time with mean 40 minutes to clean the bath room. The first one to complete their task will go outside and start raking leaves, a task that takes an exponentially distributed amount of time with a mean of one hour. When the second person is done inside, they will help the other and raking will be done at rate 2. (Of course the other person may already be done raking in which case the chores are done.) What is the expected time until the chores are all done? Ans. Label the states I = the initial state, J = Jill raking, K = Kelly raking, JK = Jill and Kelly both raking, (J) = raking done and Jill still working, (K) = raking done and Kelly still working, D = done. The rates for Jill and Kelly inside are 2 and 3/2, while the raking rate = the number of people.

	I	J	K	JK	(J)	(K)	D
I	-7/2	2	3/2	0	0	0	0
J	0	-5/2	0	3/2	0	1	0
K	0	0	-3	2	1	0	0
JK	0	0	0	-2	0	0	2
(J)	0	0	0	0	-2	0	2
(K)	0	0	0	0	0	-3/2	3/2
D	0	I 0	0	0	0	0	0

Dropping the last column and inverting the matrix, we find that(- r)^(-1) times the first row is:

2/7, 8/35, 1/7, 11/35, 1/14, 16/105

implies that the total time is 1.195238.