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Math 742 Spring 2024

Piryatinska Midterm

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1. Consider a birth and death process with birth rates $\lambda_i = 3(i+1)$, $i \geq 0$, and death rates $\mu_i = 2i$, $i \geq 0$. Determine the expected time to go from state 1 to state 3.

$\lambda_i = 3(i+1)$ $\quad \mathbb{E}$ time $S\,\mathrm{I} \to S\,\mathrm{III} \Rightarrow \mathbb{E}(S_3)$

$\mu_i = 2i$ $\quad S_3 = T_1 + T_2$

$$\mathbb{E}T_i = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i}(\mathbb{E}T_{i-1}) \Rightarrow \mathbb{E}T_1 = \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1}\mathbb{E}T_0 = \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1}\left(\frac{1}{\lambda_0}\right)$$

$$\mathbb{E}T_2 = \frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2}\mathbb{E}(T_1) = \frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2}\left(\frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1\lambda_0}\right)$$

$$\mathbb{E}(T_1 + T_2) = \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1\lambda_0} + \frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2\lambda_1} + \frac{\mu_2\mu_1}{\lambda_2\lambda_1\lambda_0}$$

$$= \frac{1}{3(1+1)} + \frac{2}{3(2)3(1)} + \frac{1}{3(2+1)} + \frac{4}{9(6)} + \frac{4(2)}{54(3)}$$

$$= \frac{1}{6} + \frac{2}{18} + \frac{1}{9} + \frac{4}{54} + \frac{8}{162} = 0.5123$$

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2. The nucleotides A and G are purines while C's and T's are pyrimidines. Kimura's model takes into account that mutations that do not change the type of base (called transitions) happen at a different rate than those that do (called transversions), so the rate matrix is

$$\mathbf{R} = \begin{pmatrix} -4 & 2 & 1 & 1 \\ 2 & -4 & 1 & 1 \\ 1 & 1 & -4 & 2 \\ 1 & 1 & 2 & -4 \end{pmatrix}$$

Find transition probability matrix $\mathbf{P(t)}$

In matlab, $P(t)$ is

$$\begin{pmatrix} \frac{e^{-4t}}{4}+\frac{e^{-6t}}{2}+\frac{1}{4} & \frac{e^{-4t}}{4}-\frac{e^{-6t}}{2}+\frac{1}{4} & \frac{1}{4}-\frac{e^{-4t}}{4} & \frac{1}{4}-\frac{e^{-4t}}{4} \\ \frac{e^{-4t}}{4}-\frac{e^{-6t}}{2}+\frac{1}{4} & \frac{e^{-4t}}{4}+\frac{e^{-6t}}{2}+\frac{1}{4} & \frac{1}{4}-\frac{e^{-4t}}{4} & \frac{1}{4}-\frac{e^{-4t}}{4} \\ \frac{1}{4}-\frac{e^{-4t}}{4} & \frac{1}{4}-\frac{e^{-4t}}{4} & \frac{e^{-4t}}{4}+\frac{e^{-6t}}{2}+\frac{1}{4} & \frac{e^{-4t}}{4}-\frac{e^{-6t}}{2}+\frac{1}{4} \\ \frac{1}{4}-\frac{e^{-4t}}{4} & \frac{1}{4}-\frac{e^{-4t}}{4} & \frac{e^{-4t}}{4}-\frac{e^{-6t}}{2}+\frac{1}{4} & \frac{e^{-4t}}{4}+\frac{e^{-6t}}{2}+\frac{1}{4} \end{pmatrix}$$

10.

3. Consider two machines that are maintained by a single repairman. Machine i functions for an exponentially distributed amount of time with rate λ_i. The repair times for each unit are exponential with rate μ_i. before it fails.Suppose machine 1 is much more important than 2, so the repairman will always service 1 if it is broken.

(a) Formulate a Markov chain model for the this system with state space 0, 1, 2, 12 where the numbers indicate the machines that are broken at the time.

(b) Suppose that $\lambda_1 = 1$, $\mu_1 = 2$, $\lambda_2 = 3$, $\mu_2 = 4$. Find the limiting distribution.

$m1 \sim exp(\lambda_1)$, repair $\sim exp(\mu_1)$
$m2 \sim exp(\lambda_2)$, repair $\sim exp(\mu_2)$ $\quad m1 > m2$

0 = none broke
1 = M1 broke
2 = M2 broke
12 = both broke

a)

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 12 \\ \hline 0 & -2(\lambda_1+\lambda_2) & \lambda_1 & \lambda_2 & \cancel{\lambda_1+\lambda_2}\ 0 \\ 1 & \mu_1 & -(\mu_1+\cancel{\lambda_1}+\lambda_2) & \cancel{\lambda_1}\ 0 & \lambda_2 \\ 2 & \mu_2 & \lambda_2 & -(\mu_2+\cancel{\lambda_2}+\lambda_1) & \lambda_1 \\ 12 & \mu_1 0 + \mu_2 & \mu_1 & \cancel{\mu_2} & -2(\mu_1+\cancel{\mu_2}) \end{array}$$

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b) $P_0 + P_1 + P_2 + P_{12} = 1$

$\lambda_1 = 1$
$\lambda_2 = 3$
$\mu_1 = 2$
$\mu_2 = 4$

$$\begin{vmatrix} -2(1+3) & 1 & 3 & 4 \\ 2 & -(2+4) & 1 & 3 \\ 4 & 3 & -(4+4) & 1 \\ 6 & 2 & 4 & -12 \end{vmatrix} \begin{matrix} =0 \\ =0 \\ =0 \\ =0 \end{matrix}$$

$$\text{matlab} = \begin{pmatrix} 0.0612 \\ 0.0650 \\ 0.0785 \\ 0.1888 \end{pmatrix}$$

Cont MC

4. I am waiting for two friends to arrive at my house. The time until A arrives is exponentially distributed with rate $\lambda_a = 2$, and the time until B arrives is exponentially distributed with rate $\lambda_b = 3/2$. Once they arrive, both will spend exponentially distributed times, with respective rates $\mu_a = 1/2$ and $\mu_b = 3/4$ at my home before departing. The four exponential random variables are independent.

(a) What is the probability that A arrives before and departs after B?

(b) What is the expected time of the last departure?

$$\left.\begin{array}{l} A \sim \exp(\lambda_a = 2, \mu_a = \frac{1}{2}) \\ B \sim \exp(\lambda_b = \frac{3}{2}, \mu_b = \frac{3}{4}) \end{array}\right] \text{all indep}$$

a) $P(\lambda = A < B, \mu = B < A)$

$A_a \to A_b \to D_b \to D_a$

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5. A cocaine dealer is standing on a street corner. Customers arrive at times of a Poisson process with rate $\lambda = 4$ per hour. The customer and the dealer then disappear from the street for an amount of time with distribution G with the mean $\mu_G = 10$ min while the transaction is completed. Customers that arrive during this time go away never to return.

(a) At what rate does the dealer make sales?

(b) What fraction of customers are lost?

arrive ~ $Poi(\lambda = 4/hr)$ customer

both leave ~ $G(\mu_G = 10\,min)$ ← arrive, go away

a) Δ of sales $= \lambda p$ $\quad p$ = prob to sell @ least one deal

$\lambda = 4$ $\qquad \mu = \mu_G + \frac{1}{\lambda}$ mean bet entering cust

$p = 1 - P_0 = 1 - \frac{2}{5} = \frac{3}{5}$ $\qquad \frac{1}{\mu} = \frac{\lambda}{1+\lambda\mu_G}$ deal

no sale

$\Delta = 4\left(\frac{3}{5}\right) = \frac{12}{5}$ $\qquad \frac{\cancel{\lambda}(1+\lambda\mu_G)}{\lambda} = \frac{\cancel{\lambda}}{1+\lambda\mu_G}$

↓ correct

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b) $= \frac{1}{1+\frac{4}{hr}\left(\frac{1\,hr}{60\,min}\right)10\,min} = \frac{1}{1+\frac{1}{15}(10)} = \frac{1}{1+\frac{2}{3}} = \frac{3}{5}$

$1 - \frac{3}{5} = \frac{2}{5}$ customer loss

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