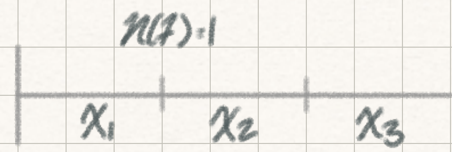
Ch 7.4: Renewal Reward Process

$\{N(t), t \geq 0\}$
renewal process



X_n = time bet renewals

$n \geq 1$

iid

denote R_n ($n \geq 1$, iid)

reward earned during nth renewal

can be dependence bet R_n & X_n

$R(t) = \sum_{n=1}^{N(t)} R_n$

total reward earned by time t

$\mathbb{E}X = \mathbb{E}(X_n)$

$\mathbb{E}R = \mathbb{E}(R_n)$

Prop 7.3:

If $\mathbb{E}R < \infty$ & $\mathbb{E}X < \infty$, then

i. $\lim_{t \to \infty} \frac{R(t)}{t} = \frac{\mathbb{E}R}{\mathbb{E}X}$ w/ prob 1

ii. $\lim_{t \to \infty} \frac{\mathbb{E}R(t)}{t} = \frac{\mathbb{E}R}{\mathbb{E}X}$

Proof (i)

$$\lim_{t \to \infty} \frac{R(t)}{t} = \lim_{t \to \infty} \frac{\sum_{n=1}^{N(t)} R_n}{t} \left[\frac{N(t)}{N(t)}\right] = \lim_{t \to \infty} \frac{\sum_{n=1}^{N(t)} R_n}{N(t)} \left[\frac{N(t)}{t}\right]$$

$$= \lim_{t \to \infty} \frac{\sum_{n=1}^{N(t)} R_n}{N(t)} \lim_{t \to \infty} \frac{N(t)}{t} = \mathbb{E}R\left(\frac{1}{\mathbb{E}X}\right) = \frac{\mathbb{E}R}{\mathbb{E}X}$$

$$\lim \frac{\text{reward}}{\text{time}} = \frac{\text{expected reward}}{\text{cycle}} \Big/ \frac{\text{expected time}}{\text{cycle}}$$

Example: Service

long run prop of customers that left

$$I = \begin{cases} 1 & \text{if system busy} \\ 0 & \text{if system empty} \end{cases}$$

$$\lim_{t\to\infty} \frac{\#\text{ of customers who left}}{t} = \frac{\mathbb{E}(R_n)}{\mathbb{E}X} = \frac{\mu_G}{\mu_G + \frac{1}{\lambda}}$$

H = successive customers deposit
those who enter bank

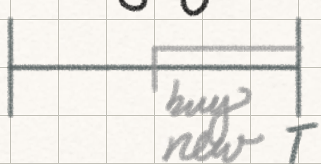
$$\lim_{t\to\infty} \frac{\text{total deposit by time } t}{t} = \frac{\mathbb{E}(\text{deposit during cycle})}{\mathbb{E}(\text{time of cycle})}$$

$$= \frac{\mu_G}{\mu_G + \frac{1}{\lambda}}$$

Example 7.14: Car buying model

life car ~ H

pdf h



C_1 = price of new car

C_2 = (+)itional cost if car broken

long run avg cost

$$\frac{\mathbb{E}(\text{cost incurring during cycle})}{\mathbb{E}(\text{length of cycle})}$$

X = lifetime of car during arbitrary cycle

C_1 if $X > T$

$C_1 + C_2$ if $X < T$

$\frac{\mathbb{E}\,\text{cost}}{\text{cycle}} = c_1 P(X > T) + (c_1 + c_2) P(X \le T)$

$= c_1 + c_2 P(X \le T) = c_1 + c_2 H(T)$

length of cycle $= \begin{cases} X; & X \le T \\ T; & X > T \end{cases}$

$\mathbb{E}$ length $= \int_0^T x h(x)\,dx + \int_T^\infty T h(x)\,dx$

$= \int_0^T x h(x)\,dx + T[1 - H(T)]$

long run avg cost $= \dfrac{c_1 + c_2 H(T)}{\int_0^T x h(x)\,dx + T[1 - H(T)]}$

$X \sim U(0, 10)$

find T that minimizes long run avg cost

$c_1 = 3$
$c_2 = 0.5$

$H = \begin{cases} \frac{x}{10}, & 0 \le x \le 10 \\ 1 & x \ge 10 \end{cases} \Rightarrow H(T) = \frac{T}{10}$

$\int_0^T \frac{x}{10}\,dx = \frac{1}{10}\left(\frac{x^2}{2}\Big|_0^T\right) = \frac{1}{10}\left(\frac{T^2}{2}\right) = \frac{T^2}{20}$

long run: $\dfrac{3 + \frac{1}{2}\left(\frac{T}{10}\right)}{\frac{T^2}{20} + T\left(1 - \frac{T}{10}\right)} = \dfrac{60 + T}{T^2 + 20T - 2T^2} = \dfrac{60 + T}{20T - T^2}$

$H(x) = F(x)$

$g(T) = \dfrac{60 + T}{20T - T^2}$

$g'(T) = \dfrac{1(20T - T^2) - (60 + T)(20 - 2T)}{20T - T^2} = 0$

$20T - T^2 = (60 + T)(20 - 2T)$

$$20T - T^2 = 1200 - 120T + 20T - 2T^2$$

$$20T - T^2 = 1200 - 100T - 2T^2$$

$$T^2 + 120T - 1200 = 0$$

$$T = \frac{-120 \pm \sqrt{120^2 + 4(1200)}}{2} = 9.25$$