$P(t) = e^{Rt}$ $\qquad R_{ij} = \begin{cases} -v_i, & i=j \\ q_{ij}, & i\neq j \end{cases}$ $\qquad q_{ij} = v_i P_{ij}$

$R = \begin{pmatrix} -v_0 & q_{01} & q_{02} & \cdots \\ & \ddots & & \end{pmatrix}$ $\qquad V = \begin{pmatrix} \text{mat of} \\ \text{eigenvec} \end{pmatrix}$ $\qquad R = V^{-1}DV$

$D = \begin{pmatrix} \lambda_0 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \Rightarrow e^{Rt} = V^{-1}e^{Dt}V$

Recitation 1

Continuous MC, definition, transition probability

Problem 1.

Consider a birth and death process with birth rates $\lambda_i = (i + 1)\lambda$, $i \geq 0$, and death rates $\mu_i = i\mu$, $i \geq 0$.

$P'(t) = RP(t)$, $P'(t) = P(t)R$ $> P(t) = e^{Rt}$

(a) Determine the expected time to go from state 0 to state 4.
(b) Determine the expected time to go from state 2 to state 5.
(c) Determine the variances in parts (a) and (b).

Problem 2

In a birth and death process with birth parameter $\lambda_n = \lambda$, n=0,1,..., and death parameters $\mu_n = n\mu$ for n=0,1,... we have $P_{0j}(t) = \frac{(\lambda p)^j e^{-\lambda p}}{j!}$

Where

$p = \frac{1}{\mu}(1 - e^{-\mu t})$.

$P_{0j}(t) = \frac{(\lambda p)^j e^{-\lambda p}}{j!}$

$p = \frac{1}{\mu}(1 - e^{-\mu t})$

$P_{0j}'(t) = \sum_{k\neq j} q_{kj} P_{0k}(t) - (\lambda_j + \mu_j)P_{0j}(t)$
$= \lambda_{j-1}P_{0,j-1}(t) + \mu_{j+1}P_{0,j+1}(t) - (\lambda_j + \mu_j)P_{0j}$

$k=1$

Verify that these transition probabilities satisfy the forward equation with i=0.

q_{01}

$$R = \begin{pmatrix} -3\mu & \mu & \mu & \mu \\ \mu & -3\mu & \mu & \mu \\ \mu & \mu & -3\mu & \mu \\ \mu & \mu & \mu & -3\mu \end{pmatrix}$$

Problem 3

(Jukes–Cantor Model). In this chain, the states are the four nucleotides A, C, G, T. Jumps, which correspond to nucleotide substitutions, occur according at rate $q_{ij} = \mu$ if $i \neq j$. Find the transition probability matrix $\mathbf{P}(t)$ using forward differential equation.

Problem 4

The nucleotides A and G are purines while C's and T's are pyrimidines. Kimura's model takes into account that mutations that do not change the type of base (called transitions) happen at a different rate than those that do (called transversions), so the transition matrix P

$$R = \begin{pmatrix} -(\alpha+2\beta) & \alpha & \beta & \beta \\ \alpha & -(\alpha+2\beta) & \beta & \beta \\ \beta & \beta & -(\alpha+2\beta) & \alpha \\ \beta & \beta & \alpha & -(\alpha+2\beta) \end{pmatrix}$$

Find $\mathbf{P}(t)$ using forward differential equation.

1. birth rate: $\lambda_i = (i+1)\lambda$, $i \geq 0$
death rate: $\mu_i = i\mu$, $i \geq 0$
(a) $E(i_{0,4}) = ?$

$E(T_i) = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} E(T_{i-1})$

$S_4 = T_0 + T_1 + T_2 + T_3$

$E(S_4)$, $V(S_4)$

$V(T_0) = \frac{1}{\lambda_0^2}$

$V(T_i) = \frac{1}{\lambda_i(\lambda_i + \mu_i)} + \frac{\mu_i}{\lambda_i} V(T_{i-1}) + \frac{\mu_i}{\mu_i + \lambda_i}[E(T_{i-1}) + E(T_i)]^2$

$V(S_4) = VT_0 + VT_1 + VT_2 + VT_3$

$E(T_0) = \frac{1}{\lambda}$

$E(T_1) = \frac{1}{2\lambda} + \frac{\mu}{2\lambda}\left(\frac{1}{\lambda}\right) = \frac{1}{2\lambda}\left(1 + \frac{\mu}{\lambda}\right)$

$E(T_2) = \frac{1}{3\lambda} + \frac{2\mu}{3\lambda}\left[\frac{1}{2\lambda}\left(1 + \frac{\mu}{\lambda}\right)\right] = \frac{1}{3\lambda}\left[1 + \frac{\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2\right]$

$E(T_3) = \frac{1}{4\lambda} + \frac{3\mu}{4\lambda}\left\{\frac{1}{3\lambda}\left[1 + \frac{\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2\right]\right\} = \frac{1}{4\lambda}\left[1 + \frac{\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2 + \left(\frac{\mu}{\lambda}\right)^3\right]$

$E(i_{0,4}) = \frac{1}{\lambda} + \frac{1}{2\lambda}\left(1 + \frac{\mu}{\lambda}\right) + \frac{1}{3}\left[1 + \frac{\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2\right] + \frac{1}{4\lambda}\left[1 + \frac{\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2 + \left(\frac{\mu}{\lambda}\right)^3\right]$

$V(T_0) = \frac{1}{\lambda_0^2}$

$V(T_1) = \frac{1}{\lambda_1(\lambda_1 + \mu_1)} + \frac{\mu_1}{\lambda_1} V(T_0) + \frac{\mu_1}{\mu_1 + \lambda_1}[E(T_0) + E(T_1)]^2$

$= \frac{1}{\lambda_1(\lambda_1 + \mu_1)} + \frac{\mu}{\lambda_1}\left(\frac{1}{\lambda_0^2}\right) + \frac{\mu_1}{\mu_1 + \lambda_1}\left[\frac{1}{\lambda} + \frac{1}{2\lambda}\left(1 + \frac{\mu}{\lambda}\right)\right]^2$

$V(T_2) = \frac{1}{\lambda_2(\lambda_2 + \mu_2)} + \frac{\mu_2}{\lambda_2} V(T_1) + \frac{\mu_2}{\mu_2 + \lambda_2}[E(T_1) + E(T_2)]^2$

$V(T_3) = \frac{1}{\lambda_3(\lambda_3 + \mu_3)} + \frac{\mu_3}{\lambda_3} V(T_2) + \frac{\mu_3}{\mu_3 + \lambda_3}[E(T_2) + E(T_3)]^2$

(b) $E(i_{2,5})$

$E(T_4) = \frac{1}{5\lambda} +$