# Exit distribution and exit time.

$\boldsymbol{P} = (P_{ij})_{ij}$ – transition probability matrix for the embedded jump chain

$q_{ij} = v_i \cdot P_{ij}$ $\quad v_i$ – rate to leave state $i$

Let $V_k = \min\{t \geq 0 : X(t) = k\}$ be time of the first visit to $k$.

$T_k = \min\{t \geq 0 : X(t) = k$ and $X(s) \neq k$ for some $s < t\}$ be the time of the first return.

The second def. is made complicated by the fact that $X_0 = k$ then the chain stays at $k$ for some amount of time that is exponential with rate $\lambda_k$.

Ex. Branching process:

$X(0) = 1$, death and birth process:

$q_{i\,i+1} = \lambda \cdot i \qquad q_{i,i-1} = \mu i.$

$0$ is an absorbing state

$\forall i \geq 0: P_{i,i+1} = \dfrac{\lambda}{\lambda+\mu} \qquad P_{i,i-1} = \dfrac{\mu}{\lambda+\mu}$

Thus absortion at $0$ is certain if $\lambda \leq \mu$

but if $\lambda > \mu$ the probability of avoiding extinction is

$$P_1(T_0 = \infty) = 1 - \frac{\mu}{\lambda}$$

$\frac{\mu}{\lambda} \rightarrow$ probability to die out.

Gambler ruin problem.

Prob. to win before going to bunctorcy

$$\frac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N}$$

Let $p = P_1(T_0 < \infty) \rightarrow$ probability to start at 1 and time until population will die out is finite.

By conditioning to the first step.

$X(0) = 1$

$$\underline{I} = \begin{cases} 1, & \text{death before birth} \\ 2, & \text{birth before death.} \end{cases}$$

$$P_1(T_0 < \infty) = P_1(T_0 < \infty \mid \underline{I} = 1)) \cdot P(\underline{I} = 1) +$$

$$+ P_1(T_0 < \infty \mid \underline{I}_2) \cdot P(\underline{I} = 2) =$$

$$= 1 \cdot \frac{\mu}{\lambda + \mu} + P_1(T_0 < \infty)^2 \cdot \frac{\lambda}{\lambda + \mu}$$

$$p = \frac{\mu}{\lambda + \mu} + p^2 \frac{\lambda}{\lambda + \mu}$$

$$\lambda p + \mu p = \mu + \lambda p^2$$

$$\lambda p^2 - (\lambda + \mu) p + \mu = (\lambda p - \lambda)\left(p - \frac{\mu}{\lambda}\right)$$

$\frac{\mu}{\lambda} < 1$ That is the root we want.

If we work with an ebedded chain we can use the approach of discrete time MC to compute exit distribution.

We can also work directly with a matrix $R$

Let $V_D = \min\{t : X(t) \in D\}$

$T = \min(V_a, V_B)$

Suppose $C = S \cap (A \cup B)'$ is finite,

$S$ → State space

$P_i(T < \infty) > 0$

probability to start at $i$ and come back to $i$ at finite time $\forall i \in C$.

If we have $h(a) = 1 \quad a \in A$

$h(b) = 0 \quad b \in B$

$$h(i) = \sum_{j \neq i} P_{ij}\, h(j) \qquad \forall i \in C$$

Then $h(i) = P_i(V_A < V_B)$ probability to visit set $A$ before $B$ if you start at $i$

$$\underbrace{v_i \cdot h(i)} = \sum_{j \neq i} \underbrace{v_i P_{ij}}_{q_{ij}} h(j)$$

$$\sum_i q_{ij}\, h(j) - v_i\, h(i) = 0$$

$$\sum_i R_{ij}\, h(j) = 0 \qquad (1)$$

Let $r = (r_{ij})_{i,j \in C}$

$r_{ij} = R_{ij} \quad i,j \in C$

Let $w_i = \sum_{j \in A} R_{ij}$

$R = (R_{ij})$

$$R_{ij} = \begin{cases} q_{ij} & i \neq j \\ -v_i & i = j \end{cases}$$

Let $h(a) = 1 \quad h(b) = 0$

(1) Can be written as $\sum_{j \in C} r_{ij} h(j) = -w_i$

$-\sum r_{ij} h(j) = w(i)$ $\qquad -R h = w$

$h(i) = (-r)^{-1} w$ $\qquad h = (-R)^{-1} w$

analogy of $(I-S)^{-1} v) = h$

Example.

*Example 4.18 (Continued).* A shop has two barbers that can cut hair at rate 3, people per hour customers arrive at times of a rate 2 Poisson process, but will leave if there are two people getting their haircut and two waiting. The state of the system to be the number of people in the shop. Find $P_i(V_0 < V_4)$ for $i = 1, 2, 3$.

The transition rate matrix is

| | **0** | **1** | **2** | **3** | **4** |
|---|---|---|---|---|---|
| **0** | −2 | 2 | 0 | 0 | 0 |
| **1** | 3 | −5 | 2 | 0 | 0 |
| **2** | 0 | 6 | −8 | 2 | 0 |
| **3** | 0 | 0 | 6 | −8 | 2 |
| **4** | 0 | 0 | 0 | 6 | −6 |

$r$

$$w = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

$(-r)^{-1} \cdot w = ?$

$$\begin{bmatrix} 5 & -2 & 0 \\ -6 & 8 & -2 \\ 0 & -6 & 8 \end{bmatrix}^{-1} \cdot w = \begin{bmatrix} \frac{13}{41} & \frac{4}{41} & \frac{1}{41} \\ \frac{12}{41} & \frac{10}{41} & \frac{5}{82} \\ \frac{9}{41} & \frac{15}{82} & \frac{7}{41} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 39/41 \\ 36/41 \\ 27/41 \end{bmatrix}$$

*Example 4.21 (Office Hours).* In Exercise 2.10. Ron, Sue, and Ted arrive at the beginning of a professor's office hours. The amount of time they will stay is exponentially distributed with means of 1, 1/2, and 1/3 hour. Part (b) of the question is to compute the probability each student is the last to leave.

If we describe the state of the Markov chain by the rates of the students that are left, with Ø to denote an empty office, then the ~~Q~~ R matrix is

$$R = \begin{array}{c|cccccccc} & \mathbf{123} & \mathbf{12} & \mathbf{13} & \mathbf{23} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \emptyset \\ \hline \mathbf{123} & -6 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ \mathbf{12} & 0 & -3 & 0 & 0 & 2 & 1 & 0 & 0 \\ \mathbf{13} & 0 & 0 & -4 & 0 & 3 & 0 & 1 & 0 \\ \mathbf{23} & 0 & 0 & 0 & -5 & 0 & 3 & 2 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ \mathbf{2} & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 2 \\ \mathbf{3} & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 3 \end{array} \tag{4.32}$$

The matrix ~~R~~ r we want to solve our problem is

$$r = \begin{array}{c|cccc} & \mathbf{123} & \mathbf{12} & \mathbf{13} & \mathbf{23} \\ \hline \mathbf{123} & -6 & 3 & 2 & 1 \\ \mathbf{12} & 0 & -3 & 0 & 0 \\ \mathbf{13} & 0 & 0 & -4 & 0 \\ \mathbf{23} & 0 & 0 & 0 & -5 \end{array}$$

$W=$

while if we want to compute the probability each student is last we note that

$$\cancel{Q(i,1)}\, R_{i1} = \begin{pmatrix} 0 \\ 2 \\ 3 \\ 0 \end{pmatrix} \qquad \cancel{Q}(i,2)\, R_{i2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \end{pmatrix} \qquad \cancel{Q(i,3)}\, R_{i3} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

Inverting the matrix gives

$$(-\cancel{R}\,r)^{-1} = \begin{pmatrix} 1/6 & 1/6 & 1/12 & 1/30 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/5 \end{pmatrix}$$

The last three rows are obvious. Starting from these three states the first jump takes us to a state with only one student left, so the diagonal elements are the mean holding times. The top left entry in the inverse is the 1/6 hour until the first student leaves. The other three on the first row are

$$\frac{1}{2} \cdot \frac{1}{3} \qquad \frac{1}{3} \cdot \frac{1}{4} \qquad \frac{1}{6} \cdot \frac{1}{5}$$

which are the probabilities we visit the state times the expected amount of time we spend there. Using (4.31) that the probabilities $\rho_i$ that the three students are last are

$$\rho_1 = 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{12} = \frac{35}{60} \quad \rho_2 = \frac{1}{6} + 3 \cdot \frac{1}{12} = \frac{35}{60} \quad \rho_3 = \frac{1}{12} + 2 \cdot \frac{1}{30} = \frac{35}{60}.$$

# Exit time

Let $C = S \cap A'$ be finite

$P_i(V_A < \infty) > 0 \quad i \in C$

probability to start at $i$ and visit $A$ in a finite time.

$$g(i) = \frac{1}{V_i} + \sum_{i \neq j} q_{ij}\, g(j)$$

$g(i) = E_i V_A$ (expected time to start at $i$ and visit $A$)

$$v_i\, g(i) = 1 + \sum_{j \neq i} q_{ij}\, g(j)$$

$$\sum_{j \neq i} q_{ij}\, g(j) - v_i\, g(i) = -1$$

$$\sum_j R_{ij}\, g(j) = -1 \qquad R_{ij} = \begin{cases} q_{ij} & i \neq j \\ v_i & i \neq j \end{cases}$$

$$-\sum_j R_{ij}\, g(j) = 1 \qquad g(a) = 0$$

$$-\sum_{j \in C} R_{ij}\, g(j) = 1$$

$$-r \cdot g(j) = \mathbf{1} \qquad \mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

*Example* 4.18 *(Continued).* A shop has two barbers that can cut hair at rate 3, people per hour customers arrive at times of a rate 2 Poisson process, but will leave if there are two people getting their haircut and two waiting. The state of the system to be the number of people in the shop. ~~Find $P_i(V_0 < V_4)$ for $i = 1, 2, 3$~~.

Compute $E_i T_0$ expected time to start at $i$ and get to $0$.

$A = \{0\}$

$$
\begin{array}{c|cccc}
 & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\
\hline
\mathbf{1} & -5 & 2 & 0 & 0 \\
\mathbf{2} & 6 & -8 & 2 & 0 \\
\mathbf{3} & 0 & 6 & -8 & 2 \\
\mathbf{4} & 0 & 0 & 6 & -6
\end{array}
$$

which has

$$
-r^{-1} = \begin{pmatrix} 1/3 & 1/9 & 1/27 & 1/81 \\ 1/3 & 5/18 & 5/54 & 5/16 \\ 1/3 & 5/18 & 7/27 & 7/81 \\ 1/3 & 5/18 & 7/27 & 41/162 \end{pmatrix}
$$

Multiplying by **1** so we have

$$
g(i) = (40/81, 119/162, 155/162, /91/81)^T
$$

where $T$ denotes transpose, i.e., $g$ is a column vector. Note that in the answer if $i > j$, then $(-r)^{-1}(i, j) = (-r)^{-1}(j, j)$ since must hit $j$ on our way from $i$ to 0.

The last calculation shows $E_1 T_0 = 40/81$. We can compute this from the stationary distribution in Example 4.18 if we note that the time spent at zero (which is exponential with mean 1/2) followed by the time to return from 1 to 0 (which has mean $E_1 T_0$) is an alternating renewal process

$$
\frac{81}{161} = \pi(0) = \frac{1/2}{1/2 + E_1 T_0}.
$$

This gives $1 + 2E_1 T_0 = 161/81$, and it follows that $E_1 T_0 = 40/81$.