Ch 6.4: Trans Prob f(x)

$P_{ij} = P\{X(t+s)=j \mid X(s)=i\}$

See Pirya's notes for breakdown

Pure birth process

$(X_1, \dots, X_n)$, exp

$(\lambda_1, \dots, \lambda_n)$; $\lambda_i \neq \lambda_j,\ i \neq j$

$$P(S>t) = \sum_{i=1}^{n} C_{in}\left(e^{-\lambda_i t}\right)$$

$$C_{in} = \prod_{\substack{j \neq i \\ i=1}}^{n} \frac{\lambda_j}{\lambda_j - \lambda_i}$$

Prop: for pure birth process, having $\lambda_i \neq \lambda_j,\ i \neq j$

$$P_{ij}(t) = \sum_{k=1}^{j} C_{ik}\left(e^{-\lambda_k t}\right) - \sum_{k=1}^{j-1} \prod_{\substack{r \neq k \\ r=i}}^{j-1} \frac{\lambda_r}{\lambda_r - \lambda_k}\left(e^{-\lambda_k t}\right)$$

$$C_{ik} = \prod_{\substack{r \neq k \\ r=i}}^{j} \frac{\lambda_r}{\lambda_r - \lambda_k} = \prod_{\substack{r \neq k \\ r=i}}^{j} \frac{r\lambda}{r\lambda - k\lambda} = \prod_{\substack{r \neq k \\ r=i}}^{j} \frac{r}{r-k}$$

$$P_{ii}(t) = e^{-\lambda_i t}$$

Gen case

Example: Yule Process   linear case

$\lambda_n = n\lambda$

$$P_{ij} = \sum_{k=1}^{j} C_{ik}\left(e^{-k\lambda t}\right)$$

$$P_{ij}(t) = e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{j-1}$$

$$P_{ij}(t) = \binom{j-1}{i-1}\left(e^{-\lambda t}\right)\left(1 - e^{-\lambda t}\right)^{j-i}$$

Gen case:

$P_{ij}(t)$ $\quad$ $T_i \sim Exp(v_i)$ $\quad$ i —— change $\quad$ $E(T_i) = \frac{1}{v_i}$

P_{ij} = prob to enter j when process is in i

$P_{ij} = q_{ij}/v_i$

v_i = rate

$q_{ij} = v_i(P_{ij})$

$\sum_{\forall j} q_{ij} = \sum_j v_i(P_{ij}) = v_i$

Metrics:

$R = (R_{ij})$

$R_{ii} = -v_i$

$R_{ij} = q_{ij}$

$$\begin{pmatrix} -v_0 & q_{01} & q_{02} & \dots & q_{0m} \\ q_{10} & -v_1 & q_{12} & \dots & q_{1m} \end{pmatrix}$$

Lemma I:

$$\lim_{h\to 0} \frac{1-P_{ii}(h)}{h} = v_i \Rightarrow \frac{1-P_{ii}(h)}{h} = \frac{\lambda h + o(h)}{h} = \lambda = v_i$$

$$\lim_{h\to 0} \frac{P_{ij}(h)}{h} = \frac{q_{ij}\cdot h + o(h)}{h} = q_{ij} + \lim_{h\to 0}\frac{o(h)}{h} = q_{ij} \qquad i \neq j$$

$N(t+h) - N(t) \geq 2$

$P(\geq 2 \text{ events in time } h) = o(h)$

$P(1 \text{ event}) = \lambda h + o(h)$

t —— $t+h$

$X(t) = i$ $\quad$ $X(t+h) = j$

$P_{ii}(h)$

Lemma II: Chapman-Kolmogorov Eq

$\forall s \geq 0,\ t \geq 0$

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s)$$

$$P_{ij}^{n+m} = \sum_{\forall k} P_{ik}^{n} (P_{kj}^{m})$$

From LII:

$$P_{ij}(h+t) - P_{ij}(t) = \sum_{k=0}^{\infty} P_{ik}(h) P_{kj}(t) - P_{ij}(t)$$
$$= \sum_{k \neq i} P_{ik}(h) P_{kj}(t) + P_{ii}(h) P_{ij}(t) - P_{ij}(t)$$
$$= \sum_{k \neq i} P_{ik}(h) P_{kj}(t) - [1 - P_{ii}(h)] P_{ij}(t)$$

$$\lim_{h \to 0} \frac{P_{ij}(h+t) - P_{ij}(t)}{h} = \lim_{h \to 0} \left\{ \sum_{k \neq i} \frac{P_{ik}(h)}{h} \cdot P_{kj}(t) - \left[\frac{1 - P_{ii}(t)}{h}\right] P_{ij}(t) \right\}$$
$$= \sum_{k \neq i} q_{ik} [P_{kj}(t)] - v_i [P_{ij}(t)]$$
$$= P_{ij}'$$

$P'(t) = R \cdot P(t)$ backward Kolmogorov eq

Theorem:

$\forall$ states i, j & times $t \geq 0$

$$P_{ij}'(t) = \sum_{k \neq i} P_{ik}(t) q_{kj} - P_{ij}(t) v_j$$

$$P'(t) = P(t) R$$

? Proof: $P'(t) = R \cdot P(t)$

$f'(t) = r \cdot f(t)$ $\qquad \dfrac{df(t)}{f(t)} = r\,dt$ $\qquad \dfrac{df}{f} = r\,dt$

$\dfrac{df(t)}{dt} = r \cdot f(t)$ $\qquad \ln f(t) = rt + \ln c$ $\qquad \ln f = rt + c$

$$f(t) = ce^{rt}$$

$P(t) = ce^{Rt}$ $\quad P(0) = C$ $\quad P(t) = P(0)e^{Rt}$

$$e^{Rt} = \sum_{n=0}^{\infty} R^n \left(\frac{t^n}{n!}\right)$$

$$P_{ij}^{n+m} = \sum_{\forall k} P_{ik}^{n} (P_{kj}^{m})$$

$$R = \left(-v_i \quad q \right)$$