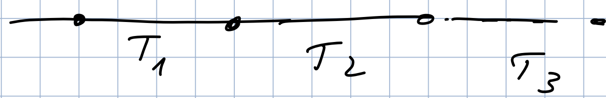
Renewal theory and its applications.

$\{N(t), t \geq 0\}$ – Poisson process.



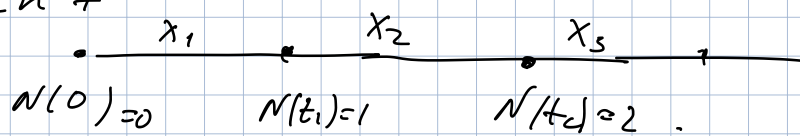
T_i – time between events.

$T_i \sim Exp(\lambda)$, λ – rate of a Poisson process.

Generalization.

Let $\{N(t), t \geq 0\}$ be a counting process.

Let X_n be the time between $(n-1)$ and n^{th} event

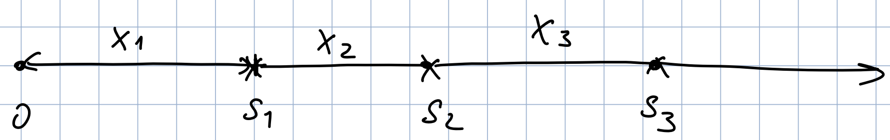


Def. If the sequence of nonnegative r.v. $\{X_1, X_2 \dots\}$ is iid then the counting process $\{N(t), t \geq 0\}$ is said to be a renewal process.

Let F be a distribution of X_i

Ex. We have an infinite supply of lightbulbs. We use a single lightbulb at a time, and when it fails we immediatly replace it with a new one.

$\{N(t)\ t \geq 0\}$ is a renewal process, $N(t)$ – # of lightbulbs had fail by time t.



Let $X_1, X_2, \dots$ be interarrival times

$S_0 = 0 \qquad S_n = \sum_{i=1}^{n} X_i$

$S_1 = X_1, \qquad S_2 = X_1 + X_2, \quad \dots$

S_n – the time of n^{th} renewal.

For the distribution F we assume:

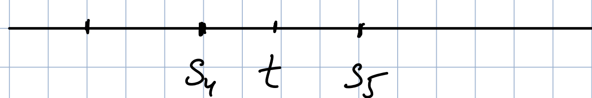
$F(0) = P(X_n = 0) < 1$

$\mu = E X_n \qquad n \geq 1$ mean time between successeful renewals

X_n is not identically 0 $\quad \mu > 0$.

$N(t)$ be ∞ for a finite t?

$N(t) = \max\{n : S_n \leq t\}$



Let $S_4 \leq t \qquad S_5 > t$

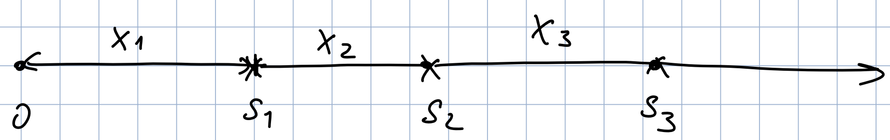
Hence, 4^{th} renewal had occured by time t

5^{th} — " — after time t

By strong law of large numbers

$\frac{S_n}{n} \to \mu \quad n \to \infty$ with probability 1.

Since $\mu > 0 \Rightarrow S_n \to \infty \quad n \to \infty$



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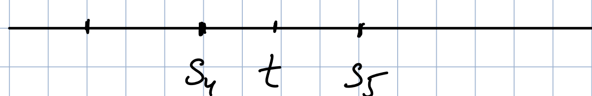

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Thus S_n can be $\leq t$ for at most a finite number of values of n $\Rightarrow$

$N(t) = \max\{n : S_n \leq t\} < \infty$ $\forall t$ with probability 1.

$$N(\infty) = \lim_{t \to \infty} N(t) = \infty$$

This follows since the only way in which $N(\infty)$, the total # of renewal that occur, can be finite is for one of the interarrival time to be infinite.

$\Rightarrow$ $P\{N(\infty) < \infty\} = P(X_n = \infty \text{ for some } n) =$

$$= P\left(\bigcup_{n=1}^{\infty} = \infty\right) \leq \sum_{n=1}^{\infty} P(X_n = \infty) = 0.$$

Convergence with probability 1 (almost surely).

Consider a sequence of r.v.s $X_1, X_2 \ldots$ all defined on the same probability space Ω. For every $s \in S$ we obtain a sample sequence (sequence of numbers) $X_1(s) \ldots X_n(s) \ldots$

A sequence $X_1, X_2, X_3 \ldots$ of r.v.s is said to converge to a r.v. X with probability 1 (almost surely) if

$$P\{\omega : \lim_{n\to\infty} X(s) = X(s)\} = 1$$

This means that the set of sample paths that convergese to $X(s)$ is sense of sequence convergance has probability 1.

Equivalently $X_1 \ldots X_n \ldots$ converges with probability 1 if for $\forall \varepsilon > 0$

$$\lim_{m\to\infty} P\{|X_n - X| < \varepsilon \quad \forall n \geq m\} = 1$$

(Strong Law of Large numbers)

Weak law of large numbers.

Convergence in probability.

A sequence of r.v. $X_1 \ldots X_n \ldots$ converges to r.v. X in probability if $\forall \varepsilon > 0$

$$\lim_{n\to\infty} P\{|X_n - X| < \varepsilon\} = 1.$$

converg. with p. 1 $\Rightarrow$ convergence in probability.

$\not\Leftarrow$