

## 6.4. The transition probab. matrix.

$$\text{Let } P_{ij}(t) = P \{ X(t+s) = j \mid X(s) = i \}$$

$$P(t) = \begin{pmatrix} P_{ij}(t) \end{pmatrix}$$

$P_{ij}$

$v_i$  - rate at which process  
transition when in state  $i$

$$g_{ij} = v_i P_{ij}$$

Backward Kolmogorov equation.

$$P'_{ij}(t) = \sum_{k \neq i} g_{ik} P_{kj}(t) - v_i P_{ij}(t) \quad (1)$$

$$P(t) = (P_{ij}(t))$$

$$R = (R_{ij})$$

$$R_{ij} = \begin{cases} g_{ij} & i \neq j \\ -v_i & i = j \end{cases}$$

$$(1) \quad P'(t) = R P(t)$$

$$P(t) = e^{Rt}$$

Forward Kolmogorov equation.

$$P'_{ij}(t) = \sum_{k \neq j} g_{kj} P_{ik}(t) - v_j P_{ij}(t)$$

$$(2) \quad P'(t) = P(t)R$$

Ex. Pure birth process.

$$v_i = \lambda_i$$

$$p_{01} = 1 \quad p_{i,i+1} = 1 \quad p_{ij} = 0 \quad i \neq i+1.$$

$$p'_{ij}(t) = \lambda_i p_{i+1,i}(t) - \lambda_i p_{ij}(t)$$

Backward Kolmogorov eq.

Forward Kolmogorov eq.

$$p'_{ij}(t) = \lambda_{j-1} p_{i,j-1}(t) - \lambda_j p_{ij}(t) \quad (3)$$

$$p_{ij} = 0 \quad j < i$$

$$(3) \quad \begin{cases} p'_{ii}(t) = -\lambda_i p_{ii}(t) \\ p'_{ij}(t) = \lambda_{i-1} p_{i,j-1}(t) - \lambda_j p_{ij}(t) \quad j \geq i+1 \end{cases}$$

Proposition. For a pure birth process

$$p_{ii}(t) = e^{-\lambda_i t}$$

$$p_{ij}(t) = \lambda_{j-1} e^{-\lambda_j t} \int_0^t e^{\lambda_j s} p_{i,j-1}(s) ds \quad j \geq i+1$$

Proof.  $p'_{ii}(t) = -\lambda_i p_{ii}(t)$

$$\frac{d p_{ii}(t)}{dt} = -\lambda_i p_{ii}(t)$$

$$\int \frac{d p_{ii}(t)}{p_{ii}(t)} = \int -\lambda_i dt$$

$$\ln P_{ii}(t) = -\lambda_i t + c$$

$$c = \ln k$$

$$\ln P_{ii}(t) = \ln e^{-\lambda_i t + c} = \ln e^{-\lambda_i t} \cdot \underbrace{e^{\ln k}}_k$$

$$P_{ii}(t) = k e^{-\lambda_i t}$$

$$P_{ii}(0) = 1 \quad k = 1$$

$$P_{ii}(t) = e^{-\lambda_i t}$$

$$P'_{ij}(t) = \lambda_{j-1} P_{ij-1}(t) - \lambda_j P_{ij}(t)$$

$$P'_{ij}(t) + \lambda_j P_{ij}(t) = \lambda_{j-1} P_{ij-1}(t) \quad | \times e^{\lambda_j t}$$

$$\underbrace{e^{\lambda_j t} [P'_{ij}(t) + \lambda_j P_{ij}(t)]}_{\frac{d}{dt} [e^{\lambda_j t} P_{ij}(t)]} = e^{\lambda_j t} \lambda_{j-1} P_{ij-1}(t)$$

$$\frac{d}{dt} [e^{\lambda_j t} P_{ij}(t)] = e^{\lambda_j t} \lambda_{j-1} P_{ij-1}(t)$$

$$d(e^{\lambda_j t} P_{ij}(t)) = e^{\lambda_j t} \underbrace{\lambda_{j-1}}_{\lambda_{j-1}} P_{ij-1}(t) dt$$

$$\underbrace{e^{\lambda_j t} P_{ij}(t)}_{\text{}} = \lambda_{j-1} \int_0^t e^{\lambda_j s} P_{ij-1}(s) ds$$

$$P_{ij}(t) = \lambda_{j-1} e^{-\lambda_j t} \int_0^t e^{\lambda_j s} P_{ij-1}(s) ds + C$$

$$P_{ij}(0) = 0$$

$$C = 0$$

$$P_{ij}(t) = \lambda_{j-1} e^{-\lambda_j t} \int_0^t e^{\lambda_j s} P_{ij-1}(s) ds$$

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Ex. The backward equation for the birth and death process.

$$\begin{aligned} v_0 &= \lambda_0 \\ v_i &= \lambda_i + \mu_i \end{aligned} \quad \begin{aligned} p_{i,i+1} &= \frac{\lambda_i}{\lambda_i + \mu_i} \\ p_{i,i-1} &= \frac{\mu_i}{\lambda_i + \mu_i} \end{aligned} \quad \begin{aligned} p_{ij} &= 0 \quad i \neq i+1 \\ &\text{or} \\ &i \neq i-1. \end{aligned}$$

$$p'_{0j}(t) = \lambda_0 p_{1j}(t) - \lambda_0 p_{0j}(t)$$

$$p'_{ij}(t) = (\lambda_i + \mu_i) \left( \frac{\lambda_i}{\lambda_i + \mu_i} p_{i+1,j}(t) + \frac{\mu_i}{\lambda_i + \mu_i} p_{i-1,j}(t) \right) - (\lambda_i + \mu_i) p_{ij}(t)$$

$$p'_{0j}(t) = \lambda_0 (p_{1j}(t) - p_{0j}(t))$$

$$p'_{ij}(t) = \lambda_i p_{i+1,j}(t) - \mu_i p_{i-1,j}(t) + (\lambda_i + \mu_i) p_{ij}(t).$$

Forward equation for Birth and Death process.

$$p'_{i0}(t) = \sum_{k \neq 0} q_{k0} p_{ik}(t) - \lambda_0 p_{i0}(t) =$$

$$= \mu_1 p_{i1}(t) - \lambda_0 p_{i0}(t)$$

$$p'_{ij}(t) = \sum_{k \neq j} q_{kj} p_{ik}(t) - (\lambda_j + \mu_j) p_{ij}(t) =$$

$$= \lambda_{j-1} p_{i,j-1}(t) + \mu_{j+1} p_{i,j+1}(t) - (\lambda_j + \mu_j) p_{ij}(t).$$

Ex. 6.11. A continuous MC consisting of two states.

$\square$  works  $T_w \sim \text{Exp}(\lambda)$   
 $T_r \sim \text{Exp}(\mu)$

States:  $w, r$ .

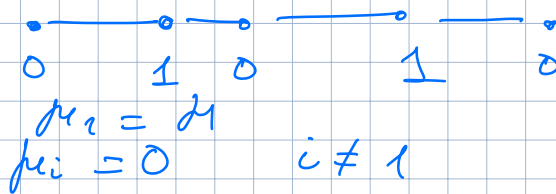
$P(X(t)=w \mid X(0)=w) - ?$

State 0 -  $w$

1 -  $r$

$$\lambda_0 = \lambda$$

$$\lambda_i = 0 \quad i \neq 0$$



$P_{00}(t) - ?$

$$P(t) = \begin{pmatrix} p_{00}(t) & p_{01}(t) \\ p_{10}(t) & p_{11}(t) \end{pmatrix} \quad R = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

$$R = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

$$P'(t) = R P(t)$$

$$\begin{pmatrix} p'_{00}(t) & p'_{01}(t) \\ p'_{10}(t) & p'_{11}(t) \end{pmatrix} = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix} \begin{pmatrix} p_{00}(t) & p_{01}(t) \\ p_{10}(t) & p_{11}(t) \end{pmatrix}$$

$$p'_{00}(t) = -\lambda p_{00}(t) + \lambda p_{10}(t) \quad \begin{array}{l} \times \mu \\ + \\ \times \lambda \end{array}$$

$$p'_{10}(t) = \mu p_{00}(t) - \mu p_{10}(t) \quad \begin{array}{l} \times \mu \\ - \\ \times \lambda \end{array}$$

$$\mu p'_{00}(t) + \lambda p'_{10}(t) = 0$$

$$\vee \mu p_{00}(t) + \lambda p_{10}(t) = C$$

$$p_{00}(0) = 1 \quad p_{10}(0) = 0$$

$$\mu \cdot 1 + \lambda \cdot 0 = C \quad \Rightarrow \quad \mu = C$$

$$\mu p_{00}(t) + \lambda p_{10}(t) = \mu$$

$$\lambda p_{10}(t) = \mu (1 - p_{00}(t))$$

$$p'_{00}(t) = \mu (1 - p_{00}(t)) - \lambda p_{00}(t)$$

$$p'_{00}(t) = \mu - (\mu + \lambda) p_{00}(t)$$

$$\text{let } h(t) = p_{00}(t) - \frac{\mu}{\mu + \lambda}$$

$$h'(t) = p'_{00}(t)$$

$$h'(t) = \mu - (\mu + \lambda) \cdot \left( h(t) + \frac{\mu}{\mu + \lambda} \right)$$

$$h'(t) = -(\mu + \lambda) h(t)$$

$$\frac{dh}{dt} = -(\mu + \lambda) h(t)$$

$$\frac{dh}{h} = -(\mu + \lambda) dt$$

$$\ln h = -(\mu + \lambda)t + C \quad C = \ln k$$

$$h(t) = k e^{-(\mu + \lambda)t}$$

$$P_{00}(t) - \frac{\mu}{\mu + \lambda} = k e^{-(\mu + \lambda)t}$$

$$P_{00}(t) = k e^{-(\mu + \lambda)t} + \frac{\mu}{\mu + \lambda}$$

$$t = 0$$

$$P_{00}(0) = 1$$

$$1 = k + \frac{\mu}{\mu + \lambda} \quad k = 1 - \frac{\mu}{\mu + \lambda} = \frac{\lambda}{\mu + \lambda}$$

$$P_{00}(t) = \frac{\lambda}{\lambda + \mu} e^{-(\mu + \lambda)t} + \frac{\mu}{\mu + \lambda}$$

$$P_{10}(t) = \frac{\mu}{\mu + \lambda} - \frac{\mu}{\mu + \lambda} e^{-(\mu + \lambda)t}$$

$$P_{00}(10) = \frac{\lambda}{\mu + \lambda} e^{-10(\mu + \lambda)} + \frac{\mu}{\lambda + \mu}$$