$X_i \sim N(\mu, \sigma^2)$, where σ^2 is known.
I) Test $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$. The space of the parameter μ is denoted by Ω and

$$\Omega = (-\infty, \infty),$$

and $\widehat{\mu}$, the MLE for μ, is $\overline{X}$.
The GLR is

$$\begin{aligned}\Lambda(\mathbf{x}) &= \frac{f(\mathbf{x}; \mu_0)}{f(\mathbf{x}; \overline{x})} \\ &= \exp[\frac{-n(\overline{x} - \mu_0)^2}{2\sigma^2}].\end{aligned}$$

Reject H_0 if $\Lambda(\mathbf{x}) \leqslant k$ which is equivalent to rejecting H_0 if

$$[\frac{(\overline{x} - \mu_0)}{\sigma/\sqrt{n}}]^2 \geqslant k_1.$$

Now $\frac{(\overline{x}-\mu_0)}{\sigma/\sqrt{n}} \sim N(0,1)$ and equivalently, reject H_0 if

$$\frac{(\overline{x} - \mu_0)}{\sigma/\sqrt{n}} \leqslant -z_{\alpha/2} \text{ or } \frac{(\overline{x} - \mu_0)}{\sigma/\sqrt{n}} \geqslant z_{\alpha/2}.$$

II) Test $H_0 : \mu = \mu_0$ versus $H_1 : \mu > \mu_0$. In this example

$$\Omega = [\mu_0, \infty).$$

The MLE for μ is

$$\widehat{\mu} = \begin{cases} \overline{x} & \overline{x} > \mu_0 \\ \mu_0 & \overline{x} \leqslant \mu_0, \end{cases}$$

and GLR is

$$\Lambda(\mathbf{x}) = \begin{cases} \exp[\frac{-n(\overline{x}-\mu_0)^2}{2\sigma^2}] & \overline{x} > \mu_0 \\ 1 & \overline{x} \leqslant \mu_0. \end{cases}$$

But under H_0

$$P[\Lambda(\mathbf{x}) < 1] = P[\overline{X} > \mu_0] = 0.05.$$

Therefore, for $\alpha < 0.05$, and $k < 1$, the critical region will not contain any $\mathbf{x}$ such that $\Lambda(\mathbf{x}) = 1$.

The GLR test is to reject H_0 if $\overline{x} > \mu_0$ and $\Lambda(\mathbf{x}) \leqslant k$, which is equivalent to

$$\overline{x} > \mu_0 \text{ and } [\frac{(\overline{x}-\mu_0)}{\sigma/\sqrt{n}}]^2 \geqslant k_1 \text{ when } \overline{x} > \mu_0.$$

Now

$$[\frac{(\overline{x}-\mu_0)}{\sigma/\sqrt{n}}]^2 \geqslant k_1 \text{ if and only if } \frac{(\overline{x}-\mu_0)}{\sigma/\sqrt{n}} \geqslant \sqrt{k_1}.$$

Consequently, a size α test ($\alpha < 0.05$) is to reject H_0 if

$$\frac{(\overline{x}-\mu_0)}{\sigma/\sqrt{n}} \geqslant z_\alpha.$$

III) Test $H_0 : \mu \leqslant \mu_0$ versus $H_1 : \mu > \mu_0$. In this example

$$\Omega = (-\infty, \infty).$$

Note

$$\widehat{\mu}_0 = \begin{cases} \overline{x} & \overline{x} \leqslant \mu_0 \\ \mu_0 & \overline{x} > \mu_0. \end{cases}$$

Hence

$$\Lambda(\mathbf{x}) = \begin{cases} 1 & \overline{x} \leqslant \mu_0 \\ \exp[\frac{-n(\overline{x}-\mu_0)^2}{2\sigma^2}] & \overline{x} > \mu_0. \end{cases}$$