Math 760

Chapter 2 HW

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4. When A^{-1} and B^{-1} exist, prove each of the following.

Hint: Part (a) can be proved by noting that $AA^{-1} = I$), I = I', and $(AA^{-1}) = (A^{-1})'A'$. Part (b) follows from $(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}B = I$

(a) $(A')^{-1} = (A^{-1})'$

We know:
I = I'($AA^{-1}) = I = A^{-1}A$
Therefore, $I' = I = (AA^{-1})' = (A^{-1})'A'$ and $I = (A^{-1}A)' = A'(A^{-1})'$. And because of this, this means $(A')^{-1}$ is the inverse of A', $(A')^{-1} = (A^{-1})'$

(b) $(AB)^{-1} = B^{-1}A^{-1}$

We know:
If ∃ a matrix B s.t. BA = AB = I, then B is called the inverse of A and is denoted by A^{-1}. And, $(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}B(I) = I(I) = I$
So following all that, AB has inverse $(AB)^{-1} = B^{-1}A^{-1}$

5. Check that

$$Q = \begin{bmatrix} \frac{5}{13} & \frac{12}{13} \\ -\frac{12}{13} & \frac{5}{13} \end{bmatrix}$$

is an orthogonal matrix.

We want to know if $Q * Q^T = I$, to prove that Q is an orthogonal matrix. The Q^T is:

```
##           [,1]       [,2]
## [1,] 0.3846154 -0.9230769
## [2,] 0.9230769  0.3846154
```

Now, we'll multiply the two matrices and see if the answer is an identity matrix.

```
##      [,1] [,2]
## [1,]    1    0
## [2,]    0    1
```

Thus, Q is an orthogonal matrix.

6. Let

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{9} & \mathbf{-2} \\ \mathbf{-2} & \mathbf{6} \end{bmatrix}$$

(a) Is A symmetric?

```
## [1] TRUE
## 9 -2 -2 6  =  9 -2 -2 6
```

Yes, **A** is symmetric.

(b) Show that A is a positive definite

```
## [1] TRUE
## The eigenvalues of A are: 10 5 .
```

Because the eigenvalues are positive, A is a positive definite

14. Show that $\boldsymbol{Q'}_{(p\ x\ p)}, \boldsymbol{A}_{(p\ x\ p)}, \boldsymbol{Q}_{(p\ x\ p)}$, and $\boldsymbol{A}_{(p\ x\ p)}$ have the same eigenvalues if Q is orthogonal.

*Hint: Let λ be an eigenvalue of **A**. Then $0 = |A - \lambda I|$. By Exercise 2.13 and Result 2A.11(e), we can write $0 = |Q'||A - \lambda I||Q| = |Q'AQ - \lambda I|$, since Q'Q = I.*

With the hint in mind, we can write: $0 = |Q||A - \lambda I||Q'| = |QAQ' - \lambda I|$. If Q is orthogonal, then λ is also an eigenvalue of QAQ'.

20. Determine the square-root matrix $A^{1/2}$, using the matrix A in Exercise 2.3. Also, determine $A^{-1/2}$, and show that $A^{1/2}A^{-1/2} = A^{-1/2}A^{1/2} = \boldsymbol{I}$.

$A^{1/2}$ is

```
##           [,1]      [,2]
## [1,] 1.3763819 0.3249197
## [2,] 0.3249197 1.7013016
```

$A^{-1/2}$ is

```
##            [,1]       [,2]
## [1,]  0.7608452 -0.1453085
## [2,] -0.1453085  0.6155367
```

Now, we prove $A^{1/2}A^{-1/2} = A^{-1/2}A^{1/2} = I$.

```
##      [,1] [,2]
## [1,]    1    0
## [2,]    0    1
##      [,1] [,2]
## [1,]    1    0
## [2,]    0    1
```

26. Use Σ as given in Exercise 2.25

$$\Sigma =$$

```
##      [,1] [,2] [,3]
## [1,]   25   -2    4
## [2,]   -2    4    1
## [3,]    4    1    9
```

(a) Find ρ_{13}.

$\rho_{13} = \frac{\sigma_{13}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}} =$

```
## [1] 0.2666667
```

(b) Find the correlation between X_1 and $\frac{1}{2}X_2 + \frac{1}{2}X_3$.

We want to know:

$$\rho\left(X_1, \frac{1}{2}X_2 + \frac{1}{2}X_3\right)$$

We know:

$$\rho(X_1, X_2) = \frac{Cov(X_1, X_2)}{\sqrt{(}VX_1)\sqrt{(}VX_@)}$$

Then:

$1(X_1) + 0(X_2) + 0(X_3) = X_1 \Rightarrow c'_1X, \text{where } c'_1 = [1,0,0]$

$0(X_1) + \frac{1}{2}(X_2) + \frac{1}{2}(X_3) = \frac{1}{2}X_2 + \frac{1}{2}X_3 \Rightarrow c'_2X, \text{where } c'_2 = \left[0, \frac{1}{2}, \frac{1}{2}\right]$

From there, we can find the variances and covariance.

$$V(X_1) = \sigma_{11} =$$

```
## [1] 25
```

Because we have the X's multiplied by a constant for (X_2, X_3), the variance will be in this form: $V(c'X) = c'\Sigma c$ (2-43).

$V\left(\frac{1}{2}X_2 + \frac{1}{2}X_3\right) = \frac{1}{4}V(X_2 + X_3) = \frac{1}{4}(\sigma_{22} + 2\sigma_{23} + \sigma_{33}) =$

```
## [1] 3.75
```

The same applies for the covariance: $\Sigma_z = Cov(Z) = Cov(CX) = C\Sigma_X C'$

$Cov\left(X_1, \frac{1}{2}X_2 + \frac{1}{2}X_3\right) = \frac{1}{2}Cov(X_1, X_2 + X_3) = \frac{1}{2}(\sigma_{12} + \sigma_{13}) =$

```
## [1] 1
```

Finally, we can plug everything in.

```
## [1] 0.1032796
```

32. You are given the random vector $X' = [X_1, X_2, \ldots, X_5]$ with mean vector $\mu_X{}' = [2, 4, -1, 3]$, and variance-covariance matrix

$$\Sigma_X = \begin{bmatrix} 4 & -1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ -1 & 3 & 1 & -1 & 0 \\ \frac{1}{2} & 1 & 6 & 1 & -1 \\ -\frac{1}{2} & -1 & 1 & 4 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix}$$

Partition X as

$$X = \begin{bmatrix} X_1 \\ X_2 \\ --- \\ X_3 \\ X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} X^{(1)} \\ --- \\ X^{(2)} \end{bmatrix}$$

Let

$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

and consider the linear combinations $AX^{(1)}$ and $BX^{(2)}$. Find

(a) $E(X^{(1)})$

$$E(X^{(1)}) \Rightarrow \mu_X^{(1)}$$

```
##      [,1]
## [1,]    2
## [2,]    4
```

(b) $E(AX^{(1)})$

$$E(AX^{(1)}) \Rightarrow AE(X^{(1)}) \Rightarrow A\left(\mu_X^{(1)}\right)$$

```
##      [,1]
## [1,]   -2
## [2,]    6
```

(c) $Cov(X^{(1)})$

$$Cov(X^{(1)}) \Rightarrow \Sigma_{11}$$

```
##      [,1] [,2]
## [1,]    4   -1
## [2,]   -1    3
```

(d) $Cov(AX^{(1)})$

$$Cov(AX^{(1)}) \Rightarrow A^2 Cov(X^{(1)}) \Rightarrow A(\Sigma_{11})A'$$

```
##      [,1] [,2]
## [1,]    9    1
## [2,]    1    5
```

(e) $E(X^{(2)})$

$$E(X^{(2)}) \Rightarrow \mu_X^{(2)}$$

```
##      [,1]
## [1,]   -1
## [2,]    3
## [3,]    0
```

(f) $E(BX^{(2)})$

$$E(BX^{(2)}) \Rightarrow BE(X^{(2)}) \Rightarrow B\left(\mu_X^{(2)}\right)$$

```
##      [,1]
## [1,]    2
## [2,]    2
```

(g) $Cov(X^{(2)})$

$$Cov(X^{(2)}) \Rightarrow \Sigma_{22}$$

```
##      [,1] [,2] [,3]
## [1,]    6    1   -1
## [2,]    1    4    0
## [3,]   -1    0    2
```

(h) $Cov(BX^{(2)})$

$$Cov(BX^{(2)}) \Rightarrow B^2 Cov(X^{(2)}) \Rightarrow B(\Sigma_{22})B'$$

```
##      [,1] [,2]
## [1,]   12    9
## [2,]    9   24
```

(i) $Cov(X^{(1)}, X^{(2)})$

```
##      [,1] [,2] [,3]
## [1,]  0.5 -0.5    0
## [2,]  1.0 -1.0    0
```

(j) $Cov(AX^{(1)}, BX^{(2)})$

$$Cov(AX^{(1)}, BX^{(2)}) \Rightarrow A(\Sigma_{12})B'$$

```
##      [,1] [,2]
## [1,]    0    0
## [2,]    0    0
```

34. Consider the vectors $b' = [2, -1, 4, 0]$ and $d' = [-1, 3, -2, 1]$. Verify the Cauchy-Schwarz inequality $(b, d)^2 \leq (b'b)(d'd)$.

$$(b, d)^2 \leq (b'b)(d'd)$$

Let's find the right side of the inequality first.

(b'b) is:

```
##      [,1]
## [1,]   21
```

(d'd) is:

```
##      [,1]
## [1,]   15
```

The product of (b'b)(d'd) is:

```
##      [,1]
## [1,]  315
```

Now, the left side:

```
##      [,1]
## [1,]  169
```

Finally, let's plug it all in.

$$(b, d)^2 \leq (b'b)(d'd) \Rightarrow 169 \leq 315$$

The Cauchy-Schwarz inequality holds!

42. Repeat Exercise 2.41, but with

$$\boldsymbol{\Sigma_X} = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 1 & -3 \end{bmatrix}$$
$$\boldsymbol{X'} = [X_1 \quad X_2 \quad X_3 \quad X_4]$$
$$\boldsymbol{\mu'_X} = [3 \quad 2 \quad -2 \quad 0]$$

(a) Find E(AX), the mean of AX.

We know: $E(AX) = AE(X) = A(\mu'_X)$. Therefore, the mean of **AX** is:

```
##      [,1]
## [1,]    1
## [2,]    9
## [3,]    3
```

(b) Find Cov(AX), the variances and covariances of AX.

We know: $Cov(AX) = ACov(X)A' = A(\Sigma_X)A'$. Therefore the variances and covariances of **AX** is:

```
##      [,1] [,2] [,3]
## [1,]    4    0    0
## [2,]    0   12    0
## [3,]    0    0   24
```

(c) Which pairs of linear combinations have zero covariances?

All pairs of linear combos have zero covariances as shown above. This is because the covariance matrix is structured as such:

$$\begin{bmatrix} V(X_1) & Cov(X_1, X_2) & Cov(X_1, X_3) \\ Cov(X_2, X_1) & V(X_2) & Cov(X_2, X_3) \\ Cov(X_3, X_1) & Cov(X_3, X_2) & V(X_3) \end{bmatrix}$$

Code

```
knitr::opts_chunk$set(echo = FALSE)
library(expm)
library(Matrix)
library(matrixcalc)
Qmat <- c(5/13,12/13,-12/13,5/13)
Q <- matrix(Qmat, nrow = 2, ncol = 2, byrow = TRUE)
QT <- t(Q)
print(QT)
Q%*%QT
aMat <- c(9,-2,
          -2,6)
A <- matrix(aMat, nrow = 2, ncol = 2, byrow = TRUE)
isSymmetric(A)
AT <- t(A)
cat(A," = ", AT)
is.positive.definite(A)
ev <- eigen(A)
value <- ev$values
cat("The eigenvalues of A are:", value, ".")
aMat <- c(2,1,1,3)
A <- matrix(aMat, nrow = 2, ncol = 2, byrow = TRUE)
sqrtMat <- sqrtm(A)
sqrtMat
neg <- solve(sqrtMat)
neg
AAneg <- sqrtMat%*%neg
negAA <- neg%*%sqrtMat
AAneg
negAA
sigMat <- c(25, -2, 4,
            -2, 4, 1,
            4, 1, 9)
sigma <- matrix(sigMat, nrow = 3, ncol = 3, byrow = TRUE)
sigma
p13 <- 4/(sqrt(25)*sqrt(9))
p13
var1 <- sigma[1,1]
var1
var2 <- 0.25*(sigma[2,2] + 2*sigma[2,3] + sigma[3,3])
var2
cov <- 0.5*(sigma[1,2] + sigma[1,3])
cov
rho <- cov/(sqrt(var1)*sqrt(var2))
rho
# A
aMat <- c(1,-1,1,1)
A <- matrix(aMat, nrow = 2, ncol = 2, byrow = TRUE)
```

```r
# B
bMat <- c(1,1,1,
          1,1,-2)
B <- matrix(bMat, nrow = 2, ncol = 3, byrow = TRUE)

# mu
mu1Mat <- c(2,4)
mu1 <- matrix(mu1Mat, nrow = 1, ncol = 2, byrow = TRUE)
mu2Mat <- c(-1,3,0)
mu2 <- matrix(mu2Mat, nrow = 1, ncol = 3, byrow = TRUE)

# sigma
sigMat <- c(4,-1,0.5,-0.5,0,
            -1,3,1,-1,0,
            0.5,1,6,1,-1,
            -0.5,-1,1,4,0,
            0,0,-1,0,2)
sigma <- matrix(sigMat, nrow = 5, ncol = 5, byrow = TRUE)
t(mu1)
A %*% t(mu1)
sig11 <- sigma[1:2,1:2]
sig11
A %*% sig11 %*% t(A)
t(mu2)
B %*% t(mu2)
sig22 <- sigma[3:5,3:5]
sig22
B %*% sig22 %*% t(B)
sig12 <- sigma[3:5,1:2]
t(sig12)
A %*% t(sig12) %*% t(B)
bMat <- c(2,-1,4,0)
dMat <-  c(-1,3,-2,1)

# matrix
b <- matrix(bMat, nrow = 1, ncol = 4, byrow = TRUE)
d <- matrix(dMat, nrow = 1, ncol = 4, byrow = TRUE)
b2 <- b %*% t(b)
b2
d2 <- d %*% t(d)
d2
bd2 <- b2 %*% d2
bd2
bd <- b %*% t(d)
bd^2
sigMat <- c(3,1,1,1,
            1,3,1,1,
            1,1,3,1,
            1,1,1,3)
aMat <- c(1,-1,0,0,
```

```
          1,1,-2,0,
          1,1,1,-3)
muMat <- c(3,2,-2,0)

# matrix
sigma <- matrix(sigMat, nrow = 4, ncol = 4, byrow = TRUE)
A <- matrix(aMat, nrow = 3, ncol = 4, byrow = TRUE)
mu <- matrix(muMat, nrow = 1, ncol = 4, byrow = TRUE)
A %*% t(mu)
A %*% sigma %*% t(A)
```