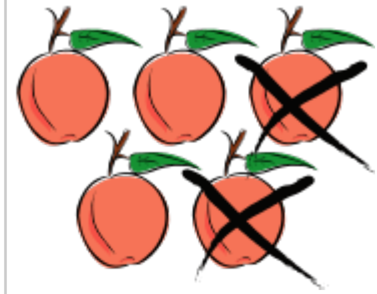
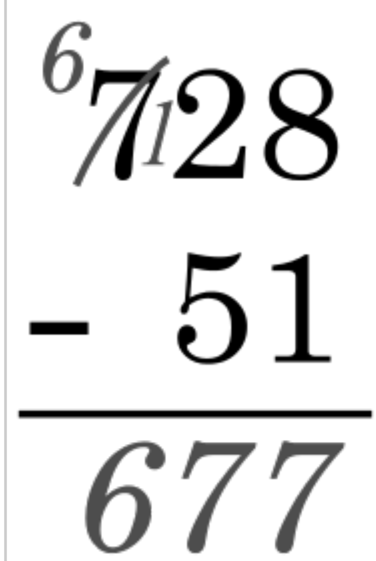
Subtraction



"5 − 2 = 3" (verbally, "five minus two equals three")

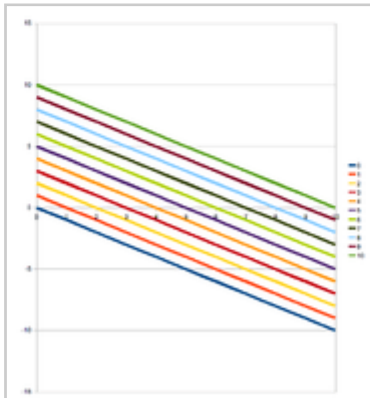
Subtraction is a mathematical operation that represents the operation of removing objects from a collection. It is signified by the minus sign (−). For example, in the picture on the right, there are 5 − 2 apples—meaning 5 apples with 2 taken away, which is a total of 3 apples. Therefore, 5 − 2 = 3. Besides counting fruits, subtraction can also represent combining other physical and abstract quantities using different kinds of objects including negative numbers, fractions, irrational numbers, vectors, decimals, functions, and matrices.



An example problem

Subtraction follows several important patterns. It is anticommutative, meaning that changing the order changes the sign of the answer. It is not associative, meaning that when one subtracts more than two numbers, the order in which subtraction is performed matters. Subtraction of 0 does not change a number. Subtraction also obeys predictable rules concerning related operations such as addition and multiplication. All of these rules can be proven, starting with the subtraction of integers and generalizing up through the real numbers and beyond. General binary operations that continue these patterns are studied in abstract algebra.

Performing subtraction is one of the simplest numerical tasks. Subtraction of very small numbers is accessible to young children. In primary education, students are taught to subtract numbers in the decimal system, starting with single digits and progressively tackling more difficult problems.



Subtraction of numbers 0–10. Line labels = minuend. X axis = subtrahend. Y axis = difference.

Notation and terminology

Subtraction is written using the minus sign "−" between the terms; that is, in infix notation. The result is expressed with an equals sign. For example,

$2 - 1 = 1$ (verbally, "two minus one equals one")
$4 - 2 = 2$ (verbally, "four minus two equals two")
$6 - 3 = 3$ (verbally, "six minus three equals three")
$4 - 6 = -2$ (verbally, "four minus six equals negative two")

There are also situations where subtraction is "understood" even though no symbol appears:

- A column of two numbers, with the lower number in red, usually indicates that the lower number in the column is to be subtracted, with the difference written below, under a line. This is most common in accounting.

Formally, the number being subtracted is known as the *subtrahend*,[1][2] while the number it is subtracted from is the *minuend*.[1][2] The result is the *difference*.[1][2]

All of this terminology derives from Latin. "Subtraction" is an English word derived from the Latin verb *subtrahere*, which is in turn a compound of *sub* "from under" and *trahere* "to pull"; thus to *subtract* is to *draw from below, take away*.[3] Using the gerundive suffix *-nd* results in "subtrahend", "thing to be subtracted".[4] Likewise from *minuere* "to reduce or diminish", one gets "minuend", "thing to be diminished".

Calculation results

Addition (+)

$$\left.\begin{matrix}\text{summand}+\text{summand}\\ \text{addend (broad sense)}+\text{addend (broad sense)}\\ \text{augend}+\text{addend (strict sense)}\end{matrix}\right\}=\text{sum}$$

Subtraction (−)

$$\text{minuend}-\text{subtrahend}=\text{difference}$$

Multiplication (×)

$$\left.\begin{matrix}\text{factor}\times\text{factor}\\ \text{multiplier}\times\text{multiplicand}\end{matrix}\right\}=\text{product}$$

Division (÷)

$$\left.\begin{matrix}\frac{\text{dividend}}{\text{divisor}}\\ \\ \frac{\text{numerator}}{\text{denominator}}\end{matrix}\right\}=\begin{matrix}\text{fraction}\\ \text{quotient}\\ \text{ratio}\end{matrix}$$

Modulo (mod)

$$\text{dividend} \bmod \text{divisor}=\text{remainder}$$

Exponentiation

$$\text{base}^{\text{exponent}}=\text{power}$$

***n*th root (√)**

$$\sqrt[\text{degree}]{\text{radicand}}=\text{root}$$

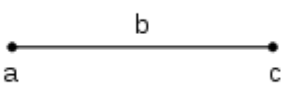
Logarithm (log)

$$\log_{\text{base}}(\text{antilogarithm})=\text{logarithm}$$

Placard outside shop in Bordeaux advertising subtraction of 20% from the price of a second perfume

Of integers and real numbers

Integers

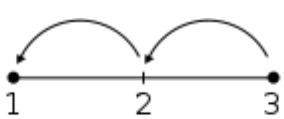


Imagine a line segment of length *b* with the left end labeled *a* and the right end labeled *c*. Starting from *a*, it takes *b* steps to the right to reach *c*. This movement to the right is modeled mathematically by addition:

$$a + b = c.$$

From *c*, it takes *b* steps to the *left* to get back to *a*. This movement to the left is modeled by subtraction:

$$c - b = a.$$



Now, a line segment labeled with the numbers 1, 2, and 3. From position 3, it takes no steps to the left to stay at 3, so 3 − 0 = 3. It takes 2 steps to the left to get to position 1, so 3 − 2 = 1. This picture is inadequate to describe what would happen after going 3 steps to the left of position 3. To represent such an operation, the line must be extended.

To subtract arbitrary natural numbers, one begins with a line containing every natural number (0, 1, 2, 3, 4, 5, 6, ...). From 3, it takes 3 steps to the left to get to 0, so 3 − 3 = 0. But 3 − 4 is still invalid since it again leaves the line. The natural numbers are not a useful context for subtraction.

The solution is to consider the integer number line (..., −3, −2, −1, 0, 1, 2, 3, ...). From 3, it takes 4 steps to the left to get to −1:

$3 - 4 = -1.$

Natural numbers

Subtraction of natural numbers is not closed. The difference is not a natural number unless the minuend is greater than or equal to the subtrahend. For example, 26 cannot be subtracted from 11 to give a natural number. Such a case uses one of two approaches:

1. Say that 26 cannot be subtracted from 11; subtraction becomes a partial function.
2. Give the answer as an integer representing a negative number, so the result of subtracting 26 from 11 is -15.

Real numbers

Subtraction of real numbers is defined as addition of signed numbers. Specifically, a number is subtracted by adding its additive inverse. Then we have $3 - \pi = 3 + (-\pi)$. This helps to keep the ring of real numbers "simple" by avoiding the introduction of "new" operators such as subtraction. Ordinarily a ring only has two operations defined on it; in the case of the integers, these are addition and multiplication. A ring already has the concept of additive inverses, but it does not have any notion of a separate subtraction operation, so the use of signed addition as subtraction allows us to apply the ring axioms to subtraction without needing to prove anything.

Properties

Anticommutativity

Subtraction is anti-commutative, meaning that if one reverses the terms in a difference left-to-right, the result is the negative of the original result. Symbolically, if a and b are any two numbers, then

$a - b = -(b - a).$

Non-associativity

Subtraction is non-associative, which comes up when one tries to define repeated subtraction. Should the expression

"$a - b - c$"

be defined to mean $(a - b) - c$ or $a - (b - c)$? These two possibilities give different answers. To resolve this issue, one must establish an order of operations, with different orders giving different results.

Predecessor

In the context of integers, subtraction of one also plays a special role: for any integer a, the integer $(a - 1)$ is the largest integer less than a, also known as the predecessor of a.

Units of measurement

When subtracting two numbers with units of measurement such as kilograms or pounds, they must have the same unit. In most cases the difference will have the same unit as the original numbers.

Percentages

Changes in percentages can be reported in at least two forms, percentage change and percentage point change. Percentage change represents the relative change between the two quantities as a percentage, while percentage point change is simply the number obtained by subtracting the two percentages.[5][6][7]

As an example, suppose that 30% of widgets made in a factory are defective. Six months later, 20% of widgets are defective. The percentage change is -33 1/3%, while the percentage point change is -10 percentage points.

In computing

The method of complements is a technique used to subtract one number from another using only addition of positive numbers. This method was commonly used in mechanical calculators and is still used in modern computers.

Binary digit	Ones' complement
0	1
1	0

To subtract a binary number *y* (the subtrahend) from another number *x* (the minuend), the ones' complement of *y* is added to *x* and one is added to the sum. The leading digit '1' of the result is then discarded.

The method of complements is especially useful in binary (radix 2) since the ones' complement is very easily obtained by inverting each bit (changing '0' to '1' and vice versa). And adding 1 to get the two's complement can be done by simulating a carry into the least significant bit. For example:

```
  01100100  (x, equals decimal 100)
- 00010110  (y, equals decimal 22)
```

becomes the sum:

```
  01100100  (x)
+ 11101001  (ones' complement of y)
+        1  (to get the two's complement)
——————————
 101001110
```

Dropping the initial "1" gives the answer: 01001110 (equals decimal 78)

The teaching of subtraction in schools

Methods used to teach subtraction to elementary school vary from country to country, and within a country, different methods are in fashion at different times. In what is, in the United States, called traditional mathematics, a specific process is taught to students at the end of the 1st year or during the 2nd year for use with multi-digit whole numbers, and is extended in either the fourth or fifth grade to include decimal representations of fractional numbers.

In America

Almost all American schools currently teach a method of subtraction using borrowing or regrouping (the decomposition algorithm) and a system of markings called crutches.[8][9] Although a method of borrowing had been known and published in textbooks previously, the use of crutches in American schools spread after William A. Brownell published a study claiming that crutches were beneficial to students using this method.[10] This system caught on rapidly, displacing the other methods of subtraction in use in America at that time.

In Europe

Some European schools employ a method of subtraction called the Austrian method, also known as the additions method. There is no borrowing in this method. There are also crutches (markings to aid memory), which vary by country.[11][12]

Comparing the two main methods

Both these methods break up the subtraction as a process of one digit subtractions by place value. Starting with a least significant digit, a subtraction of subtrahend:

$$s_j\, s_{j-1} \ldots s_1$$

from minuend

$$m_k\, m_{k-1} \ldots m_1,$$

where each s_i and m_i is a digit, proceeds by writing down $m_1 - s_1$, $m_2 - s_2$, and so forth, as long as s_i does not exceed m_i. Otherwise, m_i is increased by 10 and some other digit is modified to correct for this increase. The American method corrects by attempting to decrease the minuend digit m_{i+1} by one (or continuing the borrow leftwards until there is a non-zero digit from which to borrow). The European method corrects by increasing the subtrahend digit s_{i+1} by one.

Example: 704 − 512.

$$\begin{array}{rrrl}
\color{red}{-1} & & & \color{red}{\longleftarrow \mathrm{carry}} \\
C & D & U & \\
7 & 0 & 4 & \longleftarrow \text{Minuend} \\
5 & 1 & 2 & \longleftarrow \text{Subtrahend} \\
\hline
1 & 9 & 2 & \longleftarrow \text{Rest or Difference}
\end{array}$$

The minuend is 704, the subtrahend is 512. The minuend digits are $m_3 = 7$, $m_2 = 0$ and $m_1 = 4$. The subtrahend digits are $s_3 = 5$, $s_2 = 1$ and $s_1 = 2$. Beginning at the one's place, 4 is not less than 2 so the difference 2 is written down in the result's one place. In the ten's place, 0 is less than 1, so the 0 is increased by 10, and the difference with 1, which is 9, is written down in the ten's place. The American method corrects for the increase of ten by reducing the digit in the minuend's hundreds place by one. That is, the 7 is struck through and replaced by a 6. The subtraction then proceeds in the hundreds place, where 6 is not less than 5, so the difference is written down in the result's hundred's place. We are now done, the result is 192.

The Austrian method does not reduce the 7 to 6. Rather it increases the subtrahend hundred's digit by one. A small mark is made near or below this digit (depending on the school). Then the subtraction proceeds by asking what number when increased by 1, and 5 is added to it, makes 7. The answer is 1, and is written down in the result's hundred's place.

There is an additional subtlety in that the student always employs a mental subtraction table in the American method. The Austrian method often encourages the student to mentally use the addition table in reverse. In the example above, rather than adding 1 to 5, getting 6, and subtracting that from 7, the student is asked to consider what number, when increased by 1, and 5 is added to it, makes 7.

Subtraction by hand

Austrian method

Example:

- $$\begin{array}{r} 753 \\ -491 \\ \hline \end{array}$$

  1 + … = 3

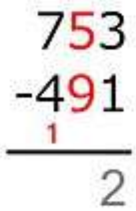
- $$\begin{array}{r} 753 \\ -491 \\ \hline 2 \end{array}$$

  The difference is
  written under the line.

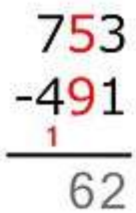
- $$\begin{array}{r} 753 \\ -491 \\ \hline 2 \end{array}$$

  9 + … = 5
  The required sum (5) is

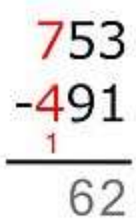
too small!

- 

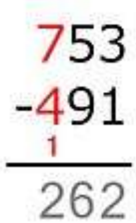
So, we add 10 to it and put a 1 under the next higher place in the subtrahend.

- 

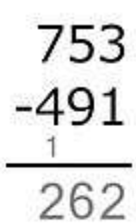
9 + … = 15
Now we can find the difference like before.

- 

(4 + 1) + … = 7

- 

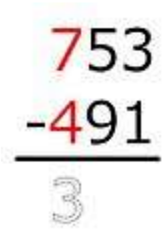
The difference is written under the line.

- 

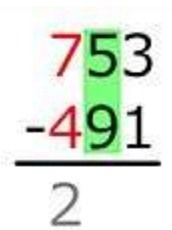
The total difference.

Subtraction from left to right

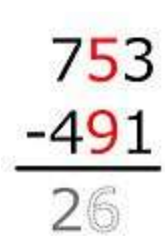
Example:

- 

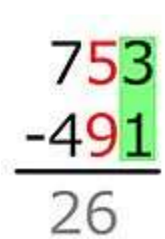
  7 − 4 = 3
  This result is only penciled in.

- 

  Because the next digit of the minuend is smaller than the subtrahend, we subtract one from our penciled-in-number and mentally add ten to the next.

- 

  15 − 9 = 6

- 

  Because the next digit in the minuend is not smaller than the subtrahend, We keep this number.

- $$\begin{array}{r} 753 \\ -491 \\ \hline 262 \end{array}$$

  3 − 1 = 2

American method

In this method, each digit of the subtrahend is subtracted from the digit above it starting from right to left. If the top number is too small to subtract the bottom number from it, we add 10 to it; this 10 is 'borrowed' from the top digit to the left, which we subtract 1 from. Then we move on to subtracting the next digit and borrowing as needed, until every digit has been subtracted. Example:

- $$\begin{array}{r} 753 \\ -491 \\ \hline \end{array}$$

  3 − 1 = …

- $$\begin{array}{r} 753 \\ -491 \\ \hline 2 \end{array}$$

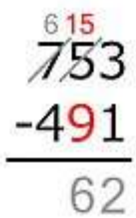
  We write the difference under the line.

- $$\begin{array}{r} 753 \\ -491 \\ \hline 2 \end{array}$$

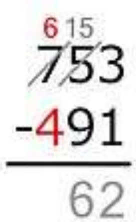
  5 − 9 = …
  The minuend (5) is too small!

- 

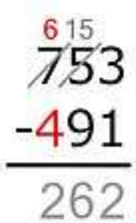
  So, we add 10 to it. The 10 is 'borrowed' from the digit on the left, which goes down by 1.

- 

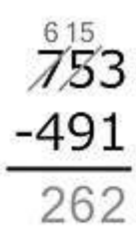
  15 − 9 = …
  Now the subtraction works, and we write the difference under the line.

- 

  6 − 4 = …

- 

  We write the difference under the line.

- 

The total difference.

Trade first

A variant of the American method where all borrowing is done before all subtraction.[13]

Example:

- 

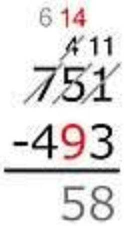
  1 − 3 = not possible.
  We add a 10 to the 1.
  Because the 10 is
  'borrowed' from the
  nearby 5, the 5 is
  lowered by 1.

- 

  4 − 9 = not possible.
  So we proceed as in
  step 1.

- 

  Working from right to
  left:
  11 − 3 = 8

- 

  14 − 9 = 5

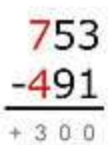
- 

  6 − 4 = 2

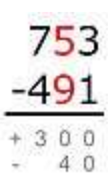
Partial differences

The partial differences method is different from other vertical subtraction methods because no borrowing or carrying takes place. In their place, one places plus or minus signs depending on whether the minuend is greater or smaller than the subtrahend. The sum of the partial differences is the total difference.[14]

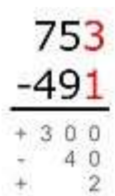
Example:

- 

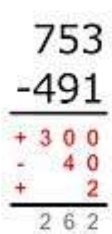
  The smaller number is subtracted from the greater:
  700 − 400 = 300
  Because the minuend is greater than the subtrahend, this difference has a plus sign.

- 

The smaller number is subtracted from the greater:
90 − 50 = 40
Because the minuend is smaller than the subtrahend, this difference has a minus sign.

- 

The smaller number is subtracted from the greater:
3 − 1 = 2
Because the minuend is greater than the subtrahend, this difference has a plus sign.

- 

+ 300 − 40 + 2 = 262

Nonvertical methods

Counting up

Instead of finding the difference digit by digit, one can count up the numbers between the subtrahend and the minuend.[15]

Example:

1234 − 567 = can be found by the following steps:

- 567 + **3** = 570
- 570 + **30** = 600

- 600 + **400** = 1000
- 1000 + **234** = 1234

Add up the value from each step to get the total difference: 3 + 30 + 400 + 234 = 667.

Breaking up the subtraction

Another method that is useful for mental arithmetic is to split up the subtraction into small steps.[16]

Example:

1234 − 567 = can be solved in the following way:

- 1234 − **500** = 734
- 734 − **60** = 674
- 674 − **7** = 667

Same change

The same change method uses the fact that adding or subtracting the same number from the minuend and subtrahend does not change the answer. One adds the amount needed to get zeros in the subtrahend.[17]

Example:

"1234 − 567 =" can be solved as follows:

- 1234 − 567 = 1237 − 570 = 1267 − 600 = 667

See also

- Decrement
- Elementary arithmetic
- Method of complements
- Negative number

References

1. 1 2 3 Schmid, Hermann (1974). *Decimal Computation* (1 ed.). Binghamton, New York, USA: John Wiley & Sons. ISBN 0-471-76180-X.
2. 1 2 3 Schmid, Hermann (1983) [1974]. *Decimal Computation* (1 (reprint) ed.). Malabar, Florida, USA: Robert E. Krieger Publishing Company. ISBN 0-89874-318-4.
3. ↑ "Subtraction". *Oxford English Dictionary* (3rd ed.). Oxford University Press. September 2005. (Subscription or UK public library membership required.)
4. ↑ "Subtrahend" is not a Latin word; in Latin it must be further conjugated, as in *numerus subtrahendus* "the number to be subtracted".
5. ↑ Paul E. Peterson, Michael Henderson, Martin R. West (2014) *Teachers Versus the Public: What Americans Think about Schools and How to Fix Them* Brookings Institution Press, p.163
6. ↑ Janet Kolodzy (2006) *Convergence Journalism: Writing and Reporting across the News Media* Rowman & Littlefield Publishers, p.180

7. ↑ David Gillborn (2008) *Racism and Education: Coincidence Or Conspiracy?* Routledge p.46
8. ↑ Paul Klapper (1916). *The Teaching of Arithmetic: A Manual for Teachers*. pp. 80–.
9. ↑ Susan Ross and Mary Pratt-Cotter. 2000. "Subtraction in the United States: An Historical Perspective," *The Mathematics Educator* 8(1):4-11. P. 8: "This new version of the decomposition algorithm [i.e., using Brownell's crutch] has so completely dominated the field that it is rare to see any other algorithm used to teach subtraction today [in America]."
10. ↑ Ross, Susan C.; Pratt-Cotter, Mary (1999). "Subtraction From a Historical Perspective". *School Science and Mathematics*. **99** (7): 389–393.
11. ↑ Klapper 1916, p. 177-.
12. ↑ David Eugene Smith (1913). *The Teaching of Arithmetic*. Ginn. pp. 77–.
13. ↑ The Many Ways of Arithmetic in UCSMP Everyday Mathematics Subtraction: Trade First
14. ↑ Partial-Differences Subtraction; The Many Ways of Arithmetic in UCSMP Everyday Mathematics Subtraction: Partial Differences
15. ↑ The Many Ways of Arithmetic in UCSMP Everyday Mathematics Subtraction: Counting Up
16. ↑ The Many Ways of Arithmetic in UCSMP Everyday Mathematics Subtraction: Left to Right Subtraction
17. ↑ The Many Ways of Arithmetic in UCSMP Everyday Mathematics Subtraction: Same Change Rule

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- Subtraction in the United States: An Historical Perspective, Susan Ross, Mary Pratt-Cotter, *The Mathematics Educator*, Vol. 8, No. 1 (original publication) and Vol. 10, No. 1 (reprint.) PDF

External links

- Hazewinkel, Michiel, ed. (2001), "Subtraction", *Encyclopedia of Mathematics*, Springer, ISBN 978-1-55608-010-4
- Printable Worksheets: Subtraction Worksheets, One Digit Subtraction, Two Digit Subtraction, Four Digit Subtraction, and More Subtraction Worksheets
- Subtraction Game at cut-the-knot
- Subtraction on a Japanese abacus selected from Abacus: Mystery of the Bead

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Elementary arithmetic

Addition (+)	Subtraction (−)	Multiplication (× or ·)	Division (÷ or /)