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ON AN INEQUALITY OF MEAN CURVATURES OF HIGHER DEGREE[1]

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1. **Introduction.** Let $x: M \to E^{n+N}$ be an immersion of an n-dimensional closed manifold M immersed in a euclidean space E^{n+N} of dimension $n+N$. Let B_v be the bundle of unit normal vectors of $x(M)$ so that a point of B_v is a pair (p, e), where e is a unit normal vector to $x(M)$ at $x(p)$. Then B_v is a bundle of $(N-1)$-dimensional spheres over M and is a manifold of dimension $n+N-1$. Let dV be the volume element of M. There is a differential form $d\sigma$ of degree $N-1$ on B_v such that its restriction to a fibre is the volume element of the sphere of unit normal vectors at a point $p \in M$; then $d\sigma \wedge dV$ is the volume element of B_v. For each $(p, e) \in B_v$, there corresponds a symmetric linear transformation $A(p, e)$ of the tangent space $T_p(M)$ of M at p, called the second fundamental form at (p, e). The eigenvalues $k_1(p, e), \cdots, k_n(p, e)$, of the second fundamental form $A(p, e)$ are called the principal curvatures at (p, e). The ith mean curvature $K_i(p, e)$, $i = 1, 2, \cdots, n$, are defined by the elementary symmetric functions as follows:

$$\binom{n}{i} K_i(p, e) = \sum k_1(p, e) \cdots k_i(p, e), \qquad i = 1, 2, \cdots, n, \tag{1}$$

where $\binom{n}{i} = n!/i!(n-i)!$.

We call the integral $K_i^*(p) = \int |K_i(p, e)|^{n/i} \, d\sigma$ over the sphere of unit normal vectors at $x(p)$, the ith total absolute curvature of the immersion x at p, and we define as the *ith total absolute curvature* of M itself the integral $\int_M K_i^*(p) dV$.

In this note, I would like to announce the following results:

THEOREM 1. *Let $x: M \to E^{n+N}$ be an immersion of a closed manifold of dimension n into E^{n+N}. Then we have the following inequality*:

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$$\int_M K_i^*(p)dV \geqq 2c_{n+N-1}, \qquad i = 1, 2, \cdots, n, \tag{2}$$

where c_{n+N-1} denotes the area of the unit $(n+N-1)$-sphere. The equality sign of (2) *holds when and only when M is imbedded as a hypersphere in an $(n+1)$-dimensional linear subspace of E^{n+N} if $i<n$, and as a convex hypersurface in an $(n+1)$-dimensional linear subspace of E^{n+N} if $i=n$.*

REMARK. If $i=n$, then this theorem is the well-known Fenchel-Borsuk-Chern-Lashof's theorem [1], [4], [5], and if $i=1$, this theorem was proved by Willmore-Chen [2], [3], [6].

THEOREM 2. *Under the hypothesis of Theorem* 1, *if the mean curvature normal $H(p)$ has constant length; $|H(p)| = (c_n/v(M))^{1/n}$, then M is immersed as a hypersphere with radius $(v(M)/c_n)^{1/n}$ in an $(n+1)$-dimensional linear subspace of E^{n+N}, where $v(M)$ denotes the volume of M.*

THEOREM 3. *Under the hypothesis of Theorem* 1, *if $N=1$ and $|K_i(p, e)|^n = (c_n/v(M))^i$, then M is immersed as a hypersphere with radius $(v(M)/c_n)^{1/n}$.*

Theorem 2 and Theorem 3 follow immediately from Theorem 1.

2. **Sketch of the proof of Theorem 1.** Fix a unit vector $e \in S_0^{n+N-1}$, S_0^{n+N-1} the unit hypersphere in E^{n+N}, the scalar product $e \cdot x(p)$ as a continuous function on M has at least one maximum and one minimum, say q and q', respectively. Since at (q, e) and (q', e), the second fundamental form $A(p, e)$ is semidefinite. Let $d\Sigma$ denote the volume element of S_0^{n+N-1} and define

$$\bar{v}: B_v \to S_0^{n+N-1} \tag{3}$$

by $\bar{v}(p, e) = e$. Then we have

$$\bar{v}^* d\Sigma = K_n(p, e) dV \wedge d\sigma. \tag{4}$$

Therefore, by Sard's theorem, we know that for almost all e in S_0^{n+N-1}, the second fundamental form $A(p, e)$ is definite for extreme points of the function $e \cdot x(p)$ on M. Hence, if we let U^* denote the set of all elements (p, e) in B_v such that $A(p, e)$ is definite, then we have

$$\int_{U^*} |K_n(p, e)| \, d\sigma \wedge dV \geqq 2c_{n+N-1}. \tag{5}$$

On the other hand, we have

$$
\begin{aligned}
(6)\qquad |K_1(p, e)|^n &\geqq |K_2(p, e)|^{n/2} \geqq \cdots \geqq |K_i(p, e)|^{n/i} \\
&\geqq \cdots \geqq |K_n(p, e)|,
\end{aligned}
$$

on U^*. Therefore, by (5) and (6), we get

$$
(7)\qquad \int_M K_i^*(p)dV \geqq \int_{U^*} |K_i(p, e)|^{n/i} d\sigma \wedge dV \geqq 2c_{n+N-1}.
$$

This proves (2).

Furthermore, if the equality sign of (2) holds, then we have

$$
(8)\qquad |K_i(p, e)|^n = |K_n(p, e)|^i, \quad \text{on } U^*
$$

and

$$
(9)\qquad K_i(p, e) = 0, \quad \text{on } B_v - U^*.
$$

By using (8), (9) and the continuity of $K_i(p, e)$ on M, we can prove that $B_v - U^* = \{(p, e) \in B_v : k_1(p, e) = \cdots = k_n(p, e) = 0\}$. Using these facts and Theorem 3 of [4], we can prove the remaining part of theorem without difficulty.

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