

Measure the Expectation
Value of Pauli Matrices

Review

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Review

ket:

colum vector

$$|\psi\rangle = \frac{4}{5}|0\rangle + \frac{3}{5}i|1\rangle = \begin{bmatrix} \frac{4}{5} \\ \frac{3}{5}i \end{bmatrix}$$

bra:

Hermitian conjugate of ket

$$\langle\psi| = \left[\frac{4}{5} \quad -\frac{3}{5}i \right]$$

projection operator:

ket and bra

$$|\psi\rangle\langle\psi|$$

Expectation value:

$$\langle\psi|Z|\psi\rangle = \left[\frac{4}{5} \quad -\frac{3}{5}i \right] \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{4}{5} \\ \frac{3}{5}i \end{bmatrix} = \frac{7}{25}$$

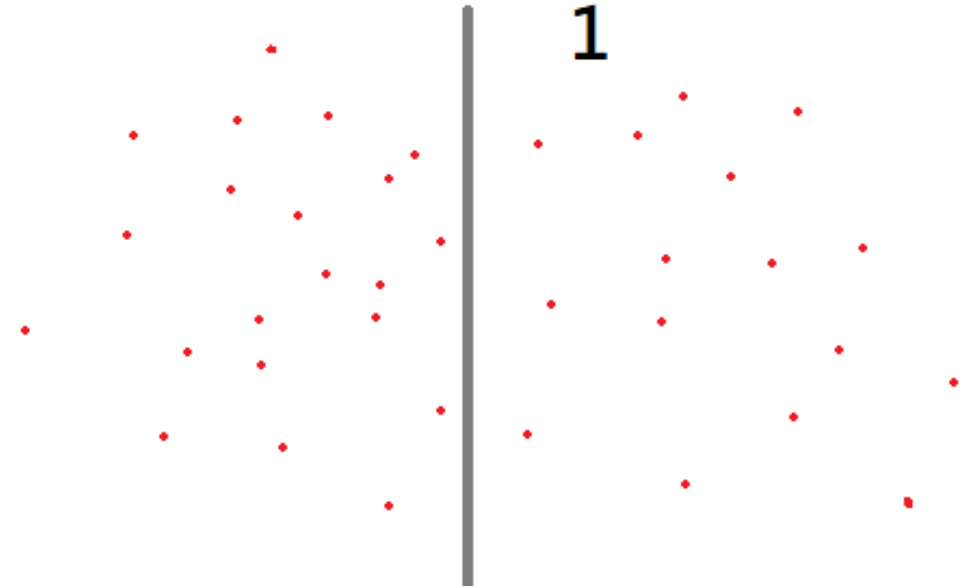
Measurement

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

$$\mathbb{P}_0 = \langle\psi|0\rangle\langle 0|\psi\rangle = |a|^2$$

$$\mathbb{P}_1 = \langle\psi|1\rangle\langle 1|\psi\rangle = |b|^2$$

0



Pauli Z

$$Z = \begin{array}{cc} \begin{array}{cc} |0\rangle & |1\rangle \\ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{array} \\ \begin{array}{c} |0\rangle \\ |1\rangle \end{array} \end{array}$$
$$= |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$|0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$|1\rangle\langle 1| = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Expectation Value of Z

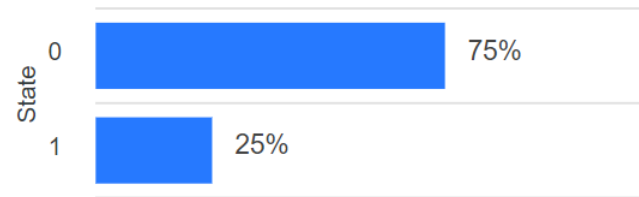
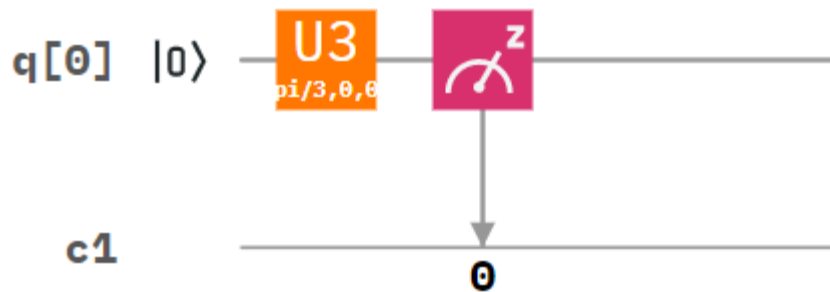
$$\begin{aligned}\langle\psi|Z|\psi\rangle &= \langle\psi|0\rangle\langle 0|\psi\rangle - \langle\psi|1\rangle\langle 1|\psi\rangle \\ &= \mathbb{P}_0 - \mathbb{P}_1\end{aligned}$$

EX: $|\psi\rangle = \frac{1}{2}(\sqrt{3}|0\rangle + |1\rangle)$

Classical computer:

$$\langle\psi|Z|\psi\rangle = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2}$$

Quantum computer:



$$\begin{aligned}\langle\psi|Z|\psi\rangle &= 0.75 - 0.25 \\ &= 0.5\end{aligned}$$

Expectation Value of X&Y

Classical computer:

$$|\psi\rangle = \frac{1}{2}(\sqrt{3}|0\rangle + |1\rangle)$$

$$\langle\psi|X|\psi\rangle = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{\sqrt{3}}{2} = 0.8660\dots$$

$$\langle\psi|Y|\psi\rangle = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} = 0$$

Expectation Value of X

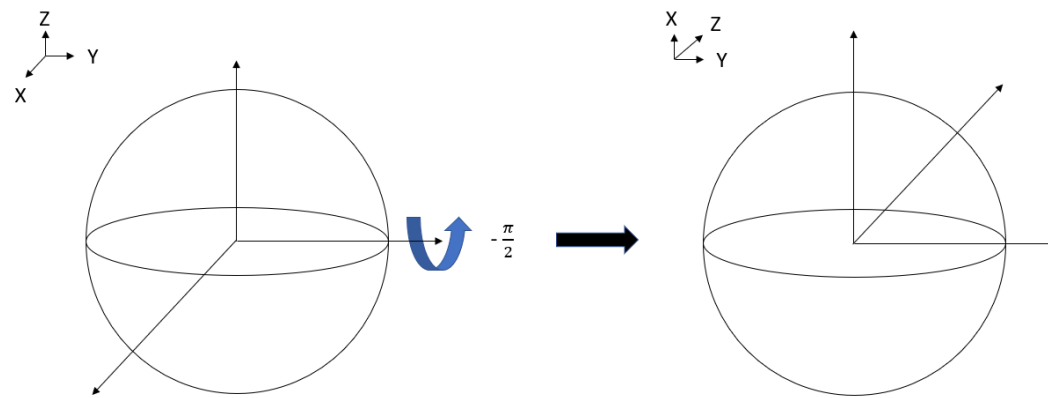
$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\langle \psi | X | \psi \rangle = \langle \psi' | Z | \psi' \rangle$$

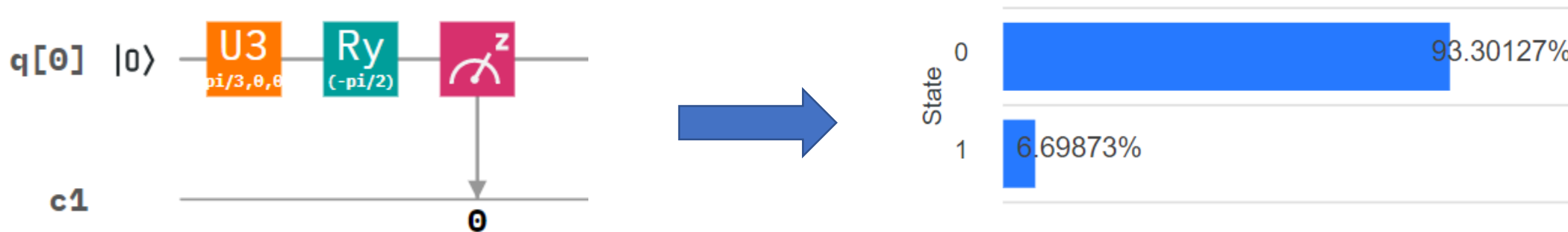
$$|\psi'\rangle = R_y(-\pi/2)|\psi\rangle$$

The rotation on the Bloch sphere:

$$R_y(-\pi/2) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$



Expectation Value of X



$$\langle \psi | X | \psi \rangle = 0.933... - 0.067... = 0.866...$$

Expectation Value of Y

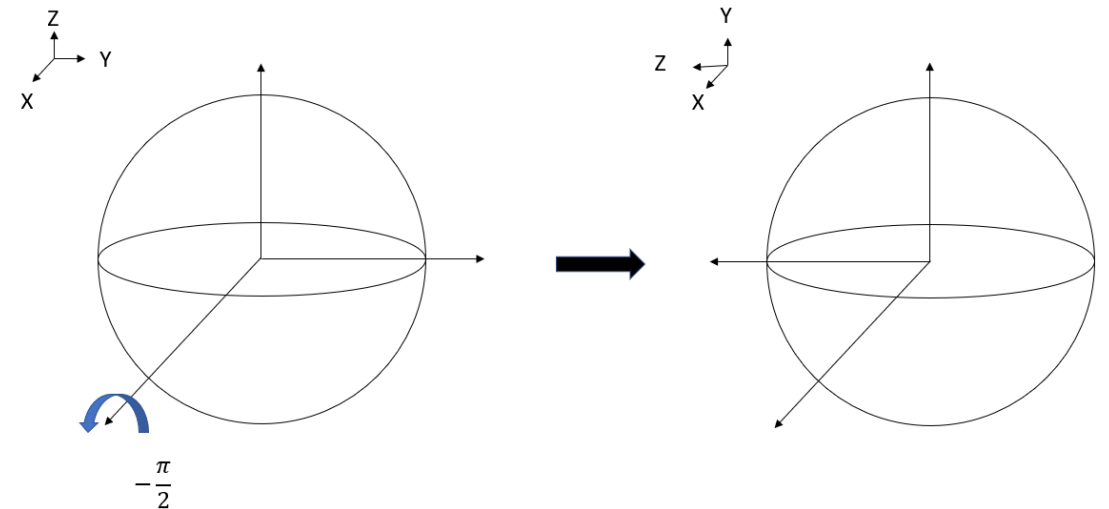
$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\langle \psi | \mathbf{Y} | \psi \rangle = \langle \psi' | \mathbf{Z} | \psi' \rangle$$

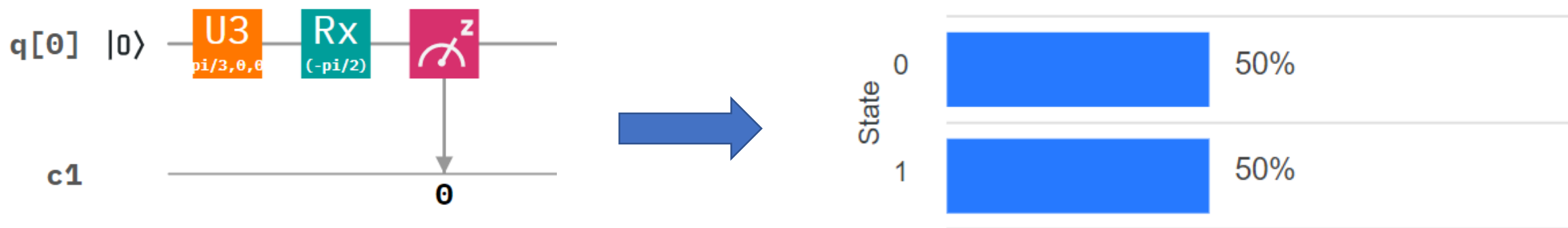
$$R_x(-\pi/2) | \psi \rangle = | \psi' \rangle$$

The rotation on the Bloch sphere:

$$R_x(-\pi/2) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

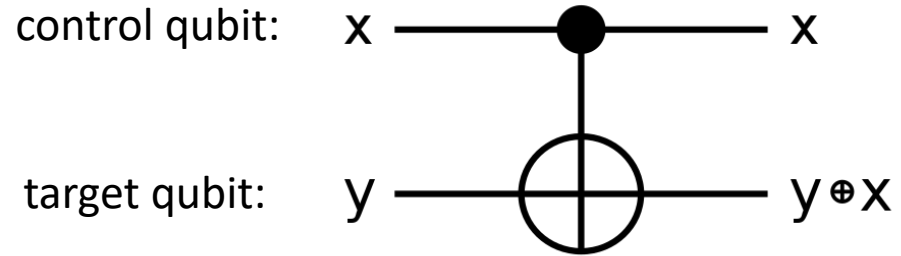


Expectation Value of X



$$\langle \psi | Y | \psi \rangle = 0.5 - 0.5 = 0$$

CNOT gate



This is a gate that can sum the information and store it in the target qubit.

input		output	
x	y	x	y+x
0⟩	0⟩	0⟩	0⟩
0⟩	1⟩	0⟩	1⟩
1⟩	0⟩	1⟩	1⟩
1⟩	1⟩	1⟩	0⟩

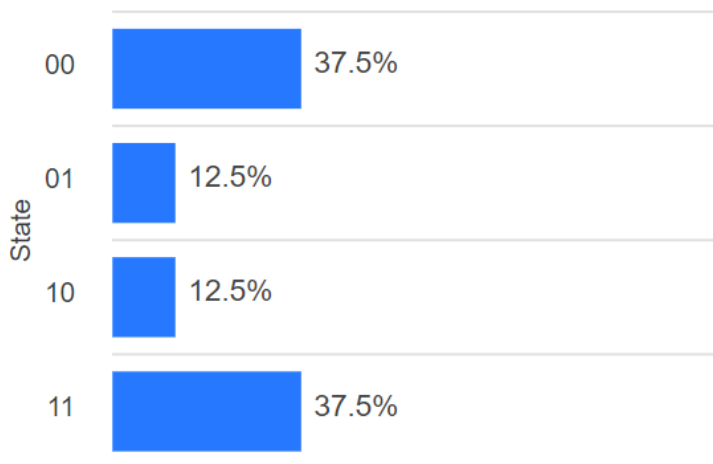
Scale UP

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

If the sum of the bit string is even(odd), the eigenvalue is 1(-1).

$$\langle\psi|ZZ|\psi\rangle = |a|^2 - |b|^2 - |c|^2 + |d|^2$$

Two qubit case



Five qubit case



N qubit case

efficient?



可回收 Recyclable



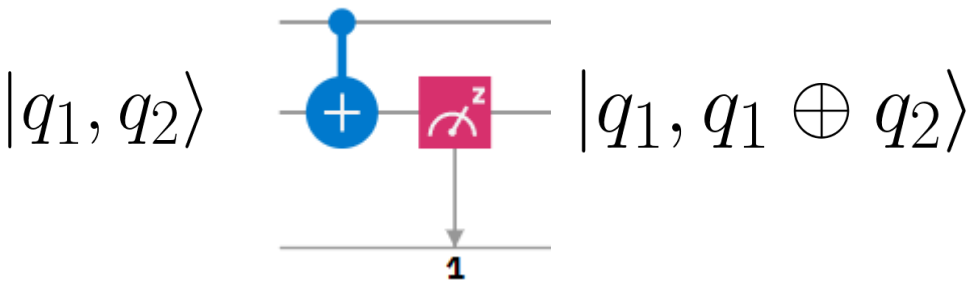
不可回收 Unrecyclable



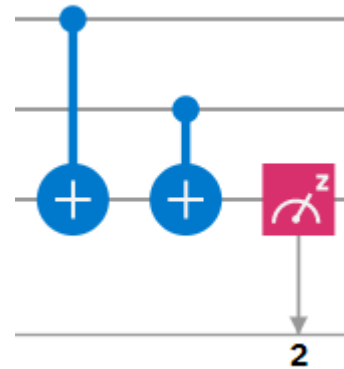
Scale UP!

We can classify the sum of the bit string even or odd via the CNOT gates.

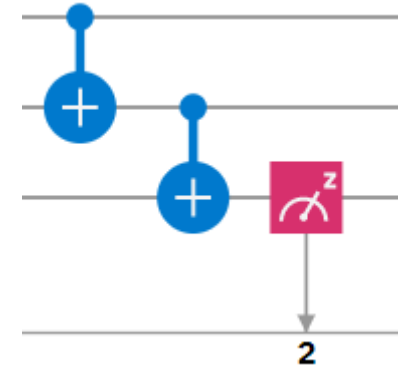
$$\langle \psi | ZZ | \psi \rangle$$



$$\langle \psi | ZZZ | \psi \rangle$$



or



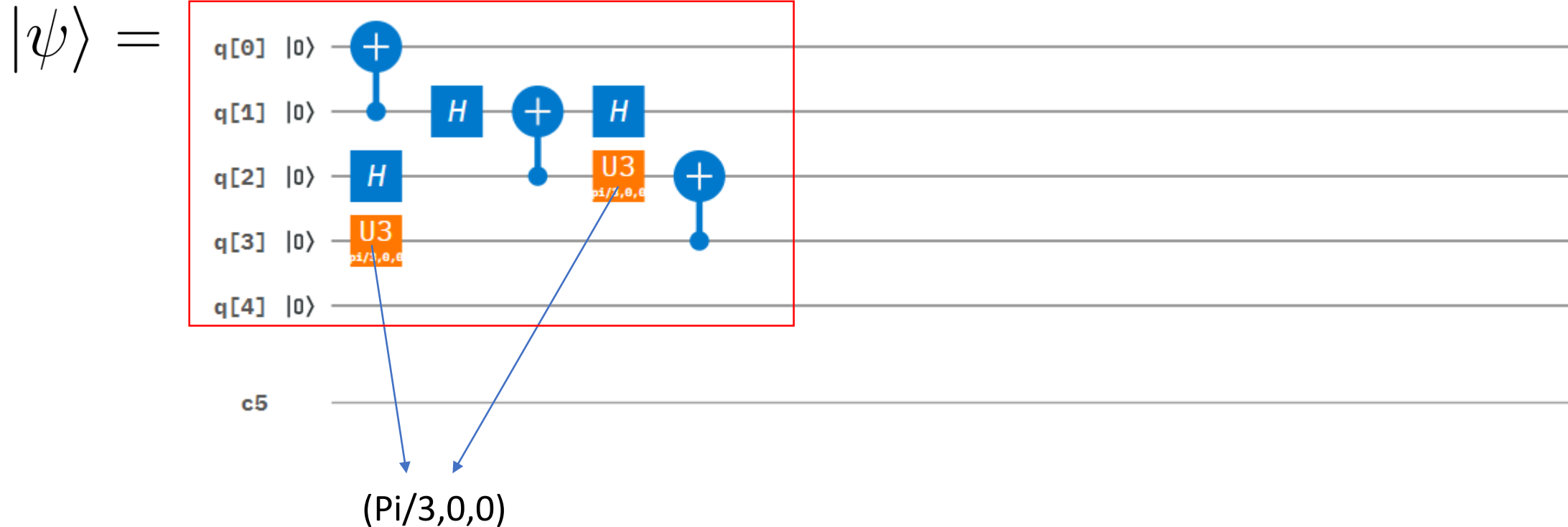
$$\langle \psi | ZZ \dots | \psi \rangle = \mathbb{P}_0 - \mathbb{P}_1$$

If you want to measure other Pauli matrices, operate the corresponding rotation gate before the CNOT gates.

Exercise

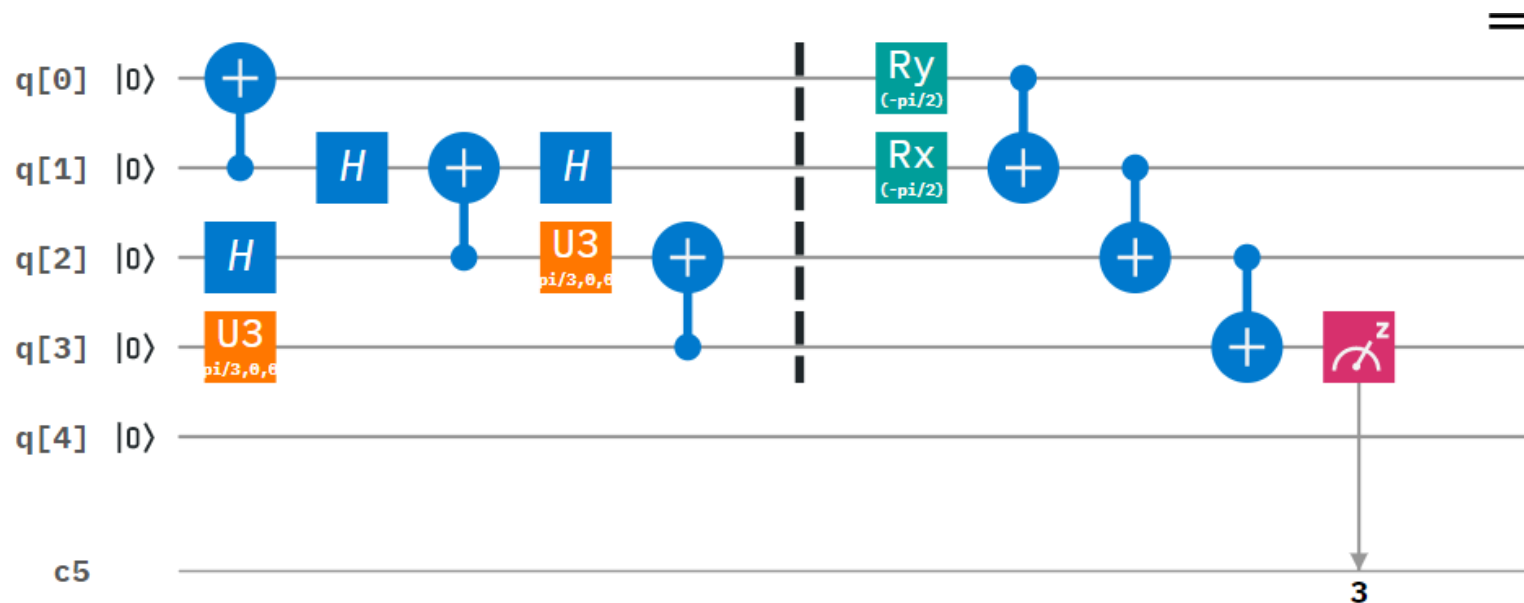
$$\langle \psi | X_0 Y_1 Z_2 Z_3 | \psi \rangle =$$

1. ibmq_qasm_simulator
2. ibmqx4
3. error mitigation

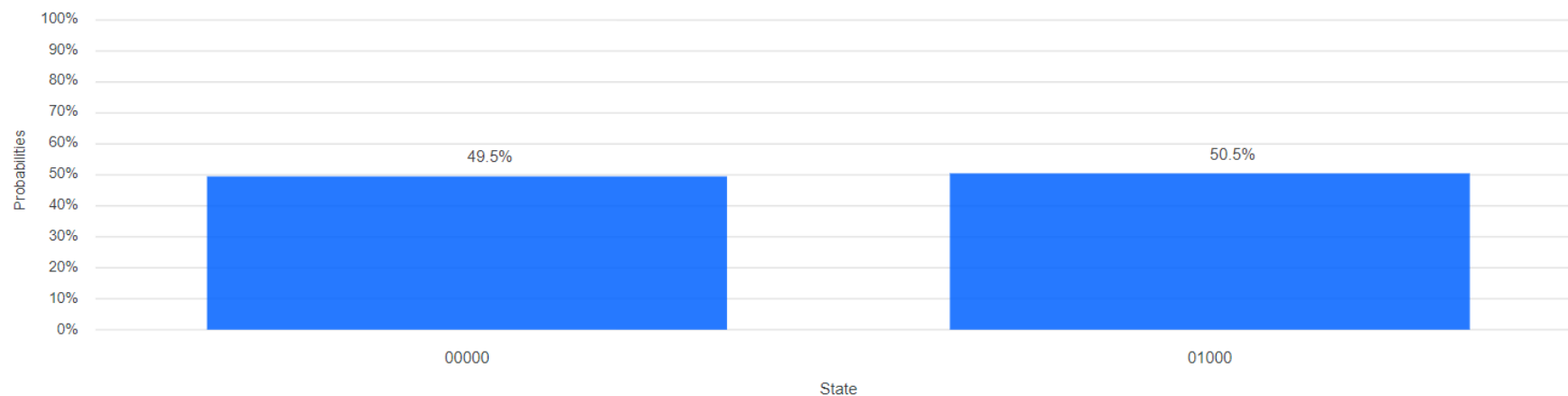


Result

Exact expectation value=0



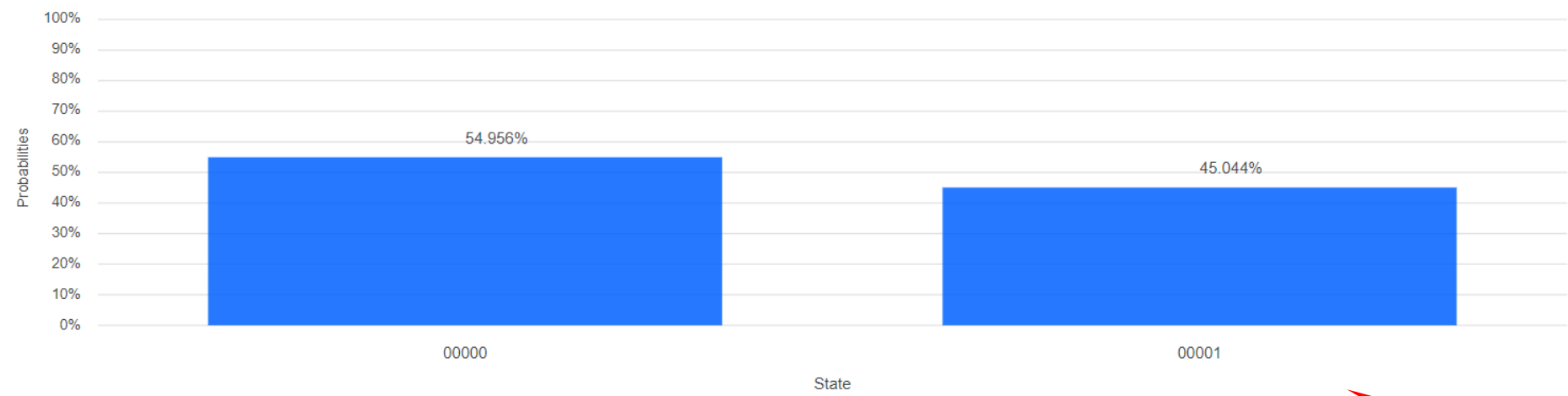
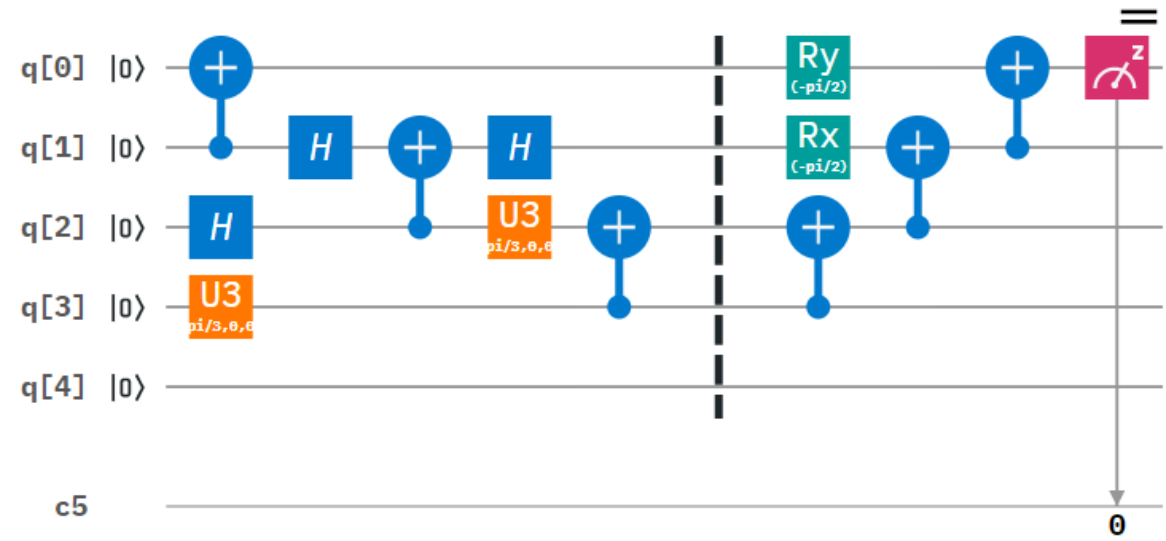
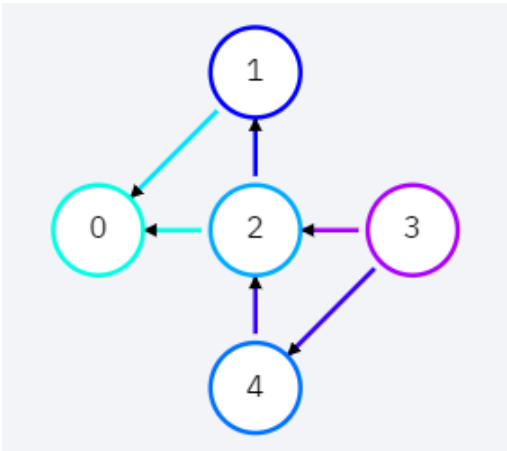
Histogram



ibmq_qasm_simulator:

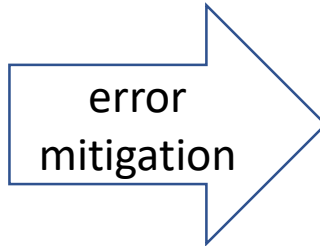
$$\langle \psi | X_0 Y_1 Z_2 Z_3 | \psi \rangle = 0.01$$

Real device



ibmqx4:

$$\langle \psi | X_0 Y_1 Z_2 Z_3 | \psi \rangle = 0.09912$$



0.030