Reconstructing Propositional Proofs in Type Theory

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Research

Goal

Formalization in type theory, classical propositional derivations generated by the `Metis` theorem prover.

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Formalization in type theory, classical propositional derivations generated by the `Metis` theorem prover.

Topics

- Automatic reasoning using automatic theorem provers (ATPs) (e. g., `Metis`, `EProver`)
- Interactive proving using proof-assistants (e. g., `Agda`, `Coq`)
- Proof-reconstruction for proofs generated by ATPs in proof-assistants

Research Outcomes

Academic result: paper (work in progress)
Software related results:

- `Athena`: a translator tool for `Metis` proofs to `Agda` in `Haskell`[1]
- `Agda` libraries:
  - `agda-metis`: `Metis` prover reasoning for propositional logic[2]
  - `agda-prop`: intuitionistic propositional logic + PEM[3]
- Bugs found in `Metis`: see Issues No. 2, No. 4, and commit `8a3f11e` in `Metis` official repository[4]

In parallel, we develop:

- `Online-ATPs`: a client for the `TPTP` world in `Haskell`[5]. This tool allowed us to use `Metis` without installing it
- `Prop-Pack`: compendium of `TPTP` problems in classical propositional logic used to test `Athena`[6]

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1 https://github.com/jonaprieto/athena.
2 https://github.com/jonaprieto/agda-metis.
3 https://github.com/jonaprieto/agda-prop.
4 https://github.com/gilith/metis.
5 https://github.com/jonaprieto/online-atps.
6 https://github.com/jonaprieto/prop-pack.

Bug in the Printing of the Proof

Fixed in `Metis` v2.3 (release 20161108)

$$\varphi := \neg p \land (\neg q \Leftrightarrow \neg r) \land (\neg p \Leftrightarrow (\neg q \Leftrightarrow \neg r))$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\varphi}\ \text{canonicalize}}{\neg p \Leftrightarrow (\neg q \Leftrightarrow \neg r)}\ \text{conjunct} \qquad \dfrac{\dfrac{\vdots}{\varphi}\ \text{canonicalize}}{\neg q \Leftrightarrow \neg r}\ \text{conjunct} \qquad \dfrac{\dfrac{\vdots}{\varphi}\ \text{canonicalize}}{\neg p}\ \text{conjunct}}{\bot}\ \text{simplify}$$

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The bug was caused by the conversion of `Xor` sets to `Iff` lists. After reporting this, `Metis` developer fixed the printing of `canonicalize` inference rule

$$\varphi := \neg p \wedge (\neg q \Leftrightarrow \neg r) \wedge (\neg p \Leftrightarrow (\neg q \Leftrightarrow \textcolor{red}{r}))$$

Soundness Bug in Splitting Goals

Fixed in `Metis` v2.3 (release 20170810)

Consider this `TPTP` problem

```
$ cat issue.tptp
fof(goal, conjecture, (~ (p <=> q)) <=> ((p => ~ q) & (q => ~p))).
```

`Metis` found a proof when other ATPs do not. Indeed, the problem is not a tautology.

```
$ metis issue.tptp
SZS status Theorem for issue.tptp
```

Testing with `EProver` with a client for `SystemOnTPTP` (`Online-ATPs`).

```
$ online-atps --atp=e issue.tptp
...
# No proof found!
# SZS status CounterSatisfiable
...
```

p	q	$\neg$	(	p	$\Leftrightarrow$	q	)	$\Leftrightarrow$	((	p	$\supset$	$\neg$	q	)	&	(	q	$\supset$	$\neg$	p	))
$\top$	$\top$	$\bot$		$\top$	$\top$	$\top$		$\top$		$\top$	$\bot$	$\bot$	$\top$		$\bot$		$\top$	$\bot$	$\bot$	$\top$	
$\top$	$\bot$	$\top$		$\top$	$\bot$	$\bot$		$\top$		$\top$	$\top$	$\top$	$\bot$		$\top$		$\bot$	$\top$	$\bot$	$\top$	
$\bot$	$\top$	$\top$		$\bot$	$\bot$	$\top$		$\top$		$\bot$	$\top$	$\bot$	$\top$		$\top$		$\top$	$\top$	$\top$	$\bot$	
$\bot$	$\bot$	$\bot$		$\bot$	$\top$	$\bot$		$\bot$		$\bot$	$\top$	$\top$	$\bot$		$\top$		$\bot$	$\top$	$\top$	$\bot$	

Soundness Bug in Splitting Goals

Fixed in `Metis` v2.3 (release 20170810)

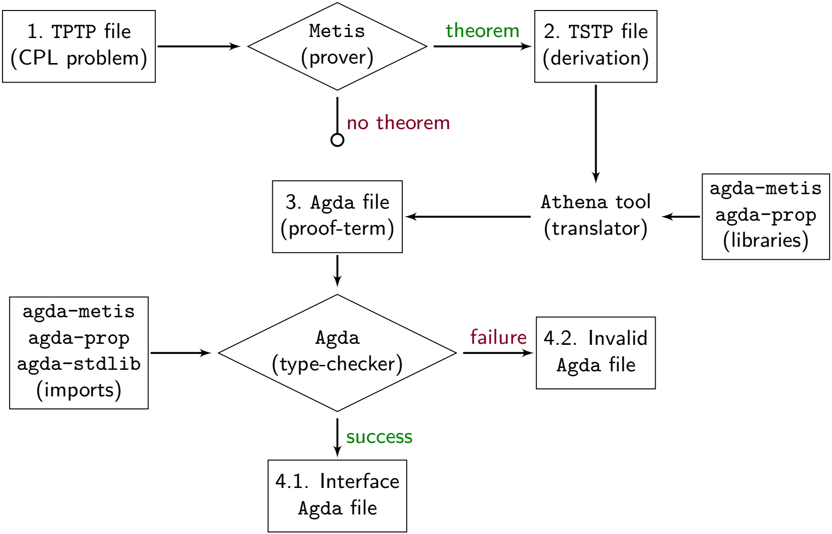
The bug was in the `strip` inference rule:

$$\neg (p \Leftrightarrow q) \Leftrightarrow ((p \Rightarrow \neg q) \land (q \Rightarrow \neg p))$$

Solved with:

$$\neg (p \Leftrightarrow q) \Leftrightarrow ((p \Rightarrow \neg q) \land (\neg q \Rightarrow p))$$

Proof Reconstruction: Overview



Inference Rules of Metis

TSTP derivations by Metis exhibit the following inferences:

Metis rule	Purpose
strip	Strip a goal into subgoals
conjunct	Takes a formula from a conjunction
resolve	A general form of the resolution theorem
canonicalize	Normalization of the formula
clausify	Performs clausification
simplify	Simplify definitions and theorems

Proposition Type in Agda

A data type for formulas

```
data PropFormula : Set where
  Var : Fin n → Prop
  ⊤   : Prop
  ⊥   : Prop
  _∧_ : (φ ψ : Prop) → Prop
  _∨_ : (φ ψ : Prop) → Prop
  _⇒_ : (φ ψ : Prop) → Prop
  _⇔_ : (φ ψ : Prop) → Prop
  ¬_  : (φ : Prop)   → Prop
```

Inference Rules For Propositional Logic I

Intuitionistic Propositional Logic + PEM ($\Gamma \vdash \varphi \vee \neg\varphi$)

$$\frac{}{\Gamma, \varphi \vdash \varphi}\ \textsf{assume} \qquad \frac{}{\Gamma \vdash \top}\ \top\textsf{-intro} \qquad \frac{}{\Gamma \vdash \varphi \vee \neg\, \varphi}\ \textsf{PEM}$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}\ \bot\textsf{-elim} \qquad \frac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg\, \varphi}\ \neg\,\textsf{-intro} \qquad \frac{\Gamma \vdash \neg\, \varphi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \bot}\ \neg\,\textsf{-elim}$$

$$\frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi}\ \wedge\textsf{-intro} \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi}\ \wedge\textsf{-proj}_1 \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}\ \wedge\textsf{-proj}_2$$

Inference Rules For Propositional Logic II

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi}\ \vee\text{-intro}_1 \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}\ \vee\text{-intro}_2$$

$$\frac{\Gamma, \varphi \vdash \gamma \qquad \Gamma, \psi \vdash \gamma}{\Gamma, \varphi \vee \psi \vdash \gamma}\ \vee\text{-elim}$$

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \Rightarrow \psi}\ \Rightarrow\text{-intro} \qquad \frac{\Gamma \vdash \varphi \Rightarrow \psi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \psi}\ \Rightarrow\text{-elim}$$

Other Rules

- Weakening: to extend the hypotheses with additional formulas

$$\frac{\Gamma \vdash \varphi}{\Gamma, \psi \vdash \varphi}\ \textsf{weaken}$$

- The RAA rule is the formulation of the principle of proof by contradiction:

$$\frac{\Gamma, \neg\, \varphi \vdash \bot}{\Gamma \vdash \varphi}\ \textsf{RAA}$$

Syntactical Consequence Relation in Agda

- Inductive family $\_ \vdash \_$ with two indexes: a set of propositions Γ (the premises) and a proposition φ (the conclusion)

Example

In [**AgdaProp**] we define $\_ \vdash \_$ as follows

```
data _⊢_ : (Γ : Ctxt)(φ : Prop) → Set
  ...
  ∧-intro
    : ∀ {Γ} {φ ψ}
    → Γ ⊢ φ → Γ ⊢ ψ
    → Γ ⊢ φ ∧ ψ

  ∧-proj₁
    : ∀ {Γ} {φ ψ}
    → Γ ⊢ φ ∧ ψ
    → Γ ⊢ φ

  ∧-proj₂
    : ∀ {Γ} {φ ψ}
    → Γ ⊢ φ ∧ ψ
    → Γ ⊢ ψ
  ...
```

Reconstructing Metis Rules in Type Theory

Let $\mathrm{metisRule}$ be a `Metis` inference rule. We define the function metisRule in type theory which has the following pattern[7]:

$$\text{metisRule} : \textsc{Premise} \to \textsc{Conclusion} \to \textsc{Prop}$$

$$\text{metisRule}\ \varphi\ \psi \quad = \begin{cases} \psi, & \text{if metisRule built } \psi \text{ by applying inference} \\ & \text{rules to } \varphi; \\ \varphi, & \text{otherwise;} \end{cases}$$

To justify all transformations done by the metisRule rule, we prove its soundness with a theorem like the following:

$$\text{If } \Gamma \vdash \varphi \text{ then } \Gamma \vdash \text{metisRule}\ \varphi\ \psi, \text{ where } \psi : \textsc{Conclusion}.$$

[7] Premise and Conclusion as synonyms of the Prop type to describe in the function types the role of the arguments

Reconstructing a Metis Inference Rule

The `clausify` rule transforms a formula into its clausal normal form.

Example

In the following TSTP derivation by Metis, we see how `clausify` transforms the $\texttt{norm}_0$ formula to get $\texttt{norm}_1$ formula.

$\texttt{fof(norm}_0\texttt{,}\ \neg\ \texttt{p} \vee \texttt{(q} \wedge \texttt{r)}\ \ldots$
$\texttt{fof(norm}_1\texttt{,}\ \texttt{(}\neg\ \texttt{p} \vee \texttt{q)} \wedge \texttt{(}\neg\ \texttt{p} \vee \texttt{r), inf(clausify, norm}_0\texttt{)).}$

Theorem

Let $\psi : \textsc{Conclusion}$. *If* $\Gamma \vdash \varphi$ *then* $\Gamma \vdash \text{clausify}\ \varphi\ \psi$, *where*

$$\text{clausify} : \textsc{Premise} \to \textsc{Conclusion} \to \textsc{Prop}$$

$$\text{clausify}\ \varphi\ \psi \quad = \begin{cases} \psi, & \textit{if } \varphi \equiv \psi; \\ \text{reorder}_{\wedge\vee}\ (\text{cnf}\ \varphi)\ \psi, & \textit{otherwise.} \end{cases}$$

Sketch of the `Metis` Algorithm

Algorithm 1 `Metis` refutation strategy

procedure METIS
input: the goal and a set of *premises* $a_1, \cdots, a_n$
output: maybe a derivation when $a_1, \cdots, a_n \vdash \mathrm{goal}$, otherwise nothing.

strip the goal into a list of *subgoals* s_i
for each subgoal s_i **do**
try to find by a refutation for $\neg s_i$:
apply clausification for the negated subgoal $\neg\, s_i$
if a premise a_j is relevant **then**
apply clausification to a_j
end if
application of `Metis` inference rules
if a contradiction can be derived from the assumptions **then**
keep the refutation and continue with the others subgoals
else
exit without a proof.
end if
end for
print the conjecture and the premises
print each refutation for each negated subgoal
end procedure

Some Challenges

- Formalization
  - Understanding the `Metis` reasoning without a proper documentation or description from the `Metis` author
  - Terminating of functions that reconstruct `Metis` inference rules
  - Intuitionistic logic implementation
- Software related
  - Parsing of `TSTP` derivations
  - Printing valid `Agda` files

Complete Example

The problem[8]:

$$(p \Rightarrow q) \wedge (q \Rightarrow p) \vdash (p \vee q) \Rightarrow (p \wedge q)$$

In TPTP syntax:

```
fof(a₁, axiom, (p ⇒ q) ∧ (q ⇒ p)).
fof(goal, conjecture, (p ∨ q) ⇒ (p ∧ q)).
```

Its TSTP solution using Metis:

```
fof(a₁, axiom, (p ⇒ q) ∧ (q ⇒ p)).
fof(goal, conjecture, (p ∨ q) ⇒ (p ∧ q))).
fof(s₁, (p ∨ q) ⇒ p, inf(strip, goal)).
fof(s₂, ((p ∨ q) ∧ p) ⇒ q, inf(strip, goal)).
...
```

[8] Problem No. 13 in Disjunction Section in [**Prieto-Cubides2017**]

fof(premise, axiom, (p q) ∧ (q p)).
fof(goal, conjecture, (p ∨ q) (p ∧ q)).
fof(s_0, (p ∨ q) p, inf(strip, goal)).
fof(s_1, ((p ∨ q) ∧ p) q, inf(strip, goal)).
fof(neg_0, ¬ ((p ∨ q) p), inf(negate, s_0)).
fof(n_{00}, (¬ p ∨ q) ∧ (¬ q ∨ p), inf(canonicalize, premise)).
fof(n_{01}, ¬ q ∨ p, inf(conjunct, n_{00})).
fof(n_{02}, ¬ p ∧ (p ∨ q), inf(canonicalize, neg_0)).
fof(n_{03}, p ∨ q, inf(conjunct, n_{02})).
fof(n_{04}, ¬ p, inf(conjunct, n_{02})).
fof(n_{05}, q, inf(simplify, [n_{03}, n_{04}])).
cnf(r_{00}, ¬ q ∨ p, inf(canonicalize, n_{01})).
cnf(r_{01}, q, inf(canonicalize, n_{05})).
cnf(r_{02}, p, inf(resolve, q, [r_{01}, r_{00}])).
cnf(r_{03}, ¬ p, inf(canonicalize, n_{04})).
cnf(r_{04}, ⊥, inf(resolve, p, [r_{02}, r_{03}])).
fof(neg_1, ¬ ((p ∨ q) ∧ p) q), inf(negate, s_1)).
fof(n_{10}, ¬ q ∧ p ∧ (p ∨ q), inf(canonicalize, neg_1)).
fof(n_{11}, (¬ p ∨ q) ∧ (¬ q ∨ p), inf(canonicalize, premise)).
fof(n_{12}, ¬ p ∨ q, inf(conjunct, n_{11})).
fof(n_{13}, ⊥, inf(simplify, [n_{10}, n_{12}])).
cnf(r_{10}, ⊥, inf(canonicalize, n_{13})).

TSTP Refutation of Subgoal No. 1

fof(premise, axiom, (p   q) $\land$ (q   p)).
fof(goal, conjecture, (p $\lor$ q)   (p $\land$ q)).
fof(s_0, (p $\lor$ q)   p, inf(strip, goal)).
...
fof(neg_0, $\neg$ ((p $\lor$ q)   p), inf(negate, s_0)).
fof(n_{00}, ($\neg$ p $\lor$ q) $\land$ ($\neg$ q $\lor$ p), inf(canonicalize, premise)).
fof(n_{01}, $\neg$ q $\lor$ p, inf(conjunct, n_{00})).
fof(n_{02}, $\neg$ p $\land$ (p $\lor$ q), inf(canonicalize, neg_0)).
fof(n_{03}, p $\lor$ q, inf(conjunct, n_{02})).
fof(n_{04}, $\neg$ p, inf(conjunct, n_{02})).
fof(n_{05}, q, inf(simplify, [n_{03}, n_{04}])).
cnf(r_{00}, $\neg$ q $\lor$ p, inf(canonicalize, n_{01})).
cnf(r_{01}, q, inf(canonicalize, n_{05})).
cnf(r_{02}, p, inf(resolve, q, [r_{01}, r_{00}])).
cnf(r_{03}, $\neg$ p, inf(canonicalize, n_{04})).
cnf(r_{04}, $\bot$, inf(resolve, p, [r_{02}, r_{03}])).

Refutation Tree for s_0

```
fof(premise, axiom, (p   q) ∧ (q   p)).
...
fof(n00, (¬ p ∨ q) ∧ (¬ q ∨ p), inf(canonicalize, premise)).
fof(n01, ¬ q ∨ p, inf(conjunct, n00)).
...
```

$$(\mathcal{D}_1) \qquad \cfrac{\cfrac{\cfrac{\cfrac{}{\Gamma \vdash (p \Rightarrow q) \wedge (q \Rightarrow p)}\ \text{axiom premise}}{\Gamma, \neg s_1 \vdash (p \Rightarrow q) \wedge (q \Rightarrow p)}\ \text{weaken}}{\Gamma, \neg s_1 \vdash (\neg p \vee q) \wedge (\neg q \vee p)}\ \text{canonicalize}}{\Gamma, \neg s_1 \vdash \neg q \vee p}\ \text{conjunct}$$

```
...
fof(s₁, (p ∨ q) ⇒ p, inf(strip, goal)).
fof(neg₁, ¬ ((p ∨ q) ⇒ p), inf(negate, s₁)).
...
fof(n02, ¬ p ∧ (p ∨ q), inf(canonicalize, neg₁)).
fof(n03, p ∨ q, inf(conjunct, n02)).
fof(n04, ¬ p, inf(conjunct, n02)).
...
```

$$(\mathcal{D}_2) \qquad \cfrac{\cfrac{\overline{\Gamma, \neg s_1 \vdash \neg s_1}\ \text{assume}}{\Gamma, \neg s_1 \vdash \neg p \wedge (p \vee q)}\ \text{canonicalize}}{\Gamma, \neg s_1 \vdash p \vee q}\ \text{conjunct}$$

$$(\mathcal{D}_3) \qquad \cfrac{\cfrac{\overline{\Gamma, \neg s_1 \vdash \neg s_1}\ \text{assume } \neg s_1}{\Gamma, \neg s_1 \vdash \neg p \wedge (p \vee q)}\ \text{canonicalize}}{\Gamma, \neg s_1 \vdash \neg p}\ \text{conjunct}$$

$$(\mathcal{D}_4) \qquad \dfrac{\dfrac{\mathcal{D}_2}{\Gamma, \neg s_1 \vdash p \vee q} \qquad \dfrac{\mathcal{D}_3}{\Gamma, \neg s_1 \vdash \neg p}}{\Gamma, \neg s_1 \vdash q}\ \textsf{simplify}$$

$$(\mathcal{R}_1) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma, \neg s_1 \vdash \neg q \vee p} \qquad \dfrac{\mathcal{D}_4}{\Gamma, \neg s_1 \vdash q}}{\Gamma, \neg s_1 \vdash p}\ \textsf{resolve } q \qquad \dfrac{\mathcal{D}_3}{\Gamma, \neg s_1 \vdash \neg p}}{\Gamma, \neg s_1 \vdash \bot}\ \textsf{resolve } p}{\Gamma \vdash s_1}\ \textsf{RAA}$$

Tree for the Subgoal No. 2: $((p \vee q) \wedge p) \Rightarrow q$

```
fof(s₂, ((p ∨ q) ∧ p) ⇒ q, inf(strip, goal)).
fof(neg₂, ¬ (((p ∨ q) ∧ p) ⇒ q), inf(negate, s2)).
fof(n10, ¬ q ∧ p ∧ (p ∨ q), inf(canonicalize, neg₂)).
fof(n11, (¬ p ∨ q) ∧ (¬ q ∨ p), inf(canonicalize, a₁)).
fof(n12, ¬ p ∨ q, inf(conjunct, n11)).
fof(n13, ⊥, inf(simplify,[n10, n12])).
cnf(r10, ⊥, inf(canonicalize, n13)).
```

$$(\mathcal{R}_2) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \neg s_2 \vdash \neg s_2}\ \text{assume } (\neg s_2)}{\Gamma, \neg s_2 \vdash \neg q \wedge p \wedge (p \vee q)}\ \text{canonicalize} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash (p \Rightarrow q) \wedge (q \Rightarrow p)}\ \text{axiom } a_1}{\Gamma, \neg s_2 \vdash (p \Rightarrow q) \wedge (q \Rightarrow p)}\ \text{weaken}}{\Gamma, \neg s_2 \vdash (\neg p \vee q) \wedge (\neg q \vee p)}\ \text{canonicalize}}{\Gamma, \neg s_2 \vdash \neg p \vee q}\ \text{conjunct}}{\dfrac{\Gamma, \neg s_2 \vdash \bot}{\Gamma \vdash s_2}\ \text{RAA}}\ \text{simplify}$$

Summarizing the Example

The problem was:

$$(p \Rightarrow q) \wedge (q \Rightarrow p) \vdash (p \vee q) \Rightarrow (p \wedge q)$$

Its TSTP solution using Metis was:

```
fof(a_1, axiom, (p ⇒ q) ∧ (q ⇒ p)).
fof(goal, conjecture, (p ∨ q) ⇒ (p ∧ q))).
fof(s_1, (p ∨ q) ⇒ p, inf(strip, goal)).
fof(s_2, ((p ∨ q) ∧ p) ⇒ q, inf(strip, goal)).
...
```

The proof is:

$$\dfrac{\dfrac{}{\Gamma \vdash (s_1 \wedge s_2) \Rightarrow \text{goal}}\,\text{strip} \qquad \dfrac{\dfrac{\mathcal{R}_1}{\Gamma \vdash s_1} \qquad \dfrac{\mathcal{R}_2}{\Gamma \vdash s_2}}{\Gamma \vdash s_1 \wedge s_2}\,\wedge\text{-intro}}{\Gamma \vdash \text{goal}}\,\Rightarrow\text{-elim}$$

(Live example using Agda and Athena)

Future Work

Further research directions include, but are not limited to:

- improve the performance of the `canonicalize` rule
- extend the proof-reconstruction presented in this paper to
  - support the identity theory
  - support other ATPs for propositional logic like `EProver` or `Z3`. See Kanso's Ph.D. thesis [**Kanso2012**]
  - support `Metis` first-order proofs

Related Work

In type theory:

- **Kanso2012** in [**Kanso2012**] reconstructs in `Agda` propositional proofs generated by `EProver` and `Z3`
- **foster2011integrating** in [**foster2011integrating**] describe proof-reconstruction in `Agda` for equational logic of `Waldmeister` prover
- **Bezem2002** in [**Bezem2002**] transform a proof produced by the first-order prover `Bliksem` in a `Coq` proof-term

In classical logic:

- **paulson2007source** in [**paulson2007source**] introduce `SledgeHammer`, a tool ables to reconstructs proofs of well-known ATPs: `EProver`, `Vampire`, among others using `SystemOnTPTP` server
- **Hurd1999** in [**Hurd1999**] integrates the first-order resolution prover `Gandalf` prover for `HOL` proof-assistant
- **kaliszyk2013** in [**kaliszyk2013**] reconstruct proofs of different ATPs for `HOL Light`

References I

BONUS SLIDES

TPTP Syntax

Thousands of Problems for Theorem Provers

- Is a language[9] to encode problems
- Is the input of the ATPs
- Annotated formulas with the form

  `language(name, role, formula).`

`language` FOF or CNF
`name` to identify the formula within the problem
`role` axiom, definition, hypothesis, conjecture
`formula` formula in TPTP format

[9] http://www.cs.miami.edu/~tptp/TPTP/SyntaxBNF.html.

Metis Theorem Prover

Metis is an automatic theorem prover for first-order logic with equality.

- Open source implemented
- Reads problems in TPTP format
- Outputs *detailed* proofs in TSTP format
- For the propositional logic, Metis has only three inference rules:

$$\frac{}{\Gamma \vdash \varphi_1 \vee \cdots \vee \varphi_n} \text{ axiom } \varphi_1, \cdots, \varphi_n$$

$$\frac{}{\Gamma \vdash \varphi \vee \neg\varphi} \text{ assume } \varphi$$

$$\frac{\Gamma \vdash \varphi_1 \vee \cdots \vee \boldsymbol{l} \vee \cdots \vee \varphi_n \qquad \Gamma \vdash \psi_1 \vee \cdots \vee \neg\, \boldsymbol{l} \vee \cdots \vee \psi_m}{\Gamma \vdash \varphi_1 \vee \cdots \vee \varphi_n \vee \psi_1 \vee \cdots \vee \psi_m} \text{ resolve } l$$

TSTP Syntax

A TSTP derivation[10]

- Is a **D**irected **A**cyclic **G**raph where

  `leaf` is a formula from the TPTP input
  `node` is a formula inferred from parent formula
  `root` the final derived formula

- Is a list of annotated formulas with the form

```
language(name, role, formula, source [,useful info]).
```

where `source` typically is an inference record

```
inference(rule, useful info, parents).
```

[10] http://www.cs.miami.edu/~tptp/TPTP/QuickGuide/Derivations.html.

Another TSTP Example

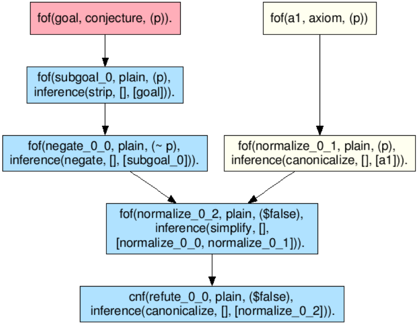
- Proof found by Metis for the problem $p \vdash p$

```
$ metis --show proof problem.tptp
fof(a, axiom, p).
fof(goal, conjecture, p).
fof(subgoal_0, plain, p),
  inference(strip, [], [goal])).
fof(negate_0_0, plain, ~ p,
  inference(negate, [], [subgoal_0])).
fof(normalize_0_0, plain, ~ p,
  inference(canonicalize, [], [negate_0_0])).
fof(normalize_0_1, plain, p,
  inference(canonicalize, [], [a])).
fof(normalize_0_2, plain, $false,
  inference(simplify, [],
    [normalize_0_0, normalize_0_1])).
cnf(refute_0_0, plain, $false,
    inference(canonicalize, [], [normalize_0_2])).
```

DAG Example

By refutation, we proved $p \vdash p$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg p}\,\text{assume}}{\neg p}\,\text{strip} \qquad \dfrac{\dfrac{}{p}\,\text{axiom}}{p}\,\text{canonicalize}}{\bot}\,\text{simplify}}{\bot}\,\text{canonicalize}$$



Athena tool

Is an Haskell program that translates proofs given by Metis in TSTP format to Agda code

- Parsing of TSTP language
- Creation and analysis of **DAG** derivations
- Analysis of inference rules used in the TSTP derivation
- Agda code generation

Library	**Purpose**
agda-prop	axioms and theorems of classical propositional logic
agda-metis	versions of the inference rules used by Metis

agda-metis: Conjunct Inference [11]

- Definition

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- Inference rules involved

$$\frac{\varphi_1 \land \varphi_2}{\varphi_1}\ \land\text{-}\mathtt{proj}_1 \qquad \frac{\varphi_1 \land \varphi_2}{\varphi_2}\ \land\text{-}\mathtt{proj}_2$$

- Example

$$\varphi := \varphi_1 \land \overbrace{(\varphi_3 \land \varphi_4)}^{\varphi_2}$$

  - conjunct $(\varphi, \varphi_3 \land \varphi_1) \equiv \varphi$
  - conjunct $(\varphi, \varphi_3) \equiv \varphi_3$
  - conjunct $(\varphi, \varphi_2) \equiv \varphi_2$

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