CHAPTER 4

Probability distributions of Random variables

RANDOM VARIABLES

Random variable(RV): real valued function that assigns a numerical value to each possible outcome in a sample space.

i. Discrete RVs : can take countable distinct values.

<u>Examples of discrete RVs</u>

- X= **number of heads** in ***n*** Tosses of a coin: X=0,1,……n
- Y=**number of hits on a website per hour**: Y= 0,1,2…..

ii. Continuous RVs: can take any values in a given interval.

- Let X be the waiting time of a person for a bus which arrives every five minutes.
- X is a continuous rv, since it takes any value b/n 0 and 5 i.e $\{X: 0<x<5\}$

Probability Distributions of one Dimensional RVs

Probability distribution of discrete and continuous RVs.

Discrete RVs

- The **Probability mass function (pmf)** of a discrete Rv **X** is the function P(x) which satisfies the following two conditions:

  i) $p(x) \geq 0$ ii) $\sum_x P(x) = 1$

- The **expected value** and **variance** of a discrete RV

$$E(X) = \mu_X = \sum_x xp(x)$$

$$Var(X) = \sigma_X^2 = E(X^2) - [E(X)]^2$$

$$where;\ E(X^2) = \sum_x x^2 p(x)$$

Continuous RVs

- **Probability density function(pdf)** of a continuous r.v X, is a real valued function, *f* (*x*) which satisfies the following conditions:

  i) $f(x) \geq 0$ ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

- The **expected value** and **variance** of a continuous RV

$$E(X) = \mu_X = \int_{-\infty}^{\infty} xf(x)\,dx$$

$$Var(X) = \sigma_X^2 = E(X^2) - [E(X)]^2$$

$$where;\ E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

EXAMPLE ON PROBABILITY DISTRIBUTION OF DISCRETE RV

Example 1 : Let X be the number of heads in three tosses of a fair coin.

a)Construct the *Probability mass function (pmf) of X*.

b)Find the mean number of heads and the variance

c)Find $P(1<X \leq 3)$

DISCRETE PROBABILITY DISTRIBUTION *CONT…..*

Example 1: Solution

a) S={TTT, HTT, THT, TTH, HHT, HTH, THH, HHH}

Number of heads: X={0,1,2,3}

The pmf of X:

x	0	1	2	3
P(X=x)	1/8	3/8	3/8	1/8

b) The mean number of heads:

$$\boldsymbol{E(X) = \mu_X = \sum_x xp(X = x)}$$

$$= 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} = 1.5$$

DISCRETE PROBABILITY DISTRIBUTION *CONT…..*

Example 1: Solution cont'd..

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \sum_x x^2 P(X = x) - [E(X)]^2$$

=3-$(1.5)^2$ =0.75

c) $P(1 < X \leq 3) = P(X = 2) + P(X = 3)$

$$= \frac{3}{8} + \frac{1}{8} = 0.5$$

Exercise: In previous example, find *P(X≤3)* , *P(X<3)* , *P(1≤X≤3)*

EXAMPLE ON PROBABILITY DISTRIBUTION OF CONTINUOUS RV

Example 2: Let X be a continuous R.V with *Probability density function(pdf)*,

$$f(x) = \begin{cases} \frac{1}{2}x, & 0 \le x \le 2 \\ 0, & otherwise \end{cases}$$

Then find

a) Verify if f(x) is a pdf.
b) $P\ (1<x<1.5)$
c) E(x)
d) Var (x)

Example 2: Solution

a) Since, i) f(x) ≥0 and

ii) $\int_0^2 f(x)\,dx=1$ i.e $\int_0^2 \frac{1}{2}x\,dx = \left.\frac{x^2}{4}\right|_0^2 = 1$, then f(x) is valid pdf.

b) $P(1 < x < 1.5) = \int_1^{1.5} \frac{1}{2}x\,dx = \left.\frac{x^2}{4}\right|_1^{1.5}$ =0.3125

c) $E(X) = \int_0^2 xf(x)\,dx = \int_0^2 \frac{x^2}{2}\,dx = \left.\frac{x^3}{6}\right|_0^2$ =4/3

Example 2: Solution cont.

d) $$Var(X) = E(X^2) - [E(X)]^2$$

$$= \int_0^2 x^2 \frac{1}{2} x \, dx - \left[\frac{4}{3}\right]^2$$

$$= \left.\frac{x^4}{8}\right|_0^2 - \frac{16}{9}$$

$$=2 - \frac{16}{9} = 2/9$$

Some Properties of Expectation and Variance of RVs

Let X be a r.v. Let a and b be constants. Then

- E(a)=a
- Var(a)=0
- E(aX +b)=aE(x) + b
- Var(aX + b)=$a^2 Var(x)$

Exercise: Refer to example 2 and find

E(3x^2-2)

Conditional probability of Continuous RV

Example 3:

Example: The diameter of an electric cable is assumed to be a continuous random variable, say X, with P.d.f $f(x)=\begin{cases}6x(1-x) & \text{if } 0\le x\le 1\\ 0 & \text{otherwise}\end{cases}$

Compute $P\left(x\le\frac{1}{2}\middle|\frac{1}{3}<x<\frac{2}{3}\right)$

Solution: Recall $\mathrm{P}(A|B)=\dfrac{P(A\cap B)}{P(B)}$

$$P\left(x\le\frac{1}{2}\middle|\frac{1}{3}<x<\frac{2}{3}\right)=\frac{P\left((x\le\frac{1}{2})\cap(\frac{1}{3}<x<\frac{2}{3})\right)}{P\left(\frac{1}{3}<x<\frac{2}{3}\right)}=\frac{P\left(\frac{1}{3}<x<\frac{1}{2}\right)}{P\left(\frac{1}{3}<x<\frac{2}{3}\right)}=\frac{\int_{1/3}^{1/2}6x(1-x)dx}{\int_{1/3}^{2/3}6x(1-x)dx}=1/2$$

Probability Distributions of Two Dimensional RVs

Joint and marginal Probability distributions of RVs

Discrete RVs

- Joint **pmf** of (X,Y): is the function, $\boldsymbol{p(x,y)}$ which satisfies the following two conditions:

  $\textbf{i)}\, \boldsymbol{p(x,y) \geq 0}$ $\qquad$ $\textbf{ii)} \displaystyle\sum_{x}\sum_{y} \boldsymbol{p(x,y)} = \mathbf{1}$

- Marginal **pmf** of (X,Y)

$$p_X(x) = \sum_{y} p(x,y)$$

$$p_Y(y) = \sum_{x} p(x,y)$$

Continuous RVs

- Joint **pdf** of (X,Y): is the function, $f(x,y)$ which satisfies the following two conditions:

  $\text{i)}\ f(x,y) \geq 0$ $\qquad$ $\textbf{ii)} \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$

- Marginal **pdf** of (X,Y)

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Joint and marginal probability distributions (discrete r.v)

Example 4 : A fair coin is tossed three times. Let X denotes 1 or 0 according as "heads" or "tails" occurs on the **first toss**, respectively and let Y denotes the **number of heads**. Find

The joint distribution of X and Y? The marginal distribution of X and Y?

Example 4: Solution

S	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
X	1	1	1	1	0	0	0	0
Y	3	2	2	1	2	1	1	0

The **joint** and **marginal** distribution of X and Y

X \ Y	0	1	2	3	$p_X(x)$
0	1/8	1/4	1/8	0	0.5
1	0	1/8	1/4	1/8	0.5
$p_Y(y)$	1/8	3/8	3/8	1/8	1

Joint and marginal probability distributions (continuous RVs) cont.

Example 5: Two characteristics of a rocket engine' s performance are thrust X and mixture ratio Y. Suppose that (X, Y) is a two-dimensional continuous random variable with joint P.d.f given by

$$f_{XY}^{(x,y)} = \begin{cases} 2(x+y-2xy) & if\ 0<x<1, 0<y<1 \\ 0 & otherwise \end{cases}$$

Find the marginal P.d.f of X and Y ?

Joint and marginal probability distributions (continuous RVs) cont.

Example 5: Solution

- **Marginal P.d.f of X**

$$f_X^{(x)} = \int_{-\infty}^{\infty} f_{XY}^{(x,y)}\, dy = \int_0^1 2(x + y - 2xy)dy = 2[xy + \frac{y^2}{2} - xy^2]\Big|_0^1 = 1$$

Thus the marginal P.d.f of X is given as follows

$$f_X^{(x)} = \begin{cases} 1, & 0 < x < 1 \\ 0, & otherwise \end{cases}$$

- **Marginal P.d.f of Y**

$$f_Y^{(Y)} = \int_{-\infty}^{\infty} f_{XY}^{(x,y)}\, dx = \int_0^1 2(x + y - 2xy)dx = 2[\frac{x^2}{2} + xy - yx^2]\Big|_0^1 = 1$$

Thus the marginal P.d.f of Y is given as follows

$$f_Y^{(y)} = \begin{cases} 1, & 0 < y < 1 \\ 0, & otherwise \end{cases}$$

Joint and marginal probability distributions (continuous RVs) cont.

Example 6: Let X and Y be continuous random variables with joint PDF given below:

$$f_{XY}^{(x,y)} = \begin{cases} cx^2y, & 0 \leq y \leq x \leq 1 \\ 0, & otherwise \end{cases}$$

a) Find the constant c.

b) Find marginal PDFs, $f_X(x)$and $f_Y(y)$?

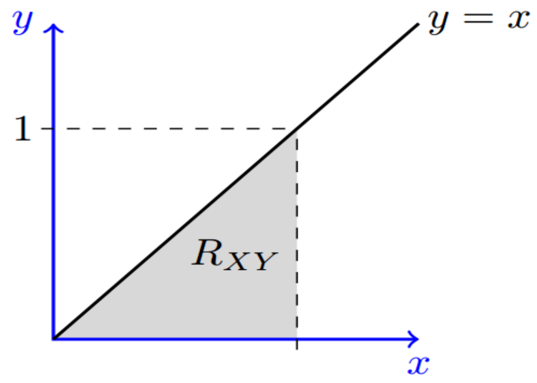
c) Find $P\left(Y < \frac{X}{2}\right)$

Joint and marginal probability distributions (continuous RVs) cont.

Example 6 : Solution

a) To find the constant c, we'll use the property of joint pdf i.e

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$$



$$\textbf{but} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\,dx = \int_{0}^{1}\int_{0}^{x} cx^2y\,dy\,dx = \int_{0}^{1} \frac{c}{2}x^4\,dx = \frac{c}{10}$$

$$\Longrightarrow c = 10$$

Joint and marginal probability distributions (continuous RVs) cont.

Example 6 : Solution cont.

b) **Marginal P.d.f of X and Y**

Marginal pdf of X: $f_X^{(x)} = \int_{-\infty}^{\infty} f_{XY}^{(x,y)}\, dy = \int_{0}^{x} 10x^2 y\, dy = 5x^4$

$$\text{Thus, } f_X^{(x)} = \begin{cases} 5x^4, & 0 < x < 1 \\ 0, & otherwise \end{cases}$$

Marginal pdf of Y: $f_Y^{(Y)} = \int_{-\infty}^{\infty} f_{XY}^{(x,y)}\, dx = \int_{y}^{1} 10x^2 y\, dx = \frac{10}{3} y(1 - y^3)$

$$\text{Thus, } f_Y^{(y)} = \begin{cases} \frac{10}{3} y(1 - y^3), & 0 < y < 1 \\ 0, & otherwise \end{cases}$$

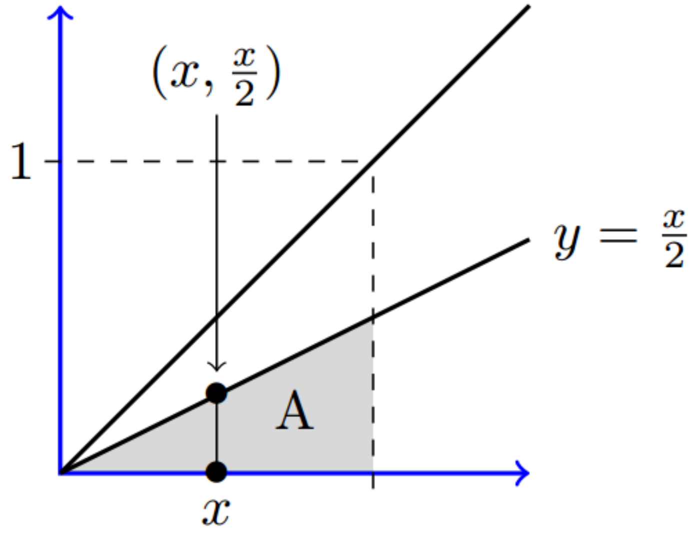
Example 6 : Solution cont

c) To find $P\left(Y < \frac{X}{2}\right)$, we need to integrate $f(x,y)$ over region A as shown below.

$$P\left(Y < \frac{X}{2}\right) = \int_0^1 \int_0^{\frac{x}{2}} f(x,y)\,dy\,dx$$

$$= \int_0^1 \int_0^{\frac{x}{2}} 10x^2 y\,dy\,dx$$

$$= \int_0^1 \frac{5}{4} x^4\,dx = \frac{1}{4}$$



Conditional Probability distributions

i. Discrete RVs X and Y

- Conditional pmf of X given Y=y : $p_{X|Y}(x|y) = \frac{p(x,y)}{p_Y(y)}$
- Conditional pmf of Y given X=x :$p_{Y|X}(y|x) = \frac{p(x,y)}{p_X(x)}$
- Two discrete r.vs X and Y are independent if $p(x,y) = p_X(x) \times p_Y(y)$

ii. Continuous RVs X and Y

- Conditional pdf of X given Y=y : $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$
- Conditional pdf of Y given X=x : $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$
- Two continuous r.vs X and Y are independent if $f(x,y) = f_X(x) \times f_Y(y)$

Covariance and Correlation

- The **Covariance** of X and Y is given by:

$$\mathrm{Cov}(X,Y) = \mu_{XY} - \mu_X \mu_Y$$

Where;

-For discrete RVs:

$$\mu_{XY} = \sum_x \sum_y xyp(x,y) \qquad \mu_X = \sum_x xp_X(x) \qquad \mu_Y = \sum_y yp_Y(y)$$

- For continuous RVs:

$$\mu_{XY} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf(x,y)\, dy\, dx \qquad \mu_X = \int_{-\infty}^{\infty} xf_X(x)\, dx \qquad \mu_Y = \int_{-\infty}^{\infty} yf_Y(y)\, dy$$

- The **correlation** of Rvs X and Y is given by:

$$\rho_{XY} = \frac{COV(X,Y)}{\sigma_X \sigma_Y} \qquad where, \ -1 \le \rho_{XY} \le 1$$

Note that $\sigma_X = \sqrt{\sigma_X^2}$ and $\sigma_Y = \sqrt{\sigma_Y^2}$

Example on Covariance and conditional pmf of discrete R.Vs

Example 7: In a certain community, levels of air pollution may exceed federal standards for ozone or for particulate matter on some days. For a particular summer season, let X be the number of days on which the ozone standard is exceeded and let Y be the number of days on which the particulate matter standard is exceeded. The joint probability mass function p(x,y) of X and Y is presented in the following table. The marginal probability mass functions are presented as well, in the margins of the table.

X \ Y	0	1	2	$p_X(x)$
0	0.10	0.11	0.05	0.26
1	0.17	0.23	0.08	0.48
2	0.06	0.14	0.06	0.26
$p_Y(y)$	0.33	0.48	0.19	1

a) Find the covariance of X and Y , where $E(X) = \mu_X$= 1 and $E(Y) = \mu_Y$=0.86.

b) Compute the conditional pmf $p_{X|Y}(x|1)$

Example on Covariance and conditional pmf of discrete cont.

Example 7: Solution

a) Covariance of X and Y is given by: $\mathrm{Cov}(X, Y) = \mu_{XY} - \mu_X \mu_Y$. But,

$$E(XY) = \mu_{XY} = \sum_{x=0}^{2} \sum_{y=0}^{2} xyp(x, y)$$

$$= (1)(1)(0.23) + (1)(2)(0.08) + (2)(1)(0.14) + (2)(2)(0.06)$$

$$= 0.91 \ (omitting\ terms\ equal\ to\ 0)$$

Thus, $\mathrm{Cov}(X, Y) = 0.91 - (1)(0.86) = 0.05$

Example on Covariance and Conditional pmf of discrete cont.

Example 7: Solution cont.

b) The conditional pmf of X given Y=1 is $p_{X|Y}(x|1) = \frac{p(x,1)}{p_Y(1)}$, and

possible values of X are 0,1 and 2. Thus , $p_{X|Y}(x|1)$ can be obtained as:

$$p_{X|Y}(0|1) = \frac{p(0,1)}{p_Y(1)} = \frac{0.11}{0.48} = 0.229$$

$$p_{X|Y}(1|1) = \frac{p(1,1)}{p_Y(1)} = \frac{0.23}{0.48} = 0.479$$

$$p_{X|Y}(2|1) = \frac{p(2,1)}{p_Y(1)} = \frac{0.14}{0.48} = 0.292$$

Example on Covariance and conditional pdf of continuous RVs

Example 8: The joint probability density function for (X , Y) and marginal PDFs, $f_X(x)$and $f_Y(y)$ are given below.

Joint pdf of X and Y: $$f(x,y) = \begin{cases} \frac{x+y}{3} & 0 < x < 2,\ 0 < y < 1 \\ 0, & otherwise \end{cases}$$

Marginal PDF of X and Y:

$$f_X^{(x)} = \begin{cases} \frac{2x+1}{6}, & 0 < x < 2 \\ 0, & otherwise \end{cases} \quad \text{and} \quad f_Y^{(y)} = \begin{cases} \frac{2+2y}{3}, & 0 < y < 1 \\ 0, & otherwise \end{cases}$$

$E(X) = \mu_X$=11/9 , $E(Y) = \mu_Y$ =5/9 and E(XY)= μ_{XY}=2/3

a) Are X and Y independent? If not, find Cov (X, Y).

a) Find the conditional pdf of X given Y , $f_{X|Y}(x|y)$

Example on Covariance and conditional pdf of continuous RVs cont…

Example 8: Solution

a) Since : $\boldsymbol{f(x,y)} \neq f_X(x) \times f_Y(y)$, X and y are not independent.

and $\boldsymbol{Cov(X,Y)} = \boldsymbol{\mu_{XY}} - \boldsymbol{\mu_X \mu_Y} = \frac{2}{3} - \frac{11}{9} \times \frac{5}{9} = \frac{1}{81} \approx 0.0123$

b) We know that $\boldsymbol{f(x,y) = \frac{x+y}{3}}$ **and** $\boldsymbol{f_Y(y) = \frac{2+2y}{3}}$

$$\textbf{Thus, } \boldsymbol{f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{x+y}{2+2y}}$$

Hence, for $\boldsymbol{0 < y < 1}$, we have

$$\boldsymbol{f_{X|Y}(x|y) = \begin{cases} \frac{x+y}{2+2y} & , 0 < x < 2 \\ 0, & otherwise \end{cases}}$$

Some Properties of Expectation and Variance of RVs

Let X and Y be RVs. Then

- $E(X \pm Y) = E(X) \pm E(Y)$
- $Var(X+Y) = Var(X-Y) = Var(X) + Var(Y)$; **if X & Y are independent:**
- $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$; **if X and Y are dependent**
- $Var(X-Y) = Var(X) + Var(Y) - 2Cov(X,Y)$; **if X and Y are dependent**