Representations of the oriented Brauer category

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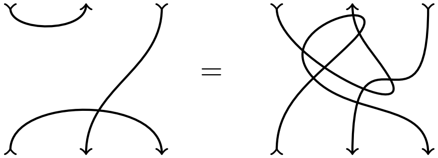
Oriented Brauer diagrams

$\langle\downarrow,\uparrow\rangle$ = the set of all words in the alphabet $\{\downarrow,\uparrow\}$, including the empty word $\varnothing$.

Let $a, b \in \langle\downarrow,\uparrow\rangle$. An *oriented Brauer diagram* of type $a \to b$ is drawn by aligning the words a, b in two rows, b above a, and drawing consistently oriented strands between the two rows connecting pairs of vertices.

Example



is a diagram of type $\uparrow\downarrow\downarrow \to \downarrow\uparrow\downarrow$.

The category $\mathcal{OB}$

Let $\Bbbk$ be an algebraically closed field, and let $\delta \in \Bbbk$.
Let $\mathcal{OB}$ be the hom-finite $\Bbbk$-linear category with

Objects: $\langle \downarrow, \uparrow \rangle$

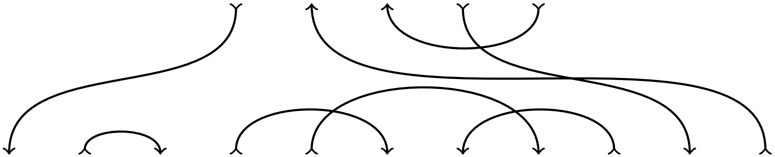
Morphisms: $\mathrm{Hom}_{\mathcal{OB}}(\mathrm{a}, \mathrm{b})$ has basis given by oriented Brauer diagrams $\mathrm{a} \to \mathrm{b}$ (finite dimensional).

Composition: D_1, D_2 diagrams. Stacking D_1 on top of D_2 produces an oriented Brauer diagram, with some number of "bubbles". $D_1 \circ D_2$ is obtained from this new diagram by removing all such bubbles, multiplying by δ for each bubble removed.

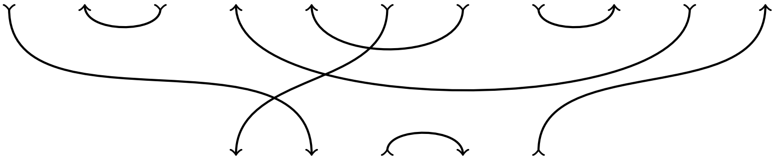
The category $\mathcal{OB}$

Example

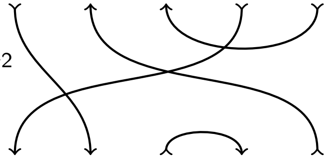
$D_1 =$



$D_2 =$



$D_1 \circ D_2 = \delta^2$



History

$\mathcal{OB}$ is the free $\Bbbk$-linear symmetric monoidal category generated by a single object $\uparrow$ of dimension δ and its dual $\downarrow$.

The endomorphism algebras in $\mathcal{OB}$ are isomorphic to the walled Brauer algebras $B_{r,s}(\delta)$ for various r, s (Turaev, Koike - 1989).

The Karoubi envelope of $\mathcal{OB}$ is Deligne's tensor category $\underline{\text{Re}}\text{p}(GL_\delta)$.

Goal

Study representations of $\mathcal{OB}$, ie. $\Bbbk$-linear functors $\mathcal{OB} \to \Bbbk\text{-Vec}_{fd}$.

Think of these as locally finite dimensional modules over the locally unital algebra OB associated to this $\Bbbk$-linear category: $OB = \bigoplus_{\mathrm{a},\mathrm{b} \in \langle \downarrow, \uparrow \rangle} \mathrm{Hom}_{\mathcal{OB}}(\mathrm{a}, \mathrm{b})$ (multiplication induced by composition).

System of idempotents: $\{1_\mathrm{a} : \mathrm{a} \in \langle \downarrow, \uparrow \rangle\}$

$$1_\mathrm{a} = \Big| \quad \Big| \quad \Big| \; \ldots \; \Big| \quad \Big| \quad \Big|$$

(orientations determined by a)

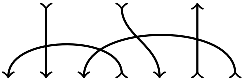
Main results of my thesis

- I classify the simple OB-modules.
- I construct a categorical action of a certain (type A) Kac-Moody algebra on OB-mod in the sense of Rouquier.
- I verify that one obtains a categorification of the tensor product $V(-\varpi_{m'}) \otimes V(\varpi_m)$ in this way.
- I give applications to the crystal graph structure and character formulae in characteristic zero.

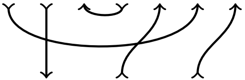
Triangular decomposition

$\mathbb{K} = \bigoplus_{\mathsf{a} \in \langle \downarrow, \uparrow \rangle} \Bbbk \cdot 1_{\mathsf{a}}$

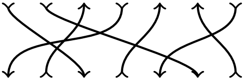
$OB^{+} =$ span of diagrams with no cups and no crossings among vertical strands.



$OB^{-} =$ span of diagrams with no caps and no crossings among vertical strands.



$OB^{0} =$ span of diagrams with no cups or caps.

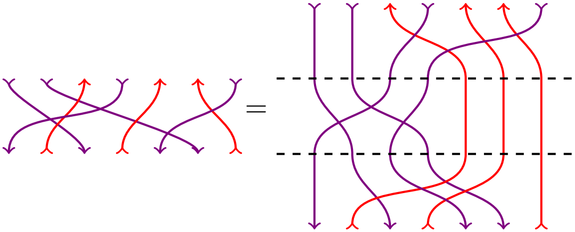


Proposition

$OB \cong OB^{-} \otimes_{\mathbb{K}} OB^{0} \otimes_{\mathbb{K}} OB^{+}$ as vector spaces.

OB^0 and the symmetric groups

$OB^0 \cong \bigoplus_{r,s\geq 0} \mathrm{Mat}_{\binom{r+s}{r}}(\Bbbk S_r \otimes \Bbbk S_s)$



OB^0 is Morita equivalent to $\bigoplus_{r,s\geq 0} \Bbbk S_r \otimes \Bbbk S_s$.

Simple OB^0-modules

- A partition is identified with its Young diagram as usual. It is called p-regular if no p rows have the same length (every partition is 0-regular).
- $\Lambda =$ the set of p-regular bipartitions, ie. pairs of partitions, both of which are p-regular.
- Simple OB^0-modules: $\{D(\lambda) : \lambda \in \Lambda\}$

Verma construction

V an OB^0-module.

Let caps act as zero to make into $OB^0 \otimes_{\mathbb{K}} OB^+$-module.

Then induce to an action of $OB^- \otimes_{\mathbb{K}} OB^0 \otimes_{\mathbb{K}} OB^+ \cong OB$.

This construction defines an exact functor

$$\Delta : OB^0\text{-mod} \to OB\text{-mod}.$$

$\overline{\Delta}(\lambda) = \Delta D(\lambda)$ (*proper standard modules*)

Remark

If $\delta \notin \mathbb{Z} \cdot 1_{\Bbbk}$, then Δ is an equivalence of categories.
Assume $\delta \in \mathbb{Z} \cdot 1_{\Bbbk}$ from now on.

Classification of simple OB-modules

Theorem

Let $\lambda \in \Lambda$. Then $\overline{\Delta}(\lambda)$ is an indecomposable module which has a unique maximal submodule. Let $L(\lambda)$ denote its unique simple quotient. Then $\{L(\lambda) : \lambda \in \Lambda\}$ is a complete set of inequivalent simple OB-modules.

Remarks

- $\overline{\Delta}(\lambda)$ often does not possess a composition series.
- OB-mod is a locally standardly stratified category, which is a generalization of the notion of a highest weight category.

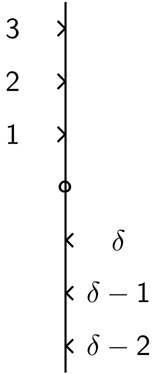
Applications ($p = 0$)

Markers

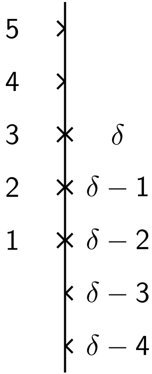
Another way to represent a bipartition visually.

Decorate the integer vertices of the y-axis with the symbols $>, <, \circ, \times$ as follows:

"Ground state" ($\delta < 1$)



"Ground state" ($\delta \geq 1$)

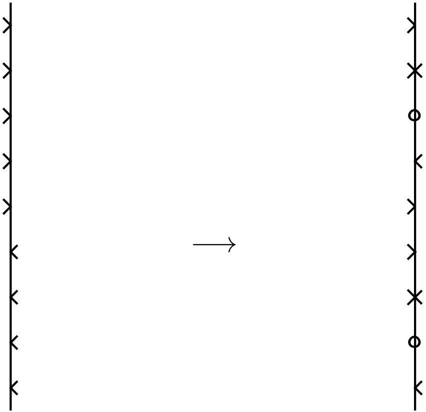


Marker of a bipartition

Given a bipartition (λ, μ), move the › marks downward according to the parts of λ, and move the ‹ marks upward according to the parts of μ.

Example

If $\delta = 0$, the marker of the bipartition $\left(\ydiagram{2,2,2}, \ydiagram{4,3,1}\right)$ is



Left arc diagram

Given the marker for a bipartion λ, its left arc diagram (denoted by $(\!\lambda$) is obtained by drawing non-crossing rays and arcs in the left half plane incident to some subset of the vertices in the marker in such a way that

- vertices at the bottom ends of arcs are labelled $>$;
- vertices at the top ends of arcs are labelled $<$;
- vertices at the right ends of rays are labelled either by $<$ or by $>$ in such a way that all rays labelled $>$ appear above all rays labelled $<$;
- all vertices not at the ends of arcs or rays are labelled either by $\circ$ or by $\times$.

Example

$>$

$<$

$>$

$<$

$<$

$<$

$>$

$\circ$

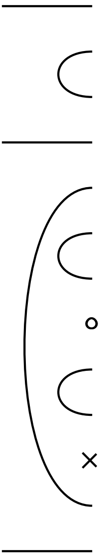
$<$

$>$

$\times$

$>$

$<$



Composite diagram

If λ, μ are two bipartitions whose markers have $\circ, \times$ in the same positions, then we can place the left arc diagram of λ over the marker representing μ to obtain their composite diagram $\lambda\boxed{\mu}$.

$\lambda\boxed{\mu}$ is well-oriented if

- each arc has exactly one label $<$ and one label $>$ making it into either a *counterclockwise* or *clockwise* arc;
- all rays labelled $>$ are above all rays labelled $<$.

Composition multiplicities

Theorem

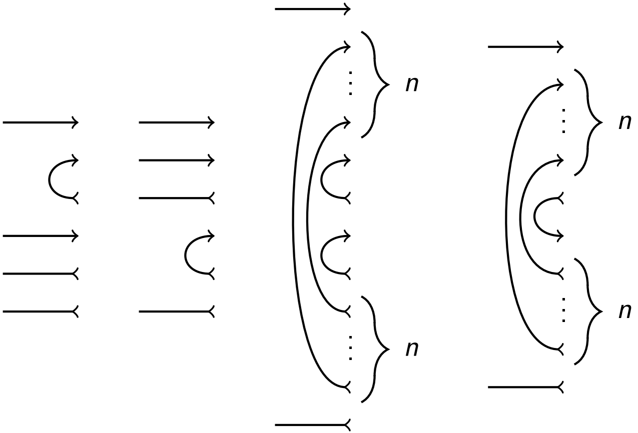
$$[\overline{\Delta}(\lambda) : L(\mu)] = \begin{cases} 1 & \text{if } \mu\lambda \text{ is well-oriented} \\ 0 & \text{otherwise} \end{cases}$$

Proof

This is an application of the functors constructed to exhibit the categorical action and the underlying crystal graph.

Example

Let $\delta = 0$. We determine all the composition factors of $\overline{\Delta}(\Box, \Box)$.
The marker for $\lambda = (\Box, \Box)$ is $\substack{<\\ >}$ (where all vertices above these are $>$ and all vertices below these are $<$). The possible well-oriented composite diagrams $\mu\lambda$ are



where $n \geq 0$ and all vertices not shown are at the ends of rays.

Example

The corresponding μ's are

$$
\begin{array}{cccc}
\begin{matrix} > \\ < \\ > \\ > \\ < \\ < \end{matrix}
&
\begin{matrix} > \\ > \\ < \\ < \\ > \\ < \end{matrix}
&
\begin{matrix} > \\ \left.\begin{matrix} < \\ \vdots \\ < \end{matrix}\right\} n+1 \\ > \\ < \\ \left.\begin{matrix} > \\ \vdots \\ > \end{matrix}\right\} n+1 \\ < \end{matrix}
&
\begin{matrix} > \\ \left.\begin{matrix} < \\ \vdots \\ < \end{matrix}\right\} n+1 \\ \left.\begin{matrix} > \\ \vdots \\ > \end{matrix}\right\} n+1 \\ < \end{matrix}
\end{array}
$$