Constraining the functional form of the Carnot Efficiency

Disclaimer: I do not take credit for this explanation. A fellow classmate told me this in passing, so I decided to write it up for my own benefit.

Setup

One can deduce the following constraint about the efficiency of a Carnot engine, η

$$1-\eta(T_1,T_3)=(1-\eta(T_1,T_2))(1-\eta(T_2,T_3)) \tag{1}$$

From this, we can conclude that $1-\eta(T_1,T_2)=\frac{G(T_1)}{G(T_2)}$, for some single-variable function G. But how?

Proof

First of all, let's simplify the problem. I will relabel T_1, T_2, T_3 to the familiar x, y, z. I will also define another function f

to be $1-\eta$. Thus,

$$f(x,z)=f(x,y)f(y,z) \tag{2}$$

Second, let's understand the setup. x, y, and z vary in a certain way, and we are supplying them as inputs to the function. This means that if we choose a particular value of one variable, this constrains the way the functions behave. This becomes more clear once we rearrange the expression.

$$\frac{f(x,z)}{f(y,z)}=f(x,y) \tag{3}$$

All we have done here is divide through by $f(y,z)$. Now, let's consider the variable z. The right hand side of the expression does *not* depend on z. This is important. This means we can choose a particular value for z ,and nothing will happen to the right hand side. To be more precise, we can define a new function that takes in three inputs,

$$g(x,y,z)\equiv\frac{f(x,z)}{f(y,z)} \tag{4}$$

and our expression becomes

$$g(x,y,z)=f(x,y) \tag{5}$$

This tells us that z is spurious! No matter what we choose, the function on the right hand side will be the same. That means

$$g(x,y,0)=g(x,y,1)=g(x,y,2)=f(x,y) \tag{6}$$

As goes for any number we put in that third slot. Therefore, we might as well just write down

$$g(x, y) = f(x, y) \tag{7}$$

And drop that third argument. And we'll do just that, with one small caveat. We cannot *really* drop the second argument from $f(x, z)$, since it is a function of two variables. However, when we choose a particular value of z we can turn it into a single variable function that only depends on x. In other words, we are defining

$$h_z(x) \equiv f(x, z) \tag{8}$$

Remember that z is a number that we have fixed in advance! Also remember that it might *not* be the case that $h_z(x)$ is the same for all values of z. All we know is that the z dependence drops out when we divide them, so it doesn't even matter what we chose z to be in the first place.

$$g(x, y, z) = \frac{h_z(x)}{h_z(y)} = f(x, y) \tag{9}$$

Since the right hand side of this expression, $f(x, y)$ exhibits no z dependence, we might as well drop the z from every other term.

$$g(x, y) = \frac{h(x)}{h(y)} = f(x, y) \tag{10}$$

And we are done! Note that the individual function $h(x)$ still *does* depend on z, but we might as well leave it out when we write it down as a ratio because it will produce the same function no matter what z was in the first place.