Notes on

Analysis

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My journey of spending a year learning analysis, starting on December 18th, 2022, using these references.

- *Principles of Mathematical Analysis* by Walter Rudin. Every exercise is included.
- *Understanding Analysis* by Stephen Abbott. Source of analogies and explanations.
- *Calculus* by Michael Spivak. Selected exercises and thoughts.
- *The Cauchy-Schwarz Masterclass* by Michael Steele. Excellent practice for inequalities and entertaining read.

Contents

1 Foundations

Here's a cute proof of the Schwarz inequality. We want to prove that $x_1y_1+x_2y_2 \leq \sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}$. These are the steps ahead.

- $b^2 - 4c < 0 \implies x^2 + bx + c > 0$
- In the following two cases, equality holds:
  1. $\exists\lambda$ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$
  2. $x = y = 0$
- Otherwise, the following must hold for all λ: $(x_1 - \lambda y_1)^2 + (x_2 - \lambda y_2)^2 > 0$.
- After expanding and rearranging the above equation into a quadratic in λ, we can use the first lemma above to derive a constraint that happens to be the Schwarz inequality!

1.1 Problems

Problem. (Spivak, Chapter 1) Prove that if x and y are not both 0, then

$$x^2 + xy + y^2 > 0 \tag{1}$$
$$x^4 + x^3y + x^2y^2 + xy^3 + y^4 > 0 \tag{2}$$

Proof. First, observe that $x = y$ trivially satisfies both inequalities ($3x^2 > 0$ and $5x^4 > 0$).

Then, recall that $x^3 - y^3 = (x-y)(x^2+xy+y^2)$. We then have $x^2 + xy + y^2 = \frac{x^3-y^3}{x-y}$ for $x \neq y$. This is always positive since x^3, y^3, x, and y all increase monotonically on their respective domains, and thus $x > y \iff x^3 > y^3$ and vice versa.

The same line of reasoning applies for the second inequality, as $x^4+x^3y+x^2y^2+xy^3+y^4 = \frac{x^5-y^5}{x-y}$. $\square$

2 Real numbers

The rational numbers have gaps.

- There exists no $q \in \mathbb{Q}$ such that $q^2 = 2$
- Between any two rational numbers, we can find another.

Theorem. Two real numbers a, b are equal iff for every real $\epsilon > 0$, $|a - b| < \epsilon$.

Proof. We consider both directions of the if and only if.

$\implies$ Trivial.

$\impliedby$ Suppose there exists a positive real number that is greater than or equal to $|a-b|$. Let ϵ saturate this bound. Without loss of generality, let $a > b$. Then, $a = b + \epsilon \neq b$. $\square$

2.1 Problems

Problem. (Abbott, Chapter 1) Prove that $\sqrt{3}$ is irrational.

Proof. Assume for the sake of contradiction that there exists $m, n \in \mathbb{Z}$ such that $m/n = \sqrt{3}$.

$$m^2/n^2 = 3 \tag{3}$$
$$m^2 = 3n^2 \tag{4}$$
$$\implies 3 \mid m^2 \tag{5}$$
$$\implies 3 \mid m \tag{6}$$

(4) is true because 3 is prime.

□

Problem. (Rudin, Chapter 1) Let E be a nonempty subset of an ordered set. Suppose α is a lower bound of E and β is an upper bound of E. Prove $\alpha \leq \beta$.

Proof. Suppose $\alpha > \beta$. By the definition of a lower bound, for every $x \in E$, $x \geq \alpha$. Take a particular $x \geq \alpha$. $x \geq \alpha \implies x > \beta$, which violates the definition of an upper bound, since every $x \in E$ must be less than or equal to β. Contradiction. □