Moments of Inertia

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1 Introduction

The **moment of inertia** of a rigid body R is defined as

$$I = \int_R r^2 dm$$

Where r is the perpendicular distance from the axis of rotation.

2 Solid sphere

We proceed in spherical coordinates.

$$I = \int_R r^2 dm$$

Here, $dm = \rho\, dV$ where ρ is the density of the sphere and $dV = r^2 \sin\theta dr d\theta d\phi$, which is the volume element in spherical coordinates. Roughly, we are dividing $R \subset \mathbb{R}^3$ into infinitesimal pieces of volume and weighting each piece by its square distance from the axis of rotation and the density at that point (in this case its uniform everywhere).

Importantly, r is the perpendicular distance from the axis of rotation, which is directly through the poles of the sphere. This distance is actually $r \sin\theta$.

$$\begin{aligned}
I &= \int_R (r\sin\theta)^2 \rho\, dV \\
&= \rho \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^R dr\, (r^2 \sin^2\theta)(r^2 \sin\theta) \\
&= \rho \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^R dr\, r^4 \sin^3\theta \\
&= \rho \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \left[\frac{1}{5} r^5 \sin^3\theta\right]_0^R \\
&= \frac{\rho R^5}{5} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin^3\theta
\end{aligned}$$

With a little trig and a simple change of variables, the inner integral

$$\int_0^\pi d\theta \sin^3\theta = \frac{4}{3}$$

As an exercise, carefully evaluate this integral.

$$\begin{aligned} &= \frac{4\rho R^5}{15}\int_0^{2\pi} d\phi \\ &= \frac{8}{15}\pi\rho R^5 \end{aligned}$$

Substituting $\rho = \frac{M}{\frac{4}{3}\pi R^3}$, we get the familiar result

$$\boxed{I = \frac{2}{5}MR^2}$$