Understanding phase transitions and coupled oscillations in complex systems

A study using hypergrpahs

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I would like to dedicate this thesis to my loving parents . . .

Declaration

I declare that this thesis has not been submitted as an exercise for a degree at this or any other university and it is entirely my own work. It contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements.

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July 2024

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Abstract

This thesis acts as a brief introduction into complex systems modelling using hypergraphs. We try to study the phase transitions of a coupled system and observe various properties pertaining to it. We discuss the results obtained and the metrics used to study them.

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CHAPTER 1

Introduction

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1.1 Brief Overview of the problem

The primary goal of this project is to attain a better understanding of non-linear dynamical systems through complex systems modeling, optimization and analysis.

Complex systems are not just a collection of simple components, but rather an intricate web of interactions between different elements. Understanding how these systems evolve over time and exhibit emergent behaviors is a crucial area of research, encompassing diverse fields such as spin systems[15], societies[3], and neural networks[10]. In this project, our aim is to delve into the evolution of such complex systems and explore their evolution over time under constraints, with the ultimate goal of identifying patterns and underlying principles.

For this, we define a system which we will talk about in detail in the following sections, which is used to study the dynamics of coupled systems including and not limited to properties like their evolution in time, synchronization, and emergent computation.
They involve intricate interactions and relationships between various elements, leading to emergent behaviors and patterns. Understanding how these systems evolve over time and exhibit such emergent behaviors is a significant area of research, spanning across

various fields such as physics (spin systems), sociology (societies), and neuroscience (neural networks).
The goal of the project is to explore the evolution of these complex systems under constraints and identify underlying principles and patterns that govern their behavior. We aim to capture the essence that "the whole is other than the sum of the parts," as famously stated by Aristotle, meaning that the system's behavior cannot be fully understood by analyzing individual components in isolation. To achieve this goal, the researchers plan to model the complex systems using mathematical tools such as hypergraphs. Hypergraphs offer a more generalized representation compared to classical graphs, where an edge can connect any number of vertices, allowing for a more flexible and comprehensive description of interactions within the system.
We aim to gain insights into the dynamics of this coupled system and its behavior over time. Understanding the properties of such systems can have implications for various fields, including autonomous systems and biological systems, broadening the scope of the research.

To facilitate the study, we define a specific system and delve into its dynamics in detail in the subsequent sections. They plan to investigate properties such as the evolution of the system over time, synchronization, and emergent computation. The theoretical framework will be rigorously developed to provide a solid foundation for analyzing and understanding the complexities of the coupled systems.
We can classify the core objectives of the project in the following manner:

1. Synchronization: Complex dynamics on hypergraphs can exhibit synchronization phenomena, where nodes or subsystems in the network become correlated or synchronized over time. Synchronization can emerge in various contexts, such as coupled oscillators or information exchange networks. The study of synchronization on hypergraphs helps uncover how interconnected components influence each other's behavior, leading to coordinated or coherent dynamics.

2. Self-Organization: Hypergraphs also offer insights into self-organizing systems, where global patterns and structures arise from local interactions without external control. Such systems can adapt and reconfigure themselves based on the dynamics of their constituent elements. Self-organization is a fundamental aspect of many natural and artificial systems, and hypergraph-based modeling facilitates its exploration.

3. Optimal Control: Optimal control seeks to find control inputs that optimize certain performance criteria. For hypergraph-based systems, this involves determining

the most efficient way to influence the dynamics of the system to achieve desired outcomes by reconfiguring various parameters involved.

4. Stability Analysis: This is vital to assess the behavior of controlled hypergraph-based systems. It helps determine whether a system converges to a desired state or exhibits oscillatory or chaotic behavior.

Analyzing complex dynamics on hypergraphs helps to evaluate the resilience and robustness of interconnected systems. The ability to model and understand the response of systems to perturbations or external influences is crucial in designing more resilient and adaptable systems. The project represents a comprehensive effort to gain a deeper understanding of the behavior and evolution of complex systems, offering potential insights into diverse applications and opening up new avenues for future research in various scientific disciplines. The use of hypergraphs as a mathematical tool enhances the capability to capture the intricate interactions within the systems, making it a promising approach to address the research objectives.

1.2 Literature Review

1.2.1 Complex Systems, Phase transitions and Emergent computation

Throughout history complex systems were defined in various ways with most ideas reflecting the lack of complete order and uncertainty with multiple parameters affecting the system studied. But they still lack a global quantitative description of necessary features which makes it a really interesting topic of research. We have gone through the basic definitions previously in the Interim report and we will be looking more closely at the same but with more focus on the problem at hand.

We have mentioned phase transitions as the transformation of a system from one state to the other with a particular interest in the parameters that govern these transitions and the critical points at which these transitions occur. And the behavior of these systems close to the critical point might give us more insight into understanding and establishing this quantitative definition [14].
Solé Et al.[14] propose that the study of phase transitions in simple, nonlinear models can provide a way to understand complex systems. They argue that many complex systems, such as ecosystems and social systems, operate at the "edge of chaos," where they exhibit a high degree of self-organization and nonlinearity. In this region, the system is sensitive

to initial conditions and small perturbations, and it can exhibit a wide variety of behaviors, including self-organization, pattern formation, and the emergence of structures.

Langton[9] evaluated the relationship between phase transitions in complex systems and emergent computation. He drew an analogy between computational complexity and the halting problem within the phenomenology of the phase transition. Langton's paper suggests that complex systems that operate at the edge of chaos exhibit a unique form of computation known as "emergent computation" that arises from the interactions between the components of the system. emergent computation arises from the interactions between the components of the system, and the behavior of the system is not determined by a central processor but emerges spontaneously from the interactions between the components. He used a cellular automata model to study the behavior of these systems, as it is a simple yet powerful enough method to model the emergence of patterns and structures, allows him to study the effects of small perturbations and initial conditions on the system's behavior, and permits him to study the transition between ordered and disordered states, which is relevant to the edge of chaos.

The above two papers are our primary reference points for trying to understand the analysis of these dynamical systems. Our goal is to develop a mathematical model for complex systems that will allow us to observe the phase transition of these systems when subjected to perturbations and how they evolve over time. This would lead to a better understanding of these systems, better predictions, and the ability to observe any pattern formation or synchronization. These insights could help us compute and retrieve real-world data more efficiently when such systems are studied and subjected to similar conditions.

1.2.2 Coupled systems, Synchronicity and Hypergraphs

There has been a lot of interest in understanding dynamical systems through network science and graph theory. Graphs have been ubiquitous in trying to establish relations between objects of concern. They represent a direct link between two nodes using edges which can be further utilized to understand the properties of these nodes and the system as a whole. But this often falls short when trying to model a real-world scenario that goes beyond simple pairwise connections. To accurately assess real-world systems, we need higher-order links that can be used to connect more than just two nodes. Higher-order nodes have been increasingly used to model systems in multiple domains including biology[8], communication[12], and physics[4].

Indeed, network science and graph theory have been instrumental in understanding dynamical systems in various fields. Graphs provide a powerful framework to represent and analyze relationships between objects or entities of interest. The nodes in a graph represent the objects, and the edges represent direct connections or interactions between the nodes. This approach has been widely used to study diverse systems such as social networks, transportation networks, and biological interactions.

However, traditional graphs that rely solely on pairwise connections have limitations when trying to model complex real-world scenarios. Many real-world systems involve interactions and relationships that go beyond simple pairwise connections. In such cases, higher-order links, represented by hyperedges in hypergraphs, become more appropriate for capturing the complexity of the relationships.

Hypergraphs extend the concept of graphs by allowing hyperedges to connect more than just two nodes. Hyperedges enable us to represent relationships involving multiple entities simultaneously. This increased expressive power makes hypergraphs a more suitable tool for modeling systems that exhibit higher-order interactions and dependencies.

1. In biology, hypergraphs have been used to model and study complex biological networks, such as protein-protein interactions and genetic regulatory networks[5], where multiple genes or proteins can interact together to influence biological processes.

2. In communication networks[17], hypergraphs have been employed to analyze and optimize data transmission and routing schemes, as these systems often involve groups of nodes collaborating to transmit information.

3. In physics, hypergraphs have found applications in various contexts, including the study of phase transitions[6] and critical phenomena, where interactions between multiple elements are crucial to understanding emergent behaviors.

The use of hypergraphs offers a more nuanced perspective on the structure and dynamics of complex systems. By considering higher-order interactions, researchers gain a deeper understanding of the emergent properties and collective behaviors of the system as a whole.

We move to Hypergraphs[1],[16] to represent phase transitions of these complex dynamical systems. A hypergraph is a generalization of a graph in which an edge (or "hyperedge") can connect any number of vertices. In contrast, in a standard graph, an edge can only

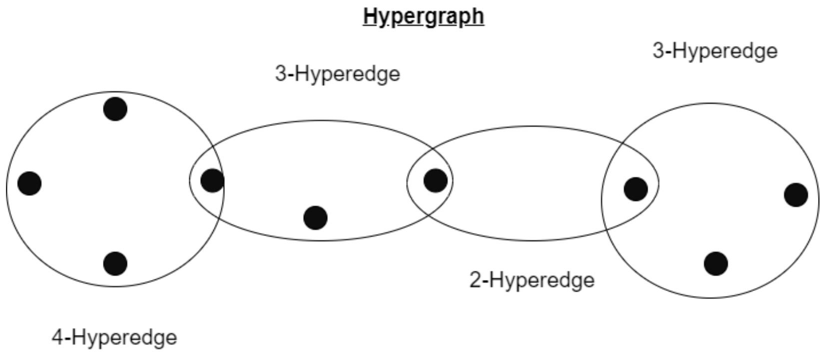

Fig. 1.1 An example of an undirected hypergraph

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connect two vertices. Hypergraphs are useful in representing relationships among a large number of entities.

The paper on phase transitions in hypergraphs[7] is of particular interest to our project, as it establishes a comprehensive mathematical framework to study the relationship between the structural organization of hypergraphs and dynamical processes beyond pairwise interactions. This framework is relevant to understanding the evolution of complex systems, such as the coupled system we are investigating.

We are particularly interested in phase transitions on hypergraphs[7] and the paper constructs a mathematical framework describing the relationship between the structural organization of hypergraphs and the class of dynamical processes beyond the pairwise interactions. The framework allows for the analysis of the stability of these systems and the identification of the role of each structural order for a given process. The authors use this framework to analyze two specific dynamics of general interest: social contagion and diffusion processes. They show that the stability conditions can be separated into structural and dynamical components and demonstrate that in social contagion processes, only pairwise interactions play a role in the stability of the absorbing state, while in diffusion dynamics, the order of interactions plays a differential role. The paper provides a general framework for further research on dynamical processes on hypergraphs which will be useful to adapt to better understand the evolution of our system and model.

In the context of our project, the paper's findings have significant implications. As we aim to explore the dynamics of a coupled system on a hypergraph, the mathematical framework can offer valuable insights into the stability of our system under different conditions. It allows us to analyze how different structural organizations within the hypergraph impact the behavior of the system as a whole. Moreover, the paper's differentiation between the structural and dynamical components of stability provides a deeper understanding of how the interactions between vertices influence the overall stability of the system. This information can aid in identifying critical elements or components that play pivotal roles in the system's behavior and evolution.

The development of a comprehensive theoretical framework combining complex dynamics on hypergraphs with control theory empowers researchers and practitioners to tackle intricate and interconnected systems. The observation of synchronicity in coupled systems provides valuable insights into the coordinated behavior of interconnected components. Synchronization phenomena are prevalent in various natural and artificial systems, ranging from biological networks and brain dynamics to information exchange in communication systems. Understanding synchronicity is vital as it allows us to characterize emergent behaviors and identify patterns of coherence and coordination within the systems. The resulting computational models offer a deeper understanding of emergent behaviors, phase transitions, synchronicity, and control strategies. This knowledge is highly valuable in addressing real-world challenges in fields such as ecology, climate science, social dynamics, and advanced communication networks. Ultimately, the application of hypergraph-based models contributes to advancements in various scientific disciplines and fosters innovative solutions for complex problems in our ever-changing world.

CHAPTER 2

Model and Methodology

2.1 Model: Overview and Construction

Overview

We now establish an appropriate model to better understand the phase transitions and dynamics for a coupled system. We will be referring to Pinero's paper [11] on Non- Equilibrium energy harvesting. We borrow one of the proposed systems from the paper and build upon it using network theory and hypergraphs. For this, we first need to understand the underpinning of the model, the subsequent systems, and the theory involved.

1. The first topic is dull
2. The second topic is duller
   (a) The first subtopic is silly
   (b) The second subtopic is stupid
3. The third topic is the dullest

2.1.1 Initial models

One of the key requirements is to establish a firm mathematical model that can be used to understand the coupled dynamical systems and their interactions. Interactions between these system components play an important role in shaping the structure and dynamics of many complex systems. Higher-order interactions, which involve more than two components at a time as discussed earlier, are particularly influential. They have been shown to significantly impact the collective behavior of such systems, leading to novel phenomena such as explosive transitions. These effects have been observed in a wide range of dynamical processes including diffusion, consensus, spreading, and evolution. But one of the key questions that arise is what type of consequences do we observe of choosing one higher-order representation over the other and how it can affect the dynamics of the system?

We briefly try to understand the two representations and how they help us better understand non-pairwise interactions.

1. **Simplicial Complex** In Algebraic Topology, a simplex is a geometric object which is used to represent higher dimensions. An n-simplex consists of n+1 vertices and has a dimension of n. For every simplicial interaction of a certain dimension n to exist, it is necessary for the $n-1^{th}$ dimension to also have existing interactions. For example, for a second-order Simplex consisting of the interaction (i,j,k) to exist, it is important for the corresponding first-order interactions (i, j), (i, k), (j, k) must also exist.
   However, the simplicial complexes approach has its limitations. One of the major restrictions is that it requires mutual inclusion for the interactions, which is too restrictive for general purposes. As a result, it may not always be the most suitable approach for studying all types of networks.

2. **Hypergraph** To address this issue, researchers have turned to hypergraphs. Think of hypergraphs as simplicial complexes but without any of the dependencies on lower orders. This allows for higher-order hypergraphs to exist without relying on the corresponding lower-order links. This helps create a more general complex system model, allowing for a better understanding of all the properties. We have extensively covered this in the Literature review section.

It is important to know if the effect of synchronization varies with these two representations and if we have a way to measure the same. For this Zhang et al. [18] have considered a simple system consisting of n identical phase oscillators whose states $\theta = (\theta_1, ..., \theta_n)$ evolve according to the equation,

$$\dot{\theta} = \omega + y_1/k^{(1)} \sum_{j=1}^{n} A_{ij} sin(\theta_j - \theta_i) + y_2/k^{(2)} \sum_{j,k=1}^{n} 1/2B_{ijk} 1/2 sin(\theta_j + \theta_k - 2\theta_i) \tag{2.1}$$

Equation 2.1 is a natural generalization of the Kuramoto model[13]that includes interactions up to order two (i.e., three-body interactions). The oscillators have natural frequency ω and the coupling strengths at each order are $y1$ and $y2$, respectively. The adjacency tensors determine which oscillators interact: $Aij = 1$ if nodes i and j have a first-order interaction, and zero otherwise. Similarly, $Bijk = 1$ if and only if nodes i, j and k have a second-order interaction. All interactions are assumed to be unweighted and undirected. The (generalized) degrees are given by $k^{(1)} = \sum_{j=1}^{n} A_{ij}$ and $k^{(2)} = \sum_{j,k=1}^{n} 1/2B_{ijk}$.
Applying synchronization here, $\theta_i = \theta_j$ for $i \neq j$, which is a solution to 2.1 we observe the stability of α which is defined by $y_1 = 1 - \alpha; y_2 = \alpha, \alpha = [0, 1]$. Further **lyapunov exponents** are used, which characterize the rate of separation of close trajectories in a complex dynamical system. They are used to characterize the stability or instability of the system, with positive exponents indicating chaos and negative exponents indicating stability. In our case, they are the opposite of the eigenvalues generated by solving the multi-order laplacian of the chosen analytical treatment of the model. We simulate the above in python using the XGI package to generate the following graph2.1.

The maximum transverse Lyapunov exponent here2.1 λ_2 measures the rate of exponential separation of nearby trajectories, which characterizes the stability of synchronization. The results show that synchronization is enhanced by higher-order interactions in random hypergraphs, as indicated by a decrease in λ_2 as α increases. In contrast, synchronization is impeded in simplicial complexes, as indicated by an increase in λ_2 as α increases. This suggests that higher-order interactions can promote synchronization in certain network topologies like hypergraphs over others which is to establish the reason for choosing them for our models.

2.1.2 Final Model

After going through a lot of literature pertaining to coupled dynamical systems and synchronicity, we looked at a recent, relevant work by Piñero et al.[11] that talks about a theoretical framework of interest. They try to find the thermodynamic cost of maintaining a steady non-equilibrium state versus the benefits of energy harvesting. Here energy harvesting refers to the concept of utilizing this additional energy in the environment to

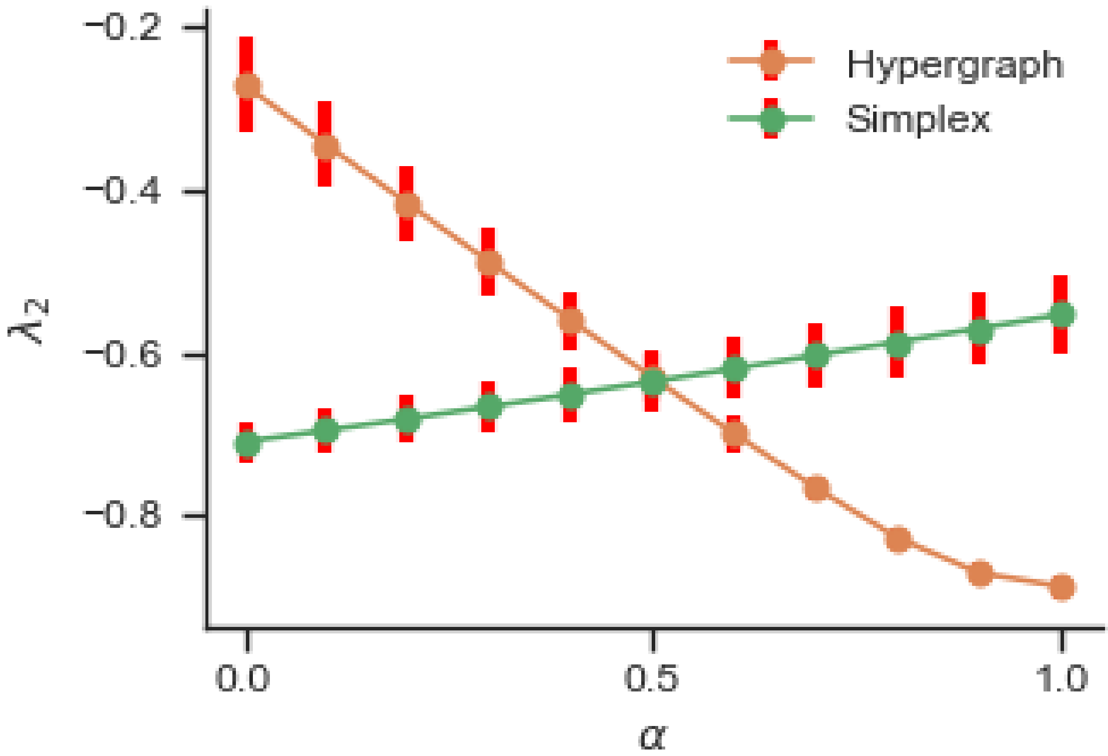

Fig. 2.1 We can observe that the synchronization is improved by higher-order interactions in random hypergraphs but is restricted in simplicial complexes.

run certain processes. The process can be bi-directional, with one direction utilizing the energy to move from one state to another while the other dissipates this energy to revert back. In short, moving from state n to n+1 releases some energy while moving from state n-1 to n uses the energy available to carry out the process. The paper discusses this system while establishing two types of processes:

1. **Baseline**: In many natural systems, there are baseline processes that are responsible for the increase of entropy or disorder, which result in a loss of information about optimal states for energy harvesting. These processes often occur spontaneously and are characterized by a tendency towards equilibrium and randomness. One common example is diffusion, where molecules or particles move randomly from areas of high concentration to areas of low concentration, leading to a gradual equilibrium in the system. Another example is degradation, where complex molecules

are broken down into simpler ones, leading to an overall decrease in order and thus increasing the entropy.

2. **Control**: These baseline processes are necessary for the functioning of many natural systems, but they also impose limits on the efficiency of energy harvesting. As the system approaches equilibrium, the availability of energy decreases, making it increasingly difficult to extract useful work. Therefore, to maintain a non-equilibrium state and sustain energy harvesting, natural systems require control processes that carry out self-maintenance and regulation. These control processes include synthesis, sensing of environmental states, and other mechanisms that actively maintain the system in a state that is favorable for energy harvesting.

We write the overall entropy of such a system as follows:

$$\partial_t S(p) = \partial_t^R S(p) + \partial_t^{R'} S(p) \tag{2.2}$$

where, $\partial_t^R S(p) := \langle Rp, -lnp + s\rangle$ and $\partial_t^{R'} S(p) := \langle R'p, -lnp + s\rangle$ are the baseline and control contributions with s being the internal entropy and R and R' are the baseline and control rate matrices.
The goal is to study the thermodynamic benefits of the cost and benefits of the control process on the system.

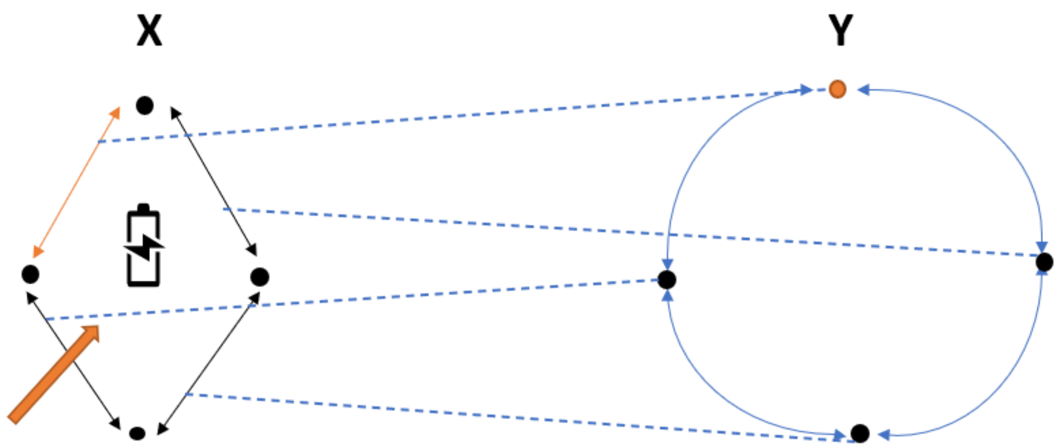

Fig. 2.2 The system under study X and the fluctuating environment Y

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The image described in 2.2 shows a system X that is able to extract and store energy from its environment, indicated by the red arrows. This means that the system is not in thermodynamic equilibrium and must maintain a non-equilibrium state to continue extracting energy. The image suggests that there is a tradeoff between the benefits of energy harvesting and the cost of maintaining the non-equilibrium state.

System X is shown to be coupled to a fluctuating environment Y. This means that the rate at which energy can be harvested by X also depends on the state of the environment Y. The coupling between X and Y introduces additional complexities to the tradeoff between energy harvesting and the cost of maintaining a non-equilibrium state. We use two different hypergraphs to model each of the systems X and Y separately to study the coupled dynamics and other properties while trying to establish the cost/benefit of the control processes.

We have a combined system consisting of two subsystems: X representing the "agent" and Y representing the "environment." Both subsystems take values in finite sets, $X = 1,\ldots,|X|$ and $Y = 1,\ldots,|Y|$. The dynamics of the joint system are governed by two rate matrices, R and R', both of size $|X||Y| \times |X||Y|$.

To model this system using hypergraphs, we can represent the elements of X and Y as vertices in two separate hypergraphs. A hypergraph, as discussed earlier, is a generalization of a graph where an edge can connect any number of vertices. In our case, we'll use hypergraphs to capture interactions involving multiple elements of X and Y.

The power gain helps us determine the overall cost and benefit of control of the system, where we define benefit as the increase of steady-state power output when control is present versus not present and we define the cost of the control processes as the minimal thermodynamic cost needed to maintain the non-equilibrium steady state. It is defined by the bound

$$\Delta \dot{W}^* = \max_{p:BR\bar{\pi}=0} -\partial_t^R \mathscr{S}(p)/\beta + \langle p-\pi, d-q\rangle \tag{2.3}$$

where the maximization is over all probability distributions. But the bound involves optimization of a convex function which has a unique property where they have a global minimum (or maximum), which is also a local minimum (or maximum). This means that any local minimum found in the optimization process is guaranteed to be the global minimum[2] and which can be solved numerically using standard techniques. However, the solution does not have a closed-form expression in general, even for a simple two-state system. Nonetheless, as we show in our second set of results, closed-form expressions may be identified in some physically meaningful regimes of interest.

1. **Linear Response Regime (LR)**:In the linear regime, the system's response to small perturbations is approximately proportional to the magnitude of the perturbation. This regime is often observed when the external forces or fluctuations are weak, and the system's dynamics remain close to its equilibrium state. In this regime, the system's behavior can be well approximated by linear differential equations or linear response theory. The linear regime is characterized by stability and predictable behavior around the equilibrium point. Small perturbations cause small changes in the system, and the dynamics remain relatively simple and tractable. We define the bound for the LR regime as follows

$$\Delta \dot{W}^{*}_{\mathrm{LR}} \approx \beta\Theta^2 \frac{k}{\xi}\frac{(n-1)}{4n^2} = \beta\Theta^2 \frac{k^2}{\left(1+\frac{\epsilon}{2}\right)}\frac{(n-1)}{4n^2} \tag{2.4}$$

   where:

   (a) β is the inverse temperature.

   (b) Θ is the energy input value.

   (c) k is the energy input parameter.

   (d) n is the number of states in the system and environment.

2. **Far from Equilibrium Regime (FE)**: The far-from-equilibrium regime refers to a situation where a system experiences strong external perturbations or fluctuations that drive it far from its equilibrium state. In this regime, the system's dynamics are complex and nonlinear, and simple linear approximations are no longer valid. Far-from-equilibrium systems often exhibit emergent behaviors, self-organization, and the potential for phase transitions.

We discuss them primarily in the context of fluctuating and static environments with respect to the system in consideration. We are more interested in the fluctuating environment as it has more scope for analysis in terms of phase transitions and coupled oscillations.

In the context of fluctuating and static environments, the far-from-equilibrium regime is observed when the external forces or fluctuations are strong and comparable to or exceed the system's intrinsic dynamics. The system is driven away from its equilibrium state, leading to complex and nonlinear behaviors. Far-from-equilibrium systems may display spontaneous pattern formation, volatile structures, and novel emergent properties which are fascinating to study.

2.2 Methodology

We now discuss the approach used to solve and simulate the model discussed in section2.1.2. We established the model across the two regimes as discussed earlier and solved for the Maximum Power Gain across various energy input values for each regime.

2.2.1 Algorithm

The system X and environment Y are modeled on hypergraphs, for both, the static environment case and the fluctuating environment case. The approach we used for solving for the maximum power gain in the Linear Response regime and the Far From Equilibrium Regime involve the following steps:

1. **Hypergraph Representation**: Represent the hypergraphs for system X and environment Y as appropriate mathematical structures. Each hypergraph can be represented by a set of nodes (vertices) and a set of hyperedges (connections) that link multiple nodes.

2. **Transition Rates**: Based on the provided transition rate scheme, define the transition rates for each hyperedge in both hypergraphs. The transition rates should capture the probabilities of transitions between states based on the hypergraph structure.

3. **Constructing the Global Rate Matrix**: The global rate matrix R can be constructed as a tensor product of the rate matrices for hypergraph X (R^X) and hypergraph Y (R^Y). This represents the combined dynamics of the system-environment interaction.

4. **Compute the Baseline Steady-State Distribution**: Calculate the baseline steady-state distribution π_{xy} for the coupled system-environment, assuming no work extraction. For a uniform steady-state, $\pi_{xy} = 1/n^2$ for all x and y.

5. **Determining the Eigenvalues and Eigenvectors**: Compute the eigenvalues and eigenvectors of the global rate matrix R. These eigenvalues and eigenvectors will help determine the probabilities of different states during work extraction.

6. **Calculate the Maximum Power Gain**: Using the approach described for the Linear Response regime, calculate the maximum power gain $\Delta \dot{W}^{*}_{\mathrm{LR}}$, which corresponds to the maximum power that can be extracted from the system while maintaining a near-equilibrium state.

We are interested in the Unicyclic environment which is a model consisting of two subsystems, system X and environment Y, each having n states. Both subsystems undergo transitions over a ring of n states. The dynamics of system X follow an unbiased random walk, as discussed in the previous context, representing its baseline dynamics. On the other hand, the environment Y follows a biased random walk.

The crucial feature of this model is that the work gained by system X is influenced by the state of the environment Y. Each time system X transitions from state $X = y$ to $X = y+1$, it gains an amount Θ of work, where y is the state of the environment Y. In other words, the work extraction process is dependent on the state of the explicitly modeled environment Y, and the dynamics of both subsystems are interconnected in this unicyclic model 2.2.

We further expand on this model and try and find out the phase transitions

CHAPTER 3

Results and Analysis

3.1 Overview

The results were run using standard parameter configurations of the Model discussed in the 2.1.2. Multiple plots were obtained for various parameters and measures about the overall stability and dynamics of the model proposed. The results were computed using Python using the NumPy and Matplotlib libraries for numerical analysis and visualization respectively. The hypergraphs were initially modeled using NetworkX and XGI but later it seemed more convenient to use self-defined functions. The code snippets have been attached to the GitHub repository of the author and can be accessed via the link provided in the appendix. User-generated data was used to model each of these instances and it is proposed that real-world data can be used to construct or study the same. The following subsections will give an overview and analysis in the order:

1. Maximum Power Gain

2. Optimal probability distribution

3. Phase Transition and Time Evolution

3.1.1 Maximum Power Gain

As discussed in the earlier section, the goal is to find the maximum power gain across the various energy input values for the unicyclic network.

Maximum power gain is an important concept in the context of optimizing power output in a coupled system, especially when the system operates in fluctuating environmental conditions. The goal is to find the optimal coupling strengths and configurations that maximize the power output of the system.

Here's how the maximum power gain is different for static vs. fluctuating environments in both the Linear Response regime and the Far from Equilibrium regime:

Static Environment

In a static environment, the parameters and rate matrices governing the system's dynamics remain constant over time. The optimization problem is to find the optimal coupling strengths that maximize the power output under these fixed conditions. The maximum power gain corresponds to the highest achievable power output in the system, and it is obtained by solving the optimization problem using the given rate matrices and hypergraph representations.

1. **Linear Response Regime**: In this regime, the system's dynamics are relatively close to equilibrium. The maximum power gain analysis focuses on finding the optimal coupling strengths that lead to the most efficient power output while staying close to the equilibrium state. The plots generated in this case would show how the power output changes with varying coupling strengths and demonstrate the optimal point for power gain.
2. **Far from Equilibrium Regime**: In this regime, the system's dynamics are significantly perturbed from equilibrium due to stronger interactions or fluctuations. The maximum power gain analysis in the Far from Equilibrium regime seeks to find the optimal coupling strengths that lead to the highest power output even under these highly perturbed conditions. The plots generated in this case would show how the power output changes with coupling strengths, indicating the most favorable range for power gain.

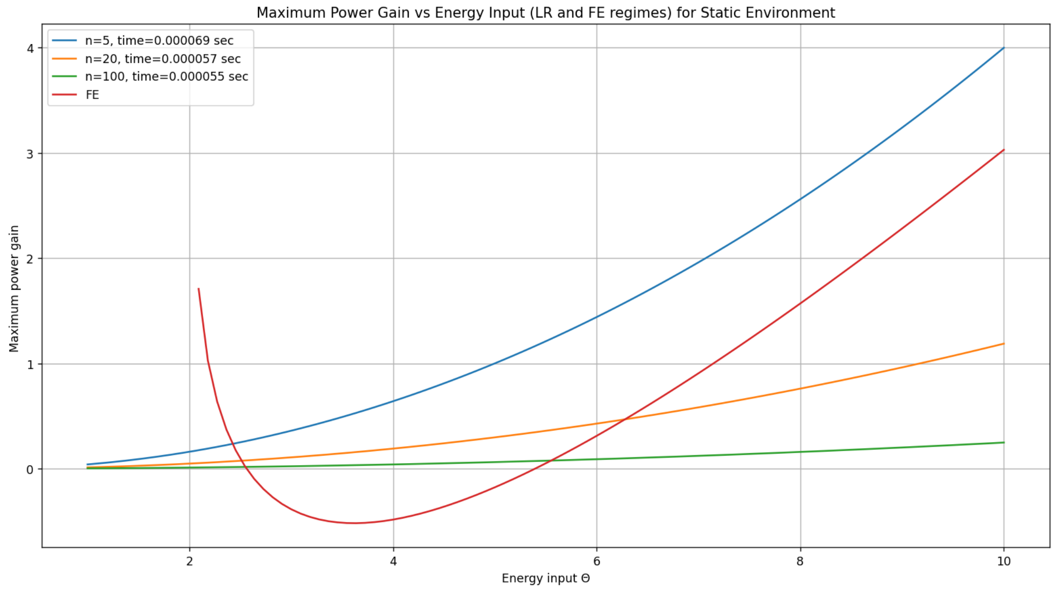

Fig. 3.1 The system under study X and the fluctuating environment Y

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Fluctuating Environment:

In a fluctuating environment, the rate matrices and system parameters change over time due to varying external conditions. The optimization problem now involves finding time-varying coupling strengths that maximize the average power output over the fluctuations in the environment. This leads to a time-dependent maximum power gain analysis.

1. **Linear Response Regime**: In this case, the system operates close to equilibrium even with environmental fluctuations. The time-dependent maximum power gain analysis seeks to find optimal coupling strengths that adjust to the varying environmental conditions to maintain high power output near equilibrium. The plots generated would show how the time-varying coupling strengths respond to environmental changes and how the power output fluctuates accordingly.

2. **Far from Equilibrium Regime**: In this regime, the system experiences strong environmental fluctuations, and the time-dependent maximum power gain analysis focuses on finding coupling strengths that adapt to these fluctuations, leading to the highest power output even far from equilibrium. The plots generated in this

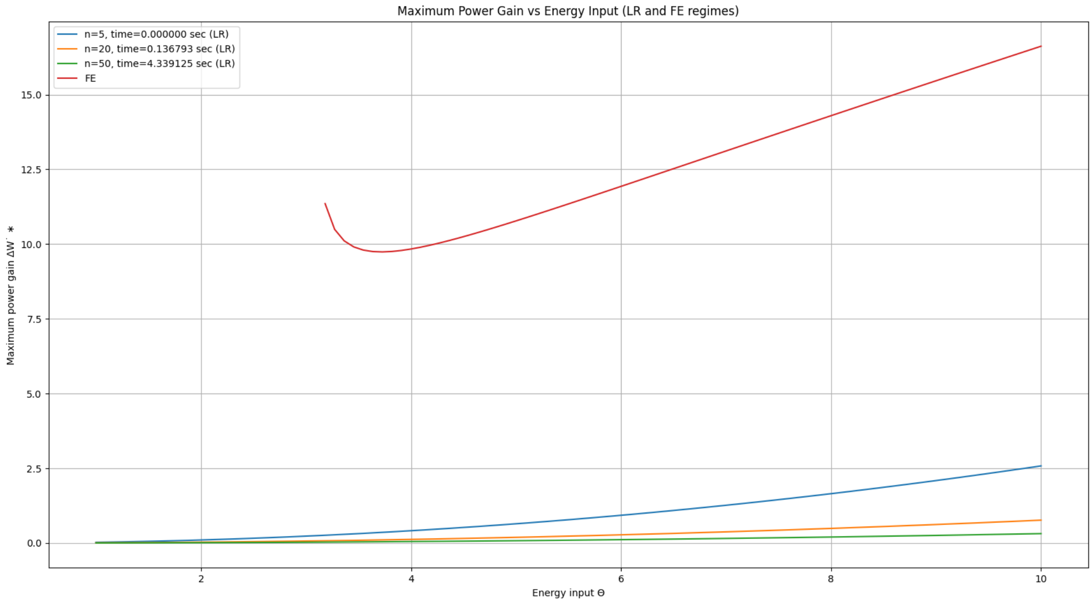

Fig. 3.2 The system under study X and the fluctuating environment Y

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case would show the dynamic changes in coupling strengths and power output as the system responds to environmental variations.

We can observe from the graphs that as the number of states increases, especially for higher n values, the power gain is relatively lower as compared to the $n = 5$ state. This is due to the change in transition rates concerning higher interactions between the two systems. Which also accounts for the increase in time complexity and computational load. The graph allows us to identify the energy input range where the maximum power gain is achieved. The peak or peaks in the graph indicate the energy input values that yield the highest power gain. This region represents the optimal operational regime where the system can extract the most work from the fluctuating environment.

In summary, the maximum power gain analysis is essential in both static and fluctuating environments to optimize the power output of a coupled system. The approach differs based on whether the system operates in the Linear Response regime (closer to equilibrium) or the Far from Equilibrium regime (strongly perturbed from equilibrium), as well as whether the environment is static or fluctuating. The plots generated in each case provide insights into the optimal coupling strengths and their time dependence, illustrating the system's power output behavior under varying conditions.

3.1.2 Optimal Probability Distribution

Optimal probabilities obtained from the Far from Equilibrium (FE) regime, a high frequency in the histogram correspond to states that are more frequently visited or more likely to occur as the system operates under non-equilibrium conditions.

When the system is operating far from thermal equilibrium, it has the potential to optimize its behavior and adapt to the given energy input (θ) to achieve maximum power gain. The optimization involves adjusting the probabilities of being in different states (e.g., states 1, 2, and n) to utilize the available energy input for useful work efficiently.

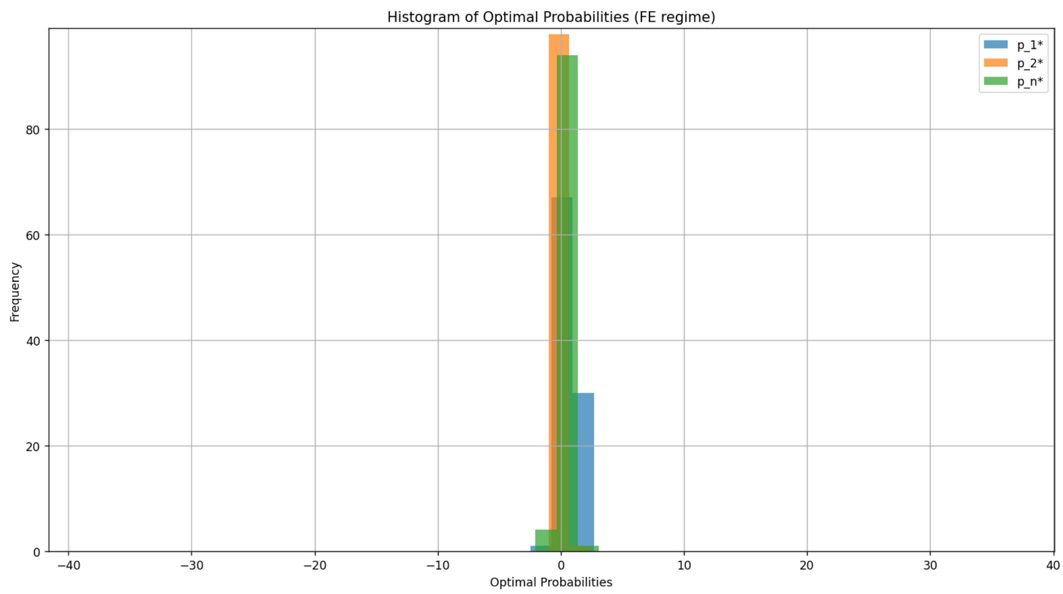

Fig. 3.3 Optimal probabilities for the Far From Equilibrium Regime

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A high frequency of a specific state's optimal probability in the histogram indicates that the system is more likely to be found in that state under the given conditions. In other words, that state is favored by the system as it enables the maximum power gain under non-equilibrium constraints. States with higher optimal probabilities have a more significant influence on the system's behavior and contribute more to the overall power

extraction and work performance.
Optimal probabilities obtained from the linear response regime, the histogram 3.4 has a more evenly distributed frequency of states. This is because the system is close to thermal equilibrium, and the probabilities of being in different states are not heavily biased towards specific states as observed in the far-from-equilibrium (FE) regime.

The system in the linear response regime may not have the same potential to optimize its behavior as in the FE regime. Instead, it tends to maintain a balanced distribution of states, making small adjustments around its equilibrium state to respond to external stimuli or changes in energy input.

As a result, the histogram of probabilities is expected to exhibit a smoother distribution, reflecting the system's tendency to explore different states in a more balanced manner.

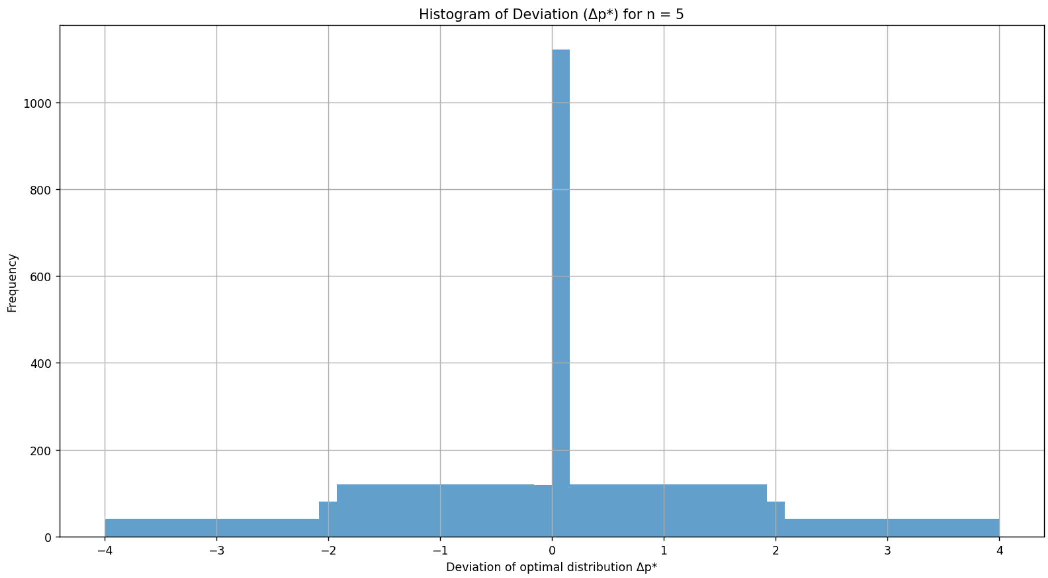

Fig. 3.4 Optimal probabilities for the Linear Response Regime

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On the other hand, states with low frequencies in the histogram have less influence on the system's behavior and are less likely to be occupied in the optimized state distribution.

3.1.3 Phase Transition and Time Evolution and Synchronization

One of the most important parts of the analysis was to observe the phase transition and synchronization of the coupled system hypergraphs. We modeled the system X and Y separately on two different hypergraphs and utilized transition rates to model their dynamics as discussed in the previous section.

The synchronization behavior of the subsystems depends on the specific parameters and dynamics of the system. In general, systems with coupled dynamics have the potential to synchronize under certain conditions. However, whether the subsystems will synchronize or not depends on factors such as the coupling strength, the initial conditions, and the specific dynamics of the subsystems.

We ran multiple simulations varying the initial phase difference and synchronization threshold which resulted in different behavior of the initial states of the system, although they eventually converged as expected. The maximum power gain equation being used 1 to govern the dynamics of the coupled system for the linear response regime is run across the various hyperedges of the system.

We can further vary the interactions on the hyperedges by running diffusion across one set of the hyperedges in parallel to the maximum power gain function using the same transition rates.

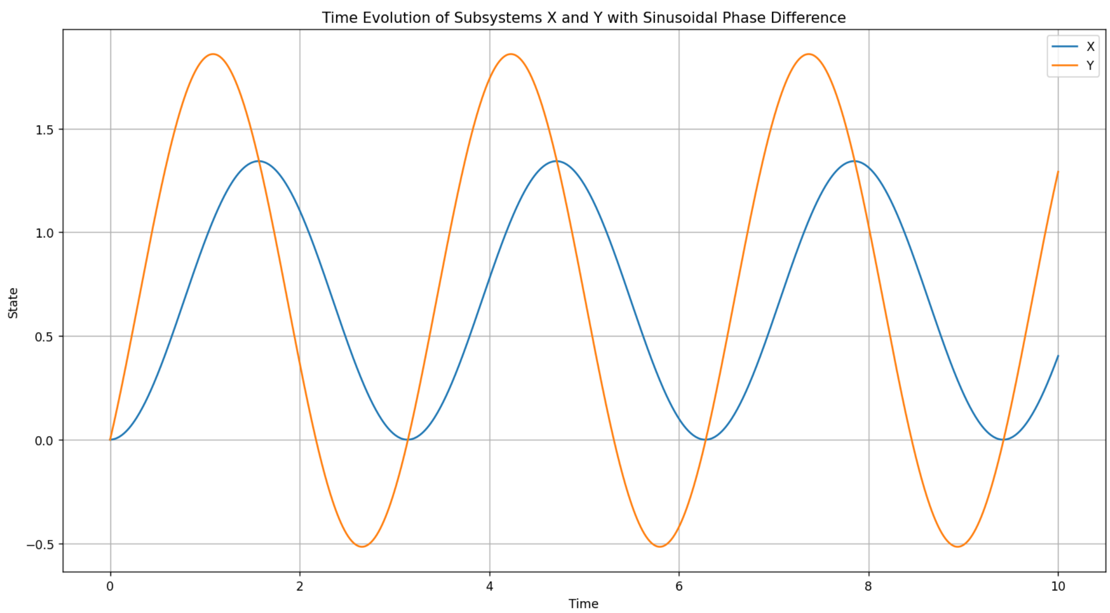

Fig. 3.5 Time Evolution with a sinusoidal Phase Difference between the environment and the system

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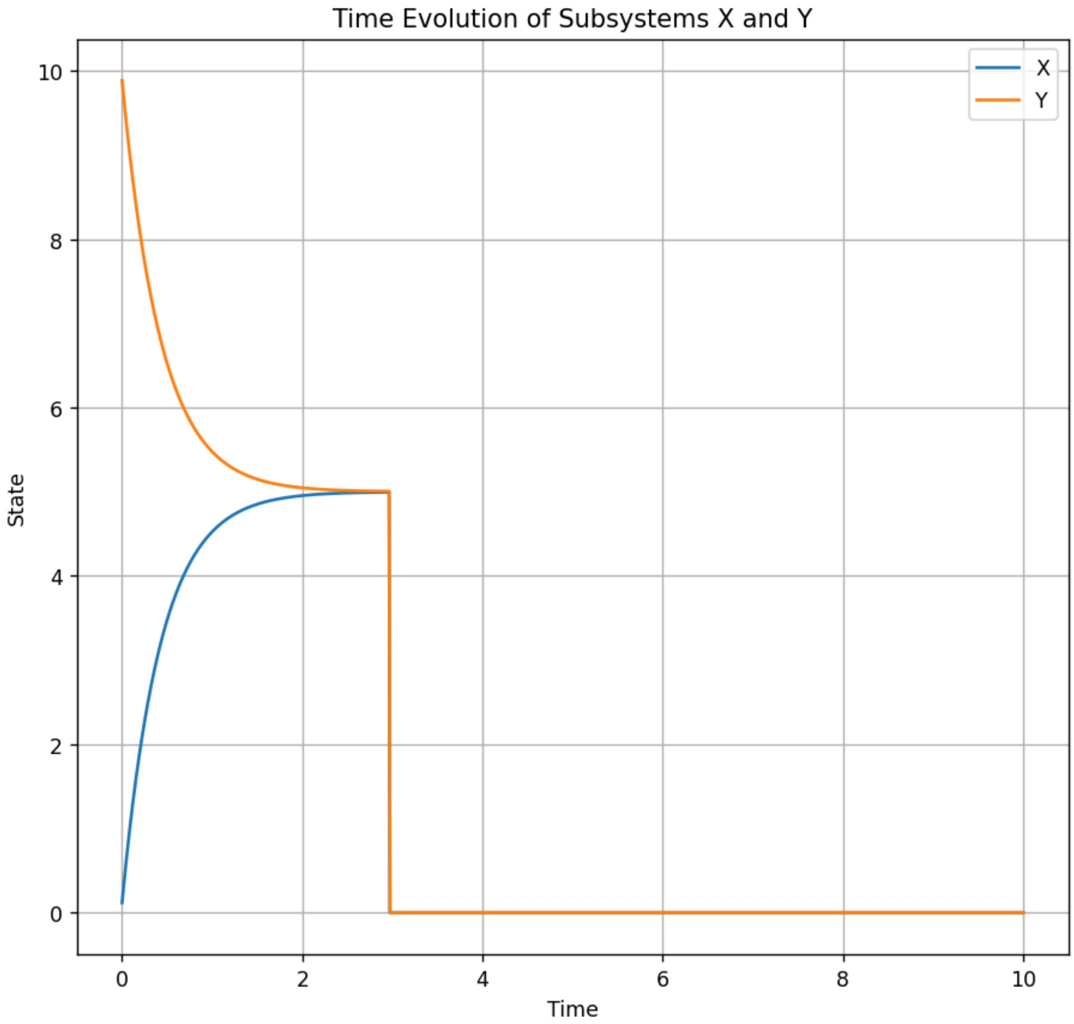

Fig. 3.6 Synchronization of the coupled systems after a regular perturbation is applied
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The process of achieving synchronization is an important phenomenon that depends on the following factors:

1. **Hypergraph Dynamics**: The hypergraph represents the connectivity structure between vertices (or nodes) in each subsystem. The way hyperedges are formed and connected influences the interaction between vertices. In this case, the hypergraphs are constructed with different connectivity patterns based on certain conditions (e.g., some edges have 2 vertices, some have 3, and some have 4). These different connectivity patterns contribute to the dynamics of the subsystems.

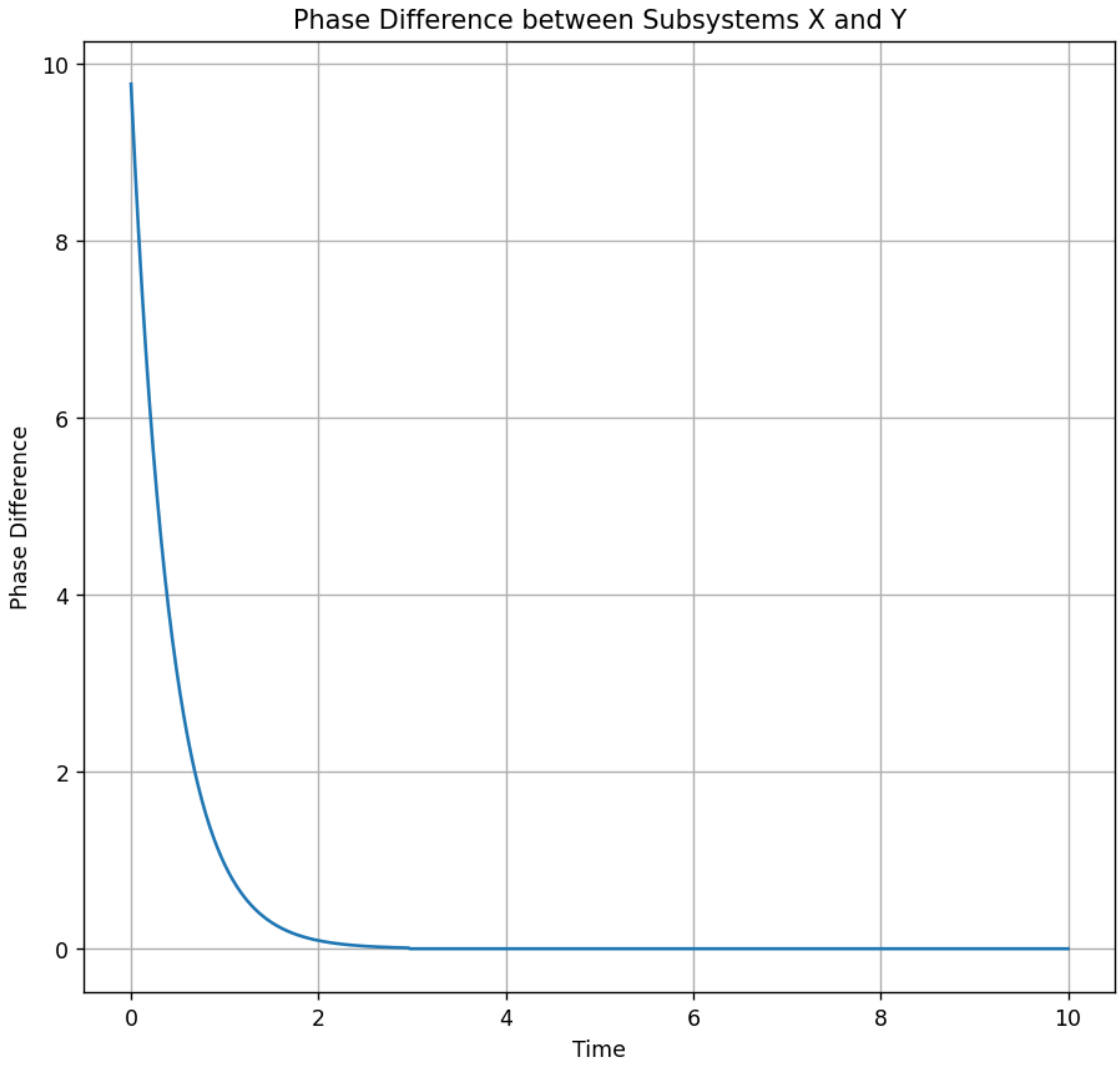

Fig. 3.7 The phase difference showing the synchronization and stability occurrence

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2. **Power Gain Mechanism**: We used the maximum power gain function to calculate the power gain between subsystems X and Y based on the states of their vertices. The power gain function helps to determine how much influence one subsystem has on the other which is a reflection of the transition rates and how the system evolves over time. It adjusts the state of each vertex in response to the influence of its adjacent vertices and other subsystem's vertices. The function aims to bring the states of the subsystems closer to each other, promoting synchronization.

3. **Diffusion Process**: We used a DiffusionHypergraph class to implement the diffusion process, which further promotes synchronization. The diffuse method updates the states of vertices based on the diffusion coefficient and the states of their adjacent vertices. Diffusion allows the states of interconnected vertices to influence each other, leading to synchronization in their behavior.

The figures 3.8 and 3.9 show how the individual systems evolve over time and attain synchronization despite initial phase difference. We tried to model a typical diffusion process on set of the hyperedges along with using the transition rates and calculating maximum power gain as the primary equation for the evolution of the systems over time. This just shows the modularity of using hypergraphs and the insight into the complex dynamics of the system.

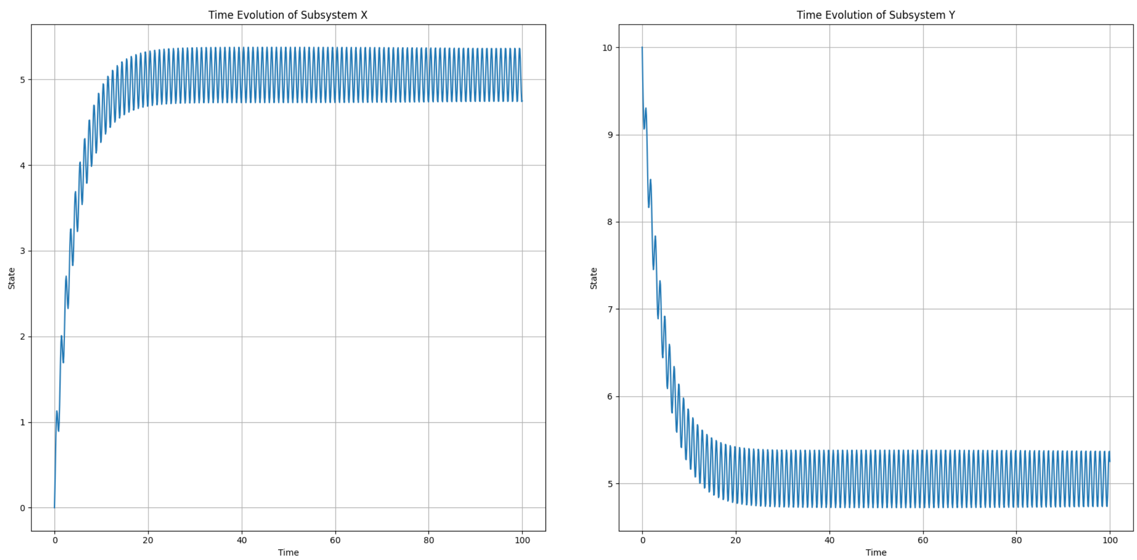

Fig. 3.8 These graphs show how the states of subsystems X and Y change over time. The x-axis represents time, and the y-axis represents the state values of the subsystems
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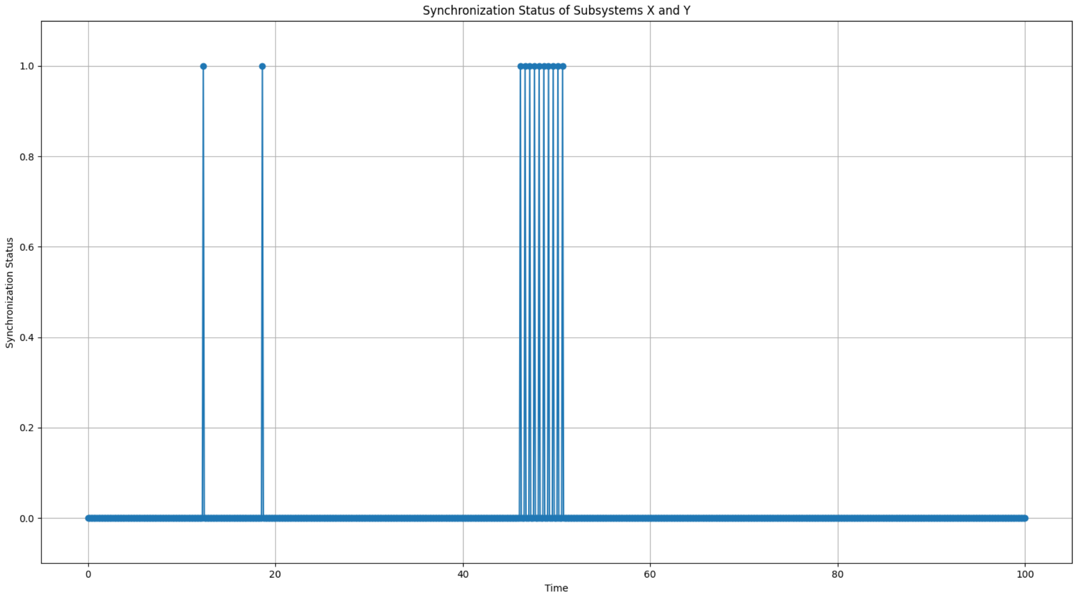

Fig. 3.9 This graph shows the synchronization status between subsystems X and Y over time. The x-axis represents time, and the y-axis represents the synchronization status (1.0 for synchronized and 0.0 for not synchronized)
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CHAPTER 4

Conclusion

4.1 Final Remarks

In this project, the main focus was to explore the dynamics of a coupled system on a hypergraph, and the objective was successfully achieved by developing a functional hypergraph model. The hypergraph model displayed synchronization over time, where the states of the subsystems X and Y became aligned with each other. Additionally, the model exhibited phase transitions, indicating shifts between different states of the system.
To capture the dynamics of the system, a particular energy harvesting model was employed, which allowed for the transition between various states of the system. The energy harvesting model played a crucial role in adjusting the states of vertices in response to their interactions with adjacent vertices, leading to a convergence towards synchronization.

By coupling the subsystems through hypergraph connectivity, the interactions between the system and its environment were effectively observed. The hypergraph structure determined the relationships between vertices, and the diffusion process further influenced the spreading of states across the interconnected vertices.
The project also highlighted the significance of various parameters in shaping the system's behavior. The choice of hypergraph structures, power gain function, diffusion coefficient, and other parameters could yield different synchronization patterns and dynamics. The ability to tune these parameters offers versatility in understanding and controlling the system's evolution, making it applicable in various real-world scenarios.

Two different processes were implemented on different sets of hyperedges to explore how the system evolved over time and achieved synchronicity. This approach provided insights into the sensitivity of the system to different configurations and allowed for a better understanding of the evolution of these complex systems.

The findings of the project enable the implementation of different control strategies tailored to specific requirements. The flexibility in changing parameters should allow us to adapt the model to diverse applications, such as information processing, communication networks, and complex system simulations.

In conclusion, this project successfully demonstrated the dynamics of a coupled system on a hypergraph, achieving synchronization and exhibiting phase transitions. By using a well-designed energy harvesting model and investigating various hypergraph configurations, the study contributed to a deeper understanding of the behavior of such complex systems. The insights gained can be leveraged to design control strategies and optimize system performance in a wide range of applications, bringing valuable contributions to the field of complex systems and network dynamics.

References

[1] Berge, C. (1984). *Hypergraphs: combinatorics of finite sets*, volume 45. Elsevier.

[2] Bertsekas, D. (1998). *Network optimization: continuous and discrete models*, volume 8. Athena Scientific.

[3] Cimini, G., Squartini, T., Saracco, F., Garlaschelli, D., Gabrielli, A., and Caldarelli, G. (2019). The statistical physics of real-world networks. *Nature Reviews Physics*, 1(1):58–71.

[4] de Arruda, G. F., Petri, G., Rodrigues, F. A., and Moreno, Y. (2020). Impact of the distribution of recovery rates on disease spreading in complex networks. *Phys. Rev. Res.*, 2:013046.

[5] Feng, S., Heath, E., Jefferson, B., Joslyn, C., Kvinge, H., Mitchell, H. D., Praggastis, B., Eisfeld, A. J., Sims, A. C., Thackray, L. B., et al. (2021). Hypergraph models of biological networks to identify genes critical to pathogenic viral response. *BMC bioinformatics*, 22(1):1–21.

[6] Ferraz de Arruda, G., Tizzani, M., and Moreno, Y. (2021a). Phase transitions and stability of dynamical processes on hypergraphs. *Communications Physics*, 4(1):24.

[7] Ferraz de Arruda, G., Tizzani, M., and Moreno, Y. (2021b). Phase transitions and stability of dynamical processes on hypergraphs. *Communications Physics*, 4:24.

[8] Giusti, C., Ghrist, R., and Bassett, D. S. (2016). Two's company, three (or more) is a simplex: Algebraic-topological tools for understanding higher-order structure in neural data.

[9] Langton, C. G. (1990). Computation at the edge of chaos: Phase transitions and emergent computation. *Physica D: Nonlinear Phenomena*, 42(1):12–37.

[10] Petri, G., Expert, P., Turkheimer, F., Carhart-Harris, R., Nutt, D., Hellyer, P., and Vaccarino, F. (2014). Homological scaffolds of brain functional networks. *Journal of The Royal Society Interface*, 11:20140873.

[11] Piñero, J., Solé, R., and Kolchinsky, A. (2023). Universal bounds and thermodynamic tradeoffs in nonequilibrium energy harvesting.

[12] Qi, G., Li, J., Xu, X., Chen, G., and Yang, K. (2022). An attack–defense game model in infrastructure networks under link hiding. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 32(11). 113109.

[13] Schimansky-Geier, L. (1986). Kuramoto, y., chemical oscillations, waves, and turbulence. berlin-heidelberg-new york-tokyo, springer-verlag 1984. viii, 156 s., 41 abb., dm 79,—. us $ 28.80. isbn 3-540-13322-4 (springer series in synergetics 19). *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik*, 66(7):296–296.

[14] Solé, R. V., Manrubia Cuevas, S., Luque, B., Delgado, J., and Bascompte, J. (1996). Phase transitions and complex systems: Simple, nonlinear models capture complex systems at the edge of chaos.

[15] Vold, R. L., Waugh, J. S., Klein, M. P., and Phelps, D. E. (2003). Measurement of Spin Relaxation in Complex Systems. *The Journal of Chemical Physics*, 48(8):3831–3832.

[16] W. G. Brown, P. Erdős, V. T. S. (1973). Some extremal problems on r-graphs. *New directions in the theory of graphs (Proc. Third Ann Arbor Conf., Univ. Michigan, Ann Arbor, Mich., 1971)*, 25:53–63.

[17] Zhang, H., Song, L., Han, Z., and Zhang, Y. (2018). *Hypergraph theory in wireless communication networks*. Springer.

[18] Zhang, Y., Lucas, M., and Battiston, F. (2023). Higher-order interactions shape collective dynamics differently in hypergraphs and simplicial complexes. *Nature Communications*, 14.