Deriving and Maximizing Common GLMs

PS 7552

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Gaussian Distribution

You observe a sample of n iid realizations $y_i \in \{y_1 \ldots y_n\}$ of a variable $Y \sim \mathcal{N}\left(\mu, \sigma^2\right)$, i.e.

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$$f\left(y_i | \mu, \sigma^2\right) = \left(2\pi\sigma^2\right)^{-1/2} \exp\left(-\frac{1}{2}\frac{\left(y_i - \mu\right)^2}{\sigma^2}\right).$$

Gaussian

What is the likelihood of having observed this sample, given that the observations were independently and identically distributed?

(1)

(2)

(3)

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$$\mathcal{L}\left(\mu, \sigma^2 | y_i\right) = \prod_{i=1}^{n} f\left(y_i | \mu, \sigma\right) \tag{1}$$

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$$
\begin{aligned}
\mathcal{L}\left(\mu, \sigma^2 | y_i\right) &= \prod_{i=1}^{n} f\left(y_i | \mu, \sigma\right) && (1) \\
&= \prod_{i=1}^{n} \left(2\pi\sigma^2\right)^{-1/2} \exp\left(-\frac{1}{2}\frac{\left(y_i - \mu\right)^2}{\sigma^2}\right) && (2) \\
&= \left(2\pi\sigma^2\right)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(y_i - \mu\right)^2\right) && (3)
\end{aligned}
$$

Gaussian

What is the log likelihood?

(4)

(5)

(6)

Gaussian

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$$\ln \mathcal{L}\left(\mu, \sigma^2 | y_i\right) = \ln\left(\left(2\pi\sigma^2\right)^{-n/2}\right) + \ln\left(\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \mu)^2\right)\right) \tag{4}$$

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$$= -\frac{n}{2}\ln\left(2\pi\sigma^2\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2 \tag{5}$$

$$= -\frac{n}{2}\ln\left(2\pi\right) - \frac{n}{2}\ln\left(\sigma^2\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2 \tag{6}$$

Gaussian

What is the score function for μ?

(7)

(8)

(9)

(10)

Gaussian

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$$S(\mu) = -\frac{\partial}{\partial\mu}\left(\frac{n}{2}\ln(2\pi)\right) - \frac{\partial}{\partial\mu}\left(\frac{n}{2}\ln(\sigma^2)\right) - \frac{\partial}{\partial\mu}\left(\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2\right) \tag{7}$$

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What is $\hat{\mu}_{MLE}$?

$$\tag{11}$$

$$\tag{12}$$

$$\tag{13}$$

$$\tag{14}$$

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(18)

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$$= -\frac{n}{2\sigma^2} - \left[\frac{1}{2}\sum_{i=1}^{n}\left(y_i-\mu\right)^2\right]\frac{\partial}{\partial\sigma^2}\left(\frac{1}{\sigma^2}\right) \tag{16}$$

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Find $\hat{\sigma}_{MLE}$.

$$\tag{19}$$

$$\tag{20}$$

$$\tag{21}$$

$$\tag{22}$$

$$\tag{23}$$

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Gaussian

How do we know $\hat{\sigma}_{MLE}$ is a maximum?

(24)

(25)

(26)

Gaussian

How do we know $\hat{\sigma}_{MLE}$ is a maximum?

$$\frac{\partial^2 \ln \mathcal{L}}{\partial \left(\sigma^2\right)^2} = \frac{\partial^2}{\partial \sigma^2} \left(\frac{1}{2\sigma^2} \left(\frac{1}{\sigma^2} \sum_{i=1}^{n} (y_i - \mu)^2 - n \right) \right) \tag{24}$$

(25)

(26)

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$$= \frac{\partial^2}{\partial \sigma^2} \left(\frac{1}{2\left(\sigma^2\right)^2} \sum_{i=1}^{n} (y_i - \mu)^2 \right) - \frac{\partial^2}{\partial \sigma^2} \left(\frac{n}{2\sigma^2} \right) \tag{25}$$

$$\tag{26}$$

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How do we know $\hat{\sigma}_{MLE}$ is a maximum?

$$\frac{\partial^2 \ln \mathcal{L}}{\partial \left(\sigma^2\right)^2} = \frac{\partial^2}{\partial \sigma^2}\left(\frac{1}{2\sigma^2}\left(\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(y_i - \mu\right)^2 - n\right)\right) \tag{24}$$

$$= \frac{\partial^2}{\partial \sigma^2}\left(\frac{1}{2\left(\sigma^2\right)^2}\sum_{i=1}^{n}\left(y_i - \mu\right)^2\right) - \frac{\partial^2}{\partial \sigma^2}\left(\frac{n}{2\sigma^2}\right) \tag{25}$$

$$= -\frac{1}{\left(\sigma^2\right)^3}\sum_{i=1}^{n}\left(y_i - \mu\right)^2 + \frac{n}{2\left(\sigma^2\right)^2} \tag{26}$$

Gaussian

$\hat{\sigma}^2$ is a maximum if equation 26, evaluated at $\hat{\sigma}^2_{MLE}$, is less than zero:

(27)

(28)

(29)

(30)

(31)

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(29)

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$$\frac{1}{2}\frac{1}{n} \sum_{i=1}^{n} (y_i - \mu)^2 = \frac{1}{n} \sum_{i=1}^{n} (y_i - \mu)^2 \tag{30}$$

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Bernoulli

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$$f(x_i \mid \pi) = \pi^{x_i} (1 - \pi)^{1 - x_i}.$$

Bernoulli

What is the likelihood?

(32)

(33)

(34)

(35)

Bernoulli

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$$\mathcal{L}(\pi|x_i) = \prod_{i=1}^{n} f(x_i\ |\pi) \tag{32}$$

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What is the log-likelihood?

$$\tag{36}$$

$$\tag{37}$$

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$$\ln \mathcal{L}\left(\pi | x_i\right) = \ln\left(\pi^{\sum_{i=1}^{n} x_i}\left(1-\pi\right)^{n-\sum_{i=1}^{n} x_i}\right) \tag{36}$$

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&= \sum_{i=1}^{n} x_i \ln\pi + \left(n - \sum_{i=1}^{n} x_i\right)\ln\left(1-\pi\right) && (37)
\end{aligned}
$$

What is the score function?

$$ \tag{38} $$

$$ \tag{39} $$

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$$\frac{\partial \ln \mathcal{L}}{\partial \pi} = \frac{\partial}{\partial \pi}\left(\sum_{i=1}^{n} x_i \ln \pi\right) + \frac{\partial}{\partial \pi}\left(\left(n - \sum_{i=1}^{n} x_i\right) \ln(1-\pi)\right) \quad (38)$$

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$$= \frac{\sum_{i=1}^{n} x_i}{\pi} - \frac{n - \sum_{i=1}^{n} x_i}{1-\pi} \quad (39)$$

Bernoulli

What is the $\hat{\pi}_{MLE}$?

$$(40)$$

$$(41)$$

$$(42)$$

$$(43)$$

$$(44)$$

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$$\frac{\sum_{i=1}^{n} x_i}{\pi} - \frac{n - \sum_{i=1}^{n} x_i}{1-\pi} = 0 \tag{40}$$

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(43)

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$$\hat{\pi}_{MLE} = \frac{1}{n} \sum_{i=1}^{n} x_i \tag{44}$$

Bernoulli

Is this estimate a maximum?

(45)

(46)

(47)

(48)

(49)

Bernoulli

Is this estimate a maximum?

$$\frac{\partial^2 \ln \mathcal{L}}{\partial \pi^2} = \frac{\partial}{\partial \pi}\left(\frac{\sum_{i=1}^{n} x_i}{\pi}\right) - \frac{\partial}{\partial \pi}\left(\frac{n - \sum_{i=1}^{n} x_i}{1-\pi}\right) \tag{45}$$

(46)

(47)

(48)

(49)

Bernoulli

Is this estimate a maximum?

$$\frac{\partial^2 \ln \mathcal{L}}{\partial \pi^2} = \frac{\partial}{\partial \pi}\left(\frac{\sum_{i=1}^{n} x_i}{\pi}\right) - \frac{\partial}{\partial \pi}\left(\frac{n - \sum_{i=1}^{n} x_i}{1-\pi}\right) \tag{45}$$

$$= -\frac{\sum_{i=1}^{n} x_i}{\pi^2} - \left(n - \sum_{i=1}^{n} x_i\right)\frac{\partial}{\partial \pi}\left(\frac{1}{1-\pi}\right) \tag{46}$$

(47)

(48)

(49)

Bernoulli

Is this estimate a maximum?

$$\frac{\partial^2 \ln \mathcal{L}}{\partial \pi^2} = \frac{\partial}{\partial \pi}\left(\frac{\sum_{i=1}^{n} x_i}{\pi}\right) - \frac{\partial}{\partial \pi}\left(\frac{n - \sum_{i=1}^{n} x_i}{1-\pi}\right) \tag{45}$$

$$= -\frac{\sum_{i=1}^{n} x_i}{\pi^2} - \left(n - \sum_{i=1}^{n} x_i\right)\frac{\partial}{\partial \pi}\left(\frac{1}{1-\pi}\right) \tag{46}$$

$$= -\frac{\sum_{i=1}^{n} x_i}{\pi^2} - \left(n - \sum_{i=1}^{n} x_i\right)\left(\frac{-\frac{\partial}{\partial \pi}(1-\pi)}{(1-\pi)^2}\right) \tag{47}$$

(48)

(49)

Bernoulli

Is this estimate a maximum?

$$\frac{\partial^2 \ln \mathcal{L}}{\partial \pi^2} = \frac{\partial}{\partial \pi}\left(\frac{\sum_{i=1}^n x_i}{\pi}\right) - \frac{\partial}{\partial \pi}\left(\frac{n - \sum_{i=1}^n x_i}{1-\pi}\right) \tag{45}$$

$$= -\frac{\sum_{i=1}^n x_i}{\pi^2} - \left(n - \sum_{i=1}^n x_i\right)\frac{\partial}{\partial \pi}\left(\frac{1}{1-\pi}\right) \tag{46}$$

$$= -\frac{\sum_{i=1}^n x_i}{\pi^2} - \left(n - \sum_{i=1}^n x_i\right)\left(\frac{-\frac{\partial}{\partial \pi}(1-\pi)}{(1-\pi)^2}\right) \tag{47}$$

$$= -\frac{\sum_{i=1}^n x_i}{\pi^2} - \left(n - \sum_{i=1}^n x_i\right)\left(\frac{-(0-1)}{(1-\pi)^2}\right) \tag{48}$$

$$\tag{49}$$

Bernoulli

Is this estimate a maximum?

$$\frac{\partial^2 \ln \mathcal{L}}{\partial \pi^2} = \frac{\partial}{\partial \pi}\left(\frac{\sum_{i=1}^{n} x_i}{\pi}\right) - \frac{\partial}{\partial \pi}\left(\frac{n - \sum_{i=1}^{n} x_i}{1-\pi}\right) \tag{45}$$

$$= -\frac{\sum_{i=1}^{n} x_i}{\pi^2} - \left(n - \sum_{i=1}^{n} x_i\right)\frac{\partial}{\partial \pi}\left(\frac{1}{1-\pi}\right) \tag{46}$$

$$= -\frac{\sum_{i=1}^{n} x_i}{\pi^2} - \left(n - \sum_{i=1}^{n} x_i\right)\left(\frac{-\frac{\partial}{\partial \pi}(1-\pi)}{(1-\pi)^2}\right) \tag{47}$$

$$= -\frac{\sum_{i=1}^{n} x_i}{\pi^2} - \left(n - \sum_{i=1}^{n} x_i\right)\left(\frac{-(0-1)}{(1-\pi)^2}\right) \tag{48}$$

$$= -\frac{\sum_{i=1}^{n} x_i}{\pi^2} - \frac{n - \sum_{i=1}^{n} x_i}{(1-\pi)^2} \tag{49}$$

Bernoulli

Find the observed Fisher information.

(50)

(51)

(52)

(53)

Bernoulli

Find the observed Fisher information.

$$I\left(\hat{\pi}\right) = -\left(-\frac{\sum_{i=1}^{n} x_i}{\pi^2} - \frac{n - \sum_{i=1}^{n} x_i}{\left(1-\pi\right)^2}\right) \tag{50}$$

(51)

(52)

(53)

Bernoulli

Find the observed Fisher information.

$$I(\hat{\pi}) = -\left(-\frac{\sum_{i=1}^{n} x_i}{\pi^2} - \frac{n - \sum_{i=1}^{n} x_i}{(1-\pi)^2}\right) \quad (50)$$

$$= \frac{\sum_{i=1}^{n} x_i}{\pi^2} + \frac{n - \sum_{i=1}^{n} x_i}{1 - 2\pi + \pi^2} \quad (51)$$

(52)

(53)

Find the observed Fisher information.

$$I\left(\hat{\pi}\right) = -\left(-\frac{\sum_{i=1}^{n} x_i}{\pi^2} - \frac{n - \sum_{i=1}^{n} x_i}{\left(1-\pi\right)^2}\right) \tag{50}$$

$$= \frac{\sum_{i=1}^{n} x_i}{\pi^2} + \frac{n - \sum_{i=1}^{n} x_i}{1 - 2\pi + \pi^2} \tag{51}$$

$$= \frac{\sum_{i=1}^{n} x_i - 2\pi \sum_{i=1}^{n} x_i + \pi^2 \sum_{i=1}^{n} x_i + \pi^2 n - \pi^2 \sum_{i=1}^{n} x_i}{\pi^2 \left(1-\pi\right)^2} \tag{52}$$

$$\tag{53}$$

Bernoulli

Find the observed Fisher information.

$$I\left(\hat{\pi}\right) = -\left(-\frac{\sum_{i=1}^{n} x_i}{\pi^2} - \frac{n - \sum_{i=1}^{n} x_i}{\left(1-\pi\right)^2}\right) \tag{50}$$

$$= \frac{\sum_{i=1}^{n} x_i}{\pi^2} + \frac{n - \sum_{i=1}^{n} x_i}{1 - 2\pi + \pi^2} \tag{51}$$

$$= \frac{\sum_{i=1}^{n} x_i - 2\pi \sum_{i=1}^{n} x_i + \pi^2 \sum_{i=1}^{n} x_i + \pi^2 n - \pi^2 \sum_{i=1}^{n} x_i}{\pi^2 \left(1-\pi\right)^2} \tag{52}$$

$$= \frac{\sum_{i=1}^{n} x_i - 2\pi \sum_{i=1}^{n} x_i + \pi^2 n}{\pi^2 \left(1-\pi\right)^2} \tag{53}$$

Observed Fisher information, cont'd.

(54)

(55)

(56)

(57)

Observed Fisher information, cont'd.

$$I\left(\hat{\pi}\right) = \frac{\left(\sum_{i=1}^{n} x_i - 2\pi \sum_{i=1}^{n} x_i + \pi^2 n\right) \frac{1}{n}}{\left(\pi^2 \left(1-\pi\right)^2\right) \frac{1}{n}} \tag{54}$$

(55)

(56)

(57)

Bernoulli

Observed Fisher information, cont'd.

$$I\left(\hat{\pi}\right) = \frac{\left(\sum_{i=1}^{n} x_i - 2\pi \sum_{i=1}^{n} x_i + \pi^2 n\right)\frac{1}{n}}{\left(\pi^2\left(1-\pi\right)^2\right)\frac{1}{n}} \tag{54}$$

$$= \frac{\pi - 2\pi^2 + \pi^2}{\left(\pi^2\left(1-\pi\right)^2\right)\frac{1}{n}} \tag{55}$$

(56)

(57)

Bernoulli

Observed Fisher information, cont'd.

$$I\left(\hat{\pi}\right) = \frac{\left(\sum_{i=1}^{n} x_i - 2\pi \sum_{i=1}^{n} x_i + \pi^2 n\right) \frac{1}{n}}{\left(\pi^2 \left(1-\pi\right)^2\right) \frac{1}{n}} \tag{54}$$

$$= \frac{\pi - 2\pi^2 + \pi^2}{\left(\pi^2 \left(1-\pi\right)^2\right) \frac{1}{n}} \tag{55}$$

$$= \frac{\pi \left(1-\pi\right)}{\left(\pi^2 \left(1-\pi\right)^2\right) \frac{1}{n}} \tag{56}$$

$$\tag{57}$$

Bernoulli

Observed Fisher information, cont'd.

$$I\left(\hat{\pi}\right) = \frac{\left(\sum_{i=1}^{n} x_i - 2\pi \sum_{i=1}^{n} x_i + \pi^2 n\right) \frac{1}{n}}{\left(\pi^2 \left(1-\pi\right)^2\right) \frac{1}{n}} \tag{54}$$

$$= \frac{\pi - 2\pi^2 + \pi^2}{\left(\pi^2 \left(1-\pi\right)^2\right) \frac{1}{n}} \tag{55}$$

$$= \frac{\pi \left(1-\pi\right)}{\left(\pi^2 \left(1-\pi\right)^2\right) \frac{1}{n}} \tag{56}$$

$$= \frac{n}{\pi \left(1-\pi\right)} \tag{57}$$

Bernoulli

Find the variance and standard error of the MLE for π.

(58)

(59)

(60)

(61)

(62)

Bernoulli

Find the variance and standard error of the MLE for π.

$$V\left(\hat{\pi}_{MLE}\right) = \left[I\left(\hat{\pi}_{MLE}\right)\right]^{-1} \tag{58}$$

(59)

(60)

(61)

(62)

Bernoulli

Find the variance and standard error of the MLE for π.

$$V\left(\hat{\pi}_{MLE}\right) = \left[I\left(\hat{\pi}_{MLE}\right)\right]^{-1} \tag{58}$$

$$= \left[\frac{n}{\pi\left(1-\pi\right)}\right]^{-1} \tag{59}$$

(60)

(61)

(62)

Bernoulli

Find the variance and standard error of the MLE for π.

$$V\left(\hat{\pi}_{MLE}\right) = \left[I\left(\hat{\pi}_{MLE}\right)\right]^{-1} \tag{58}$$

$$= \left[\frac{n}{\pi\left(1-\pi\right)}\right]^{-1} \tag{59}$$

$$= \frac{\pi\left(1-\pi\right)}{n} \tag{60}$$

$$\tag{61}$$

$$\tag{62}$$

Bernoulli

Find the variance and standard error of the MLE for π.

$$
\begin{aligned}
V(\hat{\pi}_{MLE}) &= [I(\hat{\pi}_{MLE})]^{-1} && (58)\\
&= \left[\frac{n}{\pi(1-\pi)}\right]^{-1} && (59)\\
&= \frac{\pi(1-\pi)}{n} && (60)\\
se(\hat{\pi}_{MLE}) &= \sqrt{V(\hat{\pi}_{MLE})} && (61)\\
& && (62)
\end{aligned}
$$

Bernoulli

Find the variance and standard error of the MLE for π.

$$V\left(\hat{\pi}_{MLE}\right) = \left[I\left(\hat{\pi}_{MLE}\right)\right]^{-1} \tag{58}$$

$$= \left[\frac{n}{\pi\left(1-\pi\right)}\right]^{-1} \tag{59}$$

$$= \frac{\pi\left(1-\pi\right)}{n} \tag{60}$$

$$se\left(\hat{\pi}_{MLE}\right) = \sqrt{V\left(\hat{\pi}_{MLE}\right)} \tag{61}$$

$$= \frac{\sqrt{\hat{\pi}\left(1-\hat{\pi}\right)}}{\sqrt{n}} \tag{62}$$