Hierarchical Solutions for the Stochastic Block Model

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The Stochastic Block Model

- Let G be a graph with k communities of m nodes each
- For nodes (i, j) in the same community, let the p be the probability that they share an edge on G
- For nodes (i, j) not in the same community, let the q be the probability that they *don't* share an edge on G
- Edges are drawn independently, and $p > q$
- Goal: recover the original partition

The Semidefinite Program for $k = 2$

Let A be the adjacency matrix of G. Then we need to solve:

$$\max \sum_{i,j} A_{ij} x_i x_j$$

$$\text{s.t. } x_i = \pm 1 \; \forall i$$

$$\sum_j x_j = 0$$

The Semidefinite Program for $k = 2$

Let $B = 2A - (\mathbf{1}\mathbf{1}^T - I)$. Then the original problem is equivalent to

$$\max \sum_{i,j} B_{ij} x_i x_j$$

$$\text{s.t. } x_i = \pm 1 \ \forall i$$

$$\sum_j x_j = 0$$

The Semidefinite Program for $k = 2$

By dropping the constraint $\sum_j x_j = 0$ and with a convex relaxation, we have the following semidefinite program:

$$\max Tr(BX)$$

$$\text{s.t. } X_{ii} = 1 \ \forall i$$

$$X \succeq 0$$

Under the right conditions, with high probability the solution coincides with $X = gg^T$. We obtain g by finding the leading eigenvector of X.

Conditions for Recovery

	Exact	Partial
Formula	$p = \frac{\alpha \log n}{n}$, $q = \frac{\beta \log n}{n}$	$p = \frac{a}{n}$, $q = \frac{b}{n}$
$k = 2$	$\sqrt{\alpha} - \sqrt{\beta} \geq \sqrt{2}$	$(a - b)^2 > 2(a + b)$
$k > 2$	$\sqrt{\alpha} - \sqrt{\beta} \geq \sqrt{k}$	$\frac{(a-b)^2}{k(a+(k-1)b)} > 1$

Hierarchical Approach for $k > 2$

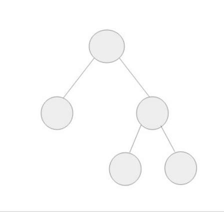


Figure 1: Hierarchy for k = 3

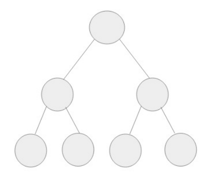
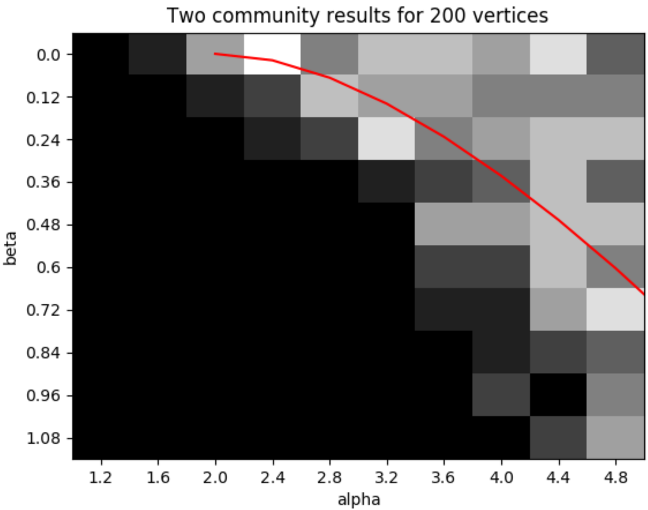
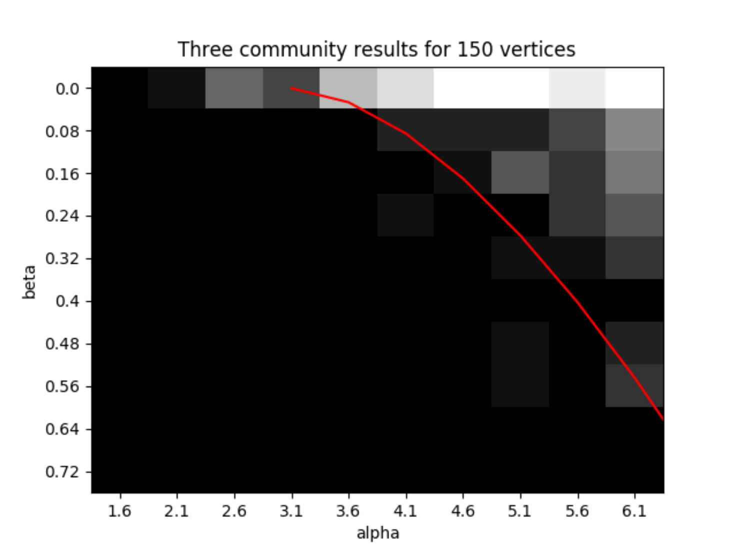


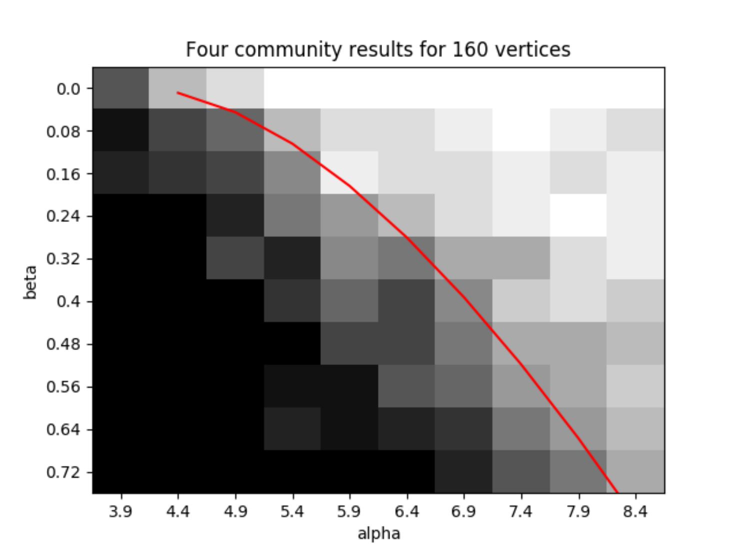
Figure 2: Hierarchy for k = 4

Exact Recovery for $k = 2$

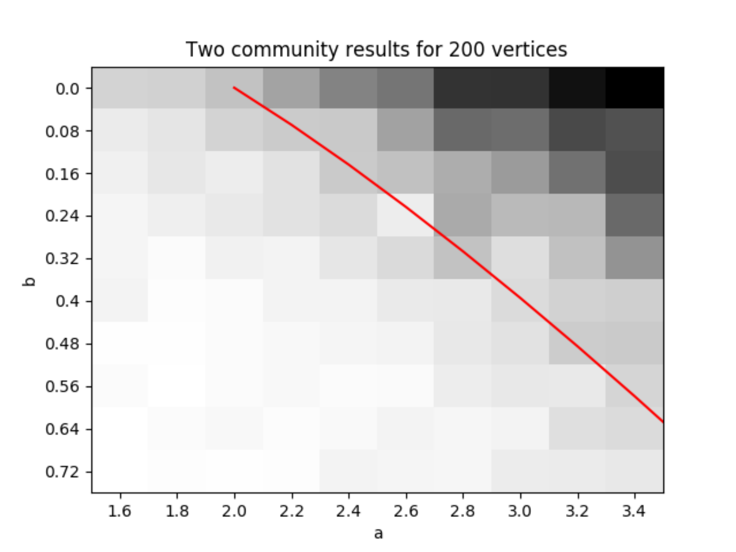


Exact Recovery for $k = 3$

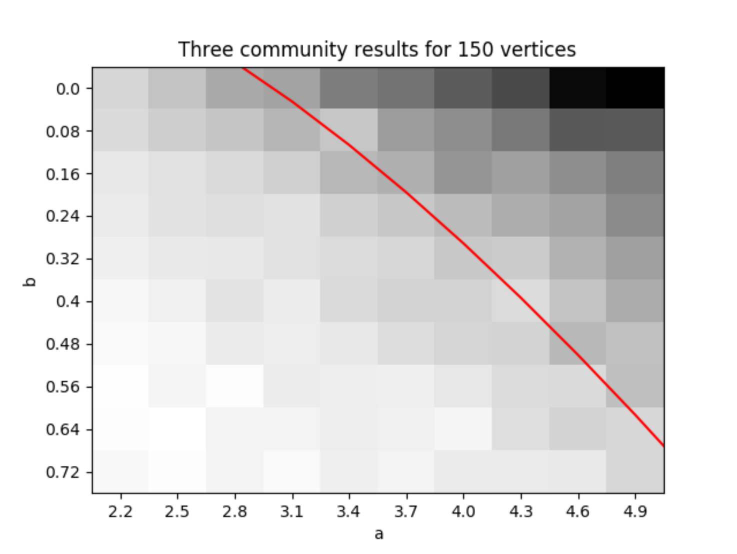



Four community results for 160 vertices
beta
0.0
0.08
0.16
0.24
0.32
0.4
0.48
0.56
0.64
0.72
3.9
4.4
4.9
5.4
5.9
6.4
6.9
7.4
7.9
8.4
alpha

Partial Recovery for $k = 2$



Partial Recovery for $k = 3$



Partial Recovery for $k = 4$