# ARIMA (0,1,1)

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## 1 ARIMA(0,1,1) Model

ARIMA(0,1,1) model with drift is defined by

$$X_t = X_{t-1} + \epsilon_t + \theta\epsilon_{t-1} + \mu \tag{1}$$

where $\epsilon_t$ is WGN, $\epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2)$, $\theta$ and $\mu$ are unknown deterministic parameters. Denote with $Z_t$ the process obtained by apply first order differencing on $X_t$

$$Z_t = X_t - X_{t-1} \tag{2}$$

Thus, we obtain from (1) ,(2)

$$Z_t = \epsilon_t + \theta\epsilon_{t-1} + \mu. \tag{3}$$

We can notice that $Z_t$ is a Gaussian random process with

$$E[Z_t] = \mu, \quad E[(Z_t - \mu)^2] = (1 + \theta^2)\sigma_\epsilon^2. \tag{4}$$

The autocorrelation function of $Z_t$ is given by

$$R(\tau) = \begin{cases} (1+\theta^2)\sigma_\epsilon^2 & \tau = 0 \\ \theta\sigma_\epsilon^2 & \tau = 1 \\ 0 & \tau >= 2 \end{cases} \tag{5}$$

## 2 MLE

Let $z_i, i = 1, ..., T$ be realizations of the random process $Z_t$. The joint PDF of $z_t$ is given by

$$p(z_1, ..., z_T|\theta, \sigma_\epsilon, \mu) = \frac{1}{(2\pi)^{T/2}\det(\Omega)^{1/2}} \exp\left[-0.5(\boldsymbol{z} - \boldsymbol{\mu})^T \Omega^{-1} (\boldsymbol{z} - \boldsymbol{\mu})\right] \tag{6}$$

and thus,

$$-\log \mathcal{L}(\theta|\boldsymbol{z}) = C + 0.5 \log\det(\Omega) + 0.5(\boldsymbol{z} - \boldsymbol{\mu})^T \Omega^{-1} (\boldsymbol{z} - \boldsymbol{\mu}) \tag{7}$$

where

$$\boldsymbol{z} = [z_1, z_2, ..., z_T], \qquad \boldsymbol{\mu} = \mathbf{1}\mu \tag{8}$$

and

$$\Omega = \sigma_\epsilon^2 \begin{pmatrix} 1+\theta^2 & \theta & 0 & \cdots & 0 \\ \theta & 1+\theta^2 & \theta & \cdots & 0 \\ 0 & \theta & 1+\theta^2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \theta \\ 0 & 0 & \cdots & \theta & 1+\theta^2 \end{pmatrix} \tag{9}$$

The ML estimates are given by

$$\hat{\mu} = \arg\min_{\mu} (\boldsymbol{z} - \boldsymbol{\mu})^T \Omega^{-1} (\boldsymbol{z} - \boldsymbol{\mu}) \tag{10}$$

$$\hat{\theta} = \arg\min_{\theta} \log\det(\Omega) + (\boldsymbol{z} - \boldsymbol{\mu})^T \Omega^{-1} (\boldsymbol{z} - \boldsymbol{\mu}) \tag{11}$$

$$\hat{\sigma_\epsilon} = \arg\min_{\sigma_\epsilon} \log\det(\Omega) + (\boldsymbol{z} - \boldsymbol{\mu})^T \Omega^{-1} (\boldsymbol{z} - \boldsymbol{\mu}) \tag{12}$$

## 2.1 ML estimate of the drift constant

The derivative of the expression in (10) is given by

$$\frac{\partial}{\partial \mu} (\boldsymbol{z} - \boldsymbol{\mu})^T \Omega^{-1} (\boldsymbol{z} - \boldsymbol{\mu}) = -2(\boldsymbol{z} - \boldsymbol{\mu})^T \Omega^{-1} 1$$

Thus,

$$\hat{\mu} = \frac{\boldsymbol{z}^T \Omega^{-1} \mathbf{1}}{\mathbf{1}^T \Omega^{-1} \mathbf{1}}$$