Training Set 1

Basic Set Operations and Combinatorics

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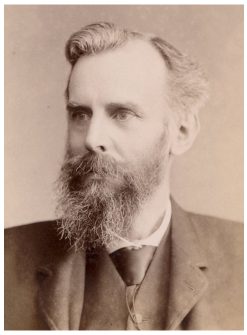
Course in Mathematical Statistics, prof. Scarsini

Set Theory Cheat Sheet:

Logic	Sets
or ($\vee$)	union ($\cup$)
and ($\wedge$)	intersection ($\cap$)
but not	subtraction ($\setminus$)
if ... then ... ($\Rightarrow$)	inclusion ($\subseteq$)
... only if ... ($\Leftarrow$)	inclusion ($\supseteq$)
if and only if ($\Leftrightarrow$)	equality ($=$)
either ... or ... ($\oplus$)	symmetric difference ($\triangle$)

We always know in advance that all elements come from a given big set S, and in this case we sometimes replace $S \setminus A$ with A^c.

Augustus De Morgan (1806 – 1871)



John Venn (1834 – 1923)

Exercise 1. In this exercise all element are taken form the set

$$S = \{\text{apple}, \text{plum}, \text{loquat}, \text{orange}, \text{pear}, \text{watermelon}, \text{grape}, \text{peach}\}.$$

1. For each fruit, determine if it is green and grows on trees.
2. Find the sets $G = \{x \in S : x \text{ is green}\}$, $T = \{x \in S : x \text{ grows on trees}\}$, $G \cap T$, $S \setminus T$, $G \triangle T$.
3. Find G^c, T^c, $G^c \cup T^c$, and verify that $G^c \cup T^c = (G \cap T)^c$.

Exercise 2. Consider the set $S = \{a, b, c, d, e, f, g\}$:

1. How many ordered subsets of S of length 4 can be obtained?
2. How many unordered subsets of S of length 4 can be obtained?

Exercise 3. How many anagrams of the word ZUZZURELLONE are there?

Exercise 4. In a Bridge game, 52 cards are distributed among 4 players (13 cards per player, 13 cards per suit):

1. How many ways are there to distribute the cards?
2. How many ways are there, for a single player, to obtain 7 diamonds card?
3. How many ways of distributing the cards are there such that every player receives an ace?

Exercise 5. A secretary has to deposit 3 letters in 5 mailboxes, all of them distinct.

1. Supposing that each mailbox can contain only one letter, how many ways are there to deposit the letters?
2. Supposing, instead, that each box can contain more than one letter, how many ways of distributing the letters are there such that at least one letter is deposited in the right box?

Exercise 6. Prove that, for any set S, the number of ways to order the elements of S is $|S|!$.

Exercise 7. ($*$ means optional) Given a set S of cardinality $|S| = n$ and any $0 \leq k \leq n$, we denote by $\binom{n}{k}$ (pronounced "n choose k") the number of subsets of S that contain exactly k elements, thus it tells how many ways there are to choose k elements out of a set of size n.

1. Prove that
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

2. Prove that
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

$*$ (3) Prove that
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

This formula is the reason that the coefficients $\binom{n}{k}$ are called the ***binomial coefficients***.

$*$ (4) Prove that
$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Hint: write $2^n = (1+1)^n$ and expand using the previous question.

Exercise 8. You wish to distribute n distinct balls in r boxes such that each box contains $n_1, n_2, ..., n_r$ balls, respectively. Assume $n_1 + n_2 + ... + n_r = n$ (explain why this assumption is necessary). How many ways are there to distribute the balls?

The next exercise is optional:

Exercise 9 (Challenge)**.** Let $S = \{0, 1, \dots, n\}$ for $n \in \mathbb{N}$. Consider the following family of sets:

$$\mathcal{A} = \left\{ A \subseteq S : \frac{1}{|A|} \sum_{x \in A} x \in \mathbb{N} \right\},$$

i.e., the subsets of S with integer average. Prove that $|\mathcal{A}|$ is even.