Data Structures

Binary Tree Formulas

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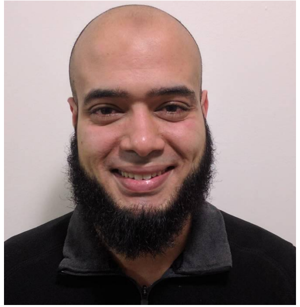
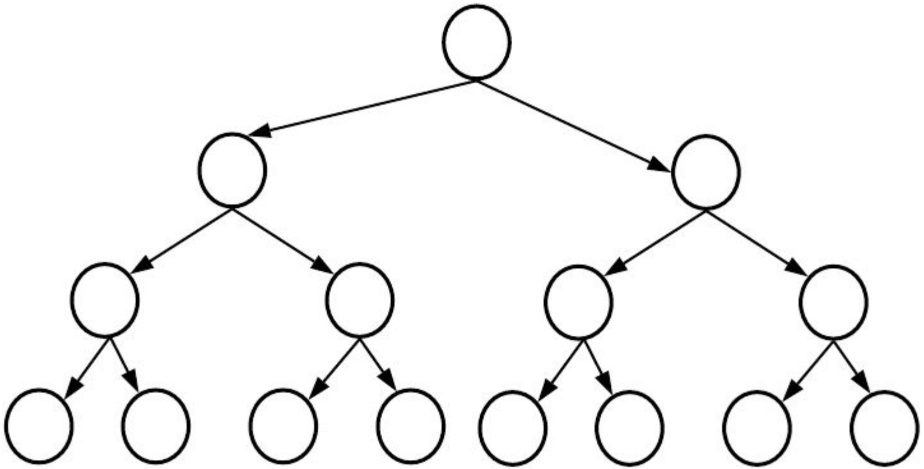


Perfect tree: From height find # nodes!

- Each level (0-based) has 2^{level} nodes (*2 * previous level nodes*).
- For N levels: $2^0 + 2^1 + 2^2 + 2^3 + \ldots.. + 2^{level} = 2^{levels} - 1 = 2^{h+1} - 1$ nodes



Leve 0: 1 node (2^0)

Leve 1: 2 node (2^1)

Leve 2: 2*2 = 4 nodes (2^2)

Leve 3: 4*2 = 8 nodes (2^3)

Perfect tree: From # nodes find height!

- We can derive this mathematically
- First recall **power rule for logarithms**

$$\log_b (M^n) = n \log_b M$$

- Also recall $\mathbf{\log_2{}^2 = 1}$

$$n = 2^{h+1} - 1$$

$$n + 1 = 2^{h+1}$$

$$\lg(n+1) = h + 1$$

$$h = \lg(n+1) - 1$$

Facts

- From the perfect tree, we can derive *upper and lower* bound on normal tree
  - As we made sure every level is complete. E.g. you can't have less levels!
- **In any binary tree, :**
- Each level has max of 2^h nodes
- For L levels, No more than $2^L - 1$ nodes
- For N nodes, the min # of levels is: ceil(log(N+1))
  - 1 Node ⇒ 1, 3 Nodes ⇒ 2, 7 Nodes ⇒ 3, 15 Nodes ⇒ 4   these are for perfect case
- For M leaves, the min # of levels is: ceil(log M) + 1
  - 1 leaves ⇒ 1, 2 leaves ⇒ 2, 4 leaves ⇒ 3, 8 leaves ⇒ 4, 16 leaves ⇒ 5

The logarithm

- Observe how the log is very small value
- This means, we can have a tree of 1 million nodes, but its height can be:
  - ~ 1 million with degenerate tree
  - ~20 only if it is perfect or complete
- In balanced trees (e.g. AVL / red-black), we put constraints that help us have such controlled height than a very deep tree

	Log	Number
$2^{0} = 1$	0	1
$2^{1} = 2$	1	2
$2^{2} = 4$	2	4
$2^{3} = 8$	3	8
$2^{4} = 16$	4	16
$2^{5} = 32$	5	32
$2^{6} = 64$	6	64
$2^{7} = 128$	7	128
$2^{8} = 256$	8	256
$2^{9} = 512$	9	512
$2^{10} = 1024$	10	1024
$2^{11} = 2048$	11	2048
$2^{12} = 4096$	12	4096
$2^{13} = 8192$	13	8192
$2^{14} = 16384$	14	16384
$2^{15} = 32768$	15	32768
$2^{16} = 65536$	16	65536
$2^{17} = 131072$	17	131072
$2^{18} = 262144$	18	262144
$2^{19} = 524288$	19	524288
$2^{20} = 1048576$	20	1048576

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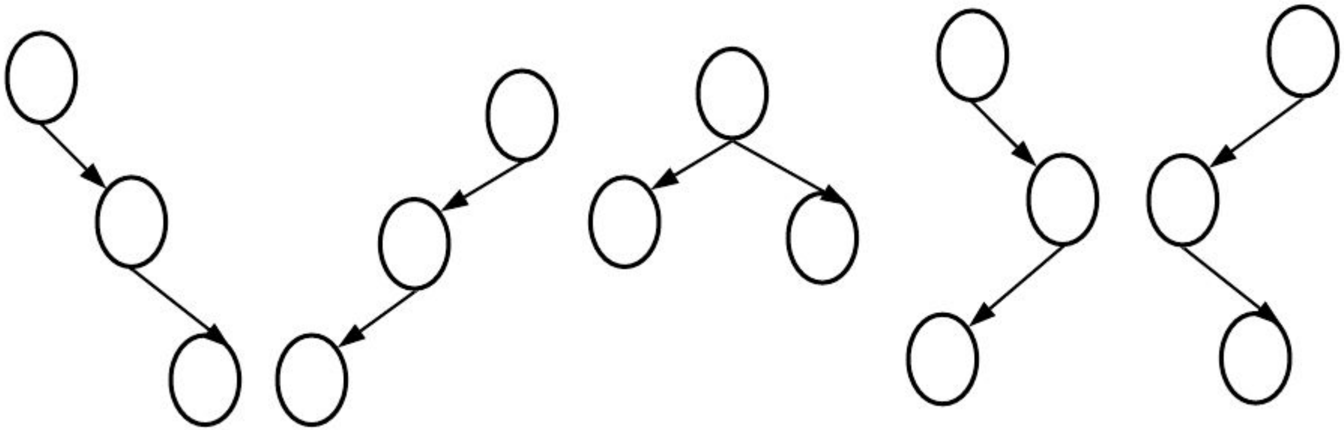
How many unlabeled binary trees of 3 nodes?

- We can draw the below 5 trees using 3 nodes
- So in general, How many unlabeled binary trees of n nodes?



How many **unlabeled** binary trees of n nodes?

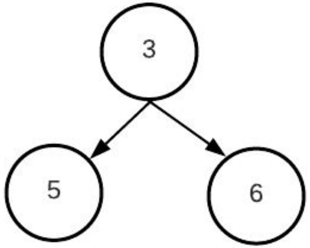
- The answer is a very interesting mathematical number!
- The Catalan Number (wiki has a lot of facts)
  - You don't need to know why

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!\,n!}$$

How many labeled binary trees of n nodes?

- Given a single tree of n nodes, we can label it in n! Ways!
- So answer is Catlan(n) * n!

$$\frac{1}{n+1}\binom{2n}{n} \times n! = \frac{(2n)!}{(n+1)!}$$



"Acquire knowledge and impart it to the people."

"Seek knowledge from the Cradle to the Grave."