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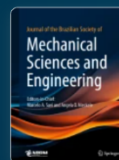
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# Static, free vibration, and buckling analysis of functionally graded plates using strain approach and Reissner–Mindlin elements

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## Abstract

The novelty of the present work lies in the development of a new four-node rectangular finite element using strain-based and Reissner–Mindlin theory. This paper is the first to apply this innovative approach to study the static, free vibration, and buckling responses of functionally graded materials (FGMs) plates. The mechanical properties of the FGM plate are considered to vary along the thickness direction by the power-law distributions. The notion of a neutral surface has been used to prevent the stretching–bending effect. The developed element has six degrees of freedom (DOFs) per node, obtained by combining two strain-based elements. The first one is a membrane which has three DOFs per node, and the second one is a Reissner–Mindlin plate which has three DOFs per node. The displacement fields of these components are represented by higher-order expressions based on the strain approach, which satisfy both rigid body modes and compatibility equations. The performance of the proposed element is evaluated through various numerical problems, and the results are compared with those published in the literature, showing good agreement. The impact of the gradient index, side-to-thickness ratio, aspect ratio, and loading types on the stresses, transverse displacements, frequency response, and critical buckling load of FGM plates is also investigated and discussed.

**Keywords** Static · Free vibration · Buckling analysis · Reissner–Mindlin · Functionally graded · Strain based

## 1 Introduction

In recent decades, the notion of functionally graded materials (FGM) has emerged from the work of a Japanese scientist in 1984 [1], which are used in industrial environments due to their excellent performance compared to conventional materials. FGM is a family of composite inhomogeneous materials consisting of a combination of isotropic materials, generally ceramics and metals, and it has many advantages, including progressive and continuous changes in their mechanical and thermal characteristics across the thickness, which prevents problems associated with traditional laminated composite structures, like higher inter-laminar stresses between the layers of a composite laminate [2].

As a result, these materials are attracting significant attention in various engineering disciplines, such as aerospace, mechanical, automotive, civil, and biomedical engineering. Many researches have been studied for static, free vibrational, and buckling behaviors of FGM beams, plates, and shells using several analytical and numerical approaches, relying on various theories, including classical plate theory (CPT), which neglects the effects of transverse shear deformation [3–7], first-order shear deformation theory (FSDT) having a linear variation in displacements [8–12], and higher-order shear deformation theory (HSDT) involving higher-order variations in displacements across the plate thickness, such as third-order shear deformation plate theory (TSDT), sinusoidal shear deformation plate theory (SSDT), and hyperbolic shear deformation plate theory (HDT) [13–23].

Several developers have employed the strain approach to design finite elements that are both efficient and durable. Initially, Ashwell et al. [24] introduced this methodology specifically for curved elements. Subsequently, this approach was employed in the analysis of shell structures [25–27]. It was later expanded to encompass plane-elasticity elements

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[28–30] and further extended to address 3D elasticity problems [31–33]; the method was also applied to plate bending [34–39]. It is important to observe that all previously formulated plate elements using the strain approach were designed specifically for isotropic materials. Thus, the researchers were motivated to develop a new finite element using the strain approach for the analysis of FG plates.

The primary objective of the current work is to contribute to the modeling of static, free vibration, and mechanical buckling of functionally graded material (FGM) plates by developing, for the first time, a new four-node rectangular element based on the strain approach and Reissner–Mindlin plate theory. The material characteristics of FGM plates, such as Young's modulus and density, are considered to vary continuously throughout the thickness of the plate with a power-law function of the volume fraction of the components, while the Poisson's ratio remains constant. To prevent the membrane-bending coupling, the notion of physical neutral surface has been presented. The developed element called SBRP24 (Strain-Based Rectangular Plate with 24 degrees of freedom) has six degrees of freedom ( $u, v, \theta_z, w, \beta_x, \beta_y$ ) for each node, and their displacement fields are derived by combining membrane and bending elements based on the strain approach. The membrane element, called SBRIER (Strain-Based Rectangular In-Plane Elasticity In-Plane Rotation), developed by Sabir [30], having three degrees of freedom at each node, two translations ( $u, v$ ) and one in-plane rotation ( $\theta_z$ ), while the bending element, named SBRP (Strain-Based Rectangular Plate), developed by Belounar and Guenoud [37], has three DOFs ( $w, \beta_x, \beta_y$ ) per node. Additionally, the potential energy and Hamilton's principle are applied to obtain the stiffness, mass, and geometrical matrices. Different numerical problems are studied and analyzed to evaluate the efficiency and performance of the current element in predicting the bending, free vibration, and buckling behavior of FGM plates. Furthermore, another objective of this paper is to investigate the impact of several parameters on the behavior of the FGM plate, including the gradient index ( $p$ ), the side-to-thickness ratio ( $l/h$ ), the types of loading (sinusoidal or uniformly distributed lateral loads, biaxial and uniaxial compression loads), the aspect ratio ( $L/l$ ), etc.

## 2 Theoretical formulation

### 2.1 FGM plate

Consider a rectangular plate of heterogeneous elastic FGM with length  $l$ , width  $L$ , and thickness  $h$ . It comprises of ceramic material on the upper surface and metallic material on the lower surface, as illustrated in Fig. 1. The material characteristics ( $E; \rho$ ) of the plate change gradually in the

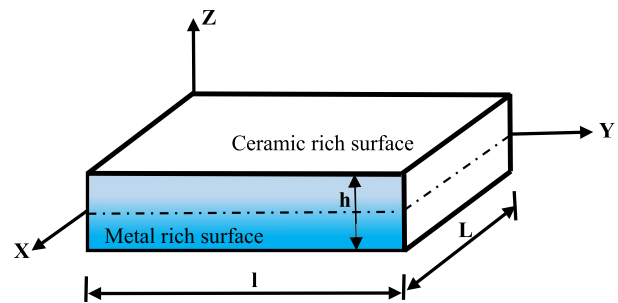


Fig. 1 FG rectangular plate geometry

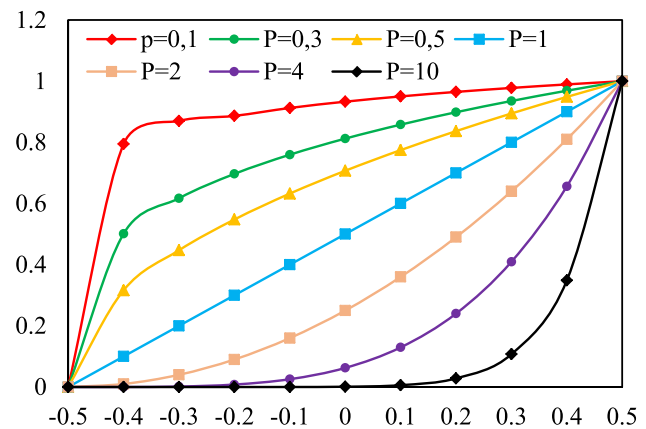


Fig. 2 Variations in the volume proportion ( $V_c$ ) through the thickness of the FGM plate

direction of thickness, characterized by the power function (P-FGM), as shown in Fig. 2:

$$V_c(z) = \left( \frac{Z}{h} + \frac{1}{2} \right)^p \quad (1)$$

$$P(z) = P_m + (P_c - P_m)V_c(z) \quad (2)$$

where the indicators  $m$  and  $c$  describe the metal and ceramic, respectively;  $P_c$  and  $P_m$  are the corresponding material characteristics, such as density ( $\rho$ ), Young's modulus ( $E$ ), and other characteristics. ( $p \geq 0$ ) represents the gradient index of martial FGM.

### 2.2 Limitations and basic assumptions

The assumptions of the current theory are as follows:

- The origin of the Cartesian coordinate system is defined at the neutral surface of the FGM plate.
- The displacement function within the element is approximated using polynomial functions based on the strain

approach, which satisfies both rigid body modes and compatibility equations ("see Appendix").

- Transverse normal stress ( $\sigma_z$ ) is assumed to be negligible compared to the in-plane stresses  $\sigma_x$  and  $\sigma_y$
- Small deformations are assumed, meaning that the displacements are small relative to the thickness of the plate, resulting in infinitesimal strains.

### 2.3 Displacement functions and strains

The displacement functions of the conventional FSDT are represented by:

$$\begin{aligned} U(x, y, z) &= u(x, y) + z\beta_x(x, y)V(x, y, z) \\ &= v(x, y) + z\beta_y(x, y)W(x, y, z) \\ &= w(x, y)\theta_z(x, y, z) = \theta_z(x, y) \end{aligned} \quad (3)$$

U, V, and W denote the displacements of each point in the (x, y, z) coordinate system; u, v and w are mid-plane displacements.  $\beta_x$ ,  $\beta_y$ , and  $\theta_z$  represent the rotations in yz, xz, and xy planes, respectively.

The drilling rotation  $\theta_z$  is defined by the following [40]:

$$\theta_z(x, y) = \frac{1}{2} \left( \frac{\partial v(x, y)}{\partial x} - \frac{\partial u(x, y)}{\partial y} \right) \quad (4)$$

The relations between displacement and strain for the FGM plate are defined as follows:

$$\varepsilon = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \{\varepsilon_m\} + z\{k\} + \{\varepsilon^{nl}\} \quad (5)$$

where

$$\{\varepsilon_m\} = \begin{Bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \end{Bmatrix} \quad (\text{Membrane strains}) \quad (6)$$

$$\{k\} = \begin{Bmatrix} k_{xx} \\ k_{yy} \\ k_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \beta_x}{\partial x} \\ \frac{\partial \beta_y}{\partial y} \\ \left( \frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x} \right) \end{Bmatrix} \quad (\text{Bending strains}) \quad (7)$$

$$\{\varepsilon^{nl}\} = \begin{Bmatrix} \varepsilon_{xx}^{nl} \\ \varepsilon_{yy}^{nl} \\ \gamma_{xy}^{nl} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \\ \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \\ \left( \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \end{Bmatrix} \quad (\text{Nonlinear strains}) \quad (8)$$

$$\{\gamma\} = \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial w}{\partial x} + \beta_x \\ \frac{\partial w}{\partial y} + \beta_y \end{Bmatrix} \quad (\text{Shear strains}) \quad (9)$$

### 2.4 Constitutive relations

The constitutive relationships in linear form for (FG) plate can be articulated as:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \quad (10)$$

Or

$$\{\sigma\} = [C]\{\varepsilon\} \quad (11)$$

$$\begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} k_s C_{44} & 0 \\ 0 & k_s C_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \quad (12)$$

where ( $\sigma_{xx}$ ,  $\sigma_{yy}$ ,  $\tau_{xy}$ ,  $\tau_{xz}$ ,  $\tau_{yz}$ ) and ( $\varepsilon_{xx}$ ,  $\varepsilon_{yy}$ ,  $\gamma_{xy}$ ,  $\gamma_{xz}$ ,  $\gamma_{yz}$ ) are the stresses and strains, respectively. The shear correction value  $k_s = 5/6$ . The rigidity coefficients  $C_{ij}$  can be formulated according to the material properties given in Eq. 2 as follows:

$$\begin{aligned} C_{11}(z) = C_{22}(z) &= \frac{E(z)}{1 - \nu^2}; C_{12}(z) = C_{21}(z) = \nu \frac{E(z)}{1 - \nu^2}; \\ C_{44}(z) = C_{55}(z) &= C_{66}(z) = \frac{E(z)}{2(1 + \nu)} \end{aligned} \quad (13)$$

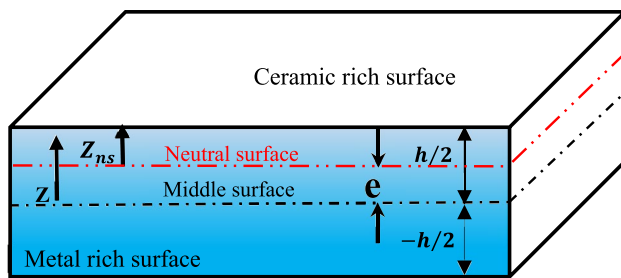
### 2.5 Neutral surface location

The neutral surface of a plate is the surface within the thickness where no longitudinal strain occurs during bending; it experiences zero normal stress under pure bending.

To prevent the coupling of the stretching and bending effects produced by the asymmetric material characteristics of functionally graded plates with respect to the central surface, the force and moments are determined using the neutral surface position  $z_{Ns}$ , which is not coincident with the central surface of the plate, as illustrated in Fig. 3 [20]:

$$z_{ns} = z - e \quad (14)$$

In order to obtain the neutral surface position, ( $e$ ) is chosen so that the membrane force caused by bending would be zero at ( $z=e$ ) [22].



**Fig. 3** Neutral surface location for the FGM plate

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx} dz_{ns} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[ C_{11}(z-e) \frac{\partial \beta_x}{\partial x} + C_{12}(z-e) \frac{\partial \beta_y}{\partial y} \right] dz = 0 \quad (15)$$

By replacing Eqs. 13 into 15, we obtain:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx} dz_{ns} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[ \frac{h}{2} \left( \frac{E(z)}{1-\nu^2} \frac{\partial \beta_x}{\partial x} + \frac{E(z)}{1-\nu^2} \frac{\partial \beta_y}{\partial y} \right) \right] dz = 0 \quad (16)$$

Simplifying Eq. 16, we get:

$$\frac{1}{1-\nu^2} \left( \frac{\partial \beta_x}{\partial x} + \nu \frac{\partial \beta_y}{\partial y} \right) \int_{-\frac{h}{2}}^{\frac{h}{2}} (E(z)(z-e)) dz = 0 \quad (17)$$

Then,

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} (E(z)(z-e)) dz = 0 \Rightarrow \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)z dz - (e) \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z) dz = 0 \quad (18)$$

Consequently, the neutral surface's position can be established as:

$$e = \frac{\int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)z dz}{\int_{-\frac{h}{2}}^{\frac{h}{2}} E(z) dz} \quad (19)$$

## 2.6 Shear locking mitigation using strain-based formulations

In traditional displacement-based finite element formulations, thin plates or shells often suffer from shear locking—an artificial stiffness caused by the inability of low-order elements to accurately represent shear deformation.

To mitigate shear locking, strain-based formulations can be used instead of displacement-based ones. In strain-based formulations:

- The strain fields (including bending and shear strains) are directly assumed or approximated, rather than being derived solely from displacement fields.
- This allows for better representation of the shear deformation without enforcing overly stiff constraints.
- It improves the element's flexibility and accuracy, especially for thin or moderately thick plates.

For functionally graded material (FGM) plates analyzed with Reissner–Mindlin theory, adopting strain-based formulations helps to avoid shear locking by accurately capturing the shear strain distribution through the thickness, leading to more precise bending and shear stress predictions.

## 2.7 Stresses resultants

The resultant stresses M, N, and T are determined by:

$$\begin{aligned} \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx}(z) \\ \sigma_{yy}(z) \\ \tau_{xy}(z) \end{Bmatrix} dz; \\ \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx}(z) \\ \sigma_{yy}(z) \\ \tau_{xy}(z) \end{Bmatrix} (z-e) dz; \\ \begin{Bmatrix} T_{xx} \\ T_{yy} \end{Bmatrix} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} dz \end{aligned} \quad (20)$$

Equation 20 is reformulated in the matrix form below:

$$\begin{Bmatrix} N \\ M \\ T \end{Bmatrix} = \begin{bmatrix} [A] & 0 & 0 \\ 0 & [D] & 0 \\ 0 & 0 & [A^s] \end{bmatrix} \begin{Bmatrix} \epsilon_m \\ k \\ \gamma \end{Bmatrix} \quad (21)$$

where

$$[A] = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}; [D] = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix}; [A^s] = \begin{bmatrix} A_{44}^s & 0 \\ 0 & A_{55}^s \end{bmatrix} \quad (22)$$

where  $A_{ij}, D_{ij}, A_{ij}^s$  represent the plate stiffness, as defined by:

$$\begin{aligned} A_{ij} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} C_{ij} dz; D_{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} C_{ij}(z-e)^2 dz (i, j = 1, 2, 6); A_{44}^s \\ &= A_{55}^s = k_s \int_{-\frac{h}{2}}^{\frac{h}{2}} C_{44} dz \end{aligned} \quad (23)$$

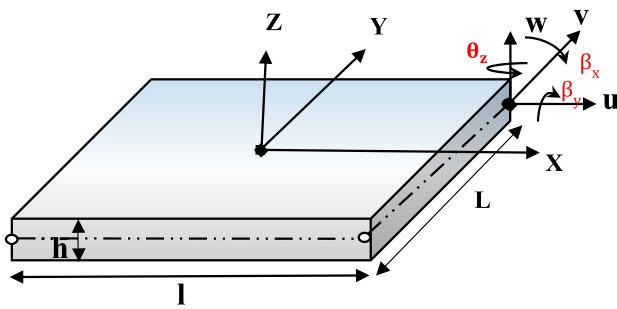


Fig. 4 Geometry of the present element SBRP24 and nodal variables

### 3 Finite element formulation

In this work, a new four-node rectangular Reissner–Mindlin plate element is formulated. This element is called SBRP24 (Strain-Based Rectangular Mindlin Plate with 24 unknowns) and possesses six DOFs ( $u, v, \theta_z, w, \beta_x, \beta_y$ ) for each node, as shown in Fig. 4, corresponding to three displacements ( $u, v, w$ ) in the  $x, y$ , and  $z$  directions, as well as three rotations ( $\beta_x, \beta_y, \theta_z$ ) in the  $yz, xz$ , and  $xy$  planes, respectively.

#### 3.1 Displacements field of the (SBRP24) element

To obtain the displacement fields for SBRP24, we combined the displacement functions derived from the thin plate element (SBRP) developed by Belouinar and Guenoud [45] and the membrane element (SBRIEIR) proposed by Sabir [30].

The displacement field defined in [30] for the stretching element (SBRIEIR) is:

$$\{U_m\} = \begin{Bmatrix} u \\ v \\ \theta_z \end{Bmatrix} = [P_m] \{\alpha_m\} \quad (24)$$

where  $\{\alpha_m\}^T = \{\alpha_1, \dots, \alpha_{12}\}$

$$[P_m] = \begin{bmatrix} 1 & 0 & -y & x & xy & 0 & 0 & \frac{y}{2} & 0 & \frac{y^2}{2} & xy^2 & x^2y^3 \\ 0 & 1 & x & 0 & 0 & y & xy & \frac{x}{2} & \frac{x^2}{2} & 0 & -x^2y & -x^3y^2 \\ 0 & 0 & 1 & 0 & -\frac{x}{2} & 0 & \frac{y}{2} & 0 & \frac{x}{2} & -\frac{y}{2} & -2xy & -3x^2y^2 \end{bmatrix} \quad (25)$$

The displacement functions for bending plate element (SBRP) are [45]:

$$\{U_b\} = \begin{Bmatrix} w \\ \beta_x \\ \beta_y \end{Bmatrix} = [P_b] \{\alpha_b\} \quad (26)$$

where  $\{\alpha_b\}^T = \{\alpha_{13}, \dots, \alpha_{24}\}$

$$[P_b] = \begin{bmatrix} 1 & -x & -y & -\frac{x^2}{2} & -\frac{x^2y}{2} & -\frac{y^2}{2} & -\frac{xy^2}{2} & -\frac{xy}{2} & \frac{x}{2} & \frac{xy}{2} & \frac{y}{2} & \frac{xy}{2} \\ 0 & 1 & 0 & x & xy & 0 & -\frac{y^2}{2} & \frac{y}{2} & \frac{1}{2} & \frac{y}{2} & 0 & -\frac{y}{2} \\ 0 & 0 & 1 & 0 & -\frac{x^2}{2} & y & xy & \frac{x}{2} & 0 & -\frac{x}{2} & \frac{1}{2} & \frac{x}{2} \end{bmatrix} \quad (27)$$

As mentioned above, the displacement functions for the current element (SBRP24) may be obtained by combining Eqs. 25 and 27 as shown here:

$$\{U_e\} = \begin{Bmatrix} \{U_m\} \\ \{U_b\} \end{Bmatrix} = \begin{Bmatrix} u \\ v \\ \theta_z \\ w \\ \beta_x \\ \beta_y \end{Bmatrix} = \begin{bmatrix} [P_m] & 0 \\ 0 & [P_b] \end{bmatrix} \begin{Bmatrix} \{\alpha_m\} \\ \{\alpha_b\} \end{Bmatrix} = [P] \{\alpha\} \quad (28)$$

where  $\{\alpha\}^T = \{\alpha_1, \dots, \alpha_{24}\}$

The 24 nodal DOFs of the element are expressed in the form of the 24 constants  $\{\alpha\}$  using the transformation matrix  $[C]$ .

$$\{\delta^e\} = [C] \{\alpha\} \quad (29)$$

where

$$\{\delta^e\}^T = \{u_i, v_i, \theta_{zi}, w_i, \beta_{xi}, \beta_{yi}\}_{i=1,2,3,4} \quad (30)$$

$$[C] = \{[P_1][P_2][P_3][P_4]\}^T \quad (31)$$

And the matrices  $[P_i]$  (for  $x_i, y_i$  from node  $i$  ( $i = 1, 2, 3$ , and  $4$ )) are described as:

$$[P_i] = \begin{bmatrix} [P_m]_i & 0 \\ 0 & [P_b]_i \end{bmatrix} \quad (32)$$

Using Eq. 29, we can now determine the constant values vector  $\{\alpha\}$

$$\{\alpha\} = [C]^{-1} \{\delta^e\} \quad (33)$$

Then, by replacing Eqs. 33 into 28, we get:

$$\{U_e\} = [P][C]^{-1} \{\delta^e\} = [N] \{\delta^e\} \quad (34)$$

where

$$[N] = [P][C]^{-1} \quad (35)$$

#### 3.2 Strain–displacement relations

The membrane strains  $\{\epsilon_m\}$  are calculated using Eq. 6:

$$\{\varepsilon_m\} = \begin{Bmatrix} \varepsilon_x^m \\ \varepsilon_y^m \\ \gamma_{xy}^m \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{Bmatrix} u \\ v \\ \theta_z \end{Bmatrix} \quad (36)$$

By inserting Eqs. 28 into 36, we obtain:

$$\{\varepsilon_m\} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} [[P_m]_{3 \times 12} [0]_{3 \times 12}]_{3 \times 24} \{\alpha\} [Q_m]_{3 \times 24} \{\alpha\} \quad (37)$$

Using Eqs. 7 and 9, the curvature and shear strain–displacement relationship given by:

$$\{\kappa\} = \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{bmatrix} 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} w \\ \beta_x \\ \beta_y \end{Bmatrix} \quad (38)$$

$$\{\gamma\} = \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 1 & 0 \\ \frac{\partial}{\partial y} & 0 & 1 \end{bmatrix} \begin{Bmatrix} w \\ \beta_x \\ \beta_y \end{Bmatrix} \quad (39)$$

Substitution of Eqs. 28 in 38 and 39, we obtain:

$$\{\kappa\} = \begin{bmatrix} 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} [[0]_{3 \times 12} [P_b]_{3 \times 12}]_{3 \times 24} \{\alpha\} = [Q_b] \{\alpha\} \quad (40)$$

$$\{\gamma\} = \begin{bmatrix} \frac{\partial}{\partial x} & 1 & 0 \\ \frac{\partial}{\partial y} & 0 & 1 \end{bmatrix} [[0]_{3 \times 12} [P_b]_{3 \times 12}]_{3 \times 24} \{\alpha\} = [Q_s] \{\alpha\} \quad (41)$$

The geometric strains can be formulated as [12]:

$$\{\varepsilon^g\} = \begin{Bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{Bmatrix} \quad (42)$$

Substituting the transverse displacement ( $w$ ) of Eqs. 27 into 42 gives:

$$\{\varepsilon^g\} = [G] \{\alpha\} \quad (43)$$

The strain–displacement relationship is obtained by replacing Eq. 33 into Eqs. (37, 40, 41), and 43, to have:

$$\{\varepsilon_m\} = [Q_m][C]^{-1} \{\delta^e\} = [B_m] \{\delta^e\} \quad (44)$$

$$\{\kappa\} = [Q_b][C]^{-1} \{\delta^e\} = [B_b] \{\delta^e\} \quad (45)$$

$$\{\gamma\} = [Q_s][C]^{-1} \{\delta^e\} = [B_s] \{\delta^e\} \quad (46)$$

$$\{\varepsilon^g\} = [G][C]^{-1} \{\delta^e\} = [B_g] \{\delta^e\} \quad (47)$$

where  $[B_m]$ ,  $[B_b]$ ,  $[B_s]$ , and  $[B_g]$  are the strain–displacement matrices given by:

$$\begin{aligned} [B_m] &= [Q_m][C]^{-1}; [B_b] = [Q_b][C]^{-1}; \\ [B_s] &= [Q_s][C]^{-1}; [B_g] = [G][C]^{-1} \end{aligned} \quad (48)$$

### 3.3 Deduction of the elementary matrices

The normal weak formulas for bending behavior can be described as [12]:

$$\begin{aligned} \int_{\Omega} \delta\{\varepsilon_m\}^T \{N\} d\Omega + \int_{\Omega} \delta\{\kappa\}^T \{M\} d\Omega \\ + \int_{\Omega} \delta\{\gamma\}^T \{T\} d\Omega = \int_{\Omega} \delta\{U_e\}^T \{q(x, y)\} d\Omega \end{aligned} \quad (49)$$

where  $\Omega$  is the area of the plate and  $q(x, y)$  is the distributed loading force.

Substituting Eqs. (21, 34, 44, 45) and Eq. 46, into the above equation one obtains:

$$\begin{aligned} \delta\{\delta^e\}^T \left( \int_{\Omega} [B_m]^T [D_m] [B_m] d\Omega \right) \{\delta^e\} \\ + \delta\{\delta^e\}^T \left( \int_{\Omega} [B_b]^T [D_b] [B_b] d\Omega \right) \{\delta^e\} \\ + \delta\{\delta^e\}^T \left( \int_{\Omega} [B_s]^T [D_s] [B_s] \{\delta^e\} d\Omega \right) \{\delta^e\} \\ = \delta\{\delta^e\}^T \left( \int_{\Omega} [N]^T \{q(x, y)\} d\Omega \right) \end{aligned} \quad (50)$$

The previous equation can be formulated as follows:

$$[K_e] \{\delta^e\} = \{F_e\} \quad (51)$$

where

$$[K_e] = \int_{-b}^b \int_{-a}^a \left( \underbrace{[B_m]^T [D_m] [B_m]}_{\text{Membrane}} + \underbrace{[B_b]^T [D_b] [B_b]}_{\text{Bending}} + \underbrace{[B_s]^T [D_s] [B_s]}_{\text{Shear}} \right) dx dy \quad (52)$$

$$\{F_e\} = \int_{\Omega} [N]^T \{q(x, y)\} d\Omega \quad (53)$$

For free vibrational and buckling response, a variation form of the virtual work principle according to the conditions of the FSDT may be written, respectively, as [12]:

$$\begin{aligned}
& \delta\{\delta^e\}^T \left( \int_{\Omega} [B_m]^T [D_m] [B_m] d\Omega \right) \{\delta^e\} \\
& + \delta\{\delta^e\}^T \left( \int_{\Omega} [B_b]^T [D_b] [B_b] d\Omega \right) \{\delta^e\} \\
& + \delta\{\delta^e\}^T \left( \int_{\Omega} [B_s]^T [D_s] [B_s] d\Omega \right) \{\delta^e\} \\
& + \delta\{\delta^e\}^T \left( \int_{\Omega} [N]^T [m] [N] d\Omega \right) \{\delta^e\} = 0
\end{aligned}$$

(54)

$$\begin{aligned}
& \delta\{\delta^e\}^T \left( \int_{\Omega} [B_m]^T [D_m] [B_m] d\Omega \right) \{\delta^e\} \\
& + \delta\{\delta^e\}^T \left( \int_{\Omega} [B_b]^T [D_b] [B_b] d\Omega \right) \{\delta^e\} \\
& + \delta\{\delta^e\}^T \left( \int_{\Omega} [B_s]^T [D_s] [B_s] d\Omega \right) \{\delta^e\} \\
& + \delta\{\delta^e\}^T \left( \int_{\Omega} [B_g]^T [\bar{N}] [B_g] d\Omega \right) \{\delta^e\} = 0
\end{aligned}$$

(55)

where the elementary matrices for mass and geometry ( $[M^e]$  and  $[K_g^e]$ ) are defined by the following equations:

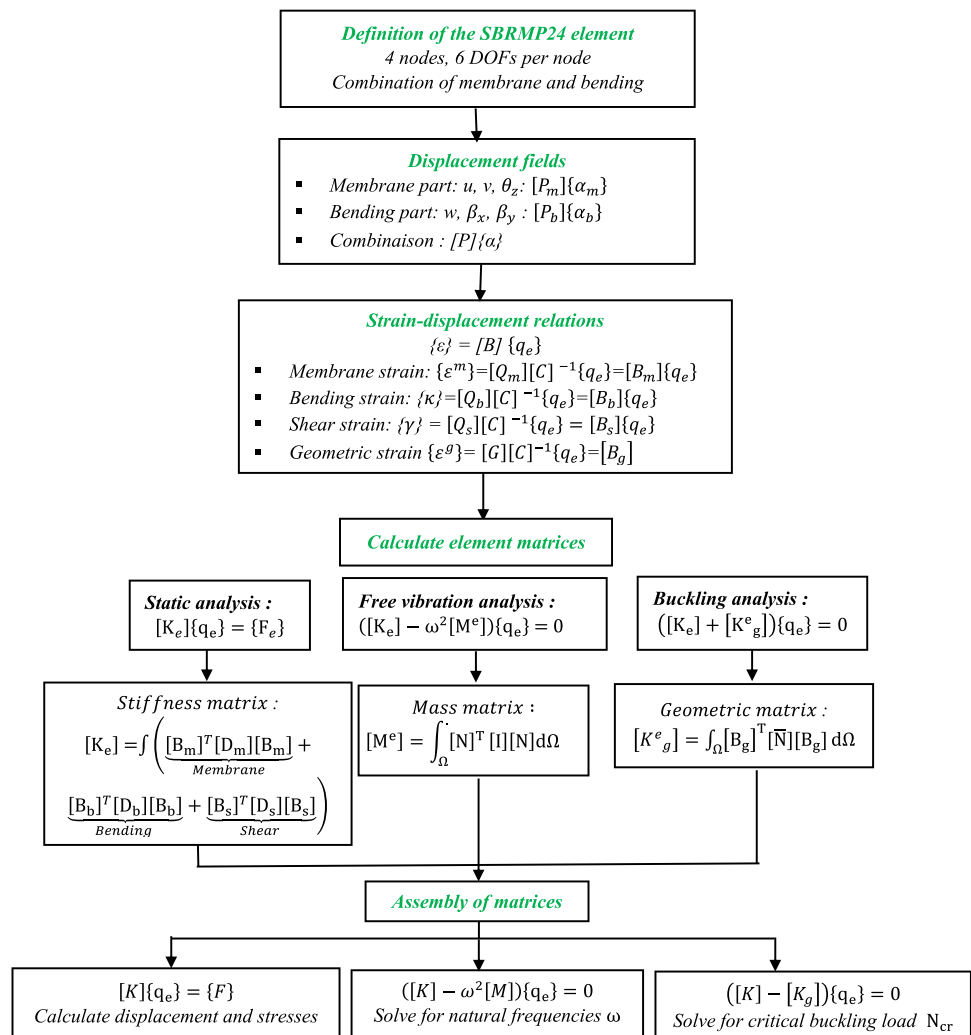
$$[M^e] = \int_{\Omega} [N]^T [m] [N] d\Omega \quad (56)$$

$$[K_g^e] = \int_{\Omega} [B_g]^T [\bar{N}] [B_g] d\Omega \quad (57)$$

in which  $[m]$  and  $[\bar{N}]$  describe the inertia matrix and the stress matrix produced by the mechanical load, respectively, as follows:

$$[m] = \begin{bmatrix} I_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_2 \end{bmatrix} \quad (58)$$

**Fig. 5** Flow diagram summarizing the steps of the finite element formulation



$$[\bar{N}] = \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix} \quad (59)$$

The matrices  $[K_e]$ ,  $[M_e]$ , and  $[K_g]$  and equivalent nodal load  $\{F_e\}$ , given in Eqs. (52, 56, 57) and (53), are calculated numerically using the Gaussian integration method. The matrices are combined through assembly to derive the rigidity, mass, and geometric matrices ( $[K]$ ,  $[M]$ , and  $[K_g]$ ) of the structure, along with the structural load vector  $\{F\}$ .

The following equations can be applied for the bending, free vibrational, and buckling behavior, respectively:

$$[K]\{\delta^e\} = \{F\} \quad (60)$$

$$[K] - \omega^2[M] = 0 \quad (61)$$

$$([K] - \lambda_{cr}[K_g])\{\delta^e\} = 0 \quad (62)$$

where  $\lambda_{cr}$  and  $\omega$  are, respectively, the critical buckling loads and natural frequency.

For better understanding, a flow diagram summarizing the steps of the finite element formulation is provided in Fig. 5.

## 4 Results and discussion

In this part, static, free vibrational, and buckling problems are examined and analyzed to verify the convergence and efficiency of the current element (SBRP24). The assumed composition of the FG plate includes metal (aluminum, Al) and ceramic (alumina,  $Al_2O_3$ ) with the specified material properties:

Metal (aluminum, Al):  $E_m=70$  Gpa;  $\nu_m=0.3$ ;  $\rho_m=2707$  kg/m<sup>3</sup>.

Ceramic (alumina,  $Al_2O_3$ ):  $E_m=380$  Gpa;  $\nu_m=0.3$ ;  $\rho_m=3800$  kg/m<sup>3</sup>.

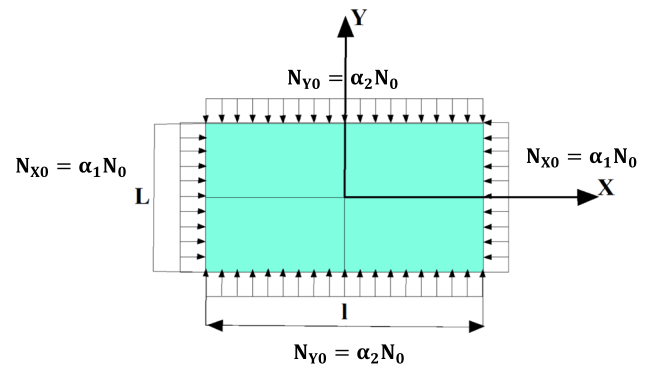
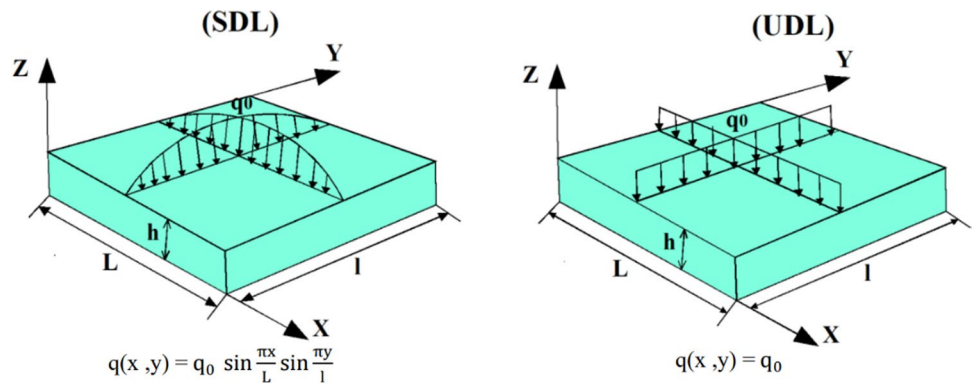
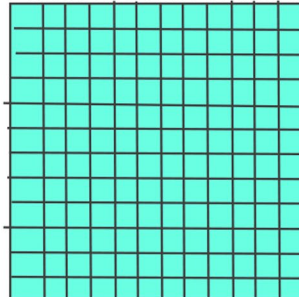


Fig. 7 Geometry and loading FGM rectangular plate

Fig. 6 Rectangular FGM plate exposed to uniform distributed load (UDL) and sinusoidal distributed load (SDL) using a mesh from  $N \times N$  elements



Mesh of  $N \times N$  elements



The boundary conditions for any edge of plate are listed below:

Simply supported (SSSS)

$$u = \beta_x = \theta_z = 1; v = w = \beta_y = 0 \text{ at } x = 0, 1$$

$$v = \beta_y = \theta_z = 1; u = w = \beta_x = 0 \text{ at } y = 0, L$$

Clamped (CCCC)

$$u = v = w = \beta_x = \beta_y = \theta_z = 0 \text{ at } x = 0, 1, \text{ and } y = 0, L.$$

For static loading, the plate has experienced a uniform (UDL) and sinusoidal lateral load (SDL) in the  $z$ -direction, as illustrated in Fig. 6. For the case of buckling loading, the plate has been exposed to various in-plane loading conditions, such as uniaxial compression ( $\alpha_1 = 1, \alpha_2 = 0$ ), biaxial compression ( $\alpha_1 = 1, \alpha_2 = 1$ ), as illustrated in Fig. 7.

For simplicity, the dimensionless formulas below are used to express deflection, in-plane stresses, natural frequencies, and critical buckling loads:

$$\begin{aligned} \bar{w} &= \frac{10h^3 E_c}{L^4 q_0} w\left(\frac{L}{2}, \frac{1}{2}\right); \bar{\sigma}_x \\ &= \frac{h}{q_0} \sigma_x\left(\frac{L}{2}, \frac{1}{2}, \frac{h}{2}\right); \bar{\sigma}_y \\ &= \frac{h}{q_0} \sigma_y\left(\frac{L}{2}, \frac{1}{2}, \frac{h}{3}\right); \bar{\tau}_{xy} \\ &= \frac{h}{q_0} \tau_{xy}\left(0, 0, -\frac{h}{3}\right) \end{aligned} \quad (63)$$

$$\bar{\omega} = \omega h \sqrt{\frac{\rho_c}{E_c}}; \bar{\beta} = \omega h \sqrt{\frac{\rho_m}{E_m}} \quad (64)$$

$$\bar{\lambda}_{cr} = \lambda_{cr} \frac{(L^2)}{(E_m h^3)}. \quad (65)$$

#### 4.1 Problem 1 bending analysis of SSSS FG plate.

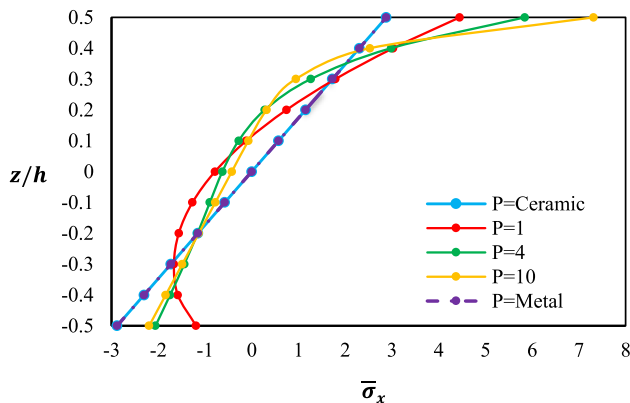
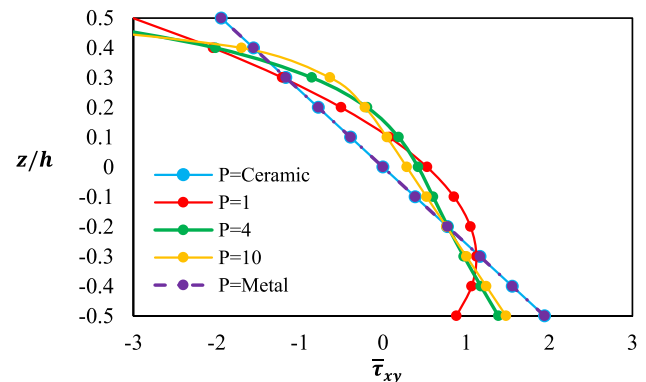
In this problem, the static response of SSSS FG square plate is studied ( $L/h = 10$ ). The non-dimensional central displacements and stresses of the SBRP24 element evaluated for the FGM plates under uniformly distributed load ( $q(x, y) = q_0$ ) and sinusoidally load ( $q(x, y) = q_0 \sin \frac{\pi x}{L} \sin \frac{\pi y}{L}$ ), as illustrated in Fig. 6, are tabulated in Tables 1 and 2, respectively, using four meshes ( $8 \times 8$ ,  $12 \times 12$ ,  $16 \times 16$  and  $20 \times 20$ ) for different values of power-law index  $p$ . The results of the present

**Table 1** Comparison of the dimensionless deflections and stresses in a square FG plate exposed to a uniformly distributed load with  $L/h = 10$

P	Mesh size	SBRP24				Tati [20]	Zenkour [15]	Reddy [13]
		8×8	12×12	16×16	20×20			
0 (Ceramic)	$\bar{w}$	0.4421	0.4559	0.4607	0.4628	0.4663	0.4665	0.4665
	$\bar{\sigma}_x$	2.7998	2.8429	2.8566	2.8627	2.8656	2.8932	2.8920
	$\bar{\sigma}_y$	1.8665	1.8953	1.9044	1.9084	1.904	1.9103	1.9106
	$\bar{\tau}_{xy}$	1.1202	1.2060	1.2410	1.2590	1.266	1.2850	1.2855
1	$\bar{w}$	0.8792	0.9074	0.9169	0.9212	0.9282	0.9287	0.9421
	$\bar{\sigma}_x$	4.3254	4.3935	4.4149	4.4244	4.415	4.4745	4.2598
	$\bar{\sigma}_y$	2.1202	2.1536	2.1641	2.1687	2.164	2.1692	2.2569
	$\bar{\tau}_{xy}$	0.9671	1.0417	1.0719	1.0874	1.093	1.1143	1.1573
2	$\bar{w}$	1.1273	1.1634	1.1756	1.1812	1.1948	1.1940	1.2228
	$\bar{\sigma}_x$	5.0506	5.1301	5.1550	5.1661	5.155	5.2296	4.8881
	$\bar{\sigma}_y$	1.9911	2.0224	2.0322	2.0366	2.032	2.0338	2.1663
	$\bar{\tau}_{xy}$	0.8619	0.9284	0.9553	0.9691	0.9745	0.9907	1.0449
10	$\bar{w}$	1.4884	1.5342	1.5498	1.5570	1.5934	1.5876	1.6054
	$\bar{\sigma}_x$	7.1120	7.2195	7.2539	7.2693	7.253	7.3689	6.9540
	$\bar{\sigma}_y$	1.2625	1.2816	1.2877	1.2904	1.288	1.2820	1.3350
	$\bar{\tau}_{xy}$	0.9360	1.0075	1.0367	1.0517	1.058	1.0694	1.1119
$\infty$ (Metal)	$\bar{w}$	2.3996	2.4749	2.5004	2.5121	—	2.5327	—
	$\bar{\sigma}_x$	2.7998	2.8429	2.8566	2.8627	—	2.8932	—
	$\bar{\sigma}_y$	1.8665	1.8953	1.9044	1.9084	—	1.9103	—
	$\bar{\tau}_{xy}$	1.1202	1.2060	1.2410	1.2590	—	1.2850	—

**Table 2** Comparison of the dimensionless deflections and stresses in a square FG plate exposed to a sinusoidal distributed load with  $l/h = 10$ 

P	Mesh size	SBRP24				Tati [20]	Zenkour [15]
		8 × 8	12 × 12	16 × 16	20 × 20	20 × 20	
0 (Ceramic)	$\bar{w}$	0.2776	0.2857	0.2884	0.2897	0.2957	0.2960
	$\bar{\sigma}_x$	2.4494	2.1603	2.0063	1.9970	1.9570	1.9955
	$\bar{\sigma}_y$	1.6330	1.4402	1.3375	1.3111	1.3050	1.3121
	$\bar{\tau}_{xy}$	0.4100	0.4377	0.4471	0.6813	0.7025	0.7065
1	$\bar{w}$	0.5573	0.5738	0.5794	0.5820	0.5882	0.5889
	$\bar{\sigma}_x$	2.4067	3.3556	3.1135	3.0545	3.0240	3.0870
	$\bar{\sigma}_y$	1.8667	1.6448	1.5261	1.4834	1.4820	1.4894
	$\bar{\tau}_{xy}$	0.3614	0.3864	0.3948	0.5989	0.6066	0.6110
2	$\bar{w}$	0.7141	0.7353	0.7425	0.7458	0.7575	0.7573
	$\bar{\sigma}_x$	4.4450	3.9168	3.6344	3.4614	3.5320	3.6094
	$\bar{\sigma}_y$	1.7523	1.5441	1.4327	1.3946	1.3920	1.3954
	$\bar{\tau}_{xy}$	0.3216	0.3439	0.3513	0.5347	0.5406	0.5441
10	$\bar{w}$	0.9258	0.9524	0.9614	0.9956	1.0125	1.0089
	$\bar{\sigma}_x$	6.1817	5.4583	5.0741	4.9388	4.9690	5.0890
	$\bar{\sigma}_y$	1.0973	0.9689	0.9007	0.8790	0.8822	0.8775
	$\bar{\tau}_{xy}$	0.3353	0.3576	0.3651	0.5685	0.5868	0.5894
$\infty$ (Metal)	$\bar{w}$	1.5067	1.5507	1.5656	1.5725	-	1.6070
	$\bar{\sigma}_x$	2.4494	2.1603	2.0063	1.9970	-	1.9955
	$\bar{\sigma}_y$	1.6330	1.4402	1.3375	1.3111	-	1.3121
	$\bar{\tau}_{xy}$	0.4100	0.4377	0.4471	0.6813	-	0.7065

**Fig. 8** Changes of the dimensionless in-plane stress ( $\bar{\sigma}_x$ ) across the thickness of the FGM plate for different parameters  $p$  ( $l/h = 10$ )**Fig. 9** Changes of the dimensionless longitudinal shear stress ( $\bar{\tau}_{xy}$ ) across the thickness of the FGM plate for different parameters  $p$  ( $l/h = 10$ )

element (SBRP24) are compared with the numerical results based on HSDT presented by Tati [20], an analytical solution of Zenkour [15] using SSDT, and those obtained by Reddy [13] using TSDT.

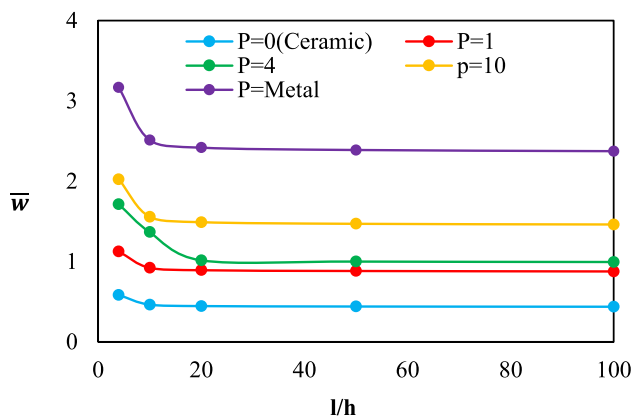
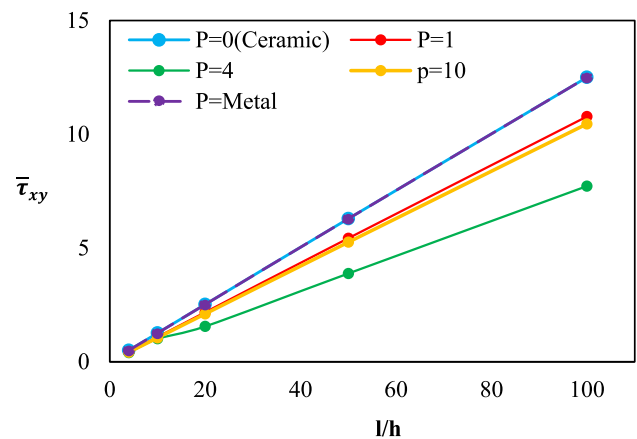
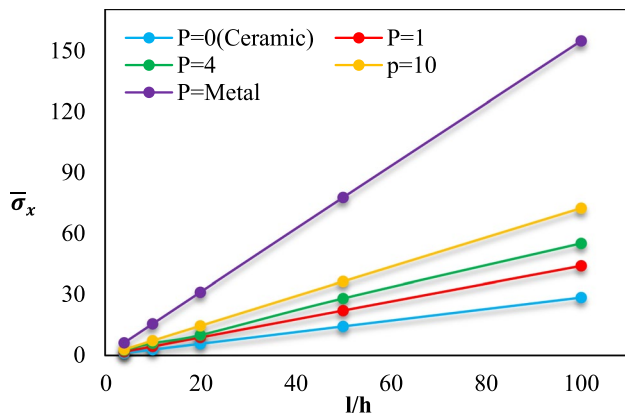
As demonstrated in Tables 1 and 2, the results obtained using the SBRP20 element generally align well with those reported in the aforementioned references for all power-law values. It can be observed that as the power-law index ( $p$ ) increases, the non-dimensional central transverse displacement ( $\bar{w}$ ) and normal stress ( $\bar{\sigma}_x$ ) both increase, while the in-plane stress ( $\bar{\sigma}_y$ ) decreases. Additionally, the in-plane shear

stress ( $\bar{\tau}_{xy}$ ) initially decreases as  $p$  ranges from 0 to 2 and then increases with a higher power-law index. It is noteworthy that the stresses for a fully ceramic plate are identical to those for a fully metal plate in this analysis, as both plates are considered fully homogeneous and the stresses are independent of Young's modulus.

Figures 8 and 9 show the variation in the dimensionless normal stress ( $\bar{\sigma}_x$ ) and in-plane shear stress ( $\bar{\tau}_{xy}$ ) across the thickness of a simply supported FG square plate subjected to a uniform distributed load (UDL), for different power-law index values ( $p$ ). It is observed that the stresses in fully

**Table 3** Impact of aspect ratio ( $l/h$ ) on the dimensionless displacement ( $\bar{w}$ ) and in-plane stress of a FGM square plate exposed to UDL

$l/h$	Theory	Model	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$
4	Present	FSDT	0.5837	1.1451	0.7634	0.5027
	Tati [20]	HSDT	0.5872	1.1425	0.7617	0.5070
	Zenkour [15]	SSDT	0.5865	1.1988	0.7534	0.4906
	Reddy [13]	TSDT	0.5868	1.1959	0.7541	0.4913
10	Present	FSDT	0.4628	2.8627	1.9084	1.2590
	Tati [20]	HSDT	0.4663	2.8560	1.9040	1.2660
	Zenkour [15]	SSDT	0.4665	2.8932	1.9103	1.2850
	Reddy [13]	TSDT	0.4666	2.8920	1.9106	1.2855
100	Present	FSDT	0.4374	28.4639	18.9760	12.6000
	Tati [20]	HSDT	0.4435	28.5600	19.0400	12.6500
	Zenkour [15]	SSDT	0.4438	28.7342	19.1543	13.0125
	Reddy [13]	TSDT	0.4438	28.7341	19.1543	12.9885

**Fig. 10** Variation in the dimensionless displacement ( $\bar{w}$ ) with the aspect ratio ( $l/h$ ) of FG square plates**Fig. 12** Variation in the dimensionless shear stress ( $\bar{\tau}_{xy}$ ) with the aspect ratio ( $l/h$ ) of FG square plates**Fig. 11** Variation in the dimensionless axial stress ( $\bar{\sigma}_x$ ) with the length/thickness ratio ( $l/h$ ) of FGM square plates

ceramic and metal (homogeneous) plates are identical because they do not depend on Young's modulus. Additionally, the through-thickness distribution of in-plane

normal and shear stresses is nearly linear for isotropic plates (with  $p=0$  and  $p=\infty$ ) but becomes nonlinear for graded plates as the volume fraction index varies through the thickness.

Figure 8 illustrates that the in-plane stress ( $\bar{\sigma}_x$ ) is tensile at the upper surface and compressive at the lower surface of the FG square plate. For  $p=0$  (homogeneous ceramic material), the stress distribution is a standard linear function with the neutral plane located at  $z/h=0$ . The maximum compressive stresses occur at the lower edge of the plate, while the maximum tensile stresses are found at the upper edge. In contrast, the in-plane shear stress ( $\bar{\tau}_{xy}$ ) is tensile at the lower surface and compressive at the upper surface. For the homogeneous plate, the maximum tensile shear stress is observed at the lower surface, while the minimum compressive shear stress is at the upper surface, as shown in Fig. 9.

Furthermore, the effect of the side-to-thickness ratio ( $l/h$ ) on the transverse displacement and stresses of FGM plates under a uniformly distributed load with  $p=0$  has

also been analyzed, as presented in Table 3. The results obtained using the present element show good agreement with those reported by Tati [20], Zenkour [15], and Reddy [13]. Table 3 also indicates that as the side-to-thickness ratio ( $l/h$ ) increases, the in-plane stresses ( $\bar{\sigma}_x$ ), ( $\bar{\sigma}_y$ ) and ( $\bar{\tau}_{xy}$ ) increase, while the non-dimensional transverse displacement ( $\bar{w}$ ) decreases.

The variation in dimensionless displacements and stresses for various side-to-thickness ratios ( $l/h$ ) and power-law index values  $p$  for the present element (SBRP24) is shown in Figs. 10, 11 and 12.

Figure 10 depicts the variation in dimensionless deflection for several gradient indexes ( $p$ ) and with the different side-to-thickness ratios ( $l/h$ ), respectively. It can be seen from the figure that the dimensionless deflection of FGM plates is between those of ceramic and metal, and the dimensionless deflection of metal-rich plates is higher than that of ceramic-rich plates because the Young's modulus of ceramic ( $Al_2O_3 = 380$  GPa) is more than that of metal ( $Al = 70$  GPa). Therefore, for the FGM plates, the dimensionless deflection increases with increasing gradient index ( $p$ ). This is due to the fact that increasing the gradient index increases the volume fraction of metal, which reduces the bending stiffness of the FG plates and makes these FG plates more flexible,

resulting in larger deflection. On the other hand, it may remain unchanged as the ratio of side to thickness increases.

As shown in Fig. 11, the dimensionless normal stress ( $\bar{\sigma}_x$ ) increase as the side-to-thickness ratio ( $l/h$ ) increases and decreases as the gradient index ( $p$ ) decreases.

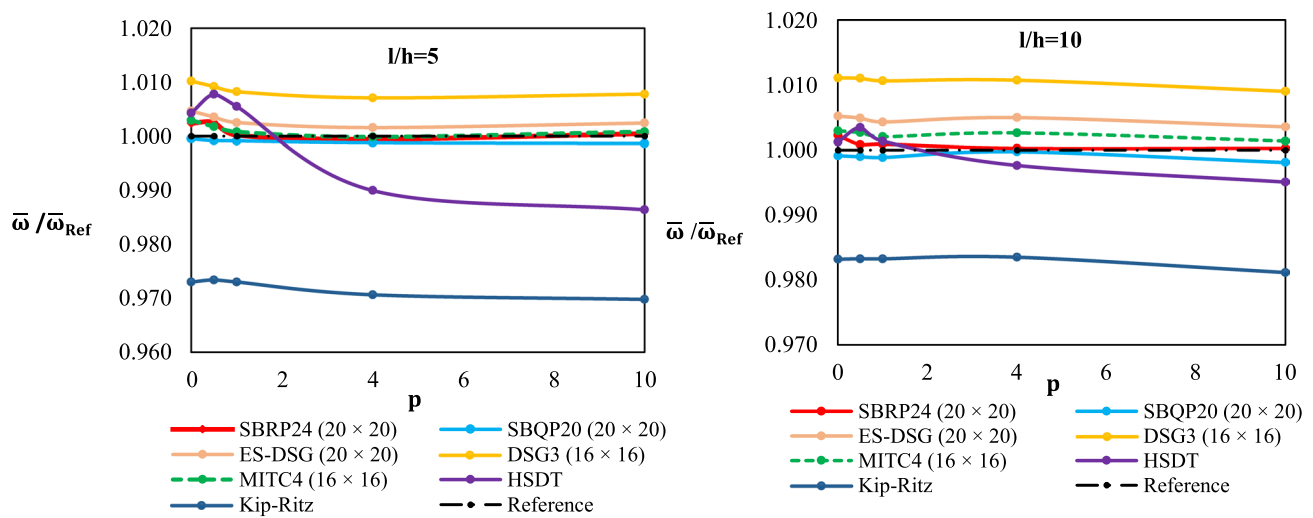
Figure 12 illustrates the variation in dimensionless in-plane shear stress ( $\bar{\tau}_{xy}$ ) for different side-to-thickness ratios ( $l/h$ ) and gradient indices ( $p$ ). The figure shows that the longitudinal shear stress ( $\bar{\tau}_{xy}$ ) increases with the side-to-thickness ratio. Additionally, it decreases as the gradient index ( $p$ ) ranges from 0 to 2 and then increases with a higher power-law index ( $p$ ). This behavior is attributed to the reduction in plate stiffness.

#### 4.2 Problem 2 free vibrational analysis of SSSS FG plate.

Consider a simply supported  $Al/Al_2O_3$  plate with three thickness-to-side ratios ( $l/h = 5, 10$  and  $20$ ) and different gradient index ( $p$ ). This problem has been studied previously by a number of researchers. For example, it was investigated by Matsunaga [16] using a HSDT. Zhao et al. [41] using kip-Ritz method based on a FSDT. Belounar et al. [11] using strain approach. Hosseini-Hashemi et al.

**Table 4** Dimensionless natural frequency ( $\bar{\omega}$ ) of FG square plate with various gradient index  $p$

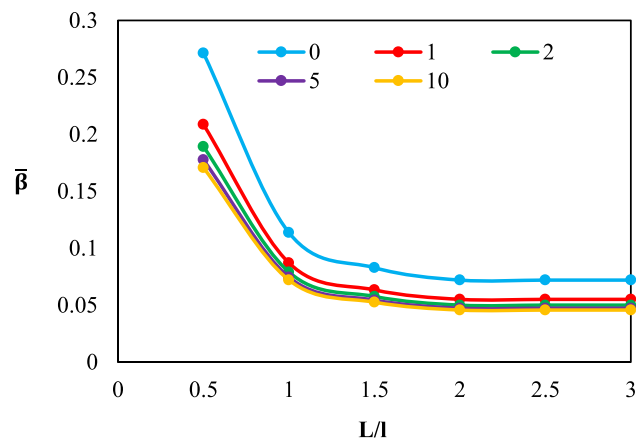
$l/h$	Methods	$P$				
		0	0.5	1	4	10
5	Present	0.21170	0.18096	0.16313	0.13962	0.13246
	SBQP [11]	0.21110	0.18035	0.16296	0.13953	0.13222
	ES-DSG [9]	0.21218	0.18114	0.16351	0.13992	0.13272
	DSG3 [9]	0.21335	0.18216	0.16444	0.14069	0.13343
	MITC4 [9]	0.21182	0.18082	0.16323	0.13968	0.13251
	HSDT [16]	0.21210	0.18190	0.16400	0.13830	0.13060
	Kip-Ritz [41]	0.20550	0.17570	0.15870	0.13560	0.12840
	Reference [8]	0.21120	0.18050	0.16310	0.13970	0.13240
10	Present	0.05783	0.04904	0.04424	0.03821	0.03661
	SBQP [11]	0.05765	0.04895	0.04415	0.03819	0.03653
	ES-DSG [9]	0.05800	0.04924	0.04439	0.03839	0.03673
	DSG3 [9]	0.05834	0.04954	0.04467	0.03861	0.03693
	MITC4 [9]	0.05787	0.04913	0.04429	0.03830	0.03665
	HSDT [16]	0.05777	0.04917	0.04426	0.03811	0.03642
	Kip-Ritz [41]	0.05673	0.04818	0.04346	0.03757	0.03591
	Reference [8]	0.05770	0.04900	0.04420	0.03820	0.03660
20	Present	0.01483	0.01257	0.01133	0.00983	0.00943
	SBQP [11]	0.01479	0.01253	0.01129	0.00980	0.00941
	ES-DSG [9]	0.01488	0.01261	0.01137	0.00986	0.00946
	DSG3 [9]	0.01498	0.01270	0.01145	0.00993	0.00952
	MITC4 [9]	0.01485	0.01258	0.01134	0.00984	0.00944
	Kip-Ritz [41]	0.01464	0.01241	0.01118	0.00970	0.00931
	Reference [8]	0.01480	0.01250	0.01130	0.00980	0.00940



**Fig. 13** Normalized frequency  $\bar{\omega} / \bar{\omega}_{Ref}$  of FG square plate with various gradient index  $p$

**Table 5** Dimensionless frequencies ( $\beta$ ) of FG square plates for various gradient  $p$  and aspect ratios ( $L/h$ )

$L/h$	$l/h$	Theory				
		$p$	Present	Zaoui et al. [43]	Sadgui and Tati [22]	Jin et al. [42]
1	10	0	0.1137	0.1137	0.1136	0.1135
		1	0.0871	0.0883	0.0870	0.0870
		2	0.0791	0.0807	0.0788	0.0789
		5	0.0746	0.0756	0.0738	0.0741
	5	0	0.4163	0.4178	0.4156	0.4169
		1	0.3214	0.3267	0.3210	0.3222
		2	0.2913	0.2968	0.2883	0.2905
		5	0.2718	0.2725	0.2632	0.2676
	2	0	1.8252	1.8583	1.8224	1.8470
		1	1.4453	1.4830	1.4435	1.4687
		2	1.3040	1.3269	1.2675	1.3095
		5	1.1815	1.1576	1.0829	1.1450
2	10	0	0.0720	0.0719	0.0722	0.0719
		1	0.0551	0.0558	0.0553	0.0550
		2	0.0500	0.0511	0.0501	0.0499
		5	0.0473	0.0480	0.0742	0.0471
	5	0	0.2714	0.2718	0.2721	0.2713
		1	0.2087	0.2119	0.2094	0.2088
		2	0.1893	0.1930	0.1888	0.1888
		5	0.1775	0.1788	0.1743	0.1754
	2	0	0.9579	1.3086	1.2943	0.9570
		1	0.7966	1.0378	1.0172	0.7937
		2	0.7195	0.9322	0.8988	0.7149
		5	0.6216	0.8250	0.7824	0.6168



**Fig. 14** Impact of aspect ratios ( $L/l$ ) on the dimensionless natural frequency ( $\bar{\beta}$ ) of FG plates for various gradient index  $p$  with ( $l/h = 10$ )

[8] suggested an analytical approach based on the FSDT to obtain exact solutions to the free vibration problem of Levy-type rectangular FGM plates. The obtained results of the first frequency parameter ( $\bar{\omega}$ ) with  $20 \times 20$  mesh are tabulated in Table 4.

Figures 13 illustrates the first frequency normalized to the reference value [8] for several different gradient index

(p). It can be seen that the present results converge to the analytical results and agree well with several other available ones [9, 11, 16].

Table 5 displays the comparisons of the dimensionless natural frequency ( $\bar{\beta}$ ) obtained by the current element with the 3D precise result suggested by Jin et al. [42], the quasi-3D results presented by Zaoui et al. [43], and a new theory of trigonometric shear deformation (TSDT) developed by Sadgui and Tati [22], for various aspect ratios ( $L/l$ ), length/thickness ratios ( $l/h$ ), and gradient index ( $p$ ). Very close agreement is observed for all conditions, from thick to thin plates. However, they are more similar to those presented by Jin et al. [42]. Part of Table 5 is also plotted in Fig. 14 for illustration purposes. Figure 14 depicts the impact of aspect ratios ( $L/l$ ) on the dimensionless natural frequencies ( $\bar{\beta}$ ). It can be seen from the figure that the dimensionless fundamental frequency decreases as aspect ratio  $l/L$  and the gradient index ( $p$ ) increase.

### 4.3 Problem 3 buckling analysis of SSSS FGM plate.

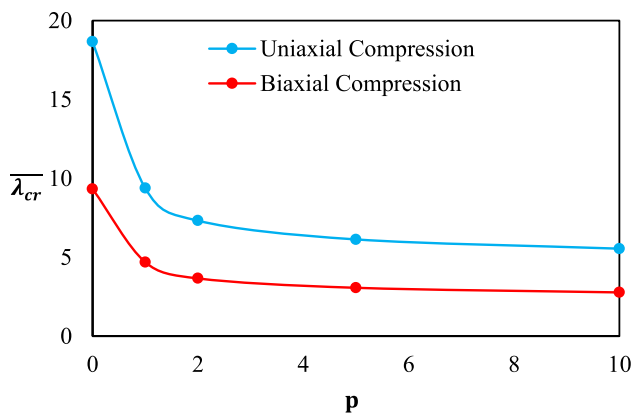
In this problem, the validity of the current element (SBRP24) is tested for buckling behavior of the SSSS FG rectangular plate. The plate is exposed to uniaxial ( $\alpha_1 = 1, \alpha_2 = 0$ ) and biaxial compression loads ( $\alpha_1 = 1, \alpha_2 = 1$ ) with ( $l/L = 1, 2$

**Table 6** Dimensionless buckling loads ( $\bar{\lambda}_{cr}$ ) of FG rectangular plate under uniaxial loading ( $l/L = 1; l/h = 10$ )

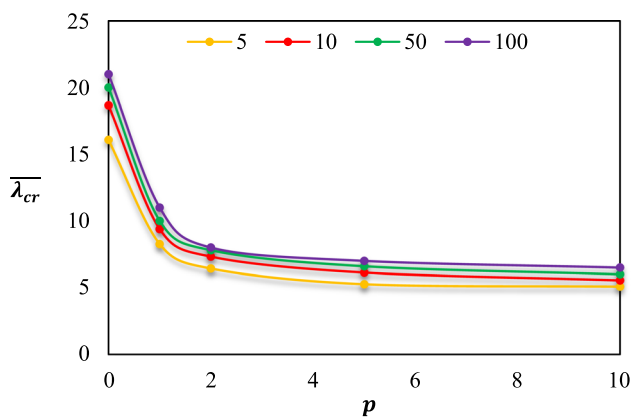
$(N_{x0}, N_{y0})$	Theory	$p=0$	$p=1$	$p=2$	$p=5$	$p=10$
(1,0)	Present	18.6717	9.3862	7.32015	6.0524	5.4667
	Tati [21]	18.6385	9.3696	–	6.0323	5.4435
	Thai and Choi [18]	18.5785	9.3391	7.2631	6.0353	5.4528
	Zenkour and Aljadani [45]	18.5785	9.3391	–	6.0353	5.4528
	Reddy et al. [44]	18.5400	9.2990	7.2100	5.9900	5.4200
(1,1)	Present	9.3358	4.6931	3.6600	3.06623	2.7733
	Tati [21]	9.3193	4.6848	3.6379	3.01613	2.7217
	Thai and Choi [18]	9.2891	4.6701	3.6321	3.0181	2.7347
	Zenkour and Aljadani [45]	9.2892	4.6695	3.6315	3.0176	2.7264
	Reddy et al. [44]	9.2730	4.6500	3.6080	2.9980	2.7150

**Table 7** Dimensionless buckling loads ( $\bar{\lambda}_{cr}$ ) of FG rectangular plate under uniaxial loading ( $l/L = 2; l/h = 10$ )

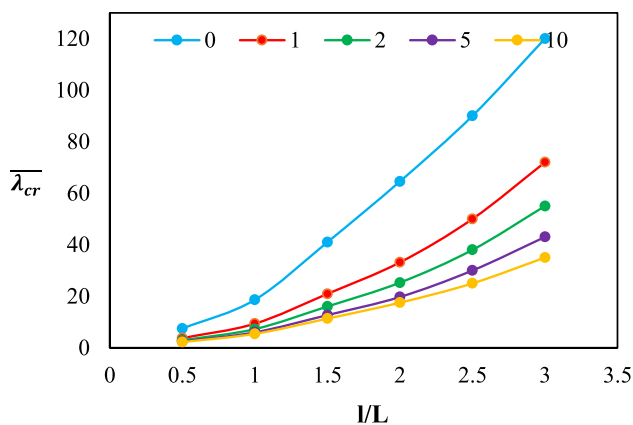
$(N_{x0}, N_{y0})$	Theory	$p=0$	$p=1$	$p=2$	$p=5$	$p=10$
(1,0)	Present	64.5074	33.1222	25.7931	21.1109	17.7719
	Tati [21]	64.3349	33.0332	25.3409	20.0340	17.6971
	Thai and Choi [18]	64.0801	32.8901	25.3701	20.2101	17.9201
	Zenkour and Aljadani [45]	64.0842	32.4600	25.3726	20.2122	17.9227
	Reddy et al. [44]	63.7800	32.8979	24.8600	19.8400	17.7200
(1,1)	Present	21.5953	10.9790	8.5556	7.0791	6.3444
	Tati [21]	21.5960	10.9800	8.4710	6.8448	6.1048
	Thai and Choi [18]	21.5051	10.9321	8.4641	6.8751	6.1481
	Zenkour and Aljadani [45]	21.5049	10.9323	8.4643	6.8749	6.1481
	Reddy et al. [44]	21.4290	10.8300	8.3450	6.7820	6.0950



**Fig. 15** Impact of the in-plane loading on the dimensionless buckling load ( $\bar{\lambda}_{cr}$ ) for a SSSS FG plate with different gradient index ( $p$ )



**Fig. 16** Impact of the gradient index ( $p$ ) on the dimensionless buckling load ( $\bar{\lambda}_{cr}$ ) exposed to uniaxial compression for a SSSS FG square plate with various aspect ratio ( $l/h$ )



**Fig. 17** Influence of aspect ratios ( $l/h$ ) on the dimensionless buckling load ( $\bar{\lambda}_{cr}$ ) exposed to uniaxial compression for a SSSS FG plate with different gradient index ( $p$ )

and  $l/h = 10$ ). The dimensionless critical buckling ( $\bar{\lambda}_{cr}$ ) load of FG plate is listed in Tables 6 and 7 using  $32 \times 16$  mesh. This problem aims to investigate the impact of aspect ratios ( $l/h$ ), gradient index ( $p$ ), on the buckling analysis of FG plate. The comparisons of current results and those given by the analytical results are made using HSDT of Reddy et al. [44], the simple refined solution of Thai and Choi [18], the refined HSDT presented by Zenkour and Aljadani [45], and a simple (HSDT) developed by Tati [21], and a very good agreement can be found. It can be noticed from Fig. 15 that the dimensionless buckling load ( $\bar{\lambda}_{cr}$ ) is greater for uniaxial compression and less for biaxial compression. This is due to applying a tensile load to the plates, which increases bending stiffness. Furthermore, the dimensionless buckling load ( $\bar{\lambda}_{cr}$ ) reduces as the gradient index ( $p$ ) value increases, and it also increases with increasing aspect ratio ( $l/h$ ), as shown in Figs. 16 and 17, respectively.

A comparative table summarizing the key findings from the reviewed studies is provided for quick reference, as shown in Table 8.

## 5 Conclusion

This paper proposes a new four-node rectangular finite element for the static, free vibrational, and buckling behavior of FG plates using the strain approach and Reissner–Mindlin theory. The properties of the FG plate are considered to vary along the thickness direction by the power-law distributions. The notion of a neutral surface has been used to prevent the membrane-bending effect. The developed element contains six DOFs per node. It is derived by combining two elements based on the strain approach: a membrane and bending elements. The displacement fields of these elements have higher-order expressions which satisfy both rigid body modes and compatibility equations. The achieved result exhibits high agreement with the existing results in the literature for thinner to thicker plates. The conclusions that can be drawn from the results obtained for various power indices ( $p$ ), thickness ratios ( $l/h$ ), aspect ratios ( $l/L$ ), and loading conditions are as follows:

- The present element (SBRP24) predicts excellent results for deflections and stresses when plates are exposed to static loading. The dimensionless central displacement increases as the gradient index  $p$  increases because the stiffer ceramic fraction decreases, which makes these FG plates more flexible.
- The dimensionless central displacement reduces with increasing thickness ratios, whereas the in-plane stresses increase.

**Table 8** A comparative summary of the key findings from the reviewed studies

Reference	Theory	Material/structure type	Analysis type	Key findings
Zenkour [15]	SSDPT	FG plates	Static response analysis	The SBRP24 element performs exceptionally well under both uniformly distributed and sinusoidal loads, converging faster than Zenkour's analytical solution, and it also demonstrates quicker convergence compared to Reddy's analytical solution for the static response of simply supported FG square plates. Moreover, the proposed element formulation is both efficient and straightforward in predicting the bending behavior of FGM plates
Reddy [13]	TSDT	FG plates	Non-linear static and dynamic finite element results	
Tati [20]	HSDT	FG plates	bending behavior analysis	
Matsunaga [16]	2D HSDT	FG plates	Vibration, buckling	SBRP24 performs well in coupled buckling and vibrational response
Zhao et al. [41]	The element-free kp-Ritz method	FG plates	Free vibration analysis	
Sadgui and Tati [22]	HSDT	FGM plates	Vibration, stability	Bending, vibration, buckling
Belounar et al. [11, 12]	FSDT + assumed strain FEM	FG square, skew, circular plates	Bending, vibration, buckling	
Reddy et al. [44]	HSDT	FGM plates	Buckling analysis	Buckling analysis
Thai and Choi [18]	RPT theory	FGM plates	Buckling analysis	
Zenkour and Aljadani [45]	Refined HSDT	FGM plates	Mechanical buckling	

- It is deduced that the in-plane and shear stresses are the same for totally metallic and totally ceramic plates.
- The dimensionless natural frequencies reduce as the gradient index increases due to the reduction in stiffness of the FGM plate.
- The dimensionless natural frequencies increase as the thickness ratios increase and decrease as the aspect ratios increase.
- The dimensionless critical buckling load reduces with increasing gradient index  $p$  and increases with increasing thickness ratios and aspect ratios.
- The dimensionless critical buckling loads for the uniaxial compression are higher than for the biaxial compression.

The current finite element formulation is a promising tool for the simulation and computation of FG plates. In the future, this element will be expanded for analyzing FG plates under thermo-mechanical buckling analysis and also for FG shell structures.

## Appendix

The membrane strains  $\{\varepsilon_m\}$  specified in Eq. (36) satisfy the compatibility conditions as follows [29, 30]:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0 \quad (1a)$$

The curvatures and shear strains  $(\kappa_x, \kappa_y, \kappa_{xy}, \gamma_{xz}, \gamma_{yz})$  specified in Eqs. (38) and (39) are satisfied by the compatibility equations [35–37]:

$$\begin{aligned} \frac{\partial^2 \kappa_x}{\partial y^2} + \frac{\partial^2 \kappa_y}{\partial x^2} &= \frac{\partial^2 \kappa_{xy}}{\partial x \partial y} \frac{\partial^2 \gamma_{xz}}{\partial x \partial y} - \frac{\partial^2 \gamma_{yz}}{\partial x^2} + \frac{\partial \kappa_{xy}}{\partial x} \\ &= 2 \frac{\partial \kappa_x}{\partial y} \frac{\partial^2 \gamma_{yz}}{\partial x \partial y} - \frac{\partial^2 \gamma_{xz}}{\partial y^2} + \frac{\partial \kappa_{xy}}{\partial y} \\ &= 2 \frac{\partial \kappa_y}{\partial x} \end{aligned} \quad (2a)$$

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## Declarations

**Conflict of interest** The authors declare that they have no conflict of interest.

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