

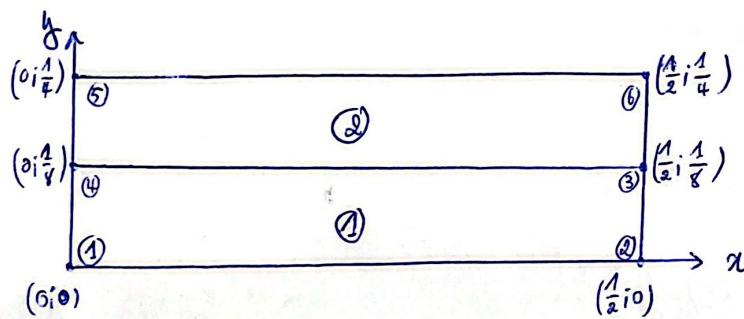
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Exercice :



1. Type d'élément : Q4 (élément plan rectangulaire à 4nœuds et 2 ddl/nœud)

2. Champ de déplacement :

$$u(x, y) = a_1 + a_2 x + a_3 y + a_4 xy$$

$$v(x, y) = b_1 + b_2 x + b_3 y + b_4 xy$$

CAL: élément 1:

$$\begin{cases} u_1 = u(0;0) = a_1 \\ u_2 = u\left(\frac{1}{2}; 0\right) = a_1 + \frac{1}{2}a_2 \\ u_3 = u\left(\frac{1}{2}; \frac{1}{8}\right) = a_1 + \frac{1}{2}a_2 + \frac{1}{8}a_3 + \frac{1}{16}a_4 \\ u_4 = u(0; \frac{1}{8}) = a_1 + \frac{1}{8}a_3 \end{cases} \Rightarrow$$

$$\begin{cases} a_1 = u_1 \\ a_2 = 2(u_2 - u_1) \\ a_3 = 8(u_4 - u_1) \\ a_4 = 16(u_1 + u_3 - u_2 - u_4) \end{cases}$$

Fonctions de forme

$$\begin{cases} N_1 = 1 - 2x - 8y + 16xy \\ N_2 = 2x - 16xy \\ N_3 = 16xy \\ N_4 = 8y - 16xy \end{cases}$$

$$[B] = \begin{bmatrix} -2+16y & 0 & 0 & 0 & -16y & 0 \\ 0 & -8+16x & 0 & -16x & 0 & 16x \\ 0 & -8+16x & 0 & 2-16y & 16x & 16y \\ -8+16x & -2+16y & -16x & 2-16y & 16x & 8-16x \end{bmatrix}$$

- Etat de contrainte plane :  $l, L \gg e$

$$[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \frac{80}{3} \begin{bmatrix} 8 & 2 & 0 \\ 2 & 8 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad (\text{GPa})$$

- Matrice de rigidité  $K$ :

$$[K] = \iiint [B]^T [D] [B] dV$$

$$K_{33} = \iint_{\Omega} \left\{ \begin{array}{c} 16x \\ 16y \\ 0 \end{array} \right\} \left\{ \begin{array}{c} 8 & 2 & 0 \\ 2 & 8 & 0 \\ 0 & 0 & 3 \end{array} \right\} \left\{ \begin{array}{c} 2-16y \\ 0 \\ -16x \end{array} \right\} \cdot e = \iint_{\Omega} \left\{ \begin{array}{c} 16-128y \\ 16-128y \\ 4-32y \end{array} \right\} \left\{ \begin{array}{c} 2-16y \\ 0 \\ -16x \end{array} \right\} \cdot e$$

$$K_{33} = \iint_{\Omega} (32-256y - 256y + 2048y^2 + 768x^2) dx dy \cdot e = \frac{7}{60}$$

$$K_{34} = \iint \left\{ \begin{array}{c} 16-128y \\ 4-32y \\ -48y \end{array} \right\} \left\{ \begin{array}{c} 0 \\ -16x \\ 2-16y \end{array} \right\} \cdot e = \iint (-64x + 512xy - 96y + 768y^2) dx dy \cdot e$$

$$K_{34} = -\frac{1}{64}$$

$$K_{35} = \iint \left\{ \begin{array}{c} 16-128y \\ 4-32y \\ -48y \end{array} \right\} \left\{ \begin{array}{c} 16y \\ 0 \\ 16x \end{array} \right\} \cdot e = \iint (256y - 2048y^2 - 768xy) dx dy \cdot e$$

$$K_{35} = -\frac{1}{96}$$

$$K_{36} = \iint \left\{ \begin{array}{c} 16-128y \\ 4-32y \\ -48y \end{array} \right\} \left\{ \begin{array}{c} 0 \\ 16x \\ 16y \end{array} \right\} \cdot e = \iint (64x - 512xy - 768y^2) dx dy \cdot e$$

$$K_{36} = \frac{1}{160}$$

$$K_{44} = \iint \left\{ \begin{array}{c} -32x \\ -128x \\ 6-48y \end{array} \right\} \left\{ \begin{array}{c} 0 \\ -16x \\ 2-16y \end{array} \right\} \cdot e = \iint (2048x^2 + 12 - 96y - 96y + 768y^2) dx dy \cdot e$$

$$K_{44} = \frac{131}{480}$$

$$K_{45} = \iint \left\{ \begin{array}{c} -32x \\ -128x \\ 6-48y \end{array} \right\} \left\{ \begin{array}{c} 16y \\ 0 \\ 16x \end{array} \right\} \cdot e = \iint (-512xy + 96x - 768xy) dx dy \cdot e$$

$$K_{45} = -\frac{1}{160}$$

$$K_{46} = \iint \left\{ -32x \quad -128x \quad 6-48y \right\} \begin{Bmatrix} 0 \\ 16x \\ 16y \end{Bmatrix} e = \iint (-2048x^2 + 96y - 768y^2) dx dy \cdot e$$

$$K_{46} = -\frac{853}{960}$$

$$K_{55} = \iint \left\{ 128y \quad 32y \quad 48x \right\} \begin{Bmatrix} 16y \\ 0 \\ 16x \end{Bmatrix} e = \iint (2048y^2 + 768x^2) dx dy \cdot e$$

$$K_{55} = \frac{7}{60}$$

$$K_{56} = \iint \left\{ 128y \quad 32y \quad 48x \right\} \begin{Bmatrix} 0 \\ 16x \\ 16y \end{Bmatrix} e = \iint (512xy + 768xy) dx dy \cdot e$$

$$K_{56} = \frac{1}{32}$$

$$K_{66} = \iint \left\{ 32x \quad 128x \quad 48y \right\} \begin{Bmatrix} 0 \\ 16x \\ 16y \end{Bmatrix} e = \iint (2048x^2 + 768y^2) dx dy \cdot e$$

$$K_{66} = \frac{131}{480}$$

$$\left[ K_{red} \right] = \frac{80}{3} \begin{bmatrix} u_2 & v_2 & u_3 & v_3 \\ \frac{7}{60} & -\frac{1}{64} & -\frac{1}{96} & \frac{1}{160} \\ -\frac{1}{64} & \frac{131}{480} & \frac{1}{160} & -\frac{253}{960} \\ -\frac{1}{96} & \frac{1}{160} & \frac{7}{60} & \frac{1}{32} \\ \frac{1}{160} & -\frac{253}{960} & \frac{1}{32} & \frac{131}{480} \end{bmatrix} \quad (GN/m)$$

- élément 2 :

champ de déplacement :  $U(x_1 y) =$

$$v(x_1 y) =$$

$$\left\{ \begin{array}{l} U_1 = U\left(\frac{1}{2}; \frac{1}{4}\right) = a_1 + \frac{1}{2}a_2 + \frac{1}{8}a_3 + \frac{1}{16}a_4 \\ U_2 = U(0; \frac{1}{8}) = a_1 + \frac{1}{8}a_3 \\ U_3 = U(0; \frac{1}{4}) = a_1 + \frac{1}{4}a_3 \\ U_4 = U\left(\frac{1}{2}; \frac{1}{4}\right) = a_1 + \frac{1}{2}a_2 + \frac{1}{4}a_3 + \frac{1}{8}a_4 \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} a_1 = 2U_2 - U_3 \\ a_2 = 2(2U_1 + U_3 - 2U_2 - U_4) \\ a_3 = 8(U_3 - U_2) \\ a_4 = 16(U_4 + U_2 - U_1 - U_3) \end{array} \right.$$

Fonctions de Formes

$$\left\{ \begin{array}{l} N_1 = 4x - 16xy \\ N_2 = 2 - 4x - 8y + 16xy \\ N_3 = -1 + 2x + 8y - 16xy \\ N_4 = -2x + 16xy \end{array} \right. \leq N_i = 1 \checkmark$$

$$[B] = \begin{bmatrix} 4-16y & 0 & -4+16y & 0 & 2-16y & 0 & -2+16y & 0 \\ 0 & -16x & 0 & -8+16x & 0 & 8-16x & 0 & 16x \\ -16x & 4-16y & -8+16x & -4+16y & 8-16x & 2-16y & 16x & -2+16y \end{bmatrix}$$

$$[D] = \frac{80}{3} \begin{bmatrix} 8 & 2 & 0 \\ 2 & 8 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{ (GPa)}$$

- Matrice de rigidité  $K$ :  $[K] = \iiint [B]^T [D] [B] dV$

$$K_{11} = \iiint \begin{bmatrix} 4-16y & 0 & -16x \end{bmatrix} \begin{bmatrix} 8 & 2 & 0 \\ 2 & 8 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 4-16y \\ 0 \\ -16x \end{bmatrix} \iiint \begin{bmatrix} 32-128y & 8-32y & -48x \end{bmatrix} \begin{bmatrix} 4-16y \\ 0 \\ -16x \end{bmatrix}$$

$$K_{11} = \iiint 128 - 1024y + 2048y^2 + 768x^2 dx dy dz = \frac{13}{60}$$

$$K_{12} = \iiint \begin{Bmatrix} 4-16y & 0 & -16x \end{Bmatrix} \begin{bmatrix} 8 & 2 & 0 \\ 2 & 8 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} 0 \\ -16x \\ 4-16y \end{Bmatrix} = \iiint \begin{Bmatrix} 32-128y & 8-32y & -48x \end{Bmatrix} \begin{Bmatrix} 0 \\ -16x \\ 4-16y \end{Bmatrix} \cdot e$$

$$K_{12} = \iint_{0}^{16/12} (-128x + 512xy - 192x + 768xy) dx dy \cdot e = -\frac{3}{32}$$

$$K_{17} = \iint \begin{Bmatrix} 32-128y & 8-32y & -48x \end{Bmatrix} \begin{Bmatrix} -2+16y \\ 0 \\ 16x \end{Bmatrix} e = \iint (-64 + 512y + 512y - 2048y^2 - 768x^2) dx dy \cdot e$$

$$K_{17} = -\frac{7}{60}$$

$$K_{18} = \iint (32-128y & 8-32y & -48x) \begin{Bmatrix} 0 \\ 16x \\ -2+16y \end{Bmatrix} e = \iint (128x - 512xy + 96x - 768xy) dx dy \cdot e$$

$$K_{18} = \frac{9}{160}$$

$$K_{22} = \iint \begin{Bmatrix} -32x & -128x & 12-48y \end{Bmatrix} \begin{Bmatrix} 0 \\ -16x \\ 4-16y \end{Bmatrix} e = \iint (2048x^2 + 48 - 192y - 192y + 768y^2) dx dy \cdot e$$

$$K_{22} = \frac{149}{480}$$

$$K_{27} = \iint \begin{Bmatrix} -32x & -128x & 12-48y \end{Bmatrix} \begin{Bmatrix} -2+16y \\ 0 \\ 16x \end{Bmatrix} e = \iint (64x - 512xy + 192x - 768xy) dx dy \cdot e$$

$$K_{27} = \frac{11}{160}$$

$$K_{28} = \iint \begin{Bmatrix} -32x & -128x & 12-48y \end{Bmatrix} \begin{Bmatrix} 0 \\ 16x \\ -2+16y \end{Bmatrix} e = \iint (-2048x^2 - 24 + 192y + 96y - 768y^2) dx dy \cdot e$$

$$K_{28} = \frac{-271}{960}$$

$$K_{77} = \iint \begin{Bmatrix} -16+128y & -4+32y & 48x \end{Bmatrix} \begin{Bmatrix} -2+16y \\ 0 \\ 16x \end{Bmatrix} e = \iint (32 - 256y - 256y + 2048y^2 + 768x^2) dx dy \cdot e$$

$$K_{77} = \frac{7}{60}$$

$$K_{78} = \iint \begin{Bmatrix} -16+128y & -4+32y & 48x \end{Bmatrix} \begin{Bmatrix} 0 \\ 16x \\ -2+16y \end{Bmatrix} e = \iint (-64x + 512xy - 96x + 768xy) dx dy \cdot e$$

$$K_{78} = -\frac{1}{32}$$

$$K_{88} = \iint \left\{ 32x - 128x - 6 + 48y \right\} \begin{cases} 16x \\ -2 + 16y \end{cases} e = \iint (-2048x^2 + 12 - 46y - 96y + 768y^2) dx dy e$$

$$K_{88} = \frac{131}{480}$$

$$[K_{red}]^2 = \frac{q_0}{3} \begin{bmatrix} u_2 & v_3 & u_6 & v_6 \\ \frac{13}{60} & \frac{-3}{32} & \frac{-7}{60} & \frac{9}{160} \\ -\frac{3}{32} & \frac{149}{480} & \frac{11}{160} & -\frac{271}{960} \\ -\frac{7}{60} & \frac{11}{160} & \frac{7}{60} & -\frac{1}{32} \\ \frac{9}{160} & -\frac{271}{960} & -\frac{1}{32} & \frac{131}{480} \end{bmatrix} \quad (GN/m)$$

- Assemblage :

$$[K] = \frac{q_0}{3} \begin{bmatrix} u_2 & v_1 & u_3 & v_2 & u_6 & v_6 \\ \frac{7}{60} & -\frac{1}{64} & -\frac{1}{96} & \frac{1}{960} & 0 & 0 \\ -\frac{1}{64} & \frac{131}{480} & \frac{1}{480} & \frac{253}{960} & 0 & 0 \\ -\frac{1}{96} & \frac{1}{480} & \frac{1}{3} & -\frac{1}{15} & -\frac{7}{60} & \frac{9}{160} \\ \frac{1}{160} & \frac{253}{960} & -\frac{7}{64} & \frac{7}{12} & \frac{11}{160} & -\frac{271}{960} \\ 0 & 0 & -\frac{7}{60} & \frac{11}{160} & \frac{7}{60} & -\frac{1}{32} \\ 0 & 0 & \frac{9}{160} & -\frac{271}{960} & -\frac{1}{32} & \frac{131}{480} \end{bmatrix} \quad (GN/m)$$

- Calculate displacement :  $F = K \cdot U$

Vector Force :

$$F = 10 \cdot (0,025 \cdot 0,25) / 3 = \frac{125}{6} \text{ KN} = 20,83 \text{ KN}$$

$$\begin{bmatrix} 20,83 \\ 0 \\ 20,83 \\ 0 \\ 20,83 \\ 0 \end{bmatrix} = [K] \begin{bmatrix} u_2 \\ v_1 \\ u_3 \\ v_2 \\ u_6 \\ v_6 \end{bmatrix} \Rightarrow \begin{cases} u_2 = 0,009 \text{ mm} \\ v_1 = 0,01 \text{ mm} \\ u_3 = 0,008 \text{ mm} \\ v_2 = -0,01 \text{ mm} \\ u_6 = 0,018 \text{ mm} \\ v_6 = -0,01 \text{ mm} \end{cases}$$

2- Les contraintes dans l'élément :

$$\boldsymbol{\sigma} = [0] \{B\} \{u\}$$

élément 1 :

$$\begin{pmatrix} 6x^0 \\ 6y^0 \\ 2xy^0 \end{pmatrix} = [D_1] [B_1] \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \frac{80}{3} \begin{pmatrix} -80x - 16y + 18 \\ 125 \\ -640x - 8y + 9 \\ 250 \\ -12x - 240y + 15 \\ 250 \end{pmatrix}$$

élément 2 :

$$\begin{pmatrix} 6x^0 \\ 6y^0 \\ 2xy^0 \end{pmatrix} = [D_2] [B_2] \begin{pmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{pmatrix} = \frac{80}{3} \begin{pmatrix} 160y - 4 \\ 125 \\ 40y - 1 \\ 125 \\ 24x - 3 \\ 50 \end{pmatrix}$$

$$[D_1] [B_1] = \begin{bmatrix} -128y - 16 & 32x - 16 & -128 + 16 & -32x & 128y & 32x & -128y & -32x + 16 \\ 32y - 6 & 128x - 64 & -32y + 4 & -128x & 32y & 128x & -32y & -128x + 64 \\ 48x - 24 & 48y - 6 & -48x & -48 + 6 & 48x & 48y & -48 + 24 & -48y \end{bmatrix}$$

$$[D_2] [B_2] = \begin{bmatrix} -128y + 32 & -32x & 128y - 32 & 32x - 16 & -128y + 16 & -32x + 16 & 128y - 16 & 32x \\ -32y + 8 & -128x & 32y - 8 & 128x - 64 & -32y + 4 & -128x + 64 & 32y - 4 & 128x \\ -48x & -48y + 12 & 48x - 24 & 48y - 12 & -48x + 24 & -48y + 6 & 48x & 48y - 6 \end{bmatrix}$$