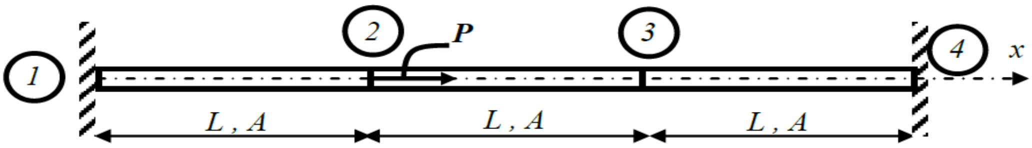
EXERCICE N°01



1. Matrice de Rigidité Globale

➢ ***Matrices de Rigidité Élémentaire***

$$[k^{(e)}] = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[\boldsymbol{k}^{(1)}] = [\boldsymbol{k}^{(2)}] = [\boldsymbol{k}^{(3)}] = \frac{\boldsymbol{AE}}{\boldsymbol{L}}\begin{bmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{1} \end{bmatrix}$$

➢ ***Assemblage des Matrices Élémentaires en Matrice Globale***

$$[K] = \sum_{i=1}^{n}[k^{(i)}] \qquad \text{Soit :} \qquad [K] = [k^{(1)}] + [k^{(2)}] + [k^{(3)}]$$

✓ ***Concept de la Superposition (Méthode de Rigidité Directe)***

$$[k_{e\,1-2}^{(1)}] = \frac{EA}{L}\begin{array}{cc} u_1 & u_2 \\ \end{array}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{array}{c} u_1 \\ u_2 \end{array} \qquad [k_{e\,2-3}^{(2)}] = \frac{EA}{L}\begin{array}{cc} u_2 & u_3 \\ \end{array}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{array}{c} u_2 \\ u_3 \end{array}$$

$$[k_{e\,3-4}^{(3)}] = \frac{EA}{L}\begin{array}{cc} u_3 & u_4 \\ \end{array}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{array}{c} u_3 \\ u_4 \end{array}$$

$$[\boldsymbol{K}] = \frac{\boldsymbol{EA}}{\boldsymbol{L}}\begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array}\begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{1} \end{bmatrix}\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array}$$

2. Déplacements u_2 et u_3

$$\{F\} = [K]\,\{U\}$$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \\ u_4 = 0 \end{Bmatrix}$$

$$\begin{Bmatrix} F_{2x} \\ F_{3x} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} P \\ 0 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$\boldsymbol{u_2 = \frac{2\,PL}{3\,EA}} \qquad ; \qquad \boldsymbol{u_3 = \frac{PL}{3\,EA}}$$

3. **Réactions aux Nœuds *1* et *4***

$$\{F\} = [K]\,\{U\}$$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\boldsymbol{F_{1x} = -\tfrac{2}{3}P \quad , \quad F_{4x} = -\tfrac{1}{3}P}$$

- ***Vérification :*** $F_{2x} = F_{1x} + F_{4x}$ ***OK !!!***

EXERCICE N°02

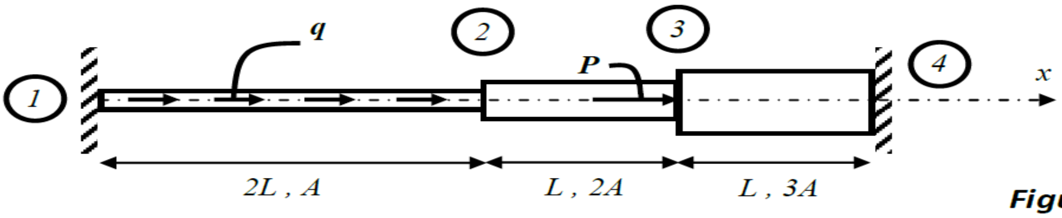


Figure 2

1. **Matrice de Rigidité Globale**

- ***Matrices de Rigidité Élémentaire***

$$\left[k^{(1)}\right] = \frac{EA}{2L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{soit :} \quad \left[\boldsymbol{k}^{(1)}\right] = \frac{\boldsymbol{EA}}{\boldsymbol{L}}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\left[k^{(2)}\right] = \frac{2EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{soit :} \quad \left[\boldsymbol{k}^{(2)}\right] = \frac{\boldsymbol{EA}}{\boldsymbol{L}}\begin{bmatrix} \boldsymbol{2} & \boldsymbol{-2} \\ \boldsymbol{-2} & \boldsymbol{2} \end{bmatrix}$$

$$\left[k^{(3)}\right] = \frac{3EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{soit :} \quad \left[\boldsymbol{k}^{(3)}\right] = \frac{\boldsymbol{EA}}{\boldsymbol{L}}\begin{bmatrix} \boldsymbol{3} & \boldsymbol{-3} \\ \boldsymbol{-3} & \boldsymbol{3} \end{bmatrix}$$

- ***Assemblage des Matrices Élémentaires en Matrice Globale***

$$[K] = \sum_{i=1}^{n}\left[k^{(i)}\right] \qquad \text{Soit :} \qquad [K] = \left[k^{(1)}\right] + \left[k^{(2)}\right] + \left[k^{(3)}\right]$$

✓ ***Concept de la Superposition (Méthode de Rigidité Directe)***

$$\left[k^{(1)}_{e\,1-2}\right] = \frac{EA}{L}\begin{array}{c} \begin{array}{cc} \boldsymbol{u_1} & \boldsymbol{u_2} \end{array} \\ \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} \end{array}\begin{array}{c} \boldsymbol{u_1} \\ \boldsymbol{u_2} \end{array} \qquad \left[k^{(2)}_{e\,2-3}\right] = \frac{EA}{L}\begin{array}{c} \begin{array}{cc} \boldsymbol{u_2} & \boldsymbol{u_3} \end{array} \\ \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \end{array}\begin{array}{c} \boldsymbol{u_2} \\ \boldsymbol{u_3} \end{array}$$

$$\left[k^{(3)}_{e\,3-4}\right] = \frac{EA}{L}\begin{array}{c} \begin{array}{cc} \boldsymbol{u_3} & \boldsymbol{u_4} \end{array} \\ \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \end{array}\begin{array}{c} \boldsymbol{u_3} \\ \boldsymbol{u_4} \end{array}$$

$$[\boldsymbol{K}] = \frac{\boldsymbol{EA}}{\boldsymbol{L}}\begin{bmatrix} \boldsymbol{0.5} & \boldsymbol{-0.5} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{-0.5} & \boldsymbol{2.5} & \boldsymbol{-2} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{-2} & \boldsymbol{5} & \boldsymbol{-3} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{-3} & \boldsymbol{3} \end{bmatrix}$$

2. Déplacements Nodaux

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 0.5 & -0.5 & 0 & 0 \\ -0.5 & 2.5 & -2 & 0 \\ 0 & -2 & 5 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\begin{Bmatrix} F_{1x} = R_1 + qL \\ F_{2x} = qL \\ F_{3x} = P = 2qL \\ F_{4x} = R_4 \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 0.5 & -0.5 & 0 & 0 \\ -0.5 & 2.5 & -2 & 0 \\ 0 & -2 & 5 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \\ u_4 = 0 \end{Bmatrix}$$

$$\begin{Bmatrix} 1 \\ 2 \end{Bmatrix} qL = \frac{EA}{L} \begin{bmatrix} 2.5 & -2 \\ -2 & 5 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$qL \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 2.5 & -2 \\ -2 & 5 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$\boldsymbol{u_2 = \frac{18}{17} \frac{qL^2}{EA}} \quad ; \quad \boldsymbol{u_3 = \frac{14}{17} \frac{qL^2}{EA}}$$

3. Forces Nodales Globales

$$\begin{Bmatrix} F_{1x} = R_1 + qL \\ F_{2x} \\ F_{3x} \\ F_{4x} = R_4 \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 0.5 & -0.5 & 0 & 0 \\ -0.5 & 2.5 & -2 & 0 \\ 0 & -2 & 5 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\boldsymbol{R_1 = -\frac{26}{17} qL} \quad ; \quad \boldsymbol{R_4 = -\frac{42}{17} qL}$$

EXERCICE N°03

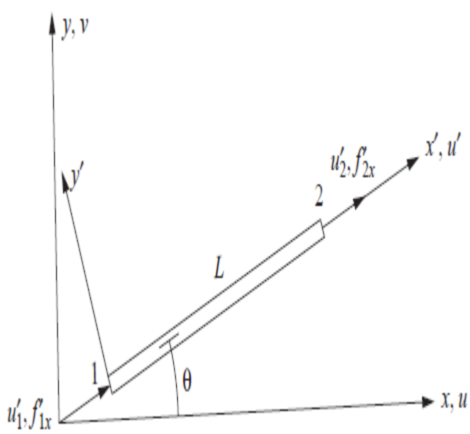
- **Matrice de Rigidité Élémentaire**

$$\begin{Bmatrix} f'_{1x} \\ f'_{2x} \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix}$$

$$[k_e] = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ f'_{2x} \\ f'_{2y} \end{Bmatrix} = [K_e] \begin{Bmatrix} u'_1 \\ v'_1 \\ u'_2 \\ v'_2 \end{Bmatrix}$$

$$[k_e] = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



- **Matrice de Rigidité Globale dans le Repère Global**

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = [K] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_1 \end{Bmatrix}$$

➢ Le changement de base est alors possible en posant que :

$$[K] = [R_e]^T [k_e][R_e]$$

avec :

$$[R_e] = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix} \qquad \text{où :} \qquad C = cos\,\theta \quad , \quad S = sin\,\theta$$

$$[\boldsymbol{K}] = \frac{\boldsymbol{EA}}{\boldsymbol{L}} \begin{bmatrix} \boldsymbol{C^2} & \boldsymbol{CS} & \boldsymbol{-C^2} & \boldsymbol{-CS} \\ \boldsymbol{CS} & \boldsymbol{S^2} & \boldsymbol{-CS} & \boldsymbol{-S^2} \\ \boldsymbol{-C^2} & \boldsymbol{-CS} & \boldsymbol{C^2} & \boldsymbol{CS} \\ \boldsymbol{-CS} & \boldsymbol{-S^2} & \boldsymbol{CS} & \boldsymbol{S^2} \end{bmatrix}$$

$$\theta = 30 \quad , \quad C = cos\,30 = \frac{\sqrt{3}}{2} \quad , \quad S = sin\,30 = \frac{1}{2}$$

$$[\boldsymbol{K}]_{(\boldsymbol{KN/m})} = \boldsymbol{7.10^4} \begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}$$

EXERCICE N°04

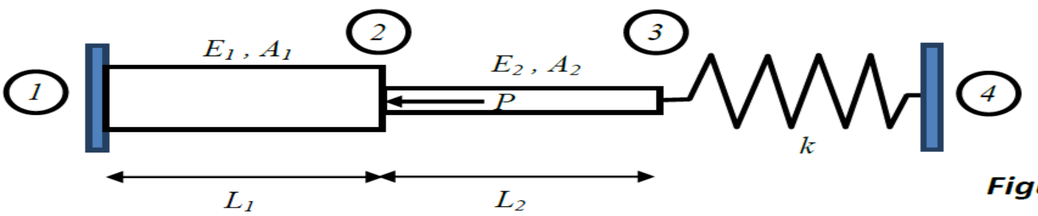


Figure 3

1. Matrice de rigidité

➢ ***Elément 1***

$$[k^{(1)}] = \frac{E_1\,A_1}{L_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

➢ ***Elément 2***

$$[k^{(2)}] = \frac{E_2\,A_2}{L_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- ***Elément 3***

$$\left[k^{(3)}\right] = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$k = \frac{E_1\,A_1}{L_1}$$

$$\left[k^{(3)}\right] = \frac{E_1 A_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$E_1 = 3E_2 = 3E \quad \text{et} \quad L_1 = L_2 = L$$

- Les sections droites sont de diamètres d_1 et d_2 avec $d_1=2d_2=2d$

$$A_1 = \frac{\pi\,d_1^2}{4} \qquad \text{donc :} \qquad A_1 = \pi\,d_2^2$$

et $$A_2 = \frac{\pi\,d_2^2}{4} \qquad \text{d'où :} \qquad A_1 = 4\,A_2$$

$$\left[\boldsymbol{k}^{(1)}\right] = \frac{\boldsymbol{3\pi\,E\,d^2}}{\boldsymbol{L}}\begin{bmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{1} \end{bmatrix}$$

$$\left[\boldsymbol{k}^{(2)}\right] = \frac{\boldsymbol{\pi\,E\,d^2}}{\boldsymbol{4\,L}}\begin{bmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{1} \end{bmatrix}$$

$$\left[\boldsymbol{k}^{(3)}\right] = \frac{\boldsymbol{3\pi\,E\,d^2}}{\boldsymbol{L}}\begin{bmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{1} \end{bmatrix}$$

$$[\boldsymbol{K}] = \frac{\boldsymbol{\pi\,E\,d^2}}{\boldsymbol{L}}\begin{bmatrix} \mathbf{3} & \mathbf{-3} & \mathbf{0} & \mathbf{0} \\ \mathbf{-3} & \mathbf{13/4} & \mathbf{-1/4} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1/4} & \mathbf{13/4} & \mathbf{-3} \\ \mathbf{0} & \mathbf{0} & \mathbf{-3} & \mathbf{3} \end{bmatrix}$$

$$\begin{Bmatrix} F_{1x} = R_1 \\ F_{2x} = -P \\ F_{3x} = 0 \\ F_{4x} = R_4 \end{Bmatrix} = \frac{\pi\,E\,d^2}{4L}\begin{bmatrix} 12 & -12 & 0 & 0 \\ -12 & 13 & -1 & 0 \\ 0 & -1 & 13 & -12 \\ 0 & 0 & -12 & 12 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \\ u_4 = 0 \end{Bmatrix}$$

$$\begin{Bmatrix} -P \\ 0 \end{Bmatrix} = \frac{\pi\,E\,d^2}{4L}\begin{bmatrix} 13 & -1 \\ -1 & 13 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{cases} -P = \dfrac{\pi\,E\,d^2}{4L}(13u_2 - u_3) \\ 0 = \dfrac{\pi\,E\,d^2}{4L}(-u_2 + 13u_3) \end{cases}$$

$$\boldsymbol{u_2} = -\frac{\mathbf{13}}{\mathbf{42}}\frac{\boldsymbol{PL}}{\boldsymbol{\pi\,E\,d^2}}$$

$$\boldsymbol{u_3} = -\frac{\mathbf{1}}{\mathbf{42}}\frac{\boldsymbol{PL}}{\boldsymbol{\pi\,E\,d^2}}$$

- ***Calcul des réactions***

$$\begin{Bmatrix} F_{1x} = R_1 \\ F_{2x} = -P \\ F_{3x} = 0 \\ F_{4x} = R_4 \end{Bmatrix} = \frac{\pi\, E\, d^2}{4L} \begin{bmatrix} 12 & -12 & 0 & 0 \\ -12 & 13 & -1 & 0 \\ 0 & -1 & 13 & -12 \\ 0 & 0 & -12 & 12 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = \cdots \\ u_3 = \cdots \\ u_4 = 0 \end{Bmatrix}$$

$$F_{1x} = R_1 = \frac{\pi\, E\, d^2}{4L}(-12u_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \boldsymbol{R_1 = \frac{13\,P}{14}}$$

$$F_{4x} = R_4 = \frac{\pi\, d_2^2 E_2}{4L}(-12u_3) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \boldsymbol{R_4 = \frac{P}{14}}$$

2. Les efforts internes dans chaque élément

$$\{f_e\} = [K_e]\{u_e\}$$

- ***Elément 1***

$$\begin{Bmatrix} f_{1x}^1 \\ f_{2x}^1 \end{Bmatrix} = \frac{3\,\pi\, E\, d^2}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} \boldsymbol{f_{1x}^1} \\ \boldsymbol{f_{2x}^1} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{\frac{13\,P}{14}} \\ \boldsymbol{-\frac{13\,P}{14}} \end{Bmatrix}$$

Effort de Compression

- ***Elément 2***

$$\begin{Bmatrix} f_{2x}^1 \\ f_{3x}^2 \end{Bmatrix} = \frac{\pi\, E\, d^2}{4\,L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} \boldsymbol{f_{2x}^1} \\ \boldsymbol{f_{3x}^2} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{-\frac{P}{14}} \\ \boldsymbol{\frac{P}{14}} \end{Bmatrix}$$

Effort de Traction

- ***Elément 3***

$$\begin{Bmatrix} f_{3x}^3 \\ f_{4x}^3 \end{Bmatrix} = \frac{3\,\pi\, E\, d^2}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_3 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \boldsymbol{f_{3x}^3} \\ \boldsymbol{f_{4x}^3} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{-\frac{P}{14}} \\ \boldsymbol{\frac{P}{14}} \end{Bmatrix}$$

Effort de Traction