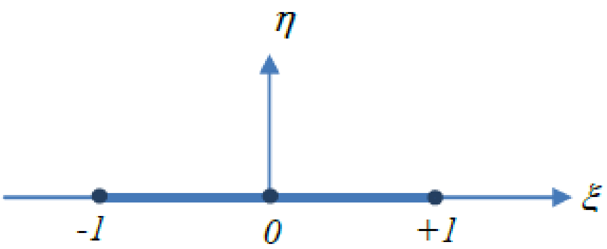
Exercice N°01 :

- Donner les fonctions de formes ainsi que la relation géométrique entre les coordonnées (x, y) et $\xi, \eta)$ permettant le passage de l'élément de référence aux éléments physiques.
- Evaluer la matrice Jacobienne associée à chaque transformation géométrique.

Exemple 1 :



- Fonctions de forme

$$u(\xi) = a_0 + a_1\xi + a_2\xi^2$$

$$\begin{cases} u(-1) = a_0 - a_1 + a_2 = u_1 \\ u(0) = a_0 = u_3 \\ u(1) = a_0 + a_1 + a_2 = u_2 \end{cases}$$

$$\begin{cases} a_0 = u_3 \\ a_1 = \dfrac{u_2 - u_1}{2} \\ a_2 = \dfrac{u_1 + u_2 - 2u_3}{2} \end{cases}$$

$$u(\xi) = u_3 + \frac{u_2 - u_1}{2}\xi + \frac{u_1 + u_2 - 2u_3}{2}\xi^2$$

$$u(\xi) = \langle \frac{\xi^2 - \xi}{2} \quad \frac{\xi^2 + \xi}{2} \quad 1 - \xi^2 \rangle \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$u(\xi) = \langle N_1 \quad N_2 \quad N_3 \rangle \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$N_1 = \frac{\xi^2 - \xi}{2}, \quad N_2 = \frac{\xi^2 + \xi}{2}, \quad N_3 = 1 - \xi^2$$

- Relation géométrique

$$x(\xi) = \langle \frac{\xi^2 - \xi}{2} \quad \frac{\xi^2 + \xi}{2} \quad 1 - \xi^2 \rangle \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

- Jacobien :

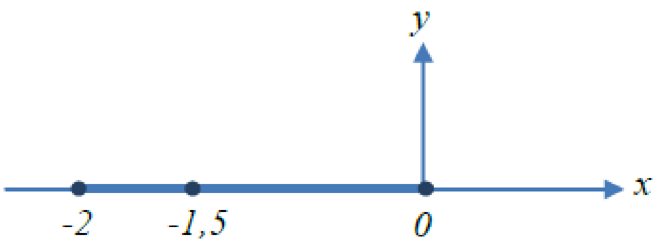
$$j = \frac{\delta x}{d\xi}$$

Vérification :

Pour $\xi = -1 \ \rightarrow x = x_1, \quad \xi = 1 \ \rightarrow x = x_2, \quad \xi = 0 \ \rightarrow x = x_3$

$$j = \frac{\delta x}{d\xi} = \langle \frac{2\xi - 1}{2} \quad \frac{2\xi + 1}{2} \quad -2\xi \rangle \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

Exemple 1a :

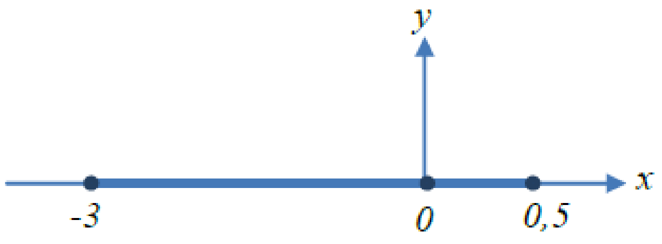


$$x(\xi) = \langle \frac{\xi^2 - \xi}{2} \quad \frac{\xi^2 + \xi}{2} \quad 1 - \xi^2 \rangle \begin{Bmatrix} -2 \\ 0 \\ -3/2 \end{Bmatrix}$$

$$j = \langle \frac{2\xi - 1}{2} \quad \frac{2\xi + 1}{2} \quad -2\xi \rangle \begin{Bmatrix} -2 \\ 0 \\ -3/2 \end{Bmatrix}$$

$$j = 1 - 2\xi + 3\xi = 1 + \xi$$

Exemple 1b :

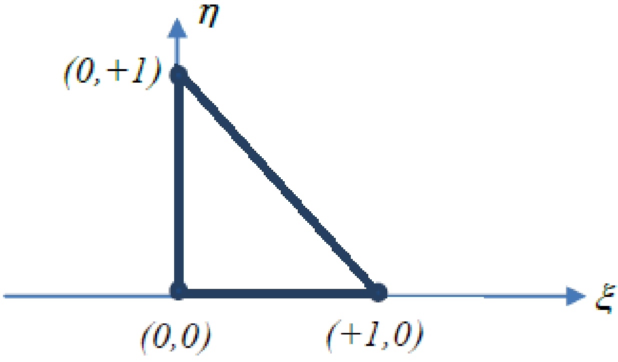


$$x(\xi) = \langle \frac{\xi^2 - \xi}{2} \quad \frac{\xi^2 + \xi}{2} \quad 1 - \xi^2 \rangle \begin{Bmatrix} -3 \\ 1/2 \\ 0 \end{Bmatrix}$$

$$j = \langle \frac{2\xi - 1}{2} \quad \frac{2\xi + 1}{2} \quad -2\xi \rangle \begin{Bmatrix} -3 \\ 1/2 \\ 0 \end{Bmatrix}$$

$$j = \frac{3 - 6\xi}{2} + \frac{2\xi + 1}{4} = \frac{7 - 10\xi}{4}$$

Exemple 2 :



- Fonctions de forme

$$\begin{cases} u(\xi,\eta) = a_0 + a_1\xi + a_2\eta \\ v(\xi,\eta) = a_0 + a_1\xi + a_2\eta \end{cases}$$

$$\begin{cases} u(0,0) = a_0 = u_1 \\ u(1,0) = a_0 + a_1 = u_2 \\ u(0,1) = a_0 + a_2 = u_3 \end{cases}$$

$$\begin{cases} a_0 = u_1 \\ a_1 = u_2 - u_1 \\ a_2 = u_3 - u_1 \end{cases}$$

$$u(\xi,\eta) = u_1 + (u_2 - u_1)\xi + (u_3 - u_1)\eta$$

$$u(\xi,\eta) = \langle 1-\xi-\eta \quad \xi \quad \eta \rangle \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{cases} N_1 = 1 - \xi - \eta \\ N_2 = \xi \\ N_3 = \eta \end{cases}$$

$$\begin{Bmatrix} u(\xi,\eta) \\ v(\xi,\eta) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

- Relation géométrique

$$\begin{cases} x = \sum_{i=1}^{n} N_i\, x_i \\ y = \sum_{i=1}^{n} N_i\, y_i \end{cases}$$

- La matrice Jocobienne

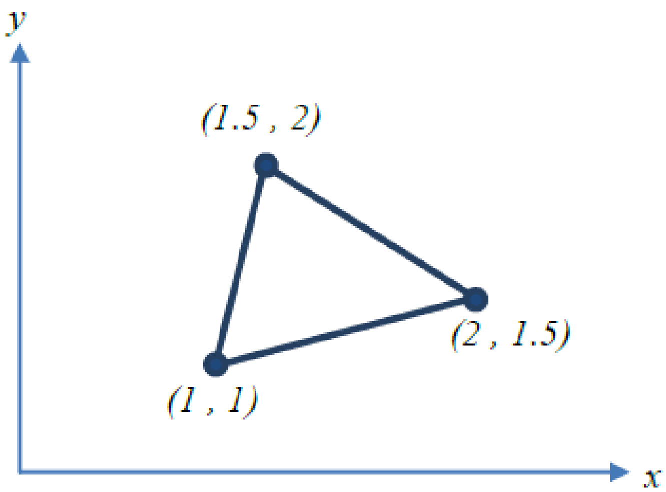
$$[J] = \begin{bmatrix} \frac{\delta x}{\delta \xi} & \frac{\delta y}{\delta \xi} \\ \frac{\delta x}{\delta \eta} & \frac{\delta y}{\delta \eta} \end{bmatrix}$$

$$[J] = \begin{bmatrix} \sum_{i=1}^{n} \frac{\delta N_i}{\delta \xi} x_i & \sum_{i=1}^{n} \frac{\delta N_i}{\delta \xi} y_i \\ \sum_{i=1}^{n} \frac{\delta N_i}{\delta \eta} x_i & \sum_{i=1}^{n} \frac{\delta N_i}{\delta \eta} y_i \end{bmatrix}$$

$$[J] = \begin{bmatrix} \frac{\delta N_1}{\delta \xi} & \frac{\delta N_2}{\delta \xi} & \frac{\delta N_3}{\delta \xi} \\ \frac{\delta N_1}{\delta \eta} & \frac{\delta N_2}{\delta \eta} & \frac{\delta N_3}{\delta \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix}$$

$$[J] = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix}$$

$$[J] = \begin{bmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{bmatrix}$$



$$x(\xi) = N_1 x_1 + N_2 x_2 + N_3 x_3$$

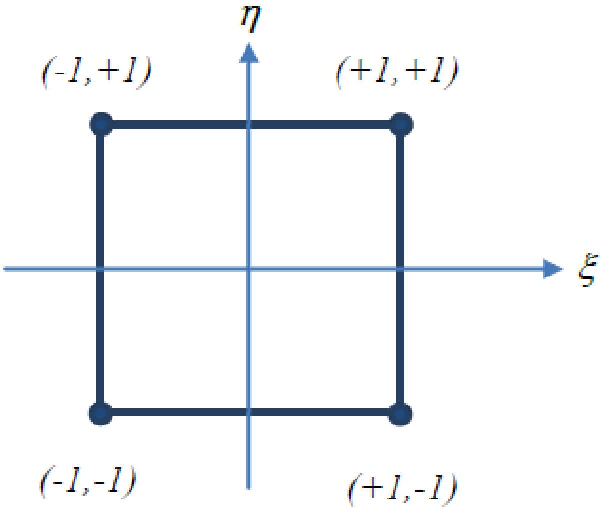
$$= \langle 1 - \xi - \eta \quad \xi \quad \eta \rangle \begin{Bmatrix} 1 \\ 2 \\ 3/2 \end{Bmatrix} = 1 + \xi + \eta/2$$

$$y(\xi) = N_1 y_1 + N_2 y_2 + N_3 y_3$$

$$= \langle 1 - \xi - \eta \quad \xi \quad \eta \rangle \begin{Bmatrix} 1 \\ 3/2 \\ 2 \end{Bmatrix} = 1 + \xi/2 + \eta$$

$$[J] = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3/2 \\ 3/2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$$

Exemple 3 :



- Fonctions de forme

$$\begin{cases} u(\xi,\eta) = a_0 + a_1\xi + a_2\eta + a_3\xi\eta \\ v(\xi,\eta) = a_0 + a_1\xi + a_2\eta + a_3\xi\eta \end{cases}$$

$$N_1 = \frac{(1-\xi)(1-\eta)}{4} \quad N_2 = \frac{(1+\xi)(1-\eta)}{4}$$
$$N_3 = \frac{(1+\xi)(1+\eta)}{4} \quad N_4 = \frac{(1-\xi)(1+\eta)}{4}$$

$$\begin{Bmatrix} u(\xi,\eta) \\ v(\xi,\eta) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

- Relation géométrique

$$\begin{cases} x = \sum_{i=1}^{n} N_i\, x_i \\ y = \sum_{i=1}^{n} N_i\, y_i \end{cases}$$

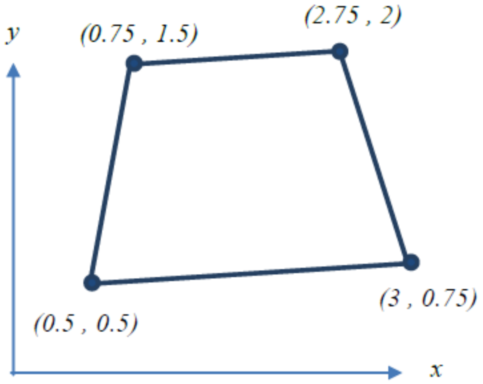
- Matrice Jacobienne

$$[J] = \begin{bmatrix} \frac{\delta x}{\delta \xi} & \frac{\delta y}{\delta \xi} \\ \frac{\delta x}{\delta \eta} & \frac{\delta y}{\delta \eta} \end{bmatrix}$$

$$[J] = \begin{bmatrix} \sum_{i=1}^{n} \frac{\delta N_i}{\delta \xi} x_i & \sum_{i=1}^{n} \frac{\delta N_i}{\delta \xi} y_i \\ \sum_{i=1}^{n} \frac{\delta N_i}{\delta \eta} x_i & \sum_{i=1}^{n} \frac{\delta N_i}{\delta \eta} y_i \end{bmatrix}$$

$$[J] = \begin{bmatrix} \frac{\delta N_1}{\delta \xi} & \frac{\delta N_2}{\delta \xi} & \frac{\delta N_3}{\delta \xi} & \frac{\delta N_4}{\delta \xi} \\ \frac{\delta N_1}{\delta \eta} & \frac{\delta N_2}{\delta \eta} & \frac{\delta N_3}{\delta \eta} & \frac{\delta N_4}{\delta \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & x_4 \end{bmatrix}$$

$$[J] = \begin{bmatrix} \frac{-1+\eta}{4} & \frac{1-\eta}{4} & \frac{1+\eta}{4} & \frac{-1-\eta}{4} \\ \frac{-1+\xi}{4} & \frac{-1-\xi}{4} & \frac{1+\xi}{4} & \frac{1-\xi}{4} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & x_4 \end{bmatrix}$$



$$\begin{cases} x(\xi,\eta) = \langle N_1 \quad N_2 \quad N_3 \quad N_4 \rangle \begin{Bmatrix} 0.5 \\ 3 \\ 2.75 \\ 0.75 \end{Bmatrix} \\ y(\xi,\eta) = \langle N_1 \quad N_2 \quad N_3 \quad N_4 \rangle \begin{Bmatrix} 0.5 \\ 0.75 \\ 2 \\ 1.5 \end{Bmatrix} \end{cases}$$

$$[J] = \begin{bmatrix} \frac{-1+\eta}{4} & \frac{1-\eta}{4} & \frac{1+\eta}{4} & \frac{-1-\eta}{4} \\ \frac{-1+\xi}{4} & \frac{-1-\xi}{4} & \frac{1+\xi}{4} & \frac{1-\xi}{4} \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 3 & 0.75 \\ 2.75 & 2 \\ 0.75 & 1.5 \end{bmatrix} = \begin{bmatrix} \frac{9-\eta}{8} & \frac{\eta+3}{16} \\ -\frac{\xi}{8} & \frac{\xi+9}{16} \end{bmatrix}$$