# Sheet 1
# Chapter One

**Tautology:** A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it.

Example: $p \vee \neg p$

**Contradiction**: A compound proposition that is always false is called a *contradiction*.

Example : $p \wedge \neg p$
A compound proposition that is neither a tautology nor a contradiction is called a ***contingency***.

TABLE 4 The Truth Table for the Exclusive Or of Two Propositions.

| $p$ | $q$ | $p \oplus q$ |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | F |

TABLE 5 The Truth Table for the Conditional Statement $p \rightarrow q$.

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

**The converse** of $p \rightarrow q$ is the proposition $q \rightarrow p$

**The contrapositive** of $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$

**The inverse** of $p \rightarrow q$ is the proposition $\neg p \rightarrow \neg q$

A conditional statement and its contrapositive are equivalent. The converse and the inverse of a conditional statement are also equivalent, but neither is equivalent to the original conditional statement

**BICONDITIONALS:** $p \leftrightarrow q$ has exactly the same truth value as $(p \rightarrow q) \wedge (q \rightarrow p)$

**TABLE 6** The Truth Table for the Biconditional $p \leftrightarrow q$.

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

**Precedence of Logical Operators:**

**TABLE 8** Precedence of Logical Operators.

| *Operator* | *Precedence* |
|---|---|
| $\neg$ | 1 |
| $\wedge$ | 2 |
| $\vee$ | 3 |
| $\rightarrow$ | 4 |
| $\leftrightarrow$ | 5 |

**Logical Equivalences**

Compound propositions that have the same truth values in all possible cases are called logically equivalent.

The compound propositions $p$ and $q$ are called *logically equivalent* if $p \leftrightarrow q$ is a tautology.

## Propositional Satisfiability

A compound proposition is **satisfiable** if there is an assignment of truth values to its variables that makes it true.

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## Logical Equivalence Laws:

**TABLE 6 Logical Equivalences.**

| *Equivalence* | *Name* |
|---|---|
| $p \wedge \mathbf{T} \equiv p$<br>$p \vee \mathbf{F} \equiv p$ | Identity laws |
| $p \vee \mathbf{T} \equiv \mathbf{T}$<br>$p \wedge \mathbf{F} \equiv \mathbf{F}$ | Domination laws |
| $p \vee p \equiv p$<br>$p \wedge p \equiv p$ | Idempotent laws |
| $\neg(\neg p) \equiv p$ | Double negation law |
| $p \vee q \equiv q \vee p$<br>$p \wedge q \equiv q \wedge p$ | Commutative laws |
| $(p \vee q) \vee r \equiv p \vee (q \vee r)$<br>$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ | Associative laws |
| $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$<br>$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ | Distributive laws |
| $\neg(p \wedge q) \equiv \neg p \vee \neg q$<br>$\neg(p \vee q) \equiv \neg p \wedge \neg q$ | De Morgan's laws |
| $p \vee (p \wedge q) \equiv p$<br>$p \wedge (p \vee q) \equiv p$ | Absorption laws |
| $p \vee \neg p \equiv \mathbf{T}$<br>$p \wedge \neg p \equiv \mathbf{F}$ | Negation laws |

**TABLE 7** Logical Equivalences Involving Conditional Statements.

| |
|---|
| $p \to q \equiv \neg p \vee q$ |
| $p \to q \equiv \neg q \to \neg p$ |
| $p \vee q \equiv \neg p \to q$ |
| $p \wedge q \equiv \neg(p \to \neg q)$ |
| $\neg(p \to q) \equiv p \wedge \neg q$ |
| $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$ |
| $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$ |
| $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$ |
| $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$ |

**TABLE 8** Logical Equivalences Involving Biconditional Statements.

| |
|---|
| $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$ |
| $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$ |
| $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$ |
| $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$ |

P = there is going to be a quiz

Q =I come to class

**State the converse, contrapositive, and inverse of each of these conditional statements.**

**a) If it snows today, I will ski tomorrow.** if I ski tomorrow then it snowed today

**b) I come to class whenever there is going to be a quiz.** if there is going to be a quiz, then I come to class

**c) A positive integer is a prime only if it has no divisors other than 1 and itself.**

**Many forms of answers for this exercise are possible.**

**a) One form of the converse that reads well in English is "I will ski tomorrow only if it snows today." We could state the contrapositive as "If I don't ski tomorrow, then it will not have snowed today." The inverse is "If it does not snow today, then I will not ski tomorrow."**

**b) The proposition as stated can be rendered "If there is going to be a quiz, then I will come to class." The converse is "If I come to class, then there will be a quiz." (Or, perhaps even better, "I come to class only if there's going to be a quiz.") The contrapositive is "If I don't come to class, then there won't be a quiz." The inverse is "If there is not going to be a quiz, then I don't come to class."**

**c) There is a variable ("a positive integer") in this sentence, so technically it is not a proposition. Nevertheless, we can treat sentences such as this in the same way we treat propositions. Its converse is "A positive integer is prime if it has no divisors other than 1 and itself." (Note that this can be false, since the number 1 satisfies the hypothesis but not the conclusion.) The contrapositive of the original proposition is "If a positive integer has a divisor other than 1 and itself, then it is**

**not prime." (We are simplifying a bit here, replacing "does have no divisors" with "has a divisor." Note that this is always true, assuming that we are talking about positive divisors.) The inverse is "If a positive integer is not prime, then it has a divisor other than 1 and itself."**

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**Show that** $[(p\to q)\wedge(q\to r)]\to(p\to r)$ **is a tautology.**

$$\begin{aligned}
& ((p \to q) \wedge (q \to r)) \to (p \to r) & \\
\equiv\ & \neg((\neg p \vee q) \wedge (\neg q \vee r)) \vee (\neg p \vee r) & \text{decomposition of } \to \\
\equiv\ & \neg(\neg p \vee q) \vee \neg(\neg q \vee r) \vee \neg p \vee r & \text{De Morgan} \\
\equiv\ & (\neg\neg p \wedge \neg q) \vee (\neg\neg q \wedge \neg r) \vee \neg p \vee r & \text{De Morgan} \\
\equiv\ & \neg p \vee (\neg\neg p \wedge \neg q) \vee (\neg\neg q \wedge \neg r) \vee r & \text{commutativity} \\
\equiv\ & ((\neg p \vee \neg\neg p) \wedge (\neg p \vee \neg q)) \vee ((\neg\neg q \vee r) \wedge (\neg r \vee r)) & \text{distributivity} \\
\equiv\ & (\top \wedge (\neg p \vee \neg q)) \vee ((\neg\neg q \vee r) \wedge \top) & \text{negation law} \\
\equiv\ & (\neg p \vee \neg q) \vee (\neg\neg q \vee r) & \text{identity law} \\
\equiv\ & \neg p \vee (\neg q \vee \neg\neg q) \vee r & \text{associativity} \\
\equiv\ & \neg p \vee \top \vee r & \text{negation law} \\
\equiv\ & \top & \text{domination law}
\end{aligned}$$

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**Show that** $(p \to q) \to r$ **and** $p \to (q \to r)$ **are not logically equivalent.**

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**Show that** $(p \wedge q) \to r$ **and** $(p \to r) \wedge (q \to r)$ **are not logically equivalent.**

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**Determine whether each of these compound propositions is satisfiable.**

**a)** $(p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$

**b)** $(p \to q) \wedge (p \to \neg q) \wedge (\neg p \to q) \wedge (\neg p \to \neg q)$

**c)** $(p \leftrightarrow q) \wedge (\neg p \leftrightarrow q)$

**61. a)** With a little trial and error we discover that setting $p = \mathbf{F}$ and $q = \mathbf{F}$ produces $(\mathbf{F}\vee\mathbf{T})\wedge(\mathbf{T}\vee\mathbf{F})\wedge(\mathbf{T}\vee\mathbf{T})$, which has the value $\mathbf{T}$. So this compound proposition is satisfiable. (Note that this is the only satisfying truth assignment.)

**b)** We claim that there is no satisfying truth assignment here. No matter what the truth values of $p$ and $q$ might be, the four implications become $\mathbf{T} \to \mathbf{T}$, $\mathbf{T} \to \mathbf{F}$, $\mathbf{F} \to \mathbf{T}$, and $\mathbf{F} \to \mathbf{F}$, in some order. Exactly one of these is false, so their conjunction is false.

**c)** This compound proposition is not satisfiable. In order for the first clause, $p \leftrightarrow q$, to be true, $p$ and $q$ must have the same truth value. In order for the second clause, $(\neg p) \leftrightarrow q$, to be true, $p$ and $q$ must have opposite truth values. These two conditions are incompatible, so there is no satisfying truth assignment.

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**Prove that** $(p \wedge \neg q) \vee q \equiv p \vee q$

**$(p \wedge \neg q) \vee q$**

$\equiv q \vee (p \wedge \neg q)$

$\equiv (q \vee p) \wedge (q \vee \neg q)$

$\equiv (q \vee p) \wedge T$

$\equiv q \vee p$

$\equiv p \vee q$

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**Prove that ¬ (p ↔ q) and p ↔ ¬q are equivalent.**

$\neg (p \leftrightarrow q)$

$\equiv \neg [(p \to q) \wedge (q \to p)]$

$\equiv \neg(p \to q) \vee \neg (q \to p)$

$\equiv (p \wedge \neg q) \vee (q \wedge \neg p)$

$\equiv [(p \wedge \neg q) \vee q] \wedge [(p \wedge \neg q) \vee \neg p]$

$\equiv [q \vee (p \wedge \neg q)] \wedge [\neg p \vee (p \wedge \neg q)]$

$\equiv [(q \vee p) \wedge (q \vee \neg q)] \wedge [(\neg p \vee p) \wedge [(\neg p \vee \neg q)]$

$\equiv [(q \vee p) \wedge T] \wedge [T \wedge [(\neg p \vee \neg q)]$

$\equiv (q \vee p) \wedge (\neg p \vee \neg q)$

$\equiv (\neg q \to p) \wedge (p \to \neg q)$

$\equiv (p \to \neg q) \wedge (\neg q \to p)$

$\equiv p \leftrightarrow \neg q$

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**Logic Puzzles**

**An island has two kinds of inhabitants, knights, who always tell the truth, and knaves, who always lie.**

**A says “B is a kight.”**

**B says “The two of us are of opposite types.”**

**What are the types of A and B?**

**Solution:**

**p: "A is a knight."**

**q: "B is a knight."**

**If A is a knight, then p is true, q must also be true. Then (p ∧ ¬q) ∨ (¬p ∧ q) would have to be true, but it is not. So, A is not a knight and therefore ¬p must be true.**

**If A is a knave, then B must not be a knight since knives always lie. So, then both ¬p and ¬q hold since both are knaves.**

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**Five friends have access to a chat room:**

- **Either Kevin or Heather, or both, are chatting**
- **Either Randy or Vijay, but not both, are chatting**
- **If Abby is chatting, then so is Randy**
- **Vijay and Kevin are either both chatting or neither is**
- **If Heather is chatting, then so are Abby and Kevin**

**Make a propositional variable for each friend:**

**K: Kevin is chatting**

**H: Heather is chatting**

**R: Randy is chatting**

**V: Vijay is chatting**

**A: Abby is chatting**

- Either Kevin or Heather, or both, are chatting $(K \vee H)$
- Either Randy or Vijay, but not both, are chatting $(R \vee V) \wedge (\neg R \vee \neg V)$
- If Abby is chatting, then so is Randy $(A \to R)$
- Vijay and Kevin are either both chatting or neither is $(V \leftrightarrow K)$
- If Heather is chatting, then so are Abby and Kevin $(H \to (A \wedge K))$

Determine if there exists K, H, R, V, A such that

$(K \vee H) \wedge ((R \vee V) \wedge (\neg R \vee \neg V)) \wedge (A \to R) \wedge (V \leftrightarrow K) \wedge (H \to (A \wedge K))$

is true.

$(K \vee H) \wedge ((R \vee V) \wedge (\neg R \vee \neg V)) \wedge (A \to R) \wedge (V \leftrightarrow K) \wedge (H \to (A \wedge K))$

$(K \vee H) \wedge ((R \vee V) \wedge (\neg R \vee \neg V)) \wedge (\neg A \vee R) \wedge (V \leftrightarrow K) \wedge (\neg H \vee (A \wedge K))$

$(K \vee H) \wedge ((R \vee V) \wedge (\neg R \vee \neg V)) \wedge (\neg A \vee R) \wedge ((V \wedge K) \vee (\neg V \wedge \neg K)) \wedge (\neg H \vee (A \wedge K))$

$(K \vee H) \wedge (R \vee V) \wedge (\neg R \vee \neg V) \wedge (\neg A \vee R) \wedge ((V \wedge K) \vee (\neg V \wedge \neg K)) \wedge (\neg H \vee (A \wedge K))$

$(K \vee H) \wedge (R \vee V) \wedge (\neg R \vee \neg V) \wedge (\neg A \vee R) \wedge ((V \wedge K) \vee (\neg V \wedge \neg K)) \wedge (\neg H \vee A) \wedge (\neg H \vee K)$

$(K \vee (H \wedge \neg H)) \wedge (R \vee V) \wedge (\neg R \vee \neg V) \wedge (\neg A \vee R) \wedge ((V \wedge K) \vee (\neg V \wedge \neg K)) \wedge (\neg H \vee A)$

$(K \vee \text{False}) \wedge (R \vee V) \wedge (\neg R \vee \neg V) \wedge (\neg A \vee R) \wedge ((V \wedge K) \vee (\neg V \wedge \neg K)) \wedge (\neg H \vee A)$

$K \wedge (R \vee V) \wedge (\neg R \vee \neg V) \wedge (\neg A \vee R) \wedge ((V \wedge K) \vee (\neg V \wedge \neg K)) \wedge (\neg H \vee A)$

$K \wedge (R \vee V) \wedge (\neg R \vee \neg V) \wedge (\neg A \vee R) \wedge ((V \wedge K) \vee (\neg V \wedge \neg K)) \wedge (\neg H \vee A)$

$((K \wedge (V \wedge K)) \vee (K \wedge (\neg V \wedge \neg K))) \wedge (R \vee V) \wedge (\neg R \vee \neg V) \wedge (\neg A \vee R) \wedge (\neg H \vee A)$

$((K \wedge V) \vee \text{False}) \wedge (R \vee V) \wedge (\neg R \vee \neg V) \wedge (\neg A \vee R) \wedge (\neg H \vee A)$

$K \wedge V \wedge (R \vee V) \wedge (\neg R \vee \neg V) \wedge (\neg A \vee R) \wedge (\neg H \vee A)$

$K \wedge V \wedge (\neg R \vee \neg V) \wedge (\neg A \vee R) \wedge (\neg H \vee A)$

$K \wedge V \wedge \neg R \wedge (\neg A \vee R) \wedge (\neg H \vee A)$

$K \wedge V \wedge \neg R \wedge \neg A \wedge (\neg H \vee A)$

$K \wedge V \wedge \neg R \wedge \neg A \wedge \neg H$

$K \wedge V \wedge \neg R \wedge \neg A \wedge \neg H$

**K = True, V = True, R = False, A = False, H = False**

**Kevin and Vijay are chatting; Randy, Abby, and Heather are not.**

*Best of luck!*