Homework 2.

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1) a. $I_k(x,y) = K(x,y) \cdot I(x,y) = \sum_{i=1}^{k} \sum_{j=1}^{k} K(i,j) \cdot I(x-i+\frac{k}{2}, y-j+\frac{k}{2})$

b. $K = g \cdot h$

$$I_k(x,y) = \sum_{i=1}^{k} \sum_{j=1}^{k} g(i) \cdot h(j) \cdot I(x-i+\frac{k}{2}, y-j+\frac{k}{2}) =$$

$$= \sum_{j=1}^{k} h(j) \sum_{i=1}^{k} g(i) \cdot I(x-i+\frac{k}{2}, y-j+\frac{k}{2}) =$$

$$= \sum_{j=1}^{k} h(j) \cdot I(x, y-j+\frac{k}{2}) = I_{gh}(x,y)$$

c. 2D filter: multiplication – k^2 operations

addition – k^2-1 operations

Total number of operations $= k^2+k^2-1 =$
$= 2k^2-1$

Total $= N \cdot N \cdot (2k^2-1) = N^2(2k^2-1)$

1D filter: multiplication – k operations

addition – $k-1$ operations

Total number of operations $= N \cdot N \cdot (2(k+k-1))$
$= N^2(2(2k-1)) =$
$= N^2(4k-2)$

Saved operations when using 1D filters rather than 2D filter $= N^2(2k^2-1) - N^2(4k-2) =$
$= N^2(2k^2-1-4k+2) = N^2(2k^2-4k+1)$

2) $S[\alpha f_i[n,m] + \beta f_j[k,l]] = \alpha S[f_i[n,m]] + \beta S[f_j[k,l]]$

Let $f_i[n,m]$ be f_1, and $f_j[k,l]$ be f_2. Then equation will be as following:

$S[\alpha f_1 + \beta f_2] = \alpha S[f_1] + \beta S[f_2]$

Multiplication of function by kernel $h[m,n]$:

$$S[\alpha f_1 + \beta f_2] = [\alpha f_1 + \beta f_2] \cdot h[m,n] = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} (\alpha \cdot f_1[i,j] + \beta \cdot f_2[i,j]) \cdot h[m-i, n-j] = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} (\alpha \cdot f_1[i,j] \cdot h[m-i, n-j] + \beta \cdot f_2[i,j] \cdot h[m-i, n-j]) =$$

$$= \alpha \cdot \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} f_1[i,j] \cdot h[m-i, n-j] + \beta \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} f_2[i,j] \cdot h[m-i, n-j] =$$

$$= \alpha \cdot f_1 \cdot h + \beta \cdot f_2 \cdot h = \alpha S[f_1] + \beta S[f_2]$$

Now, replace f_1 and f_2 back to original values:

$S[\alpha f_1 + \beta f_2] = \alpha S[f_1] + \beta S[f_2] \Rightarrow$

$S[\alpha \cdot f_i[n,m] + \beta f_j[k,l]] = \alpha S[f_i[n,m]] + \beta S[f_j[k,l]]$

Thereby, it is proven that S defined by kernel h is a Linear Invariant system, since property of superposition is held by S.

3) Since vector [0,25;0,5;0,25] is firstly applied to the rows and then the transposed vector is applied to columns we can combine it into single 2D vector as follows:

$$\begin{bmatrix} 0,25 & 0,5 & 0,25 \end{bmatrix} \cdot \begin{bmatrix} 0,25 \\ 0,5 \\ 0,25 \end{bmatrix} = \begin{bmatrix} 0,25^2 & 0,5 \cdot 0,25 & 0,25^2 \\ 0,25 \cdot 0,5 & 0,5^2 & 0,25 \cdot 0,5 \\ 0,25^2 & 0,5 \cdot 0,25 & 0,25^2 \end{bmatrix} =$$

$$= \begin{bmatrix} 0,0625 & 0,125 & 0,0625 \\ 0,125 & 0,25 & 0,125 \\ 0,0625 & 0,125 & 0,0625 \end{bmatrix} \rightarrow 3 \times 3 \text{ 2D filter}$$

4) $M = \begin{bmatrix} I_x^2 & I_x I_y \\ I_x I_y & I_y^2 \end{bmatrix}$

a. $\begin{bmatrix} I_x^2 - \lambda & I_x I_y \\ I_x I_y & I_y^2 - \lambda \end{bmatrix} = (I_x^2 - \lambda) \cdot (I_y^2 - \lambda) - I_x^2 I_y^2 = 0$

$I_x^2 I_y^2 - I_x^2 \lambda - I_y^2 \lambda + \lambda^2 - I_x^2 I_y^2 = 0$

$\lambda^2 - I_x^2 \lambda - I_y^2 \lambda = 0$

$\lambda(\lambda - (I_x^2 + I_y^2)) = 0 \quad \Rightarrow \lambda_{1,2} = 0; I_x^2 + I_y^2$

b. Yes, because according to derivative calculation, square root of $I_x^2 + I_y^2$ provides the image gradient value at each image point. This helps to detect edges on image at different scales.

5) $g(x,y) = \frac{1}{2\pi\delta^2} \cdot e^{-\frac{x^2+y^2}{2\delta^2}}$ Omit $\frac{1}{2\pi\delta^2}$ for simplicity.

$$\frac{d}{dx}(f \cdot g) = f \cdot \frac{d}{dx}(g)$$

$$\frac{d}{dx} g(x,y) = \frac{d}{dx}\left(e^{\frac{-x^2+y^2}{2\delta^2}}\right) = -\frac{x}{\delta^2} \cdot e^{-\frac{x^2+y^2}{2\delta^2}}$$

$$\frac{d^2}{d^2x} g(x,y) = \frac{x^2}{\delta^4} \cdot e^{-\frac{x^2+y^2}{2\delta^2}} - \frac{1}{\delta^2} e^{-\frac{x^2+y^2}{2\delta^2}} = \frac{x^2-\delta^2}{\delta^4} \cdot e^{-\frac{x^2+y^2}{2\delta^2}}$$

$$\frac{d^2}{d^2y} g(x,y) = \frac{y^2}{\delta^4} \cdot e^{-\frac{x^2+y^2}{2\delta^2}} - \frac{1}{\delta^2} \cdot e^{-\frac{x^2+y^2}{2\delta^2}} = \frac{y^2-\delta^2}{\delta^4} \cdot e^{-\frac{x^2+y^2}{2\delta^2}}$$

$$\text{LoG of } g(x,y) = \frac{d^2}{d^2x} g(x,y) + \frac{d^2}{d^2y} g(x,y) =$$

$$= \left(\frac{x^2-\delta^2}{\delta^4} + \frac{y^2-\delta^2}{\delta^4}\right) \cdot e^{-\frac{x^2+y^2}{2\delta^2}} =$$

$$= \frac{x^2+y^2-2\delta^2}{\delta^4} \cdot e^{-\frac{x^2+y^2}{2\delta^2}}$$

Lets return coefficient of Gaussian:

$$\text{LoG of } g(x,y) = \frac{1}{2\pi\delta^2} \cdot \frac{x^2+y^2-2\delta^2}{\delta^4} \cdot e^{-\frac{x^2+y^2}{2\delta^2}} =$$

$$= \frac{x^2+y^2-2\delta^2}{2\pi\delta^6} \cdot e^{\frac{-x^2+y^2}{2\delta^2}}$$

6) Rotation by an angle of α:

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \quad (\hat{x}, \hat{y}) = A\cdot(x,y) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} =$$

$$\Rightarrow \hat{x} = x\cos\alpha - y\sin\alpha \,; \quad \hat{y} = x\sin\alpha + y\cos\alpha$$

$$I_x = \frac{df}{dx} = \frac{df}{d\hat{x}}\cdot\frac{d\hat{x}}{dx} + \frac{df}{d\hat{y}}\cdot\frac{d\hat{y}}{dx} = \frac{df}{d\hat{x}}\cdot(x\cos\alpha)' + \frac{df}{d\hat{y}}\cdot(x\sin\alpha)' =$$

$$= \frac{df}{d\hat{x}}\cdot\cos\alpha + \frac{df}{d\hat{y}}\cdot\sin\alpha$$

$$I_y = \frac{df}{dy} = \frac{df}{d\hat{x}}\cdot\frac{d\hat{x}}{dy} + \frac{df}{d\hat{y}}\cdot\frac{d\hat{y}}{dy} = \frac{df}{d\hat{x}}\cdot(-y\sin\alpha)' + \frac{df}{d\hat{y}}\cdot(y\cos\alpha)' =$$

$$= -\frac{df}{d\hat{x}}\cdot\sin\alpha + \frac{df}{d\hat{y}}\cdot\cos\alpha$$

$$\sqrt{I_x^2 + I_y^2} = \sqrt{\left(\frac{df}{d\hat{x}}\cdot\cos\alpha + \frac{df}{d\hat{y}}\cdot\sin\alpha\right)^2 + \left(-\frac{df}{d\hat{x}}\cdot\sin\alpha + \frac{df}{d\hat{y}}\cdot\cos\alpha\right)^2} =$$

$$= \sqrt{\left(\frac{df}{d\hat{x}}\right)^2\cos^2\alpha + 2\,\frac{df}{d\hat{x}}\cos\alpha\cdot\frac{df}{d\hat{y}}\sin\alpha + \left(\frac{df}{d\hat{y}}\right)^2\sin^2\alpha + \left(\frac{df}{d\hat{y}}\right)^2\cos^2\alpha - \right.}$$

$$\overline{\left. - 2\cdot\frac{df}{d\hat{x}}\sin\alpha\cdot\frac{df}{d\hat{y}}\cos\alpha + \left(\frac{df}{d\hat{x}}\right)^2\sin^2\alpha\right.} = \sqrt{\left(\frac{df}{d\hat{x}}\right)^2(\cos^2\alpha + \sin^2\alpha) + \right.}$$

$$\overline{\left. + \left(\frac{df}{d\hat{y}}\right)^2(\sin^2\alpha + \cos^2\alpha)\right.} = \sqrt{\left(\frac{df}{d\hat{x}}\right)^2 + \left(\frac{df}{d\hat{y}}\right)^2} = \sqrt{I_{\hat{x}}^2 + I_{\hat{y}}^2} \;\blacksquare.$$

7) a. Firstly, we take an original image and subtract from it the smoothed version of original image. After that the detailed image is obtained, and then we add detailed image to the original image, thereby getting the sharpened version of original image.

b. The Derivative of Gaussian filter can be used to detect edges on image, while the Difference of Gaussian filter is used for detection of spots by approximating Laplacian of Gaussian.

c. Patch descriptor will be invariant to rotation when we take keypoints and assign to them an orientation, and after that rotate towards canonical orientation. As for SIFT, we take points close to the feature point and build Histogram of gradients. For each keypoint the direction will be corresponding largest bin.