Majority Vote and Monopolies in Social Networks

Chen Avin[1], Zvi Lotker[1], David Peleg[2], Assaf Mizrachi[1]
[1]Ben Gurion University of the Negev, Be'er-Sheva, Israel
[2] The Weizmann Institute, Rehovot, Israel
{avin, zvilo}@cse.bgu.ac.il, assafmi@bgu.ac.il, david.peleg@weizmann.ac.il

ABSTRACT

On various occasions, a society has to reach a decision regarding a question affecting the life of its members. For this, it may use a *voting* mechanism, i.e., collect the votes of the group members and output a decision which best expresses the group's will. In order to make up their minds, individuals often discuss the issue with friends, family and colleagues before they take their vote, and by that may mutually affect each other's vote. Insividuals are also, to some extent, influenced by the opinions of key figures in their culture, such as politicians, officers, publicists, writers and so on, who are commonly considered as the "elite" of the society. This work studies the "power of the elite": to what extent can the elite of a social network influence the rest of society to accept its opinion, and thus become a *monopoly*. We present an empirical study of local majority voting in social networks, where the elite forms a coalition against all other (common) nodes. The results, obtained on several social networks, show that an elite of size $\sqrt{m}$ (where m is the number of connections) has disproportionate power, relative to its size, with respect to the rest of society: it wins the majority voting and remains stable over time, in contrast to the predictions of a preferential attachment model.

1. INTRODUCTION

Since ancient times, there were numerous occasions in which tribes, nations and other groups of people had to reach a decision regarding a question affecting the life and future of their members. Nowadays, democracies use election systems in order to choose government officials and parliament members who are to set the nation's agenda for the future. In ancient Greece, people used to gather at the city center to take part in polls on issues affecting the affairs of the city (similar to a modern time referendum). In all cases, a group of people who wish to "push" their agenda has to use "politics" – the art of influencing people on a civic or individual level. In order to make up their minds on an issue which is about to be put to vote, people usually discuss the issue with our social environment, including friends, family and colleagues, before taking the vote. This process allows people to mutually affect each other's vote. In addition, individuals are, to some level, influenced by the opinions of key figures in their culture, such as politicians, publicists, writers, former top military officers, celebrities and others. These figures are commonly considered as the *elite* of the society (we discuss several definitions of the elite in section 2.2), and due to their influence on the masses it is natural to think that these "opinion leaders" have a major impact on the final result. This raises the interesting question whether one could measure how large this influence really is.

Clearly, there are also other sources affecting the final vote of individuals, such as their backgrounds, interests and personal tastes. The goal of the present work is to isolate the element of social influence from all other sources of influence, and study only the influence that members of society have on each other's opinions, and particularly, the influence of the elite's opinions on the rest of society.

In order to do so, we examine a hypothetical social voting system on a social network represented by a weighted graph, which runs as follows. In the morning of the decision day (e.g., the elections), each individual (represented by a node in the network) sets its own initial (subjective) vote. We refer to this as the *initialization* phase. Then, in the *update* phase, each individual starts an iterative process, in which it gradually modifies its vote. Specifically, in each round it gathers votes from all sources known to it directly in the network (i.e., its neighbors in the graph), representing friends and family, colleagues, mass media etc., and then changes its vote by adopting the majority of all the votes it has seen. Since one may also be influenced indirectly (e.g., by the friend of a friend), it may take more than one such update round until the system reaches "convergence"[1]. We define the convergence state of the system to be the first round when there were no vote changes from the previous round. Once convergence is reached, the voting process stops and the update phase ends. At the final phase, we count the final votes of all individuals, and the opinion that won the most votes wins the poll. The simplest case is that of a poll involving "Yes" or "No" questions and a democratic voting, i.e., in the *final* phase, each vote counts the same. However, during the *update* phase (only), we study both *uniform* and *non-uniform* influences of nodes. In particular, we may grant a

[1]Theoretically, the system might never converge, cf. [44]; for now, we assume it does.

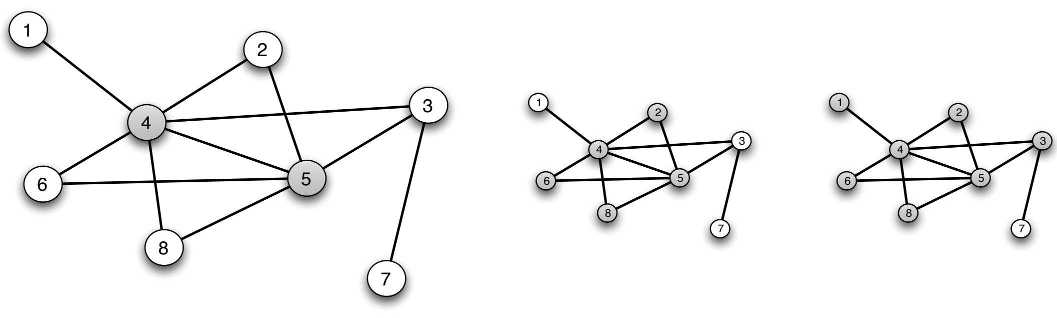

Figure 1: Example of majority voting in a social network: (A) An initial state with a coalition of nodes 4 and 5. (B) The final votes in the non-reversible model with coalition weight 1. (C) The final votes in the reversible model with coalition weight 2.

higher weight to certain subgroups of nodes (e.g., the elite), indicating their higher influence. The process of updating the vote of each node is then based on the weighted sums of its neighbors votes.

Our focus in this work is on the power of "cooperating" groups in social networks. In this context, a central notion is that of a *coalition*, i.e., a group of individuals that vote unanimously in order to influence the resulting decision. For the purpose of this study, a coalition is called a *monopoly* if at the end of the voting process, it convinces more than 50% of the population to vote according to the coalition's initial vote. As mentioned earlier, a coalition may or may not be a monopoly, and this depends to some extent on whether it is granted extra weight or not.

We also consider two types of annexation behaviors a coalition can take. In the *reversible* setting, the coalition plays *fair*, in the sense that each of its members can change its vote in each round according to the above rules (i.e., its vote is reversible). In contrast, in the *non-reversible* setting the coalition plays *unfair* and its members (and only them) stick to their initial vote and ignore the majority vote rules during the update phase.

Consider for example the network in Fig 1, where nodes colors denote a "Yes"/"No" vote. In (A), nodes 4 and 5 form a coalition with initial vote "Yes". (B) shows the result of the voting process after one round, in the uniform, non-reversible setting. Note the this is also the stable state of the voting and nodes 4 and 5 are also a monopoly, since they have succeeded in convincing the majority of the nodes in the network. In the uniform *reversible* setting (not shown in the figure), the coalition loses and upon convergence all nodes vote "No". Figure 1(C) still considers the *reversible* setting, but assumes the above coalition is granted with a weight of 2, and shows the final (stable) state of the voting, in which the coalition is also a monopoly. Note that further increasing the coalition weight will not change the stable state. To sum up the example, nodes 4 and 5 form a monopoly in the uniform, non-reversible case, and also in the reversible case provided they are granted a weight of 2.

In this setting of on-line social networks, we study the following interesting questions:

1. Can a relatively small coalition, say of size sub-linear in the population size, be a monopoly? Does providing the coalition with higher weight have a significant impact? If so, should this weight be a function of the population size or would a constant increase suffice?

2. How does the monopoly size change over time (as a function of the population size) as society grows?

3. How significant is the effect of "playing fair"? Is there a significant difference in the required monopoly size between the reversible and non-reversible settings?

In particular, we study the minimum k such that the *k-rich club* [50] (namely, the k highest degree nodes) in the network, acting as a coalition, can become a monopoly. Answering the questions above can help us understand the *power of the elite* in social networks, where by "power" we mean the level of control a unanimously voting elite has over the final results.

We focus our attention on elite size in the order of $\sqrt{m}$ (where m is the number of edges in the network) since our previous and on going work ([44], [7]) indicate that this is an interesting scale to for monopolies. We elaborate on previous findings in section .

1.1 Overview of Our Results

We studied the process of majority voting on several real social networks and models. We consider the basic case of *binary votes* and two coalitions. The first coalition consists of the members of the k-rich-club [50] of the network. The k-rich-club is defined as the sub-group of the nodes with the highest degree in the network. We used this group as it can simply approximate the elite of the network while we aware that these two groups are not necessarily the same. All other, common nodes or periphery nodes, form the second coalition. We study two cases of reversibility: In the first version, all vertices are reversible and we denote this version as *all-reversible.* In the second version, only the elite coalition is non-reversible and we denote this caseas *elite-non-reversible.* The outcome of majority voting were examined for different sizes of elites (i.e., k-rich-club) and different weights that were given to the elite. In particular, for a given size of a rich club and reversibility we found the minimum weight for it to become a monopole and vise versa for a given elite weight and reversibility we found the minimum k for the elite to become a monopole. Our main four empirical findings are:

1. The size of monopoly elites: A relatively small elite, particularly an elite of size $\sqrt{m}$, has a disproportional power toward the rest of the network and it can dramatically affect the results of majority voting. In other words, the elite has a monopoly over the final result of majority voting. Moreover, granting the elite with small weight is leading to a sharp decrease in the minimum elite size required to win the poll. In particular, for an elite of size of $\sqrt{m}$, the *maximal* observed value of all networks, of the minimum elite weight required to win the poll is 8 (Youtube). For the elite-non-reversible case, the value is even smaller: 2. Note this result is obtained for a collection of networks of several sizes and densities, hence it seems that the extra weight is independent of the network size (in terms of number of nodes and edges). However, the power of elite is bounded, meaning that from some point, granting the elite with more weight does not reduce the minimum elite size required to win the poll.

2. Reversible vs. non-reversible: *Quantitatively* speaking, an elite operating in non-reversible mode performs better compared to when operating in reversible mode. For example, a smaller elite is required in order to win the poll for the same elite weight, when operating in non-reversible mode compared to when operating in reversible mode. However, asymptotically there is no *qualitative* difference between the behavior of the elite in the two modes.

3. The power of elites over time: An elite of size $\sqrt{m_t}$ (where m_t is the number of edges in the network at time t), keeps its monopoly power on the society along the life-time of the social network. We observed that, the small extra weight that an elite of size $\sqrt{m_t}$ needs to win the poll, is a constant independent of time and as more and more nodes are joining the network .

4. The power of elite of analytical models: Real networks elites are more powerful than those reflected from the models. For example, provided with some weight, $w(e) = C > 1$, a smaller elite can win the poll in real network compared to an elite of same size for matching[2] *Erdős–Rényi* and Preferential Attachment models[3]. Moreover, in these models, the weight an elite size of $\sqrt{m_t}$ needs be granted with to become a monopol, grows asymptotically along the lifetime of a social network; Therefore there is qualitative difference between these models to real networks when it concern the power of elites in majority voting.

[2]Similar number of nodes and average degree.

[3]As expected, we also observe that Erdős-Rényi model generates weaker elite than Preferential attachment.

2. BACKGROUND AND RELATED WORK

This paper studies the power of the elite coalition in local majority voting over social networks, both for static networks and over time. Let us next briefly introduce each of the main components of the system and their basic properties.

2.1 Social Networks and Universal Properties

In the past couple of decades, the extensive study of complex systems and social networks has yielded an impressive body of knowledge on such networks and the *universal* properties they share, namely, properties and behaviors that is repeatedly seen across different networks. Among the most basic such properties are: *small world*: Short average path lengths [40, 35, 8, 8]; *Power law degree distribution and preferential attachment*: New players in the network prefer to link to strong exiting exiting players and by that shape the degree distribution of the network to follow a *power law*: the probability of a node to have degree k is proportional to $p(k) = k^{-\alpha}$, $\alpha > 1$ [9]; *Clustering*: The probability that two randomly selected individuals know each other is much higher if they happen to have a common friend (in graph theoretic terms, a social network graph is likely to have many triangles) [48]; *Navigability*: Not only is the world small, but it is possible to find short paths between individuals on the basis of purely local information (as the structure of the whole network is unknown to the individual) [33].

While these properties are defined for static graphs, some universal properties for *dynamic* graphs have recently been observed [36], like *densification* of the network over time and *shrinking diameter*.

Concerning the universal properties of the elite, in a recent, yet unpublished report [7], Avin et. al. suggest an axiomatic approach to finding the size of the elite of social networks. Bases on three axioms for the elite: *Influence*, *Stability* and *density* they proved that the elite must be of size of $\sqrt{m}$ (where m is the number of edges in the network). Empirically, they find that the $\sqrt{m}$-rich-club has the following properties:

1. **Influence:** Thinking of an edge as representing a source of influence (a natural interpretation in social networks), they find that the elite has "disproportionate" power over the rest of the network: a significant constant fraction of the edges in the network are controlled by the elite.

2. **Stability:** The elite is robust against influence from the rest of the network, as the number of links between the elite members is also a significant fraction of the number of links connecting the elite to the outside.

3. **Density:** The elite, of size $\sqrt{m}$, is a dense subgraph.

2.1.1 Random Models

The common assumption in modeling social networks is that the network links are essentially "random" (in a sense to be made more precise next).

One common model is the *Erdős–Rényi* model. In this model, the network is constructed as a randomly chosen n-node graph from the class $G_{n,p}$, where for every two nodes x, y, a link xy exists in the graph with probability p, independently of the other links. We later use this model as a base case to compare our results to.

An alternative approach is to model the network by an evolutionary process describing its growth. A well-known representative of this approach is the *preferential attachment* model (PA) [4]. In this model, nodes join the network one by one, and each new node randomly selects some existing nodes according to their degrees and attaches itself to those nodes. The higher the degree of a node, the more likely it is to attract new nodes to connect to it. The network starts as an initial network of n_0 nodes. New nodes are added to the network one at a time. Each new node is connected to $d \leq m_0$ existing nodes with a probability that is proportional to the number of neighbors that the existing nodes already have. Formally, the probability p_i that the new node is connected to node i is [4] $p_i = deg(i)/\sum_j deg(j)$, where $deg(i)$ is the degree of node i. In this work we adopt the convention $m_0 = d$ and start with an initial network forming a complete graph (clique).

2.2 Who are The Elite?

The Cambridge dictionary defines the elite as "the richest, most powerful, best educated or best trained group in a society". Wikipedia adds that "an elite, in political and sociological theory, is a small group of people who control a *disproportionate* amount of wealth or political power". In his 1957 book, "The Power Elite", the American sociologist C. Wright Mills writes: "those political, economic, and military *circles*, which as an intricate set of *overlapping small but dominant groups* share decisions having at least national consequences...the power elite are those who decide them". The Italian sociologist Vilfredo Pareto puts it succinctly [42]: "Every people is governed by an elite, by a chosen element of the population". The political scientist Robert A. Dahl [16] distinguishes between "the potential" for control and "actual" control. Potential for control is when a set of individuals has the following property: there is a high probability that if they agree on an alternative, and if they all act in some specified way, then that alternative will be chosen. "Actual" control is the active participation of this set in decision making.

Who are the people constructing the elite? What is the size of the elite and how is it formed and structured in terms of (inernal and outgoing) connections? In his book "Who's Running America?" [21], Thomas R. Dye defines the elite as those individuals who control or occupy formal positions of authority in top institutions over ten sectors: industrial (non-financial) corporations, banking, insurance, investments, mass media, law, education, foundations, civic and cultural organizations, and government. He identifies 7,314 institutional positions of power encompassing 5,778 individuals yielding an elite size of 0.0002%, or $n^{0.44}$ where n is the size of the population in the U.S. ($\sim$300 millions). Doob [19] found that most holders of top positions in the power elite possess exclusive membership in one or more social clubs. About a third belong to a small number of especially prestigious clubs in major cities like New York, Chicago, Boston, and Washington D.C. This leads to the conclusion that elite members are well connected to each other.

The above 2 paragraphs are too long and add little information relevant to the paper. Perhaps we should shorten them in the extended abstract.

2.3 Majority voting

Discrete influence systems, including ones based on voting processes, occur naturally in diverse settings, and were studied, for example, in the context of social influence [26, 17] and neural networks [27, 45]. Our voting process is also reminiscent of *epidemic* models (cf. [6] and the references therein).

A dynamic, synchronous, deterministic majority-based version of the voting process, similar to the one considered here, was studied in [45, 43, 27]. In that context, the term *monopoly* is reserved to a particularly powerful set of nodes that, if acting as a coalition, can force the process to end in consensus on its opinion after a *single* voting step; the repetitive variant considered here is referred to therein as a *dynamic monopoly*. Reversible and irreversible dynamic monopolies were studied in [43, 23, 24, 25, 13, 20]. Related problems were studied in different guises and under different names, in [31, 22, 11, 3, 46, 12, 29, 15].

Probabilistic variants of the voting process were extensively studied as well. A repetitive (synchronous) *probabilistic* voting process on weighted graphs was studied in [30]. The nodes start with their initial opinion as their vote. In each round, each node recomputes its vote by choosing at random one neighbor (with probability dependent on the edge weights) and adopting its vote. It is shown that the probability of the process ending with every node holding a vote of "yes" is proportional to the sum of the probabilities of the nodes whose initial opinion is "yes" in the stationary distribution of the process. This theorem is generalized also to processes on graphs with multiple (i.e., non-binary) opinions.

Related results concern a probabilistic *asynchronous* model called the *voter model*, introduced by [32]. This is a continuous time Markov process with a state space consisting of all possible vote distributions. This process was extensively studied in infinite grid graphs [32] and finite connected graphs [18], analyzing the convergence time and convergence probability.

3. THE MODEL

The network is modeled by a simple[4] directed weighted graph $G(V, E, W)$ where V is the set of nodes, E is the set of edges and W a weight function. Let n and m denote the number of node and edges respectively. For every two nodes, either both or none of the two directed edges appear in the graph. (The advantage of this formalism over using undirected edges is that it enables assigning different edge weights, modeling different influences of the two neighbors on each other.) Let $N(v)$ denote the set of neighbors of v in G, *including* v (i.e., $N(v) = \{u \mid uv \in E\} \cup \{v\}$).

In the context of a voting system, the nodes of the graph represent voters and the edges represent influence. A directed edge uv with weight $w_{uv} > 0$ from node u to node v represents the fact that voter u influences voter v proportionally to w_{uv}. Different votes are modeled as colors. At any time, each voter is colored with its current vote. The simplest case concerns binary votes, so each voter is colored with either white (i.e., "yes") or black ("no"). Formally, the value of a "Yes" (respectively, "no") vote is set to $+1$ (resp., -1). Neutral votes are disallowed. A *coalition* is a set of nodes that coordinates the same initial vote. A coalition is called a *monopoly* if at the end of the voting process it manages to convince more than 50% of the population to vote according to its own initial vote.

In the *initialization* phase, each voter sets its vote to its initial (subjective) opinion. Next starts the *update* phase, a synchronous process consisting of several *rounds*. In each round, each voter updates its current vote to the majority of the votes in its neighborhood, including its own vote, and considering the influence or weight of its neighbors. In case of a tie, the node keeps its current vote. A bit more formally, let $X_t = \{x_t(v_1), x_t(v_2), \ldots x_t(v_n)\}$ be the global vector state after round t, where $x_t(v_1)$ represents the value of node $v_i \in V$ after round t. Let X_0 be the vector of initial votes. For $t > 0$ let $S_t(v) = \sum_{u \in N(v)} w_{uv} \cdot x_{t-1}(u)$. Then

$$x_{t+1}(v) = \begin{cases} -1, & S_t(v) < 0, \\ 1, & S_t(v) > 0, \\ x_t(v_i), & S_t(v) = 0. \end{cases} \quad (1)$$

For example, consider a node v whose current vote is $+1$ and suppose v has 3 neighbors with influence weights $\{1, 3, 7\}$ and votes $\{1, 1, -1\}$ respectively. The updated vote for v in this case is $\text{sign}(1 + 1 \cdot 1 + 3 \cdot 1 + 7 \cdot (-1)) = -1$. The update phase continues until convergence is reached (if at all[5]). The final result of the voting process is obtained by counting uniformly the final votes of all voters. The value voted for by the majority wins the poll.

In the above definitions the coalition plays fairly (i.e., with the same rules as the rest of the player); this is refered to as the *reversible* setting. In the *non-reversible* (or irreversible) version, the coalition members play unfairly, in the sense that in the *update* phase they make one-sided moves: they "answer" when being polled for their vote but they keep their original values without executing the update rule specified in Eq. (1).

[4] with no parallel or self edges

[5] The period of the voting process is defined as $Period(X_t) = \min\{k | X_{t^*+k} = X_{t^*} \text{ for some } t^*\}$. Then for finite graphs, the period can be 1 or 2, cf. [44]. Here we only consider cases where the period is 1 (steady state).

4. EMPIRICAL RESULTS

4.1 Data Sets and Models

The WWW serves as a habitat for many social networks, whose users publicly post their view and communicate directly. We have used data on social networks from several sources. The Stanford Large Network Dataset Collection (SNAP) [1] was used for the graphs of Slashdot, Twitter, Brightkite, Gowalla and Citation (cit-HepPh). We used the collection[6] of Mislove et al. [39, 41, 47] for the graphs of Facebook, Youtube and Flickr. Finally, we have used The Koblenz Network Collection (KONECT) [2] for the graphs of Email, Epinions, Wikiusers and Slashdot2.

Table 1 lists the networks we have used and some of their basic properties. Several of these networks were used for both static and dynamic (i.e. over time) analysis. Below is a short explanation about the nature of each network and the way it was obtained. In all cases, where the original data set was directed or signed, and/or contained multiple edges, we have converted it to be undirected and simple by adding opposite edges where missing and removing parallel edges.

Youtube: Youtube is a video-sharing site that includes a social network. The graph consists of links based on a who-follows-who relationship. The network was obtained on January 2007.

Slashdot: Slashdot is a technology-related news website. In 2002 Slashdot introduced the Slashdot Zoo feature, which allows users to tag each other as friends or foes. The network contains friend/foe links between the users of Slashdot. The static network was obtained in February 2009. The network used for the dynamic analysis (Slashdot2) was obtained over the period of Aug. 2005 - Aug. 2006.

Twitter: Twitter is a well-known online microblogging service. A directed edge from node i to node j represents the fact that user i is "following" user j.

Flickr: Flickr is a photo-sharing site based on a social network. The graph consists of links based on a who-follows-who relationship. The network was obtained on January 2007.

Facebook: Facebook is the largest social network in the Internet. Its graph consists of a list of all of the user-to-user links from the Facebook New Orleans networks crawled between Sep. 2006 and Jan. 2009.

Gowalla: Gowalla is a location-based social networking website where users share their locations by checking in. The friendship network is undirected and based on the

[6] available at `http://socialnetworks.mpi-sws.org`

network name	n (WCC%)	m	avg. degree.	effective diameter	$\sqrt{m}$ (% of nodes)	min. influence factor to monopoly		duration (months)
						reversible	non-revers.	
Youtube [41]	1138499 (99.7)	2990443	5.25	6.31	1729 (0.15)	6.79	1.98	-
Slashdot [37]	82168 (100)	543381	13.23	4.70	737 (0.90)	3.64	1.48	-
Twitter [39]	81306 (100)	1242397	61.82	5.60	1115 (2.77)	3.42	0.98	-
Flickr [41]	197042	15555040	157.98	5.24	3944 (2.00)	3.32	1.20	-
Facebook [47]	63731 (99.4)	817090	25.64	5.59	904 (1.42)	6.01	2.67	29
Gowalla [14]	196591 (100)	950327	7.35	5.68	975 (0.50)	3.67	1.98	20
Brightkite [14]	58228 (97.4)	214078	7.35	5.90	463 (0.80)	4.48	1.98	30
Citation [36]	34546 (99.6)	420899	24.37	5.00	649 (1.88)	7.64	1.67	124
Slashdot2 [28]	51083 (100)	116975	4.58	5.23	342 (0.67)	-	-	12
Email [34]	87273 (96.7)	298338	6.83	5.87	546 (0.09)	-	-	73
Epinions [38]	131828 (90.4)	711496	10.79	5.49	844 (0.64)	-	-	32
Wikiusers [10]	118100 (95.8)	2036794	34.49	3.81	1427 (1.21)	-	-	8

Table 1: Examined networks and their basic properties

check-ins of these users over the period of Feb. 2009 - Oct. 2010. For the analysis of this network over time, we considered the first check-in time as the user's joining time to the network. As we've found that about 50% of Gowalla users never checked in, they were not taken in account for the over time analysis.

Brightkite: Same as Gowalla, except that almost 100% of the users did checke in. The data was obtained over the period of Apr. 2008 - Oct. 2010.

Citations: Arxiv's HEP-PH (high energy physics phenomenology) citation graph is taken from the e-print arXiv and covers all the citations within its data-set. If a paper i cites paper j, then the graph contains a directed edge from i to j (papers cited by a paper out of the data-set or citing such a paper are not contained in the graph). The data covers papers in the period from January 1993 to April 2003.

Email: The Enron email network consists of emails sent between employees of Enron between Jan. 1998 and Feb. 2004. The nodes in the network represent individual employees and the edges correspond to individual emails. The data was obtained over the period of Aug. 2001 - Mar. 2009.

Epinions: Epinions is a consumer review website where the members can decide whether to "trust" each other. The network is based on a who-trusts-who relationship. The data was obtained over the period of Jan. 2001 - Aug. 2003.

Wikiusers: This network indicates positive and negative conflicts between users of Wikipedia, for example, users involved in an edit-war. A node represents a user. An edge represents a positive or negative conflict between two users.

The SNAP Collection was also used for its software package for the creation of *Erdős–Rényi* and the Preferential Attachment models graphs. The first parameter for the generation of *Erdős–Rényi* and Preferential Attachment graphs is the number of nodes. The second parameter for *Erdős–Rényi* is the number of edges and for Preferential Attachment is the average degree, so for a given real network both models generate a random graphs with the same number of nodes and the same average degree as the real network.

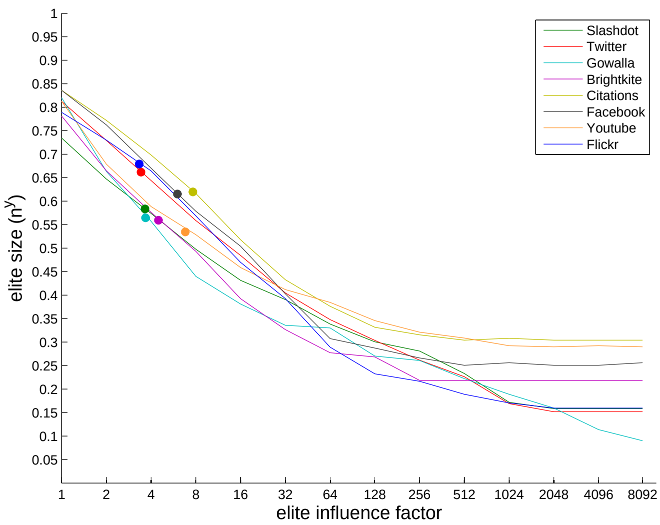

Figure 2: The power of elite. Relatively small elite coalition win the majority voting. An elite of size $\sqrt{m}$ requires a maximum influence factor of 7.64 (and 4.78 on average) to win the local majority voting.

4.2 Experimental Setup

We examined the results of majority voting on social networks. We looked at the basic case of binary votes and two coalitions, one formed by the elite (considered as the k-rich club) and the other formed by all other (common) nodes. The parameters of the setting considered are the elite size, the influence factor granted to the elite, and the reversibility of the elite. In each simulation we considered either the reversible or the irreversible setting. Then, one of the parameters – influence-factor or size – was fixed, while different values were sampled for the second parameter. By using binary search, we determined the minimum value for the free parameter that guarantees a majority for the elite, making the elite coalition a monopoly. In the following sections we discuss our findings.

4.3 The Power of the Elite

Figure 2 presents the basic results for the minimum elite sizes and influence factors that are needed for the elite coalition to become a monopoly in the *all-reversible* mode for all real social networks studied. The results should be compared with Figure 3, which demonstrates a few possible outcomes of majority voting for different types of general networks. In both figures the X-axis shows the influence factor granted to the elite (in log scale) while the Y-axis shows the value of the exponent y for which $n^y = k$ is the size of the minimum k-rich-club coalition that yields a monopoly. For example the point (4,0.6) indicate that if we would grant the elite a influence factor of 4 then the $n^{0.6}$-rich-club is the smallest rich-club coalition that become a monopoly. Each line in the figures represents a different network. The function y is in general monotonically non-increasing, since granting the elite additional influence can only decrease the minimum elite size required to win.

In Figure 2, the colored circles mark the special case where $k = \sqrt{m}$ and the minimum influence factor required by the $\sqrt{m}$-rich-club to be a monopoly. We observe similar characteristics for all real social networks depicted in the figure: (i) a relatively small influence factor granted to the elite leads to a large decrease in the elite monopoly size, (ii) granting an elite of size $\sqrt{m}$ a relatively small influence factor (between 4 and 8) makes it a monopoly, and (iii) from some point on, granting the elite coalition higher influence factor does not further reduce the minimum elite size required to win the vote.

These three characteristics point out that the elite has disproportionate power w.r.t. the rest of the society in the majority voting game. Looking at Table 1, one can see that the result are obtained for a variety of networks with distinct characteristics (i.e., number of nodes, number of edges, etc.). The results are particularly interesting for an elite of size $\sqrt{m}$, since previous studies [49] considered k-rich-clubs of linear number of nodes [5].

LO BARUR - I tried to fix a bit, Chen.

For general networks, the power of a $\sqrt{m}$ elite (or rich club) may not be the case. Figure 3 demonstrates several possible outcomes of majority voting on networks of size 1024, from ones with the weakest possible elite structure (red line – topmost) to those with the strongest (green line – lowest). Networks with the weakest elite structure require a large elite to win the poll, and granting the elite additional influence does not reduce the minimum elite size (i.e., the elite is saturated with power). A representative example for such a network and elite can be the *line graph* (or path), where the elite resides sequentially on the path[7] (i.e., the elite is a sub-path within the network). The elite of Network 1 (orange line) is stronger: although it requires the same elite size as the red network when the influence factor is 1, granting it extra influence decreases the minimum elite size, up to some point where adding more influence does not empower the elite anymore. An interesting example is the clique graph (brown line): when its elite is granted no extra influence, half of the vertices are required to form a monopoly (in this case $1024^{0.9} = n/2$). However, granting the elite additional influence reduces the required elite size linearly, until a single node granted with an influence factor of 1024 can monopolize the entire network. The blue line represents Network 2 where the elite is stronger, since the slope is sharper than this of the orange network (although it becomes saturated for a smaller elite size). In Network 3 (the pink line), the elite is stronger than in the blue one, because it is always below it (meaning, requires smaller elite size for any influence factor). The elite of the network represented by the green line is the strongest – it requires the smallest elite size to win, already with influence factor 1, although it becomes saturated from that point forward. A representative example for such elite coalitions can be a complete $(2, 1022)$-bipartite graph having 2 nodes at one side (the elite) and 1022 at the other *when the elite is irreversible*. Intuitively we may say that the smaller the area below the graph, the stronger is the elite illustrated by it.

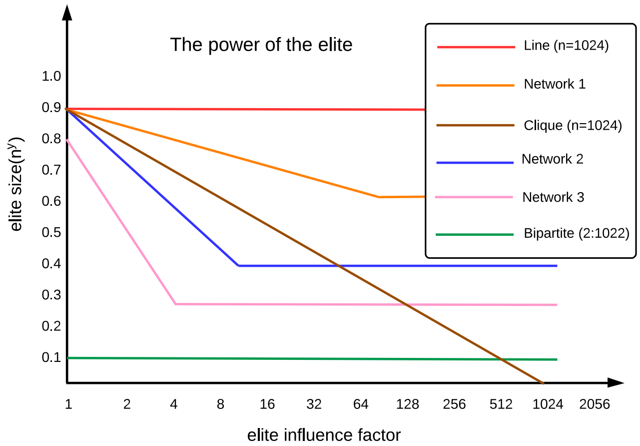

Figure 3: How to read the results

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4.4 Reversible vs. Irreversible setting

In this section we address the influence of reversibility on majority voting. Figure 4 illustrates the impact of reversibility on the outcome of majority voting for all networks. Solid (respectively, dashed) lines represent the reversible (resp., irreversible) setting. Recall that in the irreversible setting, the elite ignores its neighbors and always maintains its initial votes. The circles markers have the same meaning as in Figure 2. We observe similar characteristics for all networks: (i) the slopes in both settings are similar, (ii) for small influence factors granted to the elite, the irreversible slope is below that in the reversible setting, but from some influence and forward, both are asymptotically bounded. (iii) an elite of size $\sqrt{m}$ requires smaller influence factor to win in the

[7] Since all the vertices have the same degree, we are free to select the worst case elite.

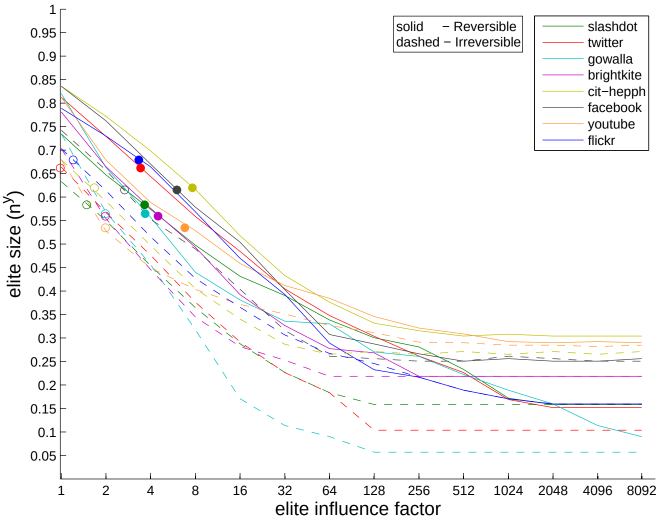

Figure 4: The power of elite. Reversible vs. irreversible setting

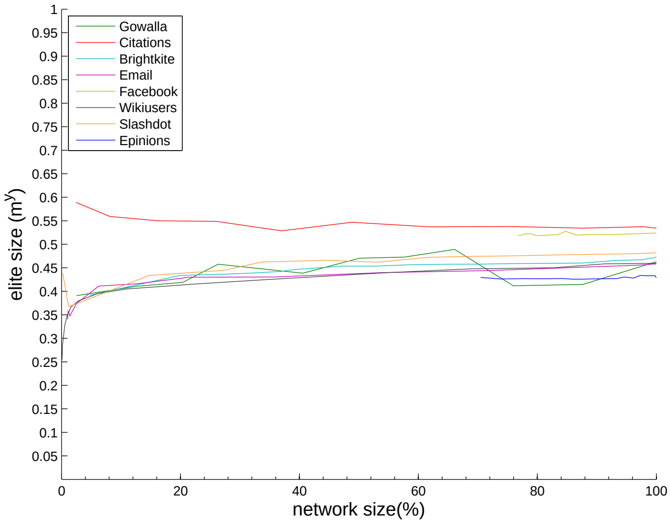

Figure 5: The power of the elite over time. For a constant influence factor ($\phi = 4$), the influence factor deriving the minimum sub-linear elite required to win the majority voting is asymptotically constant.

irreversible setting (between 1 and 2) than in the reversible setting (between 4 and 8). These three characteristics point out that the differences between the two settings are quantitative rather than qualitative.

4.5 The power of the elite over time

In the previous sections we have discussed the power of elite as reflected in static snapshots of the networks under study. In this section we address the power of the elite along the life-time of the social network. For this purpose we took multiple snapshots of the studied networks in several points in time from the start (i.e., the network creation time) to the end (i.e., the current time). Each network snapshot contained only nodes that have already joined the network by the time that snapshot was taken.

For each graph we performed two types of measurements. For the first, we granted the elite a constant influence factor throughout the network's lifetime. Then, at each point, we used binary search to find the minimum elite size required to win the vote. We observed a similar behavior for all networks: when granting the elite with a constant, unchanged, influence factor throughout its life-time, the size of the elite monopoly appears to converge. Figure 5 shows the result for all networks when the elite is granted influence factor 4 in the reversible setting. The X-axis shows the evolution of the number of nodes in the network (as a percentage of the curent size). Note that the first snapshot of some networks already contains a large fraction of the total number of nodes. The Y-axis shows the value of the factor y where $m^y = k$. We observe that different networks converge to different values between 0.45 and 0.55, i.e., the required elite size is in the order of $\sqrt{m}$.

To verify this picture, in the second experiment we took a snapshot of an elite coalition consisting of the $\sqrt{m}$-rich-club at several points of time along the life-time of the network. Then, at each point, we used binary search to find the minimum influence factor required for the elite to become a monopoly. Figure 6 shows the result for all networks. We observe similar characteristics for all networks: excluding a short initialization period at the start of the network's life-time, the (small) influence factor an elite of size $\sqrt{m}$ must be granted in order to win the poll converges to a constant with time, along the life-time of the social network.

These two observations indicate that an elite of size $\sqrt{m}$ keeps its control over the rest of the network nodes throughout the life-time of the social network. It monopolizes the outcome of majority voting in every time-step along the way. These results support the claim that $\sqrt{m}$ is the right scaling that should be considered for the size of the elites that "controls" the network.

4.6 Power of the elite in random models

We have compared the performance of the elite of the *Erdős–Rényi* (ER) and Preferential Attachment (PA) models when challenged with the same simulations done for the elite of real world networks. We generated a matching model graph for each of the networks shown in Table 1. By "matching" we mean it has the same number of nodes and average degree[8].

4.6.1 *The power of the elite*

[8]For the Preferential Attachment model, the average degree is rounded to the nearest natural number for obvious reasons.

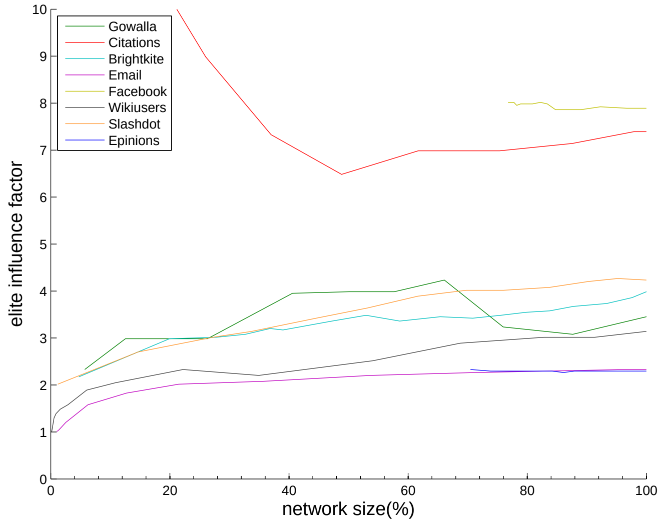

Figure 6: The power of an elite of size $\sqrt{m}$ over time. The (small) influence factor required for such an elite to win the majority voting converges to a constant with time.

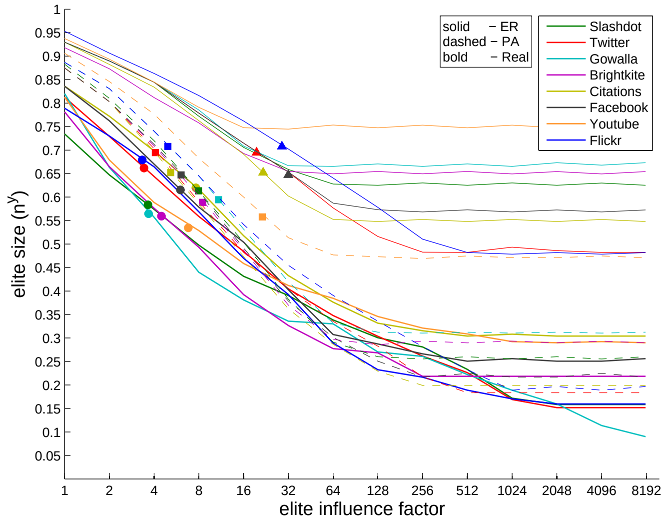

Figure 7: Real networks vs. Models, Reversible. Real networks are demonstrating stronger elites than Preferential Attachment for small influence factor. *Erdős–Rényi* model elite fail to win though granted with asymptotically large influence factor.

We observe similar characteristics for all networks: (i) Both models reflect weaker elites than their corresponding real networks, where the *Erdős–Rényi* elite is the weakest of the three. (ii) Particularly, an *Erdős–Rényi* elite of size $\sqrt{m}$ is unable to win even when granted an asymptotically large influence factor. This may be expected recalling that this model has a binomial degree distribution, unlike the power-law distribution characterizing other models. (iii) In the Preferential Attachment model, the elite is weaker than in real networks, mostly when a small influence factor is granted to the elite. When large influence factors are used, the difference is milder and for some networks, the Preferential Attachment elite is even stronger than that of the corresponding real network. Figure 7 shows the difference between the three for all networks.

4.6.2 Reversible vs. Irreversible

Similar to the behavior of real networks, both models do not present qualitative differences between the reversible and irreversible settings. In addition, both models demonstrate small, sometimes negligible, quantitative differences between the two settings.

4.6.3 Power of the elite over time

We simulated the network growth for both the *Erdős–Rényi* and Preferential Attachment model graphs, and repeated the two simulations from section 4.5 for each.

For the first simulation, using constant influence factor and searching for the minimum elite size required to win the vote, Figure 8 shows the result for both graphs when the elite is granted influence factor 4. The X-axis shows the graph size (in log scale). The Y-axis shows the value of the exponent y where $m^y = |E|$. One can see that the required minimum elite size converges for both models but the convergence is slower (and for larger scale) compared to real networks.

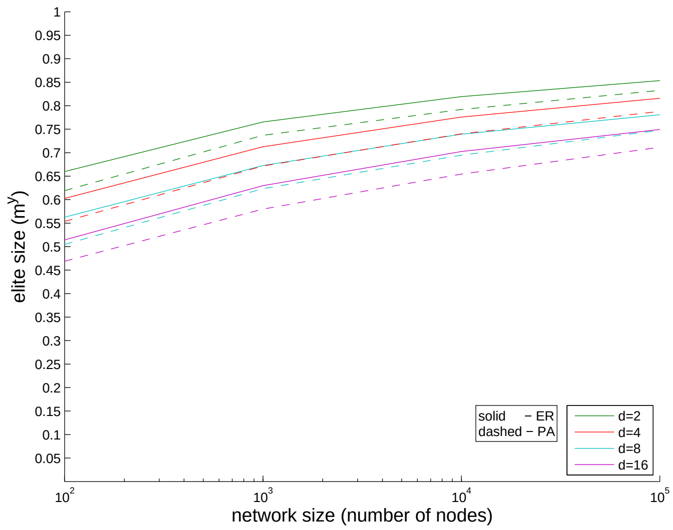

Figure 8: The power of the elite for the *Erdős–Rényi* and Preferential Attachment models. The minimum elite size to win when granted constant influence factor converges to larger scale compared to real world networks.

Similarly, for the second simulation, using an elite of size $\sqrt{m}$ and searching for the minimum influence factor required to win the vote, Figure 9 shows the result for both models. The X-axis shows the graph size (in log scale). The Y-axis shows the influence factor granted to the elite. One can see that the minimum influence factor grows super-linearly with time, unlike the observed growth for real world networks.

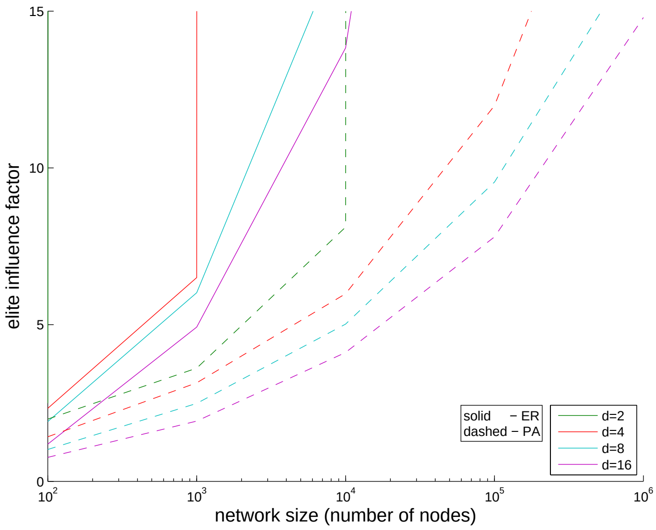

Figure 9: The power of the elite in the *Erdős–Rényi* and Preferential Attachment models. The influence factor required for an elite of size $\sqrt{m}$ grows super-linearly with time, unlike the observed growth for real world networks.

5. RANDOM MODEL

In this section we consider a class of *random social networks* denoted $\mathcal{G}(n, k, p_1, p_2, p_3, w)$, defined as follows. A random social network in this class consists of pair (G, w), where G is a random graph on n nodes, $V = \{v_1, v_2, \ldots, v_n\}$, partitioned into a *core* $\mathcal{C} = \{v_1, \ldots, v_k\}$ and a *periphery* $\mathcal{P} = \{v_{k+1}, \ldots, v_n\}$, and w is an *influence weight* assigned to the nodes of the core $\mathcal{C}$. For $i < j$, let e_{ij} denote an edge between v_i and v_j. Then each e_{ij} is selected independently at random to be in $\mathcal{G}$ with probability

$$\Pr\left(e_{ij} \in \mathcal{G}\right) = \begin{cases} p_1, & i, j \in \mathcal{C}, \\ p_2, & i \in \mathcal{C} \text{ and } j \in \mathcal{P}, \\ p_3, & i, j \in \mathcal{P}. \end{cases}$$

Let x_{ij} be the indicator variable for the event that $e_{ij} \in G$. For a periphery node $v_j \in \mathcal{P}$, let

$$X_C^j = \sum_{i=1}^{k} x_{ij} \,,$$
$$X_P^j = \sum_{i=k+1}^{j-1} x_{ij} + \sum_{i=j+1}^{n} x_{ji} \,,$$

be random variables denoting the number of neighbors of v_j in the core $\mathcal{C}$ and the periphery $\mathcal{P}$, respectively. Note that

$$\mathbb{E}[X_C^j] = p_2 k \,,$$
$$\mathbb{E}[X_P^j] = p_3(n - k - 1) \,.$$

THEOREM 1. *For $\epsilon > 0$, let $k = n^{1-\epsilon}$, $p_3(n - k - 1) \geq 16 \log n$ and $p_2 k \geq 1/2$, and consider the class $\mathcal{G}(n, k, p_1, p_2, p_3, w)$ of random social networks. There exists a constant c such that if the influence weight satisfies $w < \frac{n^\epsilon}{c \log n} \frac{p_3}{p_2}$, then w.h.p. the k-rich-club of a random social network from this class is* not *a monopoly.*

PROOF. Consider a random social network (G, w) from the class $\mathcal{G}(n, k, p_1, p_2, p_3, w)$, assigning an influence weight of w to the core. Clearly, if the core cannot influence even a single player from the set of itsa immediate neighbors in the periphery, then it is not a monopoly. Consider a node $v_j \in \mathcal{P}$. This node will keep its vote "No" if

$$w \cdot X_C^j \leq X_P^j \,.$$

Let $\mathcal{E}_P^j$ denote the event that $X_P^j > p_3(n - k - 1)/2$. Using Chernoff's bound with $\delta = 1/2$, we have:

$$\Pr(\overline{\mathcal{E}_P^j}) = \Pr(X_P^j \leq (1 - \delta)p_3(n - k - 1))$$
$$\leq e^{-\delta^2 p_3(n-k-1)/2} \leq e^{-16 \log n/8} \leq \frac{1}{n^2} \,,$$

so $\mathcal{E}_P^j$ occurs w.h.p., and

$$X_P^j > p_3(n - k - 1)/2 \geq 8 \log n. \qquad (2)$$

For X_C^j we consider two cases regarding $\mathbb{E}[X_C^j] = p_2 k$.
Case I: $p_2 k > 24 \log n$. In this case, letting $\mathcal{E}_C^{j,1}$ denote the event that $X_C^j < 3p_2 k/2$, we show (using Chernoff's bound with $\delta = 1/2$):

$$\Pr(\overline{\mathcal{E}_C^{j,1}}) = \Pr(X_C^j \geq (1 + \delta)p_2 k)$$
$$\leq e^{-\delta^2 p_2 k/3} \leq e^{-24 \log n/12} = \frac{1}{n^2} \,,$$

so $\mathcal{E}_C^{j,1}$ occurs w.h.p., and $wX_C^j < 3wp_2k/2$. Substituting $k = n^{1-\epsilon}$ and $w < \frac{n^\epsilon}{\log n} \frac{p_3}{p_2}$, we get w.h.p.

$$wX_C^j < \frac{n^\epsilon}{\log n} \cdot \frac{p_3}{p_2} \cdot p_2 n^{1-\epsilon} = \frac{n}{\log n} \cdot p_3$$
$$\leq p_3(n - n^{1-\epsilon} - 1)/2 < X_P^j \,, \qquad (3)$$

where the last inequality is by Eq. (2).
Case II: $p_2 k < 24 \log n$. We have, again using Chernoff's bound,

$$\Pr(X_C^j \geq 6 \cdot 24 \log n) \leq 2^{-2 \log n} = O\left(\frac{1}{n^2}\right) \,,$$

so w.h.p.

$$wX_C^j < w \cdot 144 \log n = w \cdot c' \cdot p_2 k \,.$$

Again, substituting $k = n^{1-\epsilon}$ and $w < \frac{n^\epsilon}{c \log n} \frac{p_3}{p_2}$ and noting that $c' \leq 288 \log n$, we get w.h.p. (for a suitable selection of

c):

$$
\begin{aligned}
wX_C^j &< w \cdot c' \cdot p_2 n^{1-\epsilon} \le \frac{n^\epsilon}{c \log n} \cdot \frac{p_3}{p_2} \cdot c' \cdot p_2 n^{1-\epsilon} \\
&= \frac{1}{3} p_3 n \le \frac{1}{2} p_3 (n-k-1) < X_P^j . \qquad (4)
\end{aligned}
$$

As Eqs. (3) and (4) both hold w.h.p., we use the union bound and get that none of the periphery nodes changes its vote. □

In the opposite direction, we focus on the following special case.

THEOREM 2. *Let $k = n^{1-\epsilon}$ and $\mathbb{E}[X_P] = \mathbb{E}[X_C] = 1$. Consider the class $\mathcal{G}(n, k, p_1, p_2, p_3, w)$ of random social networks. For $w = 3$, the core is a monopoly w.h.p.*

PROOF. We first prove the claim on the similar Possion random graph, defined by three main parameters: $\lambda_1 = p_1 k$, $\lambda_2 = p_2 k$ and $\lambda_3 = p_3(n-k-1)$. In the Possion model, the partial node degrees X_P and X_C (towards $\mathcal{C}$ and $\mathcal{P}$ respectively) are random Possion variables with the appropriate λ_i (e.g., X_C is distributed according to a Poisson distribution with parameter λ_2). After generating the degree sequence, edges are then connected according to the configuration model **??**.

Let X_C^j, X_P^j $Poisson[\lambda = 1]$ be i.i.d. r.v's with Poisson distribution for $\lambda = 1$. Recall the influence weight is $w = 3$. Next we calculate the probability that a node $v_j \in \mathcal{P}$ in the periphery changes its opinion, i.e., that $3X_C^j > X_P^j$. Letting r and s denote possible values of X_C^j and X_P^j respectively, we get that

$$
\Pr(3X_C^j > X_P^j) = \sum_{r=1}^{\infty} \frac{(\lambda_2)^{-r}}{e \cdot r!} \left(\sum_{s=0}^{3r-2} \frac{(\lambda_3)^{-s}}{e \cdot s!} \right).
$$

Plugging $\lambda_2 = \lambda_3 = 1$ yields $\Pr(3X_C^j > X_P^j) = 0.534238$, i.e., each node changes its vote with probability 0.534238. Let a_i, $i > k$, be an indicator variable for the event that node v_i changes its vote and let $A = \sum_{j=k+1}^{n} a_i$ be a r.v. denoting the number of periphery nodes that change their vote. Using Chernoff's bound for sufficiently small δ we have:

$$
\Pr(A \le n/2) \le e^{-0.53n\delta^2/2} . \qquad (5)
$$

This establishes that under the Poisson random graph model, if the core has influence weight 3 then it is a monopoly w.h.p. To conclude the proof we note that any event that occurs w.h.p. in the Poisson random graph model happen also w.h.p. in the *Erdős–Rényi* model. □

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