Nondimensionalization and Scaling

Recall dimerization

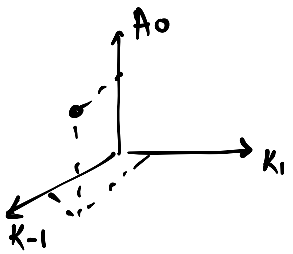
$$A + A \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} C$$

$$\frac{dA}{dt} = -2k_1A^2 + 2k_{-1}C$$

$$\frac{dC}{dt} = k_1A^2 - k_{-1}C$$

$A + 2C = \text{constant} = A_0$

$k_1,\ k_{-1}$ $\qquad (A_0, k_1, k_{-1})$



Can we reduce # parameters?

Relative magnitudes

Example 1. logistic eqn

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

$N(t)$ = population at time t

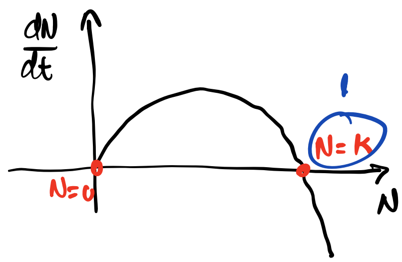
r = time scale $\rightarrow$ $1/\text{time}$

K = carrying capacity

$N \rightarrow \frac{N}{K}$

Malthusian population eqn

$$\frac{dN}{dt} = rN \qquad N(t) = N_0e^{rt}$$



$$\frac{d}{dt}\left(\frac{N}{K}\right) = r\frac{N}{K}\left(1 - \frac{N}{K}\right)$$

$y = \frac{N}{K}$

$$\frac{dy}{dt} = ry(1-y)$$

$s = rt$

$$\frac{dy}{ds} = y(1-y) \qquad \text{no free parameters}$$

Example 2 $A \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} B$

Conservation statement
$A + B = M'$ ↳ const.
$A(t=0) = A_0$

$$\frac{dA}{dt} = -\underline{(k_1' + k_{-1}')}A' + k_{-1}'M$$

$[A] \rightarrow$ dimension of A $= L^{-3}$; $k_1, k_{-1} = \frac{1}{\text{time}}$

define a dimensionless time $\boxed{t^* = k_{-1} t}$

alternate rescalings

$t^* = k_1 t$; $t^* = (k_1 + k_{-1})t$; $t^* = \sqrt{k_1 k_{-1}}\, t$

$[A] = L^{-3}$ $\quad \underline{a^*} = \frac{A}{A_0}$, $t^* = k_{-1}t$

rescaled conc.

$\frac{dA}{dt} \rightarrow \frac{da^*}{dt^*} \rightarrow$ $\quad \frac{dA}{dt} = \frac{da^*}{dt^*}\frac{dt^*}{dt}$

$A = A_0 a^*$ $\quad \frac{dA}{dt} = \frac{A_0\, da^*}{dt} \rightarrow A_0 \frac{da^*}{dt^*}\frac{dt^*}{dt}$

$$\frac{dA}{dt} = k_{-1}A_0\frac{da^*}{dt^*} = -(k_1 + k_{-1})A + k_{-1}M$$

$$\frac{da^*}{dt^*} = -\left(1 + \overset{\varepsilon}{\frac{k_1}{k_{-1}}}\right)\overset{a^*}{\frac{A}{A_0}} + \overset{\theta}{\frac{M}{A_0}}$$

$$\frac{da^*}{dt^*} = -(1+\varepsilon)a^* + \theta$$

$k_1, k_{-1}, M, A_0 \mid \varepsilon, \theta$

6 steps to non-dimensionalization and scaling