(i) $\frac{dx}{dt} = rx$

(ii) $\frac{dx}{dt} = I - \gamma x$

(iii) $\frac{dx}{dt} = K(t)\,x$

$\boxed{\text{(iv)} \quad \frac{dx}{dt} = f(t) + rx}$

$$x(t) = x_g(t) + x_p(t)$$

(a) $f(t) = \text{const}$, $x_p = C_2$ $\quad x(t) = \underbrace{c_1 e^{rt}}_{x_g} + \underbrace{C_2}_{x_p}$

(b) $\frac{dx}{dt} = rx + \underbrace{\alpha \exp(\beta t)}_{f(t)} \quad \rightarrow \quad x_p(t) = A e^{\beta t}$

$\boxed{\frac{dy}{dt} = ay} + \cancel{q(t)}$ $\quad y(t) = y_0 e^{at}$

y = money in your bank

$t = 0$ $\quad y(t=0) = y_0$

a = rate of interest

$$y(t) = \underbrace{\boxed{y_0 \exp(at)}}_{\substack{\text{general soln} \\ \text{homogeneous ODE}}} + \underbrace{\int_0^t e^{a(t-s)} q(s)\, ds}_{\text{particular soln}}$$

$$\frac{dy}{dt} = a y_0 e^{at} + \frac{d}{dt} e^{at} \int_0^t e^{-as} q(s)\, ds$$

$$\frac{dy}{dt} = a y_0 e^{at} + a e^{at} \int_0^t e^{-as} q(s)\, ds + \underbrace{e^{at} e^{-at}}_{1} q(t)$$

$$\frac{dy}{dt} = \underbrace{a\left[y_0 e^{at} + e^{at} \int_0^t e^{-as} q(s)\, ds\right]} + q(t)$$

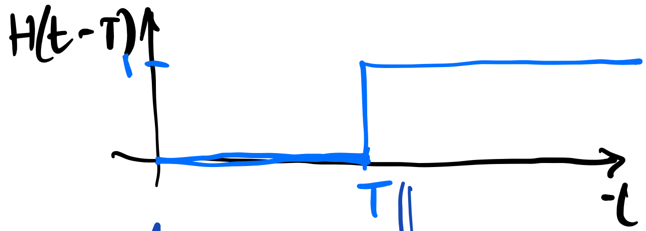
$$\frac{dy}{dt} = ay + q(t)$$

(i) $q(t) = \text{const}$

(ii) $q(t) = e^{pt}$

(iii) $q(t) = \cos(\omega t)$

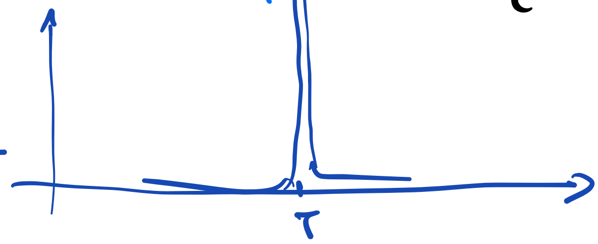
(iv) $q(t) = H(t-T)$



(v) $q(t) = \delta(t-T)$

$$\int_0^\infty f(t)\,\delta(t-T)\,dt$$

$$= f(T)$$



(vi) $q(t) = t^n$

3.2 'Primer'

linear 2^{nd} order ODE constant coefficients.

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

$$\frac{d^2x}{dt^2} - \beta\frac{dx}{dt} + \gamma x = 0 \qquad \beta = -\frac{b}{a}, \quad \gamma = \frac{c}{a}$$

Euler $\quad x(t) = e^{mt}$

$$m^2 \cancel{e^{mt}} - \beta m \cancel{e^{mt}} + \gamma \cancel{e^{mt}} = 0$$

$$m^2 - \beta m + \gamma = 0$$ → characteristic equation.

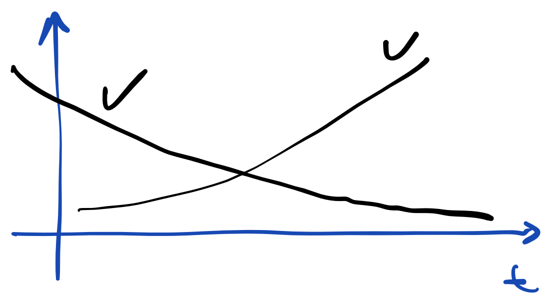
$$m_{1,2} = \frac{1}{2}\left[\beta \pm \sqrt{\beta^2 - 4\gamma}\right] \quad \Bigg| \quad e^{m_1 t},\ e^{m_2 t}$$

$$x(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}$$ c_1, c_2 are constants.

$$\beta = -1/2, \quad \gamma = -1/2$$

$$m_1 = -1, \quad m_2 = 1/2$$

$$x(t) = c_1 e^{-t} + c_2 e^{t/2}$$



$m_1, m_2 \rightarrow$ real.

Complex roots $\quad m_1 = p + iq \;;\; m_2 = p - iq.$

$$x_1(t) = e^{(p+iq)t} = e^{pt}\underbrace{e^{iq(t)}}$$

$$x_2(t) = e^{(p-iq)t} = e^{pt}\underbrace{e^{-iq(t)}}$$

De Moivres theorem

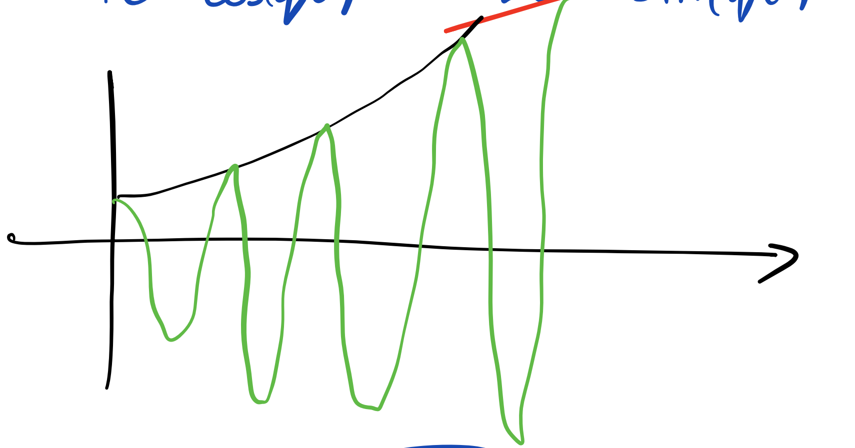
$$x_1(t) = e^{pt}(\cos qt + i\sin qt)$$

$$x_2(t) = e^{pt}(\cos qt - i\sin qt)$$

$$\frac{1}{2}\left(x_1(t) - x_2(t)\right) = i\sin q_1(t)\, e^{pt}$$

$$\frac{1}{2}\left(x_1 + x_2\right) = \cos q_1(t)\, e^{pt}$$

$$x(t) = c_1 e^{pt}\cos(qt) + c_2 e^{pt}\sin(qt)$$



$$\frac{d^2x}{dt^2} \underbrace{- \beta\frac{dx}{dt} + \gamma x}_{} = 0$$

$$y = \frac{dx}{dt}$$

$$\frac{dx}{dt} = y$$

$$\underbrace{\frac{dy}{dt} = \frac{d^2x}{dt^2} = \beta\frac{dx}{dt} - \gamma x}_{}$$

$$\frac{dy}{dt} = \beta y - \gamma x$$

$$\frac{dX}{dt} = A\,X$$

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + dy$$

$$\rightarrow \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$X(t) = X(t=0)\,\boxed{e^{At}}$$

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad e^{A} = I + A + \frac{A^2}{2!} \ldots$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix} + \cdots$$

$$\begin{bmatrix} 1+\lambda_1+\frac{\lambda_1^2}{2!}+\cdots & 0 \\ 0 & 1+\lambda_2+\frac{\lambda_2^2}{2!}+\cdots \end{bmatrix}$$

$$\begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix}$$