$$\boxed{\frac{d^2x}{dt^2} - \beta\frac{dx}{dt} + \gamma x = 0}$$

1. Convert n^{th} order ODE to n 1^{st} order ODEs

$$\boxed{\alpha_n \frac{d^n x}{dt^n}} + \alpha_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \dots + \alpha_1\frac{dx}{dt} + \alpha_0 x = 0$$

$$x_1 = \frac{dx}{dt} \; ; \quad x_2 = \frac{d^2x}{dt^2} \quad \dots \quad x_n = \frac{d^n x}{dt^n}$$

$$x_n = \frac{d^n x}{dt^n} = -\frac{\alpha_{n-1}}{\alpha_n}\frac{d^{n-1}x}{dt^{n-1}} - \dots \frac{\alpha_0}{\alpha_n}x$$

$$\frac{dx_{n-1}}{dt} = -\frac{\alpha_{n-1}}{\alpha_{new}}x_{n-2} - \frac{\alpha_{n-1}}{\alpha_{n-1}}x_{n-3} - \dots$$

$$\alpha_n\frac{d^n x}{dt^n} = \frac{dx_{n-1}}{dt} = -\frac{\alpha_{n-1}}{\alpha_n}\frac{d^{n-1}x}{dt^{n-1}} - \dots$$

2.
$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

3. decoupled system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$x_1(t) = e^{\lambda_1 t} x_1(0)$, $x_2(t) = e^{\lambda_2 t} x_2(0)$, $\ldots$, $x_n(t) = e^{\lambda_n t} x_n(0)$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\lambda_1 = a$, $\lambda_2 = -1$ $\qquad V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = x_1(t=0)\, e^{\lambda_1 t}\, V_1 + x_2(t=0)\, e^{\lambda_2 t}\, V_2$$

↳ eigenvalues .

↳ eigenvectors

4. $\dfrac{dx}{dt} = \boxed{A}x$ ← coupled

$$\Downarrow$$

$\dfrac{dz}{dt} = \boxed{D}z$ ↙ decoupled

$$A V_1 = \lambda_1 V_1$$

$$\begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_n \end{array}\underbrace{\begin{bmatrix} \text{———} \\ \text{———} \\ \text{———} \end{bmatrix}}_{A\ n \times n}\begin{bmatrix} V_1 & V_2 & \cdots & V_n \\ | & | & & | \end{bmatrix} = \underbrace{\begin{bmatrix} \boxed{\begin{matrix} a_1 \cdot V_1 \\ a_2 \cdot V_1 \\ \vdots \\ a_n \cdot V_1 \end{matrix}} & \cdots & \begin{matrix} a_1 \cdot V_n \\ \vdots \\ a_n \cdot V_n \end{matrix} \end{bmatrix}}$$

$$A\,V_1 = \boxed{\lambda_1 V_1} = \begin{bmatrix} a_1 \cdot V_1 \\ a_2 \cdot V_1 \\ \vdots \\ a_n \cdot V_1 \end{bmatrix}$$

$$\overset{AV}{\begin{bmatrix} | & | & & | \\ \lambda_1 V_1 & \lambda_2 V_2 & \cdots & \lambda_n V_n \\ | & | & & | \end{bmatrix}} = \overset{V}{\begin{bmatrix} V_1 & V_2 & \cdots & V_n \\ | & | & & | \end{bmatrix}}\overset{D}{\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}}$$

$$AV = VD$$

$$A = VDV^{-1}$$

$$\frac{dx}{dt} = Ax \quad \Rightarrow \quad x(t) = \boxed{e^{At}}\, x_0$$

$$e^{At} = \underline{I} + At + \frac{A^2t^2}{2!} + \cdots$$

$$A = VDV^{-1}$$

$$A^2 = (VD\cancel{V^{-1}})(\cancel{V}DV^{-1}) \qquad VD^2V^{-1}$$

$$\vdots$$

$$A^n = VD^nV^{-1}$$

$$e^{At} = VV^{-1} + VDV^{-1}t \quad \frac{VD^2V^{-1}t^2}{2!} + \cdots + \frac{VD^nV^{-1}t^n}{n!} + \cdots$$

$$e^{At} = V\left(I + Dt + \frac{D^2t^2}{2!} + \cdots\right)V^{-1}$$

$$e^{At} = Ve^{Dt}V^{-1}$$

$$e^{At} = V\begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} V^{-1}$$

$$x(t) = \underbrace{e^{At}}\, x_0$$

$$x(t) = V\underbrace{e^{Dt}\underbrace{V^{-1}x_0}_{z_0}}_{z(t)}$$

$$x(t) = Vz(t)$$

$$\frac{dz}{dt} = Dz$$

$$z(t) = e^{Dt}z_0$$

[Steve Brunton]

Classification of the solutions of linear ODEs

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \frac{dx}{dt} = Ax \qquad x(t) = e^{At} x_0$$

$$x(t) = e^{\lambda t} \vec{v}$$

$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = 0$$

$$\lambda^2 - \underbrace{(a+d)}_{\text{trace}(A)}\lambda + \underbrace{(ad-bc)}_{\det A} = 0$$

$$\lambda^2 - \tau\lambda + \Delta$$

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

Strogatz

Ch. 5

eg.

$$\frac{dx}{dt} = x + y$$

$$\frac{dy}{dt} = 4x - 2y$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}}^{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{bmatrix}\right) = 0$$

$$\lambda_1 = 2, \qquad \lambda_2 = -3$$

$$A \check{v}_1 = \lambda_1 \check{v}_1 \qquad A\check{v}_2 = \lambda_2 \check{v}_2$$

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

$$x(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}$$

$V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\qquad$ $V_2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$