

## Chapter 1

# Mathematical Concepts and Symbols

## 1.1 Logical Symbols

The logical symbols are a kind of symbols using for theoretical statements.

Table 1.1: Frequently-used Logical Symbols

Symbols	Meanings
$L \implies P$	Proposition $L$ is contained in proposition $P$
$L \iff P$	Proposition $L$ is equivalent to proposition $P$
$\neg P$	Not $P$
$L \wedge P$	Proposition $L$ and proposition $P$
$L \vee P$	Proposition $L$ or proposition $P$

e.g.

$$((A \implies B) \wedge (\neg B) \implies (\neg A))$$

stands for “ if  $A$  is contained in  $B$ , and  $B$  is not true, then  $A$  is not true”.

We also call  $A \iff B$  “ $A$  is the necessary and sufficient condition of  $B$ ”.

The typical math proposition is like “ $A \implies B$ ”. In order to prove this proposition, we can use the implication relationship

$$A \implies C_1 \implies \cdots \implies C_n \implies B$$

The every implication relationship in this chain is general truth or proved proposition.

Table 1.2: Truth Table

$\neg A$	$A$	0	1
	$\neg A$	1	0
$A \vee B$	$\begin{array}{c} B \\ \diagdown \\ A \end{array}$	0	1
	0	0	1
	1	1	1
$A \wedge B$	$\begin{array}{c} B \\ \diagdown \\ A \end{array}$	0	1
	0	0	0
	1	0	1
$A \implies B$	$\begin{array}{c} B \\ \diagdown \\ A \end{array}$	0	1
	0	1	1
	1	0	1

**Question 1.**  $\neg(A \wedge B) \Leftrightarrow (\neg A \vee \neg B)$ .

*Proof.* (Use the truth table)

If  $A$  is true,  $B$  is true,  $A \wedge B$  is true.  $\neg(A \wedge B)$  is false.  $\neg A$  is false,  $\neg B$  is false,  $(\neg A \vee \neg B)$  is false.

If  $A$  is true,  $B$  is false,  $A \wedge B$  is false.  $\neg(A \wedge B)$  is true.  $\neg A$  is false,  $\neg B$  is true,  $(\neg A \vee \neg B)$  is true.

If  $A$  is false,  $B$  is true,  $A \wedge B$  is false.  $\neg(A \wedge B)$  is true.  $\neg A$  is true,  $\neg B$  is false,  $(\neg A \vee \neg B)$  is true.

If  $A$  is false,  $B$  is false,  $A \wedge B$  is false.  $\neg(A \wedge B)$  is true.  $\neg A$  is true,  $\neg B$  is true,  $(\neg A \vee \neg B)$  is true.

So

$$\neg(A \wedge B) \Leftrightarrow (\neg A \vee \neg B)$$

□

**Question 2.**  $(A \Rightarrow B) \Leftrightarrow \neg A \vee B$ .

*Proof.* Firstly, we confirm the truth of

$$(A \Rightarrow B) \Rightarrow \neg A \vee B$$

If  $(A \Rightarrow B)$  is false, then  $\neg A \vee B$  is true.

If  $(A \Rightarrow B)$  is true, then we have two possibilities. The first is  $A$  is true,  $B$  is true, so  $\neg A \vee B$  is true. The second is  $A$  is false, then  $B$  can be true or false, but  $\neg A \vee B$  will be true.

Hence,  $(A \Rightarrow B) \Rightarrow \neg A \vee B$ .

Secondly, we prove

$$(A \Rightarrow B) \Leftarrow \neg A \vee B$$

If  $\neg A \vee B$  is false, then  $(A \Rightarrow B)$  is true.

If  $\neg A \vee B$  is true, we have

1.  $\neg A$  is true,  $B$  is false, then,  $A$  is false,  $(A \Rightarrow B)$  is true.
2.  $\neg A$  is false,  $B$  is true, then,  $A$  is true,  $(A \Rightarrow B)$  is true.
3.  $\neg A$  is true,  $B$  is true, then,  $A$  is false,  $(A \Rightarrow B)$  is true.

So  $(A \Rightarrow B) \Leftarrow \neg A \vee B$ .

□

**TIPS.** 1.  $\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$ ;

2.  $\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$ ;

3.  $(A \Rightarrow) \Leftrightarrow (\neg B \Rightarrow \neg A)$ ;

4.  $(A \Rightarrow) \Leftrightarrow (\neg A \vee B)$ ;

5.  $\neg(A \Rightarrow B) \Leftrightarrow A \wedge \neg B$ .

## 1.2 Sets and their Operations

A set is a collection of well-defined objects.

If  $A$  is a set, we write  $a \in A$  to express element  $a$  belongs to set  $A$ , the negative proposition of which is  $a \notin A$ . We use the symbol  $\emptyset$  to denote the empty set, that is, the set with no elements.

**Theorem 1.2.1** (Cantor). *There is no set contains all the sets.*

*Proof.* We assume  $P(M)$  represents  $M$  doesn't contain itself.

Consider  $K = \{M | P(M)\}$  which is made of **sets**  $M$  that satisfies  $P$ . Assuming  $K$  is a set, then either  $P(K)$  or  $\neg P(K)$  is true.

If  $P(K)$  is true,  $K$  doesn't contain itself, but because of the definition of  $K$ ,  $K$  is belong to  $K$ , which means  $\neg P(K)$ .

If  $\neg P(K)$  is true, it's easy to find the similar conclusion.

So to the contrary,  $K$  is not a set. This reveals a set can't contain all the sets.  $\square$

**Theorem 1.2.1** is a typical paradox called Russell's paradox.

$\forall$  and  $\exists$  are logical symbols to describe

Table 1.3: Universal and Existential Quantifications

Symbols	Meanings
$\forall x \in A$	For all elements $x$ in $A$
$\exists x \in A$	There exist at least one element $x$ in $A$

To show the inclusion relation of two sets, we often use the Symbol  $A \subset B$ , which means set  $A$  is a **subset** of set  $B$  (All the elements in  $A$  also belong to  $B$ ). We indicate that  $A$  is not a subset of  $B$  by this notation:  $A \not\subset B$ .

$$(A \subset B) := \forall x((x \in A) \Rightarrow (x \in B))$$

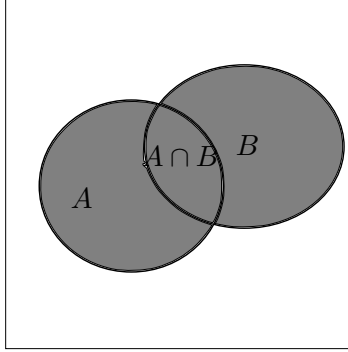
We define the equal relation between two sets, using the symbol  $=$ :

$$A = B := (A \subset B) \wedge (B \subset A)$$

We often use this definition to prove  $A = B$ . Symbol  $\neq$  denotes the negative proposition of equal.

$A$  is a **proper subset** of  $B$ , if  $A$  is a subset of  $B$ , and  $A \neq B$ , denoted by the symbol  $\subsetneq$ .

Figure 1.1: Union of two sets



If  $A$  and  $B$  are sets, then their **union**, denoted by  $A \cup B$ , is the set of all elements that are elements of either  $A$  or  $B$ :

$$(A \cup B) := \{x \in M \mid (x \in A) \vee (x \in B)\}$$

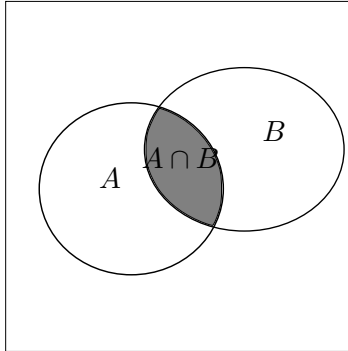
Clearly,  $A \cup B = B \cup A$ .

If  $A$  and  $B$  are sets, then their **intersection**, denoted by  $A \cap B$ , contains all the elements in both  $A$  and  $B$ :

$$(A \cap B) := \{x \in M \mid (x \in A) \wedge (x \in B)\}$$

Also, we have  $A \cap B = B \cap A$ .

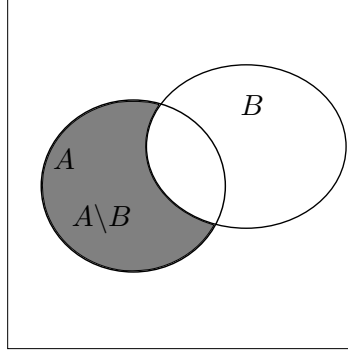
Figure 1.2: Intersection of two sets



We use the notation  $A \setminus B$  to represent the set contains all the elements which belongs to  $A$  but not belong to  $B$ .

$$A \setminus B := \{x \in M \mid (x \in A) \wedge (x \notin B)\}$$

Figure 1.3: Complement



For  $B \subset A$ , we can also denote it as the symbol  $C_A B$ .

**Question 3** (de Morgen).

$$C_M(A \cup B) = C_M A \cap C_M B$$

$$C_M(A \cap B) = C_M A \cup C_M B$$

*Proof.* We prove the first one.

$$\begin{aligned} (x \in C_M(A \cup B)) &\Rightarrow (x \notin (A \cup B)) \\ &\Rightarrow ((x \notin A) \wedge (x \notin B)) \Rightarrow ((x \in C_M A) \wedge (x \in C_M B)) \\ &\Rightarrow ((x \in C_M A) \cap (x \in C_M B)) \end{aligned}$$

So we have proved  $C_M(A \cup B) \subset C_M A \cap C_M B$ . On the other hand:

$$\begin{aligned} ((x \in C_M A) \cap (x \in C_M B)) &\Rightarrow ((x \in C_M A) \wedge (x \in C_M B)) \\ &\Rightarrow ((x \notin A) \wedge (x \notin B)) \Rightarrow (x \notin (A \cup B)) \\ &\Rightarrow (x \in C_M(A \cup B)) \end{aligned}$$

That's the same as  $C_M(A \cup B) = C_M A \cap C_M B$ . □

**Question 4.**  $(A \subset C_M B) \Leftrightarrow (B \subset C_M A)$ .

*Proof.*

$$\begin{aligned} (A \subset C_M B) &\Rightarrow ((x \in A) \Rightarrow ((x \notin B) \wedge (x \in M))) \\ &\Rightarrow (\neg(x \in A) \Leftarrow \neg((x \notin B))) \\ &\Rightarrow ((x \in B) \Rightarrow (x \notin A)) \Rightarrow (B \subset C_M A) \end{aligned}$$

The other hand of this problem is the same. □

- TIPS.** 1.  $(A \subset C) \wedge (B \subset C) \Leftrightarrow ((A \cup B) \subset C)$ ;  
 2.  $(C \subset A) \wedge (C \subset B) \Leftrightarrow (C \subset (A \cap B))$ ;  
 3.  $C_M(C_MA) = A$ ;  
 4.  $(A \subset C_MB) \Leftrightarrow (B \subset C_MA)$ ;  
 5.  $(A \subset B) \Leftrightarrow (C_MA \supset C_MB)$ .

**Question 5.**  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ .

*Proof.*

$$\begin{aligned} A \cup (B \cap C) &\Leftrightarrow ((x \in A) \vee ((x \in B) \wedge (x \in C))) \\ ((x \in A) \vee (x \in B)) \wedge ((x \in A) \vee (x \in C)) &\Leftrightarrow (A \cup B) \cap (A \cup C) \end{aligned}$$

So, we should prove:

$$((x \in A) \vee ((x \in B) \wedge (x \in C))) \Leftrightarrow ((x \in A) \vee (x \in B)) \wedge ((x \in A) \vee (x \in C))$$

That's the same as:

$$(A \vee (B \wedge C)) \Leftrightarrow ((A \vee B) \wedge (A \vee C))$$

It's easy to prove with the help of truth table. □

In this question, we also establish a relation between logical operation and sets' operation.

- TIPS.** 1.  $A \cup (B \cup C) = (A \cup B) \cup C := A \cup B \cup C$ ;  
 2.  $A \cap (B \cap C) = (A \cap B) \cap C := A \cap B \cap C$ ;  
 3.  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ ;  
 4.  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ .

We denote two sets' Cartesian product as  $A \times B$ .  $A \times B$  is a set contains ordered pairs, which means  $A \times B \neq B \times A$ .

$$A \times B := \{(x, y) | (x \in A) \wedge (y \in B)\}$$

**Question 6.**  $(X \times Y) \cup (Z \times Y) = (X \cup Z) \times Y$ .

*Proof.*

$$\begin{aligned} &(X \times Y) \cup (Z \times Y) \\ &\Rightarrow \{(x, y) | (x \in X) \wedge (y \in Y)\} \cup \{(z, y) | (z \in Z) \wedge (y \in Y)\} \\ &\Rightarrow \{(x(\text{or } z), y) | ((x \in X) \wedge (y \in Y)) \vee ((z \in Z) \wedge (y \in Y))\} \\ &\Rightarrow \{(x(\text{or } z), y) | ((x \in X) \vee (z \in Z)) \wedge (y \in Y)\} \\ &\Rightarrow (X \cup Z) \times Y \end{aligned}$$

On the other hand:

$$\begin{aligned}
& (X \cup Z) \times Y \\
& \Rightarrow \{(x(\text{or } z), y) | ((x \in X) \vee (z \in Z)) \wedge (y \in Y)\} \\
& \Rightarrow \{(x(\text{or } z), y) | ((x \in X) \wedge (y \in Y)) \vee ((z \in Z) \wedge (y \in Y))\} \\
& \Rightarrow \{(x, y) | (x \in X) \wedge (y \in Y)\} \cup \{(z, y) | (z \in Z) \wedge (y \in Y)\} \\
& \Rightarrow (X \times Y) \cup (Z \times Y)
\end{aligned}$$

In conclusion:  $(X \times Y) \cup (Z \times Y) = (X \cup Z) \times Y$ .  $\square$

**Question 7.**  $(A \times B \subset X \times Y) \Leftrightarrow (A \subset X) \wedge (B \subset Y)$ .

*Proof.*

$$(A \times B) \subset (X \times Y) \Rightarrow (((a \in A) \wedge (b \in B)) \Rightarrow ((x \in X) \wedge (y \in Y)))$$

In this formula, we will get

$$((a \in A) \Rightarrow (x \in X)) \wedge ((b \in B) \Rightarrow (y \in Y))$$

. That's because  $A \times B$  is a ordered pair, which means there is a consistent one-to-one match between  $A$  and  $X$ ,  $B$  and  $Y$ .

The proof of the other hand is similar.  $\square$

- TIPS.**
1.  $(X \times Y = \emptyset) \Leftrightarrow (X = \emptyset) \vee (Y = \emptyset)$ ,  
if  $X \times Y \neq \emptyset, A \times B \neq \emptyset$ , we have
  2.  $(A \times B \subset X \times Y) \Leftrightarrow (A \subset X) \wedge (B \subset Y)$ ;
  3.  $(X \times Y) \cup (Z \times Y) = (X \cup Z) \times Y$
  4.  $(X \times Y) \cap (X' \times Y') = (X \cap X') \times (Y \cap Y')$ .



### 1.3 Function

A function or mapping  $f$  from  $X$  to  $Y$  is a rule, or formula, or assignment, or relation of association that assigns to each  $x \in X$  a unique element  $y \in Y$ . Here we call  $X$  the domain of the function  $f$ ,. Elements  $x \in X$  are the arguments of the function. The elements  $y \in Y$  are the dependent variables (the image of  $x$ ), denoted by the symbol  $f(x)$ . The set  $Y$  is made of the value of the function, which is called the range or codomain of the function  $f$ .

$$f(X) := \{y \in Y \mid \exists x((x \in X) \wedge (y = f(x)))\}$$

We often use the symbol

$$f : X \longrightarrow Y, X \xrightarrow{f} Y$$

to denote the function  $f$ .