# Chapter 1

# Groups

## 1.1 Semigroups, monoids and groups

**Exercise 1.1.1.** Give examples other than those in the text of semigroups and monoids that are not groups.

**Answer.** Semigroup: $(\mathbf{Z}_+, +)$
Monoid: $(\mathbf{Z}_+, \times)$

**Exercise 1.1.2.** Let $G$ be a group (written additively), $S$ a nonempty set, and $M(S, G)$ the set of all functions $f : S \to G$. Define addition in $M(S, G)$ as follows: $(f + g) : S \to G$ is given by $s \to f(s) + g(s) \in G$. Prove that $M(S, G)$ is a group, which is abelian if $G$ is.

**Answer.** Firstly we check $M(S, G)$ is a group

1. $f + g : s \mapsto f(s) + g(s) \in G$, so $f + g \in M(S, G)$
2. $(f + g) + h : s \mapsto (f(s) + g(s)) + h(s)$, $G$ is a group, so $s \mapsto (f(s) + g(s)) + h(s) \Leftrightarrow s \mapsto f(s) + (g(s) + h(s))$, $(f + g) + h = f + (g + h)$.
3. Take the unit element as $e' : s \mapsto e$. $f + e' : s \mapsto f(s) + e'(s) = f(s) + e = f(s)$, so $f + e' = f$. Similarly, $e' + f = f$.
4. For any $f \in M(S, G)$, take $f^{-1} : s \mapsto (f(s))^{-1}$, whence $f(s) + (f(s))^{-1} = (f(s))^{-1} + f(s) = e$.

In conclusion, $M(S, G)$ is a group. If $G$ is abelian $f + g : s \mapsto f(s) + g(s) = g(s) + f(s)$, $f + g = g + f$, so $M(S, G)$ is abelian.

**Exercise 1.1.3.** Is it true that a semigroup which has a left identity element and in which every element has a right inverse (see Proposition 1.3) is a group?

**Answer.** If $e$ is the left identity, $\forall a \in A, ea = a$ and $\forall a \in A, \exists a^{-1} s.t. aa^{-1} = e$. We have proved that if $cc = c$, then $c = e$.

$$(a^{-1}a)(a^{-1}a) = a^{-1}(aa^{-1})a = a^{-1}(ea) = a^{-1}a \Rightarrow a^{-1}a = e$$

$a^{-1}$ is also the left inverse. $ae = a(a^{-1}a) = (aa^{-1})a = ea = a$, $e$ is also the right identity.

**Exercise 1.1.4.** Write out a multiplication table for the group $D_4^*$.

**Answer.** $D_4^* = \{R, R^2, R^3, I, T_x, T_y, T_{13}, T_{24}\}$

| | $I$ | $R$ | $R^2$ | $R^3$ | $T_x$ | $T_y$ | $T_{13}$ | $T_{24}$ |
|---|---|---|---|---|---|---|---|---|
| $I$ | $I$ | $R$ | $R^2$ | $R^3$ | $T_x$ | $T_y$ | $T_{13}$ | $T_{24}$ |
| $R$ | $R$ | $R^2$ | $R^3$ | $I$ | $T_{13}$ | $T_{24}$ | $T_y$ | $T_x$ |
| $R^2$ | $R^2$ | $R^3$ | $I$ | $R$ | $T_y$ | $T_x$ | $T_{24}$ | $T_{13}$ |
| $R^3$ | $R^3$ | $I$ | $R$ | $R^2$ | $T_{24}$ | $T_{13}$ | $T_x$ | $T_y$ |
| $T_x$ | $T_x$ | $T_{24}$ | $T_y$ | $T_{13}$ | $I$ | $R^2$ | $R^3$ | $R$ |
| $T_y$ | $T_y$ | $T_{13}$ | $T_x$ | $T_{24}$ | $R^2$ | $I$ | $R$ | $R^3$ |
| $T_{13}$ | $T_{13}$ | $T_y$ | $T_{24}$ | $T_x$ | $R^3$ | $R$ | $I$ | $R^2$ |
| $T_{24}$ | $T_{24}$ | $T_x$ | $T_{13}$ | $T_y$ | $R$ | $R^3$ | $R^2$ | $I$ |

**Exercise 1.1.5.** Prove that the symmetric group on $n$ letters, $S_n$, has ordrer $n!$.

**Answer.** For a set $A$ whose order is $n$, we prove there's $n!$ different bijections by induction

1. For $n = 1$, trivial.
2. Assume $n = k$, there's $k!$ bijections. For $n = k + 1$, fix one element in $A$, and take $a \mapsto a$, there's $k$ free elements, so there's $k! \cdot (k + 1)$ bijections in total.

By induction, we get the result.

**Exercise 1.1.6.** Write out an addition table for $Z_2 \oplus Z_2$. $Z_2 \oplus Z_2$ is called the Klein four group.

**Answer.** $Z_2 = \{1, 0\}$, $Z_2 \oplus Z_2 = \{(1, 1), (1, 0), (0, 1), (0, 0)\}$

| | $(1,1)$ | $(1,0)$ | $(0,1)$ | $(0,0)$ |
|---|---|---|---|---|
| $(1,1)$ | $(0,0)$ | $(0,1)$ | $(1,0)$ | $(1,1)$ |
| $(1,0)$ | $(0,1)$ | $(0,0)$ | $(1,1)$ | $(1,0)$ |
| $(0,1)$ | $(1,0)$ | $(1,1)$ | $(0,0)$ | $(0,1)$ |
| $(0,0)$ | $(1,1)$ | $(1,0)$ | $(0,1)$ | $(0,0)$ |

**Exercise 1.1.7.** If $p$ is prime, then the nonzero elements of $Z_p$ form a group of order $p-1$ under multiplication. Show that this statement is false if $p$ is not prime.

**Answer.** For the set $Z_p\backslash\{\bar{0}\}$

1. $Z_p\backslash\{\bar{0}\}$ is obviously associative and commutative.
2. Take $\bar{1}$ as the identity element, $\forall \bar{a} \in Z_p\backslash\{\bar{0}\}, \bar{1} \times \bar{a} = \bar{a}$.
3. We prove there is a unique element $a^{-1} \in Z_p\backslash\{\bar{0}\} s.t. aa^{-1} = \bar{1}$. Assume there exists $\bar{b}, \bar{c}$ and $\bar{a} \cdot \bar{b} = \bar{k}, \bar{a} \cdot \bar{c} = \bar{k}$, then $a(b-c) \equiv 0 \mod p$. $p$ is a prime, so $lcm(p, a) = 1, lcm(p, b-c) = 1$, so $\bar{b} = \bar{c}$. There is at most one element s.t. $\bar{a}\bar{b} = \bar{k}$. Take $\bar{b} = \bar{1}, \bar{2}, \ldots \overline{p-1}$, $\bar{k}$ travels through $\bar{b} = \bar{1}, \bar{2}, \ldots \overline{p-1}$. There exists an element $\bar{b} \in Z_p\backslash\{\bar{0}\}, \bar{a}\bar{b} = \bar{1}$.

$Z_p\backslash\{\bar{0}\}$ is a group. If $p$ is not a prime, the inverse element is not always unique. Take $a|p$, there's more than one inverse element in $Z_p\backslash\{\bar{0}\}$.

**Exercise 1.1.8.** (a) The relation given by $a \sim b \Leftrightarrow a - b \in \mathbf{Z}$ is a congruence relation on the additive group $\mathbf{Q}$ [see Theorem 1.5].

(b) The set $\mathbf{Q}/\mathbf{Z}$ of equivalence classes is an infinite abelian group.

**Answer.** (a) For group $(\mathbf{Q}, +)$, $a_1 \sim b_1 \Leftrightarrow a_1 - b_1 = k_1 \in \mathbf{Z}$, $a_2 \sim b_2 \Leftrightarrow a_2 - b_2 = k_2 \in \mathbf{Z}$, so $(a_1+a_2)-(b_1+b_2) = ((k_1+b_1)+(k_2+b_2))-(b_1+b_2) = k_1 + k_2 \in \mathbf{Z}$. $a \sim b$ is a congruence relation.

(b) 1 if $a + b \geq 1$, $\bar{a} + \bar{b} = \overline{a + b - 1}$. If $a + b < 1$, $\bar{a} + \bar{b} = \overline{a + b}$.

2 $\mathbf{Q}/\mathbf{Z}$ is obviously associative and commutative.

3 Take the identity element as $\bar{0}$, $\bar{0} + \bar{a} = \bar{a}$.

4 If $\bar{a} \neq 0$, take $(\bar{a})^{-1} = \overline{1-a}$, then $\bar{a} + \overline{1-a} = \bar{0}$

so $\mathbf{Q}/\mathbf{Z}$ is a abelian group. (Infinite remains to be certified)

**Exercise 1.1.9.** Let $p$ be a fixed prime. Let $R_p$ be the set of all those rational numbers whose denominator is relatively prime to $p$. Let $R^p$ be the set of rationals whose denominator is a power of $p(p^i, i > 0)$. Prove that both $R_p$ and $R^p$ are abelian groups under ordinary addition of rationals.

**Answer.** Trivial.

**Exercise 1.1.10.** Let $p$ be a prime and let $Z(p^\infty)$ be the following subset of the group $\mathbf{Q}/\mathbf{Z}$:

$$Z(p^\infty) = \{\overline{a/b} \in \mathbf{Q}/\mathbf{Z} | a, b \in \mathbf{Z} \text{ and } b = p^i \text{ for some } i \geq 0\}$$

Show that $Z(p^\infty)$ is an infinite group under the addition operation of $\mathbf{Q}/\mathbf{Z}$.

**Answer.** $Z(p^\infty) = \{\overline{a/b} | a, b \in \mathbf{Z}, b = p^i, i \geq 0\}$. Take $a = \frac{\bar{a_1}}{b_1}$, $b = \frac{\bar{a_2}}{b_2}$. $b^{-1} = \frac{b_2 \bar{-} a_2}{b_2}$

$$\begin{aligned} a + b^{-1} = \frac{\bar{a_1}}{b_1} + \frac{b_2 \bar{-} a_2}{b_2} &= \frac{\bar{a_1}}{p^{s_1}} + \frac{p^{s_2} \bar{-} a_2}{p^{s_2}} \\ &= \frac{a_1 \cdot p^{s_2} + \bar{p}^{s_1}(p^{s_2} - a_2)}{p^{s_1+s_2}} \in Z(p^\infty) \end{aligned}$$

Therefore, $Z(p^\infty)$ is a subgroup of $\mathbf{Q}/\mathbf{Z}$. $\frac{1}{p^i} \in Z(p^\infty)$ for any $i \in \mathbf{Z}$, so $Z(p^\infty)$ is infinite, $\mathbf{Q}/\mathbf{Z}$ is also infinite.

**Exercise 1.1.11.** The following conditions on a group $G$ are equivalent:

i $G$ is abelian;
ii $(ab)^2 = a^2b^2$ for all $a, b \in G$;
iii $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$;
iv $(ab)^n = a^nb^n$ for all $n \in \mathbf{Z}$ and all $a, b \in G$;
v $(ab)^n = a^nb^n$ for three consecutive integers $n$ and all $a, b \in G$. Show that v$\Rightarrow$ i is false if 'three' is replaced by 'two'.

**Answer.** i$\Leftrightarrow$ iii: $((ab)b^{-1})a^{-1} = (ab)(b^{-1}a^{-1}) = e$, so $(ab)^{-1} = b^{-1}a^{-1}$. If iii, $b^{-1}a^{-1} = a^{-1}b^{-1}$ for any $a, b \in G$, $G$ is abelian. If i, $G$ is abelian, $(ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1}$.
iv $\Rightarrow$ v, iv$\Rightarrow$ ii and i$\Rightarrow$ iv are trivial.
ii$\Rightarrow$ i:

$$(ab)(ab) = aabb \Rightarrow a^{-1}(ab)^2b^{-1} = a^{-1}aabbb^{-1} = ba = ab$$

so $G$ is abelian.

v $\Rightarrow$ i: $a^nb^n = (ab)^n$, $a^{n-1}b^{n-1} = (ab)^{n-1}$, $a^{n+1}b^{n+1} = (ab)^{n+1}$.

$$(b^{-1})^n(a^{-1})^n = ((ab)^n)^{-1} = ((ab)^{-1})^n$$

$$((ab)^{-1})^n(ab)^{n+1} = (b^{-1})^n ab^{n+1}$$

$$((ab)^{-1})^n(ab)^{n-1} = b^{-1}a^{-1} = (b^{-1})^n a^{-1}b^{n-1}$$

$$a = (b^{-1})^n ab^n \qquad b^{-1}a^{-1}b = (b^{-1})^n a^{-1}b^n$$

So $a^{-1} = b^{-1}a^{-1}b$, which means $G$ is abelian.
If "three" is replaced by "two": $a^nb^n = (ab)^n$, $a^{n+1}b^{n+1} = (ab)^{n+1}$.

$$(b^{-1})^n(a^{-1})^n = ((ab)^{-1})^n \qquad a = (b^{-1})^n ab^n$$

For the group $S_3 = \{(1), (12), (13), (23), (123), (132)\}$, taking any $a \in S_3$, we can check that $a^6 = (1)$. If $n = 6$, then $a = (b^{-1})^n ab^n$ for any $a, b \in S_3$. But $S_3$ is nonabelian.

**Exercise 1.1.12.** If $G$ is a group, $a, b \in G$ and $bab^{-1} = a^r$ for some $r \in \mathbf{N}$, then $b^jab^{-j} = a^{r^j}$ for all $j \in \mathbf{N}$.

**Answer.** $bab^{-1} = a^r$. We prove it by induction. For $j = 1$, its always true. Assume $j = k$ the equation is correct, $b^kab^{-k} = a^{r^k}$. $ba^{r^k}b^{-1} = (a^{r^k})^{r=a^{r^{k+1}}}$. For $j = k+1$, it's also true.

**Exercise 1.1.13.** If $a^2 = e$ for all elements $a$ of a group $G$, then $G$ is abelian.

**Answer.**

$$a^2 = e \Rightarrow a^2a^{-1} = ea^{-1} = a(aa^{-1}) = ae \Rightarrow a = a^{-1}$$

$$ab = a^{-1}b^{-1} = (ab)^{-1} = (ba)^{-1}$$

So $ab = ba \forall a, b \in G$. $G$ is abelian.

**Exercise 1.1.14.** If $G$ is a finite group of even order, then $G$ contains an element $a \neq e$ such that $a^2 = e$.

**Answer.** Suppose not. $\forall a \neq e, aa \neq e \Leftrightarrow a \neq a^{-1}$. We can classify the group into some subsets. $G = \bigcup\limits_{a \neq e} \{a, a^{-1}\} \cup \{e\}$. Notice that $\{a, a^{-1}\} \cap \{b, b^{-1}\} = \varnothing$ if $a \neq b$, so $|G| = 2n + 1$, That's contradictory!

**Exercise 1.1.15.** Let $G$ be a nonempty finite set with an associative binary operation such that for all $a, b, c \in G$, $ab = ac \Rightarrow b = c$ and $ba = ca \Rightarrow b = c$. Then $G$ is a group. Show that this conclusion may be false if $G$ is finite.

**Answer.** $G$ is a semigroup. Fix $a \in G$ and take $b$ travels through all elements in $G$, then $ab$ travels through all elements in $G$.
There exists an element $e_1$ s.t. $ae_1 = a \forall a \in G$. Similarly, we can find $e_2$ s.t. $e_2 a = a \forall a \in G$. $e_2 e_1 = e_1 = e_2 = e$. $e$ is the identity element of $G$. Easily, we can find that $\forall a \in G, \exists! a^{-1} \in G$ s.t. $a^{-1}a = aa^{-1} = e$ because $ab = ac \Rightarrow b = c$ and $ba = ca \Rightarrow b = c$.
$G$ is a group. If $G$ is infinite, $G$ may not be a group, for example: $(Z_+, \times)$.

**Exercise 1.1.16.** Let $a_1, a_2, \ldots$ be a sequence of elements in a semigroup $G$. Then there exists a unique function $\Psi : \mathbf{N}^* \to G$ such that $\Psi(1) = a_1, \Psi(2) = a_1 a_2, \Psi(3) = (a_1 a_2) a_3$ and for $n \geq 1$, $\Psi(n+1) = (\Psi(n)) a_{n+1}$. Note that $\Psi(n)$ is precisely the standard $n$ product $\prod_{i=1}^{n} a_i$.

**Answer.** Applying the Recursion Theorem with $a = a_1, S = G$ and $f_n : G \to G$ given by $x \mapsto x a_{n+2}$ yields a function $\phi : \mathbf{N} \to G$. Let $\Psi = \phi\theta$, where $\theta : \mathbf{N}^* \to \mathbf{N}$ is given by $k \mapsto k - 1$.

## 1.2 Homomorphisms and subgroups

**Exercise 1.2.1.** If $f : G \to H$ is a homomorphism of groups, then $f(e_G) = e_H$ and $f(a^{-1}) = f(a)^{-1}$ for all $a \in G$. Show by example that the first conclusion may be false if $G$, $H$ are monoids that are not groups.

**Answer.** For example, $(\mathbf{Z}_+, +)$ and $(\mathbf{N}, \times)$ are monoids. Denote $f : \mathbf{Z}_+ \to \mathbf{N}$ as $f(x) = 0 \forall x \in \mathbf{Z}_+$. $f$ is a homomorphism satisfies those conditions.

**Exercise 1.2.2.** A group $G$ is abelian if and only if the map $G \to G$ given by $x \mapsto x^{-1}$ is automorphism.

**Answer.** If $G$ is abelian, $f(x) = x^{-1}$ is a monomorphism and epimorphism. $f(a)f(b) = a^{-1}b^{-1} = (ab)^{-1} = f(ab)$
If $f(x) = x^{-1}$ is a isomorphism, $f(a)f(b) = a^{-1}b^{-1} = f(ab) = (ab)^{-1} = b^{-1}a^{-1} \forall a, b \in G$, so $G$ is abelian.

**Exercise 1.2.3.** Let $Q_8$ be the group(under ordinary matrix multiplication) generated by complex matrices $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, where $i^2 = -1$. Show that $Q_8$ is a nonabelian group of order 8. $Q_8$ is called the quaternion group.

**Answer.** The multiply operation is associative by the difinition. $A^4 = B^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ which is the identity element.

$$A^{-1} = A^3 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \in G \qquad B^{-1} = B^3 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \in G$$

So $\forall A^i B^j \in G$, $(A^i B^j)^{-1} \in G$. $G$ is a group. Now we examine the order of $G$ is 8.

$$BA = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$A^3 B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix}$$

So $BA = A^3B$. Take $X = A^{s_1}B^{s_2}A^{s_3}B^{s_4}\dots A^{s_{2n-1}}B^{s_{2n}} = A^{s_1}B^{s_2-1}A^3B$ $A^{s_3-1}B^{s_4}\dots A^{s_{2n-1}}B^{s_{2n}} = \dots$ In finite steps , we can change it into $X = A^aB^b$. $A^4 = B^4 = I$, so we only consider $1 \le a, b \le 4$. $A^2 = B^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, we list all: $Q_8 = \{A, A^2, B, BA, AB, A^2B, AB^2, I\}$. The order of $Q_8$ is 8.

**Exercise 1.2.4.** Let $H$ be the group(under ordinary matrix multiplication) of real matrices generated by $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Show that $H$ is a nonabelian group of order 8 which is not isomorphic to the quaternion group, but is isomorphic to the group $D_4^*$.

**Answer.** $C^4D^2 = I, DC = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = C^3D$.Similarly, we can prove $H$ is a nonabelian group of order 8. $H = \{C, C^2, C^3, I, D, CD, C^2D, C^3D\}$
Assume $G \cong H$ and the isomorphism is $f$, Let $f(D) = X$, $f(D^2) = X^2 = f(I) = I$, so $X^2 = I$. But $f^{-1}(I) = I \Rightarrow X \neq I \Rightarrow X = AB$ or $X = A^2$ or $X = B^2$.
If $X = A^2$, consider $f(C) = Y, f(C^2D) = Z$, we have $(Y, Z) = (B^2, AB)$ or $(Y, Z) = (AB, B^2)$. $f(C^2D) = f(C^2)f(D) \Leftrightarrow Z = XY$. That's contradictory!
If $X = B^2$, the proof is similar.
If $X = AB$, $(Y, Z) = (A, B)$ or $(Y, Z) = (B, A)$. That's contradictory! So $f$ doesn't exist. $G$ is not isomorphic to $H$.
Now we prove $H \cong D_4^*$. For any point $(x, y)^T$ inside the square

$$T_x = (x, -y)^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (x, y)^T = CD(x, y)^T$$

$$T_y = (-x, y)^T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} (x, y)^T = C^3D(x, y)^T$$

$$T_{13} = (-y, x)^T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (x, y)^T = C^3(x, y)^T$$

$$T_{24} = (y, -x)^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (x, y)^T = C(x, y)^T$$

so $D_4^* = \langle T_x, T_y, T_{13}, T_{24} \rangle = H = \langle C, D \rangle$.

**Exercise 1.2.5.** Let $S$ be a nonempty subset of a group $G$ and define a relation on $G$ by $a \sim b$ if and only if $ab^{-1} \in S$. Show that $\sim$ is an equivalence relation if and only if $S$ is a subgroup of $G$.

**Answer.** If $\sim$ is a equivalence relation

1. $a \sim b \Rightarrow b \sim a$;
2. $a \sim a$;
3. $a \sim b, b \sim c \Rightarrow a \sim c$.

$2 \Leftrightarrow aa^{-1} = e \in S$. $1 \Rightarrow a \sim e \Rightarrow e \sim a \forall a \in S$, so $ae^{-1} = a \in S, ea^{-1} = a^{-1} \in S$. If $a, b \in S, b^{-1} \in S$, so $ae^{-1} \in S, e(b^{-1})^{-1} \in S$. By 3, $a \sim e, e \sim b^{-1} \Rightarrow a \sim b^{-1} \Rightarrow ab \in S$. $S$ is a subgroup of $G$.
If $S$ is a subgroup of $G$

1. $aa^{-1} \in S \Rightarrow a \sim a$;
2. $ab^{-1} \in S \Rightarrow (ab^{-1})^{-1} = ba^{-1} \in S \Rightarrow (a \sim b \Rightarrow b \sim a)$;
3. $ab^{-1} \in S, bc^{-1} \in S \Rightarrow (ab^{-1})(bc^{-1}) = ac^{-1} \in S$, which means $a \sim b, b \sim c \Rightarrow a \sim c$

In conclusion, $\sim$ is a equivalence relation.

**Exercise 1.2.6.** A nonempty finte subset of a group is s subgroup if and only if it is closed under the product in $G$.

**Answer.** $\Rightarrow$: Trivial.
$\Leftarrow$: $S$ is apparently associative. $\forall a, b \in S, ab \in S$. $S$ is a finite set, so there exists $m > n \in \mathbf{N}$ s.t. $a^m = a^n$.

**Exercise 1.2.7.** If $n$ is a fixed integer, then $\{kn | n \in \mathbf{Z}\} \subset \mathbf{Z}$ is an additive subgroup of $\mathbf{Z}$, which is isomorphic to $\mathbf{Z}$.

**Answer.** Denote $Z^n = \{kn | k \in \mathbf{Z}\}$. We can easily check that $Z^n$ is a subgroup of $\mathbf{Z}$. Now we build a isomorphism between $Z^n$ and $\mathbf{Z}$. Take $f : Z^n \to \mathbf{Z}$ as $f(kn) = k$, $f^{-1}(n) = kn$. $f$ is a bijection so $Z^n$ and $\mathbf{Z}$ are isomorphism.

**Exercise 1.2.8.** The set $\{\sigma \in S_n | \sigma(n) = n\}$ is a subgroup of $S_n$ which is isomorphic to $S_{n-1}$.

**Answer.** Denote $S_n^{(n)} = \{\sigma \in S_n | \sigma(n) = n\}$. $\forall \sigma_1, \sigma_2 \in S_n^{(n)}, \sigma_1\sigma_2(n) = \sigma_1(\sigma_2(n)) = \sigma_1(n) = n$, so $\sigma_1\sigma_2 \in S_n^{(n)}$. By the above exercise, $S_n^{(n)}$ is a subgroup of $S_n$. Now we build an isomorphism between $S_n^{(n)}$ and $S_{n-1}$. Take $f : S_{n-1} \to S_n^{(n)}$ as $f(\sigma) = \sigma'$, where $\sigma'(x) = \begin{cases} n, & x = n \\ \sigma(n), & x \neq n \end{cases}$. $\sigma' \in S_n^{(n)}$ and $f$ is a bijection, so $S_{n-1} \cong S_n^{(n)}$.

**Exercise 1.2.9.** Let $f : G \to H$ be a homomorphism of groups, $A$ a subgroup of $G$, and $B$ a subgroup of $H$.
(a) Ker$f$ and $f^{-1}(B)$ are subgroups of $G$.
(b) $f(A)$ is a subgroup of $H$.

**Answer.** (a) $f$ is a homomorphism, so $f(e) = e', e \in$ Ker$f$. $\forall a \in$ Ker$f$, $f(aa^{-1}) = f(a)f(a^{-1}) = e'$, so $f(a^{-1}) = f(a)^{-1} = e'^{-1} = e'$. $\forall a, b \in$ Ker$f, f(ab^{-1}) = f(a)f(b^{-1}) = f(a)f(b)^{-1} = e' \Rightarrow ab^{-1} \in$ Ker$f$, which means Ker$f$ is a subgroup of $G$. The proof of $f^{-1}(B)$ is a subgroup of $G$ is similar.
(b) $f$ is a homomorphism, $f(e) = e'$. $\forall a, b \in A, ab^{-1} \in A$, so $f(ab^{-1}) = f(a)f(b^{-1}) = f(a)f(b)^{-1} \in f(A)$, $f(A)$ is a subgroup of $H$.

**Exercise 1.2.10.** List all subgroups of $Z_2 \oplus Z_2$. Is $Z_2 \oplus Z_2$ isomorphic to $Z_4$?

**Answer.** $Z_2 \oplus Z_2$: $\{\{(1,1),(1,0),(0,1),(0,0)\}, \{(1,1),(0,0)\}, \{(0,0)\}, \{(1,0),(0,0)\}, \{(0,1),(0,0)\}, \{(0,1),(1,0),(0,0)\}\}$.
$Z_4$: $\{\{\bar{0},\bar{1},\bar{2},\bar{3}\}, \{\bar{0},\bar{2}\}, \{\bar{0}\}\}$.
$Z_4$ and $Z_2 \oplus Z_2$ are not isomorphic because they have different subgroups.

**Exercise 1.2.11.** If $G$ is a subgroup, then $C = \{a \in G | ax = xa$ for all $x \in G\}$ is a abelian subgroup of $G$. $C$ is called the center of $G$.

**Answer.** Take $a, b \in C, ab = ba$, $C$ is commutative. $\forall a, b \in C, x \in G$, $b^{-1} \in G$, so $ab^{-1} = b^{-1}a$.

$$ax = axbb^{-1} = abxb^{-1} = baxb^{-1} = bxab^{=1} = abb^{-1}x = bab^{-1}x$$

so $b^{-1}ax = ab^{-1}x = xab^{-1}$, $ab^{-1} \in C$, $C$ is a subgroup of $G$.

**Exercise 1.2.12.** The group $D_4^*$ is not cyclic, but can be generated by two elements. The same is true of $S_n$(nontrivial). What is the minimal number of generators of the additive group $\mathbf{Z} \oplus \mathbf{Z}$?

**Answer.** $\mathbf{Z} \oplus \mathbf{Z} = \{(a, b)|a \in \mathbf{Z}, b \in \mathbf{Z}\} = \langle (0, 0), (1, 0), (0, 1)\rangle$. We can easily check the spanning set is the minimal.

**Exercise 1.2.13.** If $G = \langle a\rangle$ is a cyclic group and $H$ is any group, then every homomorphism $f : G \to H$ is completely determined by the element $f(a) \in H$.

**Answer.** $\forall x \in G$, there exist $m \in \mathbf{N}$ s.t. $x = a^m$, so $f(x) = f(a^m) = f(a)^m \Rightarrow \mathrm{Im} f = \langle f(a)\rangle$. $f : a^m \mapsto f(a)^m \forall m \in \mathbf{N}$. $f$ is completely determined by $f(a) \in H$.

**Exercise 1.2.14.** The following cyclic subgroups are all isomorphic: the multiplication group $\langle i\rangle$ in $\mathbf{C}$, the additive group $\mathbf{Z_4}$ and the subgroup $\left\langle \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \right\rangle$ of $S_4$.

**Answer.** $\langle i\rangle = \{i, -1, -i, 1\}$, $Z_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$,
$\langle (1234)\rangle = \{(1234), (13)(24), (1432), (1)\}$. Denote $f : \langle i\rangle \to Z_4$ as $f(i) = \bar{i}$, $g : Z_4 \to \langle (1234)\rangle$ as $g(i) = (1234)$. From the exercise above we know $f$ and $g$ aer homomorphisms, and they are bijections, so $\langle i\rangle \cong Z_4 \cong \langle (1234)\rangle$.

**Exercise 1.2.15.** Let $G$ be a group and Aut$G$ is the set of all automorphisms of $G$.

(a) Aut$G$ is a group with composition of functions as binary operation.

(b) Aut$\mathbf{Z} \cong Z_2$ and Aut$Z_6 \cong Z_2$; Aut$Z_8 \cong Z_2 \oplus Z_2$; Aut$Z_p \cong Z_{p-1}$ ($p$ prime).

(c) What is $\mathrm{AutZ_n}$ for arbitrary $n \in \mathbf{N}^*$?

**Answer.** We only prove the third question.
For $\bar{a} \in Z_n$, the order of $\bar{a}$ is $|\bar{a}| = \frac{n}{(n,a)}$. When $(n,a) = 1$, $\bar{a}$ is a generator of $Z_n$. Denote Euler function as $\varphi(x)$ and $Z_n^* = \{\bar{a} \in Z_n | (a,n) = 1\}$, then $|Z_n^*| = \varphi(n)$. For $\sigma \in \mathrm{Aut}Z_n$, $\sigma$ is completely determined by $\sigma(\bar{1}) = \bar{a}$, and we denote $\sigma$ as $\sigma_a$. For $\sigma_a, \sigma_b \in \mathrm{Aut}Z_n$, $\sigma_a(\sigma_b(\bar{1})) = \sigma_a(\bar{b}) = \bar{ab} = \sigma_{ab}(\bar{1})$. We have proved $\mathrm{Aut}Z_n \cong Z_n^*$.
Now we give out a lemma to show the structure of $Z_n^*$.

**Lemma.** *If $n = st, (s,t) = 1$, then $Z_n^* \cong Z_s^* \oplus Z_t^*$.*

The proof of this lemma is quite simple. Consider the mappping $f^* : Z_n^* \to Z_s^* \oplus Z_t^*$ which is defined by $(x \mod n) \mapsto (x \mod s, x \mod t)$. Since for any $a, b \in Z_n^*$, $f^*(a)f^*(b) = (a \mod s, a \mod t)(b \mod s, b \mod t) = (ab \mod s, ab \mod t) = f^*(ab)$, $f^*$ is a well defined homomorphism. For $x \in \mathrm{Ker}f^*$, $x \equiv 1 \mod s$, $x \equiv 1 \mod t$, so $x \equiv 1 \mod [s,t]$, $x \equiv 1 \mod n$, $f^*$ is a monomorphism. Since $|f^*(Z_n^*)| = |Z_n| = \varphi(n) = \varphi(s)\varphi(t) = |Z_s^* \oplus Z_t^*|$, $f^*$ is a epimorphism. $Z_n^* \cong Z_s^* \oplus Z_t^*$
For $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$, $Z_n^* \cong Z^*_{p_1^{k_1}} \oplus Z^*_{p_2^{k_2}} \oplus \cdots \oplus Z^*_{p_m^{k_m}}$. Now we consider the structure of $Z^*_{p^k}$.
For $p = 2$, $Z_2^* \cong Z_1$, $Z_4^* \cong Z_2$, $Z_{2^k}^* \cong Z_2 \oplus Z_{2^{k-2}}$.
For other odd prime $p$, $Z^*_{p^k} \cong Z_{(p-1)p^{k-1}}$.
In order to prove the result, we need the Lagrange theorem in number theory.

**Lemma** (Lagrange). *$f(x) \in Z[n]$, $f(x) \equiv k$ has at most $n$ solutions when* mod $p$, *where $p$ is an odd prime.*

We use induction to prove the lemma.

1. $n = 1$, the proof the trivial.
2. Assume for $n \le m-1$ the lemma is correct, and for $n = m$, $f(x) \equiv k$ has $m+1$ solutions. $f(x) - f(x_{m+1}) = (x - x_{m+1})g(x) \equiv 0 \mod p$. Take $x = x_i, i = 1, 2, \ldots, m$, $(x_i - x_{m+1})g(x_i) \equiv 0 \mod p$, $x_i \ne x_{m+1}$, so $g(x_i) \equiv 0 \mod p$, That's contradictory to the induction assumptions!

The lemma is proved.
Come back to the original question. Firstly, we consider $k=1$ and $p$ is an odd prime. For any factor $d$ of $p-1$, denote $S(d)=\{\bar{a}\in Z_p^*|\mathrm{ord}_p(a)=d\}$. $S(d)$ forms a partition of $Z_p^*$. If $S(d)\neq\varnothing$, there exists $\bar{a}\in S(d)$ and $a^d\equiv 1 \mod p$. By Largrange theorem, $a^d\equiv 1 \mod p$ has at most $d$ solutions. Notice that $\{1,a,a^2,\dots,a^{d-1}\}$ are the solutions of the equation, $a^i\not\equiv a^j \mod p$, whence $S(d)\subset\langle\bar{a}\rangle$. For $k=1,2,\dots,d-1$, $\mathrm{ord}_p(\bar{a^k})=\left|a^k\right|=\frac{d}{(d,k)}=d\Leftrightarrow(d,k)=1$. Thus $|S(d)|=\varphi(d)$.
From $Z_p^*=\bigcup\limits_{d|p-1}S(d)$, we get

$$p-1=\left|Z_p^*\right|=\sum_{d|p-1}|S(d)|\leq\sum_{d|p-1}\varphi(d)=p-1$$

If $d|p-1$, $|S(d)|=\varphi(d)$. Particularly, when $d=p-1$, $|S(p-1)|=\varphi(p-1)\neq 0$, $Z_p^*$ has a element of order $p-1$, $Z_p^*$ is a cyclic group.
Secondly, we consider $k\geq 2$. Take $a\in\mathbf{Z}$ and $\bar{a}$ is the class of $x\equiv a \mod p^k$. For $s\geq t$, we have a group homomorphism $f_{s,t}:Z_{p^s}^*\to Z_{p^t}^*$ which is defined by $(a \mod p^s)\mapsto(a \mod p^t)$. Since $a\equiv b \mod p^s\Rightarrow a\equiv b \mod p^t$, $f$ is well defined. $\mathrm{Ker}f_{s,t}=\{up^t+1 \mod p^s|u=0,1,\dots,p^{s-t}-1\}$. If $2t\geq s$, since $(up^t+1)(vp^t+1)\equiv uvp^{2t}+(u+v)p^t+1\equiv(u+v)p^t+1 \mod p^s$, $\mathrm{Ker}f_{s,t}\cong Z_{p^{s-t}}$ is a cyclic group. There exists a isomorphism $g_{s,t}:Z_{p^s}^*/\mathrm{Ker}f_{s,t}\to Z_{p^t}^*$.

$$\{\bar{1}_{p^k}\}=\mathrm{Ker}f_{k,k}<\mathrm{Ker}f_{k,k-1}<\cdots<\mathrm{Ker}f_{k,1}<Z_{p^k}^*$$

**Lemma.** *Suppose $i\geq 2$, $\bar{a}_{p^k}\in\mathrm{Ker}f_{k,i}$, but $\bar{a}_{p^k}\notin\mathrm{Ker}f_{k,i+1}$, then $\bar{a}_{p^k}^p\in\mathrm{Ker}f_{k,i+1}$ and $\bar{a}_{p^k}^p\notin\mathrm{Ker}f_{k,i+2}$.*

This lemma can be proved by LTE. Here we use the language in group theory to prove it. $f_{k,i+2}(\bar{a}_{p^k})=\bar{a}_{p^{i+2}}$, $\bar{a}_{p^{i+2}}\in f_{k,i+2}(\mathrm{Ker}f_{k,i})=\mathrm{Ker}f_{i+2,i}$. $\mathrm{Ker}f_{i+2,i}\cong Z_{p^2}$ since $2i\geq i+2$. $\bar{a}_{p^{i+2}}\notin f_{k,i+2}(\mathrm{Ker}f_{k,i+1})=\mathrm{Ker}f_{i+2,i+1}\cong Z_p$. $\mathrm{Ker}f_{i+2,i+1}$ contains all the elements whose order is $p$ in $\mathrm{Ker}f_{i+2,i}$, so $\left|\bar{a}_{p^{i+2}}\right|=p^2$. $\bar{a}_{p^{i+2}}^p\in\mathrm{Ker}f_{i+2,i+1}$, $\bar{a}_{p^{i+2}}^p\notin\mathrm{Ker}f_{i+2,i+2}$, $\bar{a}_{p^k}^p\in g_{k,i+2}^{-1}(\bar{a}_{p^{i+2}}^p)\subset g_{k,i+2}^{-1}(\mathrm{Ker}f_{i+2,i+1})=\mathrm{Ker}f_{k,i+1}$, $\bar{a}_{p^k}^p\notin g_{k,i+2}^{-1}(\mathrm{Ker}f_{i+2,i+2})=\mathrm{Ker}f_{k,i+2}$.
For $i=1$, if $p$ is an odd prime, $\mathrm{Ker}f_{3,1}=\left\langle\bar{p+1}_{p^3}\right\rangle\cong Z_{p^2}$, if $p=2$, $\mathrm{Ker}f_{3,1}=\{\bar{1}_8,\bar{3}_8,\bar{5}_8,\bar{7}_8\}\cong Z_2\oplus Z_2$. Thus, for $\bar{a}_{p^k}\in\mathrm{Ker}f_{k,2},\bar{a}_{p^k}\notin\mathrm{Ker}f_{k,3}$, using the lemma above for several times, we get $\bar{a}_{p^k}^{p^{k-2}}\in\mathrm{Ker}f_{k,2},\bar{a}_{p^k}^{p^{k-3}}\notin\mathrm{Ker}f_{k,k}$, $\left|\bar{a}_{p^k}\right|=p^{k-2}$, $\mathrm{Ker}f_{k,2}\cong Z_{p^{k-2}}$.
If $p$ is an odd prime, we can further obtain $\mathrm{Ker}f_{k,1}\cong Z_{p^{k-1}}$.

Suppose $x$ is a generator of $Z_p^*$, assume $a \in g_{k,1}^{-1}(x)$, $g_{k,1}^{-1}(x) = a\mathrm{Ker}f_{k,1}$, and $a^{p-1} \in g_{k,1}^{-1}(x^{p-1}) = g_{k,1}^{-1}(\bar{1}_p) = \mathrm{Ker}f_{k,1}$. If $a^{p-1} \notin \mathrm{Ker}f_{k,2}$, then $\left|a^{p-1}\right| = p^{k-1}$. If $a^{p-1} \in \mathrm{Ker}f_{k,2}$, $\forall h \in \mathrm{Ker}f_{k,1}, h \notin \mathrm{Ker}f_{k,2}$. Since $(ah)^{p-1} = (a^{p-1}h^p)h^{-1}$, $(ah)^{p-1} \in \mathrm{Ker}f_{k,1}, (ah)^{p-1} \notin \mathrm{Ker}f_{k,2}$, whence $\left|(ah)^{p-1}\right| = p^{k-1}$, $Z_{p^k}^* \cong Z_{(p-1)p^{k-1}}$.
If $p = 2$, $Z_{2^k}^* = \mathrm{Ker}f_{k,1} \cong Z_{2^{k-2}} \oplus Z_2$.
For Aut$\mathbf{Z}$, assume there exist $f \neq 1_G, -1_G$, $f \in \mathbf{AutZ}$. WLOG, $f(1) = x \neq \pm 1, f(-1) = y$. $f(1)+f(-1) = f(0) = x+y = 0$. Assume $af(1)+bf(-1) = f(a-b) = 1 = (a-b)x$, since $x \neq \pm 1$, there is a contradiction. Aut$\mathbf{Z} \cong Z_2$.

**Exercise 1.2.16.** For each prime $p$ the additive subgroup $Z(p^\infty)$ of $\mathbf{Q}/\mathbf{Z}$ is generated by the set $\{\overline{1/p^n} | n \in \mathbf{N}^*\}$.

**Answer.** We prove that $\left\langle \bigcup_{n=1}^{\infty} \frac{1}{p^n} \right\rangle \cong Z(p^\infty)$. $\forall x \in Z(p^\infty), x = \frac{\bar{a}}{b} = \frac{\bar{a}}{p^k}$. Expand $a$ as $a = \sum_{i=0}^{k-1} p^i a_i$, where $a_i = 1, 2, \dots n-1$. $x = \frac{\bar{a}}{b} = \sum_{i=0}^{k-1} \frac{\bar{a_i}}{p^{k-i}} = \sum_{i=1}^{k} \frac{\overline{a_{k-i}}}{p^i}$. Denote $f : \left\langle \bigcup_{n=1}^{\infty} \frac{1}{p^n} \right\rangle \to Z(p^\infty)$ as $f(\sum_{i=1}^{n} \frac{a_i}{p^i}) = \overline{\sum_{i=1}^{n} \frac{a_i}{p^i}}$. $f$ is an isomorphism because every $x \in Z(p^\infty)$ can be written in such form.

**Exercise 1.2.17.** Let $G$ be an abelian group and let $H, K$ be subgroups of $G$. Show that the join $H \vee K$ is the set $\{ab | a \in H, b \in K\}$. Extend this result to any finite number of subgroups of $G$.

**Answer.** $H \vee K = \langle H \cup K \rangle, I = \{ab | a \in H, b \in K\}$. $G$ is abelian so $I$ is a subgroup of $G$. $H < I, K < I, (H \cup K) \subset I$. $\langle H \cup K \rangle \subset I \Rightarrow \langle H \cup K \rangle$.
For any $ab \in I$, $a \in H$, $b \in K$, we prove that $ab$ is contained in any subgroup which contains $H \cup K$.
Assume $(H \cup K) \subset J$, so $a \in J, b \in J \Rightarrow ab \in J$, which means $I \subset H \vee K$.
$\langle H \cup K \rangle = I$.
$G$ is abelian group, $H_1, H_2, \dots H_n$ are $n$ subgroups. $\left\langle \bigcup_{i=1}^{n} H_i \right\rangle = \{\prod_{i=1}^{n} h_i | h_i \in H_i, i = 1, 2, \dots n\}$. This proposition can be proved by induction.

**Exercise 1.2.18.** 1. Let $G$ be a group and $\{H_i|i \in I\}$ a family of subgroups. State and prove a condition that will imply that $\bigcup_{i\in I} H_i$ is a subgroup, that is $\bigcup_{i\in I} H_i = \left\langle \bigcup_{i\in I} H_i \right\rangle$.

2. Given an example of a group $G$ and a family of subgroups $\{H_i|i \in I\}$ such that $\bigcup_{i\in I} H_i \neq \left\langle \bigcup_{i\in I} H_i \right\rangle$.

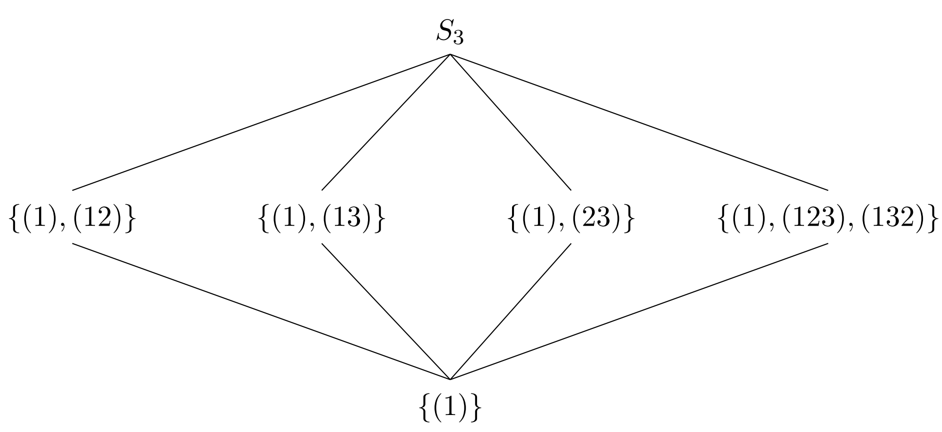
**Answer.** I didn't find a sufficient and necessary condition for this question, just choose one as you like:)

**Exercise 1.2.19.** 1. The set of all subgroups of a group $G$, partially ordered by set theoretic inclusion, forms a complete lattice in which the g.l.b of $\{H_i|i \in I\}$ is $\bigcap_{i\in I} H_i$ and the l.u.b is $\left\langle \bigcap_{i\in I} H_i \right\rangle$.

2. Exhibit the lattice of subgroups of the groups $S_3, D_4^*, Z_6, Z_{27}$ and $Z_{36}$.

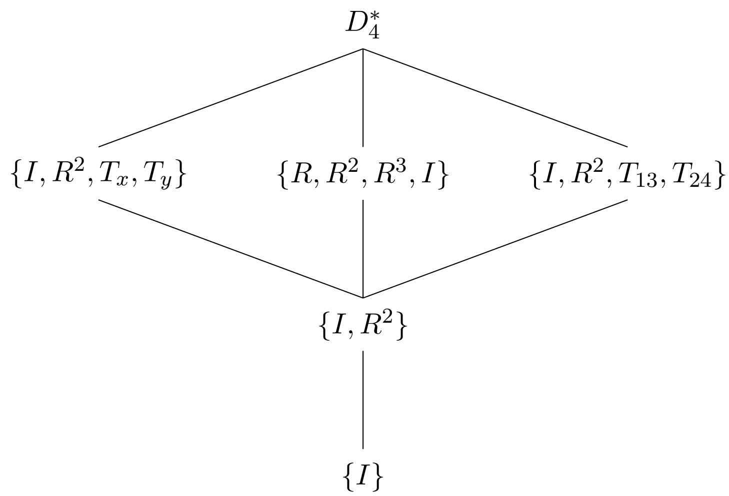
**Answer.** 1. The subset relation $<$ forms a partially ordered relation. By the difinition of $\left\langle \bigcup_{i\in I} H_i \right\rangle$, $\left\langle \bigcup_{i\in I} H_i \right\rangle$ is the smallest set contains $\bigcup_{i\in I} H_i$, so it's lup. For glb, we know that $\bigcap_{i\in I} H_i \subset H_i \ \forall i \in I$, and $\forall H \supset \bigcap_{i\in I} H_i$, there exists $x \in H, x \notin H_j \ j \in I$, so $\bigcap_{i\in I}$ is glb.

2. $S_3 = \{(1), (12), (13), (23), (123), (132)\}$.



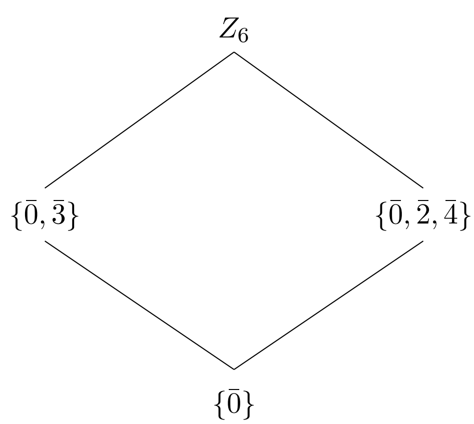

The Hasse figure of the lattice of $S_3$

$D_4^* = \{R, R^2, R^3, I, T_x, T_y, T_{13}, T_{24}\}$.




The Hasse figure of the lattice of $D_4^*$

$Z_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$.




The Hasse figure of the lattice of $Z_6$

$$
\begin{tikzcd}
Z_{27} \arrow[d, dash] \\
Z_9 \arrow[d, dash] \\
Z_3 \arrow[d, dash] \\
\{\bar{1}\}
\end{tikzcd}
$$

The Hasse figure of the lattice of $Z_{27}$

$$
\begin{tikzcd}
 & & Z_{36} \arrow[dl, dash] \arrow[dr, dash] & & \\
 & Z_{12} \arrow[dl, dash] \arrow[dr, dash] & & Z_{18} \arrow[dl, dash] \arrow[dr, dash] & \\
Z_4 \arrow[dr, dash] & & Z_6 \arrow[dl, dash] \arrow[dr, dash] & & Z_9 \arrow[dl, dash] \\
 & Z_2 \arrow[dr, dash] & & Z_3 \arrow[dl, dash] & \\
 & & \{\bar{1}\} & &
\end{tikzcd}
$$

The Hasse figure of the lattice of $Z_{36}$

## 1.3 Cyclic groups

**Exercise 1.3.1.** Let $a, b$ be elements of group $G$. Show that $|a| = \left|a^{-1}\right|$; $|ab| = |ba|$, and $|a| = \left|cac^{-1}\right|$ for all $c \in G$.

**Answer.** We only consider that $|a|, |b|, |c|$ are finite. Assume $a^k = e$, $(ab)^m = e$, $(ac^{-1})^n = e$, $kmn \neq 0$. $a^k \cdot (a^{-1})^k = e$, so $k$ sialso the order of $a^{-1}$, $\left|a^{-1}\right| = k$. $(ab)^m = e = a(ba)^{m-1}b \Rightarrow (ba)^{m-1} = a^{-1}b^{-1}$, $(ba)^m = a^{-1}b^{-1}ba = e$. $m$ is the order of $ba$. $(cac^{-1})^r = cac^{-1}cac^{-1} \cdots cac^{-1} = ca^nc^{-1} = e$, so $a^n = e$, whence $n = k$.

**Exercise 1.3.2.** Let $G$ be an abelian group containing elements $a$ and $b$ of orders $m$ and $n$ respectively. Show that $G$ contains an element whose order is the least commom multiple of $m$ and $n$.

**Answer.** If $(m, n) = 1$, we know that $\forall a^i, i = 1, 2, \ldots, m$, $b^j, j = 1, 2, \ldots, n$, $a^ib^j \neq e$, since if $a^i = b^j$, $\left|a^i\right| = n = \left|b^{-j}\right| = \left|b^j\right| = m$. $G$ is abelian, so $(ab)^k = a^kb^k \Rightarrow |ab| = mn = [m, n]$.
If $m|n$ or $n|m$, then $a$ or $b$ is the element we want. We consider $m \nmid n$ and $n \nmid m$. Factorise $n = p_1^{t_1}p_2^{t_2}\cdots p_l^{t_l}$, $m = p_1^{s_1}p_2^{s_2}\cdots p_l^{s_l}$, where $p_1, \cdots, p_l$ are primes and $t_1, \cdots, t_l, s_1, \cdots, s_l \geq 0$. We can choose a new arrangement of $p_1, \cdots, p_l$ and make $t_1 \geq s_1$, $t_2 \geq s_2$, $\ldots$, $t_i \geq s_i$, $t_{i+1} < s_{i+1}$, $\ldots$, $t_l < s_l$.

$$(m, n) = p_1^{s_1} \cdots p_i^{s_i} p_{i+1}^{t_{i+1}} \cdots p_l^{t_l}, [m, n] = p_1^{t_1} \cdots p_i^{t_i} p_{i+1}^{s_{i+1}} \cdots p_l^{s_l}$$

Take $x = a^{p_{i+1}^{s_{i+1}} \cdots p_l^{s_l}}$, $y = b^{p_1^{t_1} \cdots p_i^{t_i}}$, then $|x| = p_1^{t_1} \cdots p_i^{t_i}$, $|y| = p_{i+1}^{s_{i+1}} \cdots p_l^{s_l}$. Thus $(x, y) = 1$, the order of $xy$ is $|x| \cdot |y| = p_1^{t_1} \cdots p_i^{t_i} p_{i+1}^{s_{i+1}} \cdots p_l^{s_l} = [m, n]$.

**Exercise 1.3.3.** Let $G$ be an abelian group of order $pq$, with $(p, q) = 1$. Assume there exist $a, b \in G$ such that $|a| = p, |b| = q$ and show that $G$ is cyclic.

**Answer.** From **Exercise 1.3.2** we know $a^ib^j \neq e$ for $i < p$, $j < q$. $|G| = pq$ for all $a^ib^j$ and $a^mb^n$ with $i \neq m$, $b \neq n$, $a^ib^j \neq a^mb^n$. So $G$ can be generated by $ab$. $G$ is cyclic.

**Exercise 1.3.4.** If $f : G \to H$ is a homomorphism, $a \in G$, and $f(a)$ has finte order in $H$, then $|a|$ is infinite or $|f(a)|$ divides $|a|$.

**Answer.** Assume $|f(a)| = n$, $|a| = m$, and $n \nmid m$. Trivially, $m \geq n$. Assume $\gcd(m, n) = k \leq n$. $a^m = e \Rightarrow f(a)^m = e' = f(a)^n$. By Bezout theorem $\exists x, y \in \mathbf{Z}$ s.t. $f(a)^{mx+ny} = f(a)^k = e'$, $k \leq n$, that's contradictory!

**Exercise 1.3.5.** Let $G$ be the multiplicative group of all nonsingular $2 \times 2$ matrices with rational entries. Show that $a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has order 4 and $b = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ has order 3, but $ab$ has infinite order. Conversely, show that the additive group $Z_2 \oplus \mathbf{Z}$ contains nonzero elements $a, b$ of infinite order such that $a + b$ has finite order.

**Answer.** The verification of $|a| = 4$ and $|b| = 3$ is trivial. $ab = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\det(ab = \lambda I) = 0 \Rightarrow \lambda_1 = \lambda_2 = 1$. $ab$ is not diagnizable. By induction, we have $(ab)^n = \begin{pmatrix} 1 & 2^{n-1} \\ 0 & 1 \end{pmatrix}$ which means $(ab)$ has infinite order.
For $a = (\bar{0}, 1), b = (\bar{0}, -1) \in Z_2 \oplus \mathbf{Z}$, $a, b$ have infinite order, but $a + b = (\bar{0}, 0)$ has finite order 1.

**Exercise 1.3.6.** If $G$ is a cyclic group of order $n$ and $k|n$, then $G$ has exactly one subgroup of order $k$.

**Answer.** Assume $a^n = e$, $mk = n$, we verify that $\langle a^m \rangle$ is a subgroup of order $k$. $\forall x, y \in \mathbf{Z}_+$, $a^{xm} \cdot a^{-ym} = a^{(x-y)m} \in \langle a^m \rangle$, so $\langle a^m \rangle$ is a subgroup. $a^{km} = e$, $a^{sm} \neq e$ for $s < k$, so $|\langle a^m \rangle| = k$.

**Exercise 1.3.7.** Let $p$ be prime and $H$ a subgroup of $Z(p^\infty)$.
(a) Every element of $Z(p^\infty)$ has finite order $p^n$ for some $n \geq 0$.
(b) If at least one element of $H$ has order $p^k$ and no element of $H$ has order greater than $p^k$, then $H$ is the cyclic subgroup generated by $\overline{1/p^k}$, whence $H \cong Z_{p^k}$.

(c) If there is no upper bound on the orders of elements of $H$, then $H = Z(p^\infty)$.

(d) The only proper subgroups of $Z(p^\infty)$ are the finite cyclic groups $C_n = \langle 1/p^n \rangle$ $(n = 1, 2, \dots)$. Futhermore, $\langle 0 \rangle = C_0 < C_1 < C_2 < C_3 < \cdots$.

(e) Let $x_1, x_2, \dots$ be elements of an abelian group $G$ such that $|x_1| = p, px_2 = x_1, px_3 = x_2, \dots, px_{n+1} = x_n, \dots$. The subgroup generated by the $x_i (i \geq 1)$ is isomorphic to $Z(p^\infty)$.

**Answer.** (a) $\forall x \in Z(p^\infty)$, $x = \frac{a}{p^n}$ where $a < p^n$, $p \nmid a$. $p$ is a prime, so $\gcd(p, a) = 1$. $m \cdot a | p^n \Rightarrow m = p^n$. Thus $m \cdot \frac{a}{p^n} = e$, $p^n$ is the smallest number satisfies it. $\frac{a}{p^n}$ has order $p^n$.

(b) For all $x \in Z(p^\infty)$, if $x$ has order smaller than $p^k$, $x$ must have the form $x = \frac{a}{p^i}(i \leq k)$, $(p, a) = 1$, so $x \in \left\langle \frac{1}{p^k} \right\rangle$. If not, assume $x = \frac{a}{p^i}(i > k)$, then $p^k \cdot x = \frac{a}{p^{i-k}} \neq 1$.

(c) Assume not, $H < Z(p^\infty)$, $H \neq Z(p^\infty)$. There exist $y \in H$ s.t. $y$ has order $p^m, m \geq n$. $y = \frac{b}{p^m}$, $(p, b) = 1$, so there exists $b^{-1} \in \{1, 2, \dots, p-1\}$, $bb^{-1} \equiv 1 \mod p^m$. But $ab^{-1}p^{m-n}y = \frac{a}{p^n} = x \in H$, that's contradictory! Conversely, $H = Z(p^\infty)$.

(d) From (b), we know that if there's least upper bound $p^n$ for elements in a subgroup $S$, then $S = C_n$.

$$\langle 0 \rangle = C_0 < C_1 < C_2 < C_3 < \cdots < Z(p^\infty)$$

is easy to verify.

(e) We can verify that $f : x_i \mapsto \frac{\bar{1}}{p^i}$ is a well defined isomorphism. $f(e) = f(px_1) = 1$, $f(px_{i+1}) = f(x_i) = \frac{1}{p^i} = p \cdot \frac{1}{p^{i+1}}$. $f$ is obviously a bijection, so $H \cong Z(p^\infty)$.

**Exercise 1.3.8.** A group that has only a finite number of subgroups must be finite.

**Answer.** Suppose not. If the order of all subgroups are finite, $G$ must be finite. So there exists a infinite subgroup $H < G$. $\forall a \in G$, if $\forall n \in \mathbf{N}$, $a^n \neq e$. then we can construct infinte subgroups $\langle a \rangle$, $\langle a^2 \rangle$, $\langle a^3 \rangle \dots$. If $\forall a \in G$, $\exists n \in \mathbf{N}$, $a^n = e$, so $\langle a \rangle$ is a proper subgroup of $G$, we can take $b \in G \ni \langle a \rangle$ to construct another subgroup. By induction, there are infinte subgroups in $G$. That's contradictory, so $G$ must be finite.

**Exercise 1.3.9.** If $G$ is an abelian group, then the set $T$ of all elements of $G$ with finite order is a subgroup of $G$.

**Answer.** We can easily verify that $\forall a, b \in S$, $|a| = m$, $|b| = n$ and $\left|ab^{-1}\right| \leq mn$ is finite. $T$ is a subgroup of $G$.

**Exercise 1.3.10.** An infinite group is cyclic if and only if it is isomorphic to each of its proper subgroups.

**Answer.** If $G$ is cyclic, $G \cong \mathbf{Z}$, $S < G$. For any subgroup of $\mathbf{Z}$, it has the form $\{na\}, a \in \mathbf{Z}$. We can construct a isomorphism $f : n \mapsto na$, so $S \cong \{na\} \Rightarrow G \cong S$.
If $\forall S < G$, $G \cong S$ and $|G| = |S|$ is finite. We prove there exists $S < G$ s.t. $|S| = \aleph_0$. Take $a \in G$ and $S = \{na | n \in \mathbf{Z}\}$, $S$ is a subgroup. If there exists $ma = 0$, $S$ must be finite, contradictory! Thus, $S \cong \mathbf{Z} \cong G$. $G$ is a infinite cyclic group.

## 1.4 Cosets and counting

**Exercise 1.4.1.** Let $G$ be a group and $\{H_i | i \in I\}$ a family of subgroups. Then for any $a \in G$, $(\bigcap\limits_i H_i)a = \bigcap\limits_i H_i a$.

**Answer.** $\bigcap\limits_i H_i$ is a subgroup of $G$. Take $x \in \bigcap\limits_i H_i$, $x \in H_i$, $\forall i \in I$. Then $xa \in H_i a$, $\forall i \in I$, so $xa \in \bigcap\limits_i (H_i a)$. Thus, $(\bigcap\limits_i H_i)a = \bigcap\limits_i (H_i a)$.

**Exercise 1.4.2.** (a) Let $H$ be the cyclic subgroup (of order 2) of $S_3$ generated by $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$. Then no left cosets of $H$ (except $H$ itself) is also a right coset. There exists $a \in S_3$ such that $aH \cap Ha = \{a\}$.

(b) If $K$ is the cyclic subgroup (of order 3) of $S_3$ generated by $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, then every left coset of $K$ is also a right coset of $K$.

**Answer.** (a) $H = \{(12), (1)\}$. $S_3 = \{(12), (13), (23), (1), (123), (132)\}$. For $a \in H$, $aH = Ha = H$.
$a = (13)$, $aH = \{(13), (123)\}$, $Ha = \{(13), (132)\}$.
$a = (23)$, $aH = \{(23), (132)\}$, $Ha = \{(23), (123)\}$.
$a = (123)$, $aH = \{(123), (23)\}$, $Ha = \{(132), (13)\}$.
$a = (132)$, $aH = \{(132), (13)\}$, $Ha = \{(123), (23)\}$.

(b) $K = \{(123), (132), (1)\}$. For $a \in K$, $aK = Ka = K$.
$a = (12)$, $aK = Ka = \{(12), (23), (13)\}$.
$a = (13)$, $aK = Ka = \{(12), (23), (13)\}$.
$a = (23)$, $aK = Ka = \{(12), (23), (13)\}$.

**Exercise 1.4.3.** The following conditions on a finite group $G$ are equivalent.
(i) $|G|$ is prime.
(ii) $G \neq \langle e \rangle$ and $G$ has no proper subgroups.
(iii) $G \cong Z_p$ for some prime $p$.

**Answer.** (i)$\Rightarrow$(ii): If there exists $S < G$, $S \neq G$, then $|S| \mid |G| = p$. That's contradictory!
(ii)$\Rightarrow$(iii): $\forall a \in G$, take $S = \{na | n = 1, 2, \ldots, p\}$. If there exists $ma = na, (1 \leq m < n \leq p)$, $(n-m)a = 0$. So there exists subgroup $S$, and $|S| = n - m < p$. That's contradictory! So $S < G$, $|S| = |G| \Rightarrow S = G \cong Z_p$.

(iii)⇒(i): Trivial.

**Exercise 1.4.4.** Let $a$ be an integer and $p$ be a prime such that $p \nmid a$. Then $a^{p-1} \equiv 1 \mod p$.

**Answer.** $(Z_p \backslash \{\bar{0}\}, \times)$ is a group of order $p-1$. From **Exercise 1.1.7**, we know that $\forall \bar{a} \in Z_p \backslash \{\bar{0}\}$ and $b \in Z_p \backslash \{\bar{0}\}$, taking different $\bar{b}$ we will have different $\bar{a}\bar{b} \in Z_p \backslash \{\bar{0}\}$. $\bar{a}\bar{b}$ travels through all the elements in $Z_p \backslash \{\bar{0}\}$. So

$$\prod_{i=1}^{p-1} (\bar{i} \cdot \bar{a}) = \prod_{i=1}^{p-1} \bar{i}$$

By the definition of $Z_p \backslash \{\bar{0}\}$, $Z_p \backslash \{\bar{0}\}$ is commutative. So

$$(\bar{a})^{p-1} (\prod_{i=1}^{p-1} \bar{i}) = \prod_{i=1}^{p-1} \bar{i} \Rightarrow (\bar{a})^{p-1} = \bar{1}$$

**Exercise 1.4.5.** Prove that there are only two distinct groups of order 4 (up to isomorphism), namely $Z_4$ and $Z_2 \oplus Z_2$.

**Answer.** The only cyclic group of order 4 is $Z_4$. For a group $G$ of order 4 which is not cyclic, $\forall a \in G$, $a \neq e$, if $|a| = 2$, $G \cong Z_2 \oplus Z_2$. If there exists $a \in G$, $|a| = 4$, $G \cong Z_4$. If there exists $a \in G$, $|a| = 3$, denote $a^2 = b, a^3 = e$. Then $b^2 = a^4 = a$, $\{e, a, b\} < G$, which is contradictory to the Largrange theorem.

**Exercise 1.4.6.** Let $H, K$ be subgroups of a group $G$. Then $HK$ is a subgroup of $G$ if and only if $HK = KH$.

**Answer.** If $HK = KH$, for $a_1b_1, a_2b_2 \in HK$,

$$(a_1b_1)(a_2b_2)^{-1} = (a_1b_1)(b_2^{-1}a_2^{-1}) = (a_1b_1)(a_3b_3)$$

since $b_2^{-1}a_2^{-1} \in KH = HK$, there exists $b_2^{-1}a_2^{-1} = a_3b_3$.

$$(a_1b_1)(a_3b_3) = a_1(b_1a_3)b_3 = a_1a_4b_4b_3$$

since $b_1a_3 \in KH = HK$, there exists $b_1a_3 = a_4b_4$. $(a_1b_1)(a_2b_2)^{-1} = a_1a_4b_4b_3 = a_5b_5 \in HK$. Thus $HK$ is a subgroup of $G$.
If $HK$ is a subgroup of $G$, $\forall b_1a_1 \in KH$, there exists $(a_1^{-1}b_1^{-1}) \in HK$ s.t. $b_1a_1 = (a_1^{-1}b_1^{-1})^{-1} \in HK$. So $KH \subset HK$. $\forall a_1b_1 \in HK$, $(a_1b_1)^{-1} = b_1^{-1}a_1^{-1} \in HK$, so $\exists a_2b_2 \in HK$ s.t. $b_1^{-1}a_1^{-1} = a_2b_2$. $a_1b_1 = b_2^{-1}a_2^{-1} \in KH$. So $HK \subset KH$. Thus $HK = KH$.

**Exercise 1.4.7.** Let $G$ be a group of order $p^k m$, with $p$ prime and $(p,m) = 1$. Let $H$ be a subgroup of order $p^k$ and $K$ a subgroup of order $p^d$, with $0 < d \leq k$ and $K \not\subset H$. Show that $HK$ is not a subgroup of $G$.

**Answer.** Assume $HK < G$, $|HK| = p^k n$, $n|m$. We can get $[HK : H] = n = [K : K \cap H]$. $[K : K \cap H]\,|p^k \Rightarrow n|p^k$. That's contradictory to $(m, p^k) = 1$.

**Exercise 1.4.8.** If $H$ and $K$ are subgroups of finite index of a group $G$ such that $[G : H]$ and $[G : K]$ are relatively prime, then $G = HK$.

**Answer.** Assume $[G : H] = m$, $[G : K] = n$, $(m,n) = 1$. Then $|H| = np$, $|K| = mp$. $H \cap K < H$, $H \cap K < G \Rightarrow |H \cap K|\,|p$.

$$[G : H] = m \geq [K : H \cap K] = \frac{|K|}{|H \cap K|} \geq m$$

Thus $[G : H] = [K : H \cap K] = m$, $G = HK$.

**Exercise 1.4.9.** If $H, K$ and $N$ are subgroups of a group $G$ such that $H < N$, then $HK \cap N = H(K \cap N)$.

**Answer.** $\forall x = hk \in HK \cap N$, $\exists h_1^{-1} \in H$ s.t. $h_1^{-1}hk \in K \cap N$. $H < N$ so $\forall h_1^{-1} \in H, h_1^{-1}hk \in N$. Take $h_1^{-1} = h^{-1}$, $h_1^{-1}hk = k \in K$. So $HK \cap N \subset H(K \cap N)$.
$\forall x = hk \in H(K \cap N)$ where $h \in H$, $k \in K \cap N$. $hk \in HK, h, k \in N \Rightarrow hk \in N$. So $H(K \cap N) \subset HK \cap N$.
Thus, $HK \cap N = H(K \cap N)$.

**Exercise 1.4.10.** Let $H, K, N$ be subgroups of a group $G$ such that $H < K$, $H \cap N = K \cap N$, and $HN = KN$. Show that $H = K$.

**Answer.** Assume there exists $x \in K \backslash H$. $K \bigcup\limits_{i \in I} Ha_i$, $\forall h_i \in H$ there exists $a \in K$ s.t. $x = h_1 a$. Take $n_1 \in N$. Since $HN = KN$, $xn_1 \in HN$, there exists $h_2 \in H$, $n_2 \in N$ s.t. $xn_1 = h_2 n_2 = h_2 a n_1$. So $a = n_2 n_1^{-1} \in N$, $a \in K \cap N = H \cap N \Rightarrow a \in H, x \in H$. That's contradictory!

**Exercise 1.4.11.** Let $G$ be a group of order $2n$; then $G$ contains an element of order 2. If $n$ is odd and $G$ abelian, there is only one element of order 2.

**Answer.** The proof of the first part is exactly the same as **Exercise 1.1.14**. Assume there exists $a, b \in G$, $a^2 = b^2 = e$. We can check $H = \{e, a, b, ab\}$ is a subgroup of $G$. $|H| \,|\, |G| \Rightarrow 4|2n \Rightarrow 2|n$, which is contradictory to $n$ is odd. So there's only one element $a$ s.t. $a^2 = e$.

**Exercise 1.4.12.** If $H$ and $K$ are subgroups of a group $G$, then $[H \vee K : H]$ $\geq [K : H \cap K]$.

**Answer.** The question is a direct corollary of Proposition 4.8.

**Exercise 1.4.13.** If $p > q$ are primes, a group of order $pq$ has at most one subgroup of order $p$.

**Answer.** $H \cap K < H$, $H \cap K < K$, $H \neq K \neq H \cap K$. $|H \cap K| \,|p$ and $|H \cap K| \neq q$, so $H \cap K = \{e\}$. From **Exercise 1.3.12**,

$$[H \vee K : H] \geq [K : K \cap H] = p$$

$$|H \vee K| = |H| \cdot [H \vee K : H] \geq p^2$$

But $H \vee K \in G$, $|H \vee K| \leq pq < p^2$. That's contradictory!

**Exercise 1.4.14.** Let $G$ be a group and $a, b \in G$ such that (i) $|a| = 4 = |b|$; (ii) $a^2 = b^2$; (iii) $ba = a^3b = a^{-1}b$; (iv) $a \neq b$; (v) $G = \langle a, b \rangle$. Show that $|G| = 8$ and $G \cong Q_8$.

**Answer.** The proof is exactly the same as **Exercise 1.2.3**.

## 1.5 Normality, quotient groups, and homomorphisms

**Exercise 1.5.1.** If $N$ is a subgroup of index 2 in a group $G$, then $N$ is normal in $G$.

**Answer.** $\forall a \in G\backslash N$, $G = N \cup Na = N \cup aN$ and $N \cap Na = \varnothing$, $N \cap aN = \varnothing$. So $\forall x \in Na$, $x \in G\backslash N \Rightarrow x \in aN$, $Na \subset aN$. Similarly, $aN \subset Na$, whence $Na = aN$, $N \lhd G$.

**Exercise 1.5.2.** If $\{N_i | i \in I\}$ is a family of normal subgroups of a group $G$, then $\bigcap\limits_{i\in I} N_i$ is a normal subgroup of $G$.

**Answer.** $\bigcap\limits_{i\in I} N_i$ is a subgroup of $G$. $N_i (i \in I)$ are normal subgroups of $G$, so $\forall a \in G$, $aN_i a^{-1} = \{an_i a^{-1} | n_i \in N_i\} = N_i$. $\forall x = ana^{-1} \in a(\bigcap\limits_{i\in I} N_i)a^{-1}$, $n \in N_i \Rightarrow x \in a(\bigcap\limits_{i\in I} N_i)a^{-1} \subset \bigcap\limits_{i\in I} aN_i a^{-1} = \bigcap\limits_{i\in I} N_i$. $\bigcap\limits_{i\in I} N_i$ are normal subgroup of $G$.

**Exercise 1.5.3.** Let $N$ be a subgroup of a group $G$. $N$ is normal in $G$ if and only if (right) congruence modulo $N$ is a congruence relation on $G$.

**Answer.** If $N \lhd G$. $\forall a, b \in G$, $ab^{-1} \in N \Leftrightarrow a^{-1}b \in N$. If $a_1 \equiv b_1 \mod N$, $a_2 \equiv b_2 \mod N$, then $a_2 b_2^{-1} \in N$, $a_1 N = Na_1 = Nb_1 \Rightarrow a_1 N b_1^{-1} = N$. So $a_1 a_2 b_1^{-1} b_2^{-1} = (a_1 a_2)(b_1 b_2)^{-1} \in N$. Similarly, $(a_1 a_2)^{-1}(b_1 b_2) \in N$. Congruence modulo $N$ is a congruence relation.
If congruence modulo $N$ is a congruence relation. $\forall a_1 \equiv b_1 \mod N$, $a_2 \equiv b_2 \mod N$, we will have $a_1 a_2 \equiv b_1 b_2 \mod N$. Take $n \in N$ and fix $a_2 \in G$, define $b_2 = n^{-1} a_2$. Then $\forall n \in N$, $n$ can be expressed as $a_2 b_2^{-1}$, $a_2 \equiv b_2 \mod N$. $\forall a_1 \in G$ and $\forall b_1 \equiv a_1 \mod N$, $a_1 n b_1^{-1} = a_1 a_2 b_2^{-1} b_1^{-1} \in N$. Take $b_1 = a_1$ and $n$ varies in $N$, $a_1 n a_1^{-1} \in N \Rightarrow a_1 N a_1^{-1} \subset N$. Thus $N \lhd G$.

**Exercise 1.5.4.** Let $\sim$ be an equivalence relation on a group $G$ and let $N = \{a \in G | a \sim e\}$. Then $\sim$ is a congruence relation on $G$ if and only if $N$ is a normal subgroup of $G$ and $\sim$ is congruence modulo $N$.

**Answer.** If $G \lhd N$ and $\sim$ is congruence modulo $N$. $\forall a \in G$, $aNa^{-1} \subset N$. $\forall a_1, b_1, a_2, b_2 \in G$, $a_1 b_1^{-1} \in N$, $a_2 b_2^{-1} \in N$. $a_1 a_2 (b_1 b_2)^{-1} = a_1 a_2 b_2^{-1} b_1^{-1}$, denote $n = a_2 b_2^{-1} \in N$, $a_1 a_2 b_2^{-1} b_1^{-1} = a_1 n b_1^{-1} \in a_1 N b_1^{-1}$. $\forall n \in N$, there exists $n' = b_1^{-1} a_1, n' \in N$ s.t. $a_1 n = b_1 n'$. So $a_1 n b_1^{-1} = b_1 n' b_1^{-1} \in b_1 N b_1^{-1} \subset N$. That means $(a_1 a_2)(b_1 b_2)^{-1} \in N$, $a \sim b$ is a congruence relation.
If $a \sim b$ is a congruence relation. We first prove $N$ is a subgroup of $G$. $\forall a \in N$, $a \sim e$, $a^{-1} \sim a^{-1} \Rightarrow e \sim a^{-1}$, so $a^{-1} \sim e$, $a^{-1} \in N$. $\forall a, b \in N$, $b^{-1} \sim e$, $a \sim e \Rightarrow ab^{-1} \in e$, thus $N < G$.
$\forall x \in G$, $xN = \{xa | a \sim e\} = \{xa | xa \sim xe\} = \{ax | ax \sim e\} = Nx$, so $N$ is normal in $G$. $x \sim y \Leftrightarrow y \in xN$. $\sim$ is congruence modulo $N$.

**Exercise 1.5.5.** Let $N < S_4$ consist of all those permutations $\sigma$ such that $\sigma(4) = 4$. Is $N$ normal in $S_4$?

**Answer.** $N = \{(1), (12), (13), (23), (123), (132)\}$. Take $a = (14) \in G$, $a^{-1} = (14)$, $a^{-1}(12)a = (24) \notin N$. So $N$ is not normal in $S_4$.

**Exercise 1.5.6.** Let $H < G$; then the set $aHa^{-1}$ is a subgroup for each $a \in G$, and $H \cong aHa^{-1}$.

**Answer.** $H < G$, $aHa^{-1} = \{aha^{-1} | h \in H\}$. $\forall x, y \in aHa^{-1}$, $x = ah_1 a^{-1}$, $y = ah_2 a^{-1}$. $y^{-1} = ah_2^{-1} a^{-1}$, $xy = ah_1 h_2^{-1} a^{-1} \in aHa^{-1}$, so $aHa^{-1} < G$.
Take $f : H \to aHa^{-1}$ as $f(h) = aha^{-1}$. If $f(h_1) = f(h_2) = ah_1 a^{-1} = ah_2 a^{-1}$, then $h_1 = h_2$, so $f$ is an injection. $f$ is a surjection because $\forall x \in aHa^{-1}$, $f(a^{-1} x a) = x$, $a^{-1} x a \in H$. In conclusion, $H \cong aHa^{-1}$.

**Exercise 1.5.7.** Let $G$ be a finite group and $H$ a subgroup of $G$ of order $n$. If $H$ is the only subgroup of $G$ of order $n$, then $H$ is normal in $G$.

**Answer.** Applying **Exercise 1.5.6**, $\forall a \in G$, $aHa^{-1} \cong H$. $\left|aHa^{-1}\right| = |H| = n \Rightarrow aHa^{-1} = H$. Whence $H \lhd G$.

**Exercise 1.5.8.** All subgroups of the quaternion group are normal.

**Answer.** $Q_8 = \{a, b, a^2, ba, ab, a^2b, ab^2, a^2b^2\}$ where $a^2 = b^2$, $a_1b = ba = a^3b$ and $|a| = |b| = 4$. There are several subgroups $\{a, a^2, ab^2, a^2b^2\}$, $\{b, a^2, a^2b, a^2b^2\}$, $\{ab, a^2b^2\}$, $\{ba, a^2b^2\}$, $\{a^2, a^2b^2\}$. From **Exercise 1.5.1**, we know the first two subgroups are normal in $G$. For $\{ab, a^2b^2\}$, $\{ba, a^2b^2\}$, $\{a^2, a^2b^2\}$, we can check that $ab, ba, a^2$ is commutative in $G$, that is $\forall x \in G$, $xabx^{-1} = ab$, $xbax^{-1} = ba$, $xa^2x^{-1} = a^2$. They are all normal in $G$.

**Exercise 1.5.9.** (a) If $G$ is a group, then the center of $G$ is a normal subgroup of $G$;

(b) the center of $S_n$ is the identity subgroup for all $n > 2$.

**Answer.** (a) By the definition of center $C$, $\forall x \in G$ and $a \in C$, $ax = xa$, so $xCx^{-1} = C$. $C$ is normal in $G$.

(b) $\forall x \in S_n$, $x$ can be expressed as

$$x = (a_1a_2 \cdots a_{i_1})(a_{i_1+1}a_{i_1+2} \cdots a_{i_2}) \cdots (a_{i_{n-1}+1}a_{i_{n-1}+2} \cdots a_{i_n})$$

Those cycles $(a_1a_2 \cdots a_{i_1})$, $(a_{i_1+1}a_{i_1+2} \cdots a_{i_2})$, ..., $(a_{i_{n-1}+1}a_{i_{n-1}+2} \cdots a_{i_n})$ are all disjoint, so they are commutative.

If there exists cycles whose length is longer than 2. WLOG, assume $i_1 > 2$. Take $y = (a_1a_2)$,

$$y^{-1}xy = (a_1a_2)(a_1a_2 \cdots a_{i_1})(a_1a_2) \cdots (a_{i_{n-1}+1}a_{i_{n-1}+2} \cdots a_{i_n})$$

$(a_1a_2)(a_1a_2 \cdots a_{i_i})(a_1a_2) = (a_2a_1a_3 \cdots a_{i_1})$, so $y^{-1}xy \neq x$, $x \notin C$.

If $x = (a_1a_2)(a_3a_4) \cdots (a_{2n-1}a_{2n})$ and $n \geq 2$. Take $y = (a_1a_3)$,

$$\begin{aligned}
y^{-1}xy &= (a_1a_3)(a_1a_2)(a_3a_4) \cdots (a_{2n-1}a_{2n})(a_1a_3) \\
&= (a_1a_3)(a_1a_2)(a_3a_4)(a_1a_3) \cdots (a_{2n-1}a_{2n}) \\
&= (a_1a_4)(a_2a_3) \cdots (a_{2n-1}a_{2n}) \\
&\neq x
\end{aligned}$$

So $x \notin C$.

If $x = (a_1a_2)$. Take $y = (a_1a_3)$, $y^{-1}xy = (a_2a_3) \neq x$, so $x \notin C$.

In conclusion, $C = \{(1)\}$.

**Exercise 1.5.10.** Find subgroups $H$ and $K$ of $D_4^*$ such that $H \lhd K$ and $K \lhd D_4^*$, but $H$ is not normal in $D_4^*$.

**Answer.** $D_4^* = \{I, R, R^2, R^3, T_x, T_y, T_{13}, T_{24}\}$. Take $K = \{I, R, T_x, T_y\}$, $H = \{I, T_x\}$. We can easily verify that $H \lhd K$ and $K \lhd D_4^*$ but $K \ntriangleleft D_4^*$.

**Exercise 1.5.11.** If $H$ is a cyclic subgroup of a group $G$ and $H$ is normal in $G$, then every subgroup of $H$ is normal in $G$.

**Answer.** Assume $K < H \lhd G$, $H$ has the generator $a$, and $K$ has the generator $a^n$. Here we used: *Every subgroup of a cyclic group is cyclic.* This can be easily proved by the conclusion $H \cong Z_m$ for some $m \in \mathbf{Z}$. $\forall x \in G$, $h = a^s \in H$, $x^{-1}a^s x = a^t \in H$. Assume $x^{-1}ax = a^m$, then $x^{-1}a^n x = (x^{-1}ax)^n = a^{mn} = a^k$, so $n|k$, $a^k \in K$. $x^{-1}Kx \subset K$, $K$ is normal in $G$.

**Exercise 1.5.12.** If $H$ is a normal subgroup of a group $G$ such that $H$ and $G/H$ are finitely generated, then so is $G$.

**Answer.** Assume $A = \{a_1, a_2, \ldots, a_m\}$, $B = \{b_1, b_2, \ldots, b_n\}$. $H = \langle A \rangle$, $G/H = \langle \{Hb_i | b_i \in B\} \rangle$. We prove that $G$ can be generated by $A \cup B$. $\forall x \in G$, $x$ is in one of the right cosets of $H$, $x \in Ha$. $Ha \in G/H$ so $Ha = \prod_{b_i \in B} Hb_i^{s_i} = H(\prod_{b_i \in B} b_i^{s_i})$. Thus $a^{-1}(\prod_{b_i \in B} b_i^{s_i}) = a' \in H$. $H$ is generated by $A$ so $xa^{-1} = \prod_{a_i \in A} a_i^{t_i}$, $a' = \prod_{a_i \in A} a_i^{-r_i}$. Then

$$x = (\prod_{a_i \in A} a_i^{t_i + r_i})(\prod_{b_i \in B} b_i^{s_i}) \in \langle A \cup B \rangle$$

Thus $G \subset \langle A \cup B \rangle$ is finitely generated.

**Exercise 1.5.13.** (a) Let $H \lhd G$, $K \lhd G$. Show that $H \vee K$ is normal in $G$.

(b) Prove that the set of all normal subgroups of $G$ forms a complete lattice under inclusion.

**Answer.** (a) $\forall x \in G$, $a \in H \vee K$, we need to prove $x^{-1}ax \in H \vee K$.
$a \in H \vee K$ so $a$ can be expressed as

$$a = b_1^{n_1} b_2^{n_2} \cdots b_t^{n_t} \quad \text{where } b_i \in H \text{ or } b_i \in K, i = 1, 2, \ldots, t$$

so $x^{-1}ax = x^{-1}b_1^{n_1} \cdots b_t^{n_t} x = (x^{-1}b_1x)^{n_1}(x^{-1}b_2x)^{n_2} \cdots (x^{-1}b_tx)^{n_t}$.
$H \lhd G$, $K \lhd G$, so $x^{-1}b_ix \in H \vee K, i = 1, 2, \ldots, t$ and

$$x^{-1}ax = (x^{-1}b_1x)^{n_1}(x^{-1}b_2x)^{n_2} \cdots (x^{-1}b_tx)^{n_t} \in H \vee K$$

$H \vee K \lhd G$.

(b) Actually, in **Exercise 1.2.19** and (a), we have proved lub exists.
Now we only consider glb. For $H \lhd G$, $K \lhd G$. If $H \cap K \lhd G$, then their glb is $H \cap K$. If not, assume there exists $A < H \cap K$, $B < H \cap K$, $A$, $B$ are both normal in $H$ and $K$. And there doesn't exists $I$ s.t. $A \lhd I \lhd H$, $A \lhd I \lhd K$, $B \lhd I \lhd H$, $B \lhd I \lhd K$. Just like the figure:

$$\begin{tikzcd} H \arrow[d, dash] \arrow[drr, dash] & & K \arrow[d, dash] \arrow[dll, dash] \\ A & & B \end{tikzcd}$$

But $A < H \cap K$, $B < H \cap K \Rightarrow A \vee B < H \cap K$. So $A \vee B \lhd H$, $A \vee B \lhd K$. That's contradictory! There is only one lower bound for $\{H, K\}$. Notice that $\{e\} < H \cap K$ so there exists at least one subgroup satisfies the condition. We have proved normality forms a lattice.

**Exercise 1.5.14.** If $N_1 \lhd G_1$, $N_2 \lhd G_2$ then $(N_1 \times N_2) \lhd (G_1 \times G_2)$ and $(G_1 \times G_2)/(N_1 \times N_2) \cong (G_1/N_1) \times (G_2/N_2)$.

**Answer.** Take $a \in (N_1 \times N_2)$, $a = (n_1, n_2)$ where $n_1 \in N_1$, $n_2 \in N_2$. $\forall x \in (G_1 \times G_2)$, $x = (g_1, g_2)$ where $g_1 \in G_1$, $g_2 \in G_2$. $x^{-1} = (g_1^{-1}, g_2^{-1})$, $x^{-1}ax = (g_1^{-1}n_1g_1, g_2^{-1}n_2g_2)$. $N_1 \lhd G_1$, $N_2 \lhd G_2$, so $g_1^{-1}n_1g_1 \in N_1$, $g_2^{-1}n_2g_2 \in N_2$. $x^{-1}ax \in (N_1 \times N_2)$. Thus $(N_1 \times N_2) \lhd (G_1 \times G_2)$.
Assume $G_1 = \bigcup\limits_{i \in I} N_1a_i$, $G_2 = \bigcup\limits_{j \in J} N_2b_j$. Then $G_1 \times G_2 = \bigcup\limits_{i \in I} N_1a_i \times \bigcup\limits_{j \in J} N_2b_j$.
Denote $A = \{a_i | i \in I\}$, $B = \{b_j | j \in J\}$. We construct two bijections $(G_1 \times G_2)/(N_1 \times N_2) \to A \times B$ and $(G_1/N_1) \times (G_2/N_2)$.

$$f : N_1a_i \times N_2b_j \mapsto (a_i, b_j)$$

$$g : (N_1 a_i, N_2 b_j) \mapsto (a_i, b_j)$$

Take $h = g^{-1} \circ f$, $f, g$ are bijections, so $h$ is an isomorphism. $(G_1 \times G_2)/(N_1 \times N_2) \cong (G_1/N_1) \times (G_2/N_2)$.

**Exercise 1.5.15.** Let $N \lhd G$ and $K \lhd G$. If $N \cap K = \langle e \rangle$ and $N \vee K = G$, then $G/N \cong K$.

**Answer.** Assume $G = \bigcup\limits_{i \in I} Na_i$, we construct $f : k \to G/N$. We prove that $\forall x, y \in K$, $x, y$ belong to different cosets of $N$. Suppose not. $\exists x, y \in K$, $x, y \in Na_i$, then $xy^{-1} \in N \Rightarrow x = y$. That's contradictory! So $f$ is a monomorphism.
$G = H \vee K$, so $G = HK$. we can write $x$ as $pq$, where $p \in H$, $q \in K$. $|G/H| = [G : H] = [HK : H] = [K : K \cap H] = |K|$. $f$ is a epimorphism.
Thus, $G/N \cong K$.

**Exercise 1.5.16.** If $f : G \to H$ is a homomorphism, $H$ is abelian and $N$ is a subgroup of $G$ containing $\mathrm{Ker} f$, then $N$ is normal in $G$.

**Answer.** Assume there exists $x \in G$, $x \notin N$ s.t. $f(x) \in f(N)$. $\exists n \in N$, $f(x) = f(n)$, $f(xn^{-1}) = f(x)f(n)^{-1} = e' \Rightarrow xn^{-1} \in \mathrm{Ker} f \Rightarrow x \in N$. That's contradictory! $\forall x \in G$, $n \in N$, $f(x^{-1}nx) = f(x^{-1})f(n)f(x) = f(n) \in f(N)$, so $x^{-1}nx \in N$. Thus, $N \lhd G$.

**Exercise 1.5.17.** (a) Consider the subgroups $\langle 6 \rangle$ and $\langle 30 \rangle$ of $\mathbf{Z}$ and show that $\langle 6 \rangle / \langle 30 \rangle \cong Z_5$.

(b) For any $k, m > 0$, $\langle k \rangle / \langle km \rangle \cong Z_m$; in particular, $\mathbf{Z}/\langle m \rangle = \langle 1 \rangle / \langle m \rangle \cong Z_m$.

**Answer.** (a) $\langle 6 \rangle = \{6n | n \in \mathbf{Z}\}$, $\langle 30 \rangle = \{30n | n \in \mathbf{Z}\}$. So $\langle 6 \rangle / \langle 30 \rangle = \{\langle 30 \rangle, \langle 30 \rangle + 6, \langle 30 \rangle + 12, \langle 30 \rangle + 18, \langle 30 \rangle + 24\} \cong Z_5$

(b) $\langle km \rangle \lhd \langle k \rangle$, $\langle k \rangle = \bigcup\limits_{i \in I} (\langle km \rangle + a_i)$. For $x \in \langle k \rangle$, $x \equiv a_i \mod km$, then $x \in \langle km \rangle + a_i$. $f : \langle k \rangle / \langle km \rangle \to \{a_i | i \in I\}$ defined by $f(\langle km \rangle + a_i) = a_i$ is a bijection. We check that $g : \{a_i | i \in I\} \to Z_m$ is also a bijection. Define

$b_i \equiv \frac{a_i}{k} \mod m$, $g(a_i) = b_i$. If there exists $b_i = b_j$ for $i \neq j$, $a_i \equiv a_j \mod km$. That's contradictory! So $g$ is an injection. $g$ is obviously a surjection, so $g$ is a bijection. Take $h = g \circ f : \langle k \rangle / \langle km \rangle \to Z_m$ is a isomorphism, so $\langle k \rangle / \langle km \rangle \cong Z_m$.

**Exercise 1.5.18.** If $f : G \to H$ is a homomorphism with kernel $N$ and $K < G$, then prove that $f^{-1}(f(K)) = KN$. Hence $f^{-1}(f(K)) = K$ if and only if $N < K$.

**Answer.** Take $x \in f^{-1}(f(K))$, then there exists $k \in K$ s.t. $f(x) = f(k)$. $f(xk^{-1}) = f(x)f(k)^{-1} = e' \in f(K) \Rightarrow xk^{-1} \in \mathrm{Ker} f = N$. Thus, $x \in Nk \subset NK$, $f^{-1}(f(K)) \subset NK$.
$\forall x = nk \in NK$, where $n \in N$ and $k \in K$. $f(x) = f(n)f(k) = e'f(k) \in f(K)$, so $NK \subset f^{-1}(f(K))$.
Thus, $f^{-1}(f(K)) = NK$. Hence $f^{-1}(f(K)) = K$ if and only if $N < K$.

**Exercise 1.5.19.** If $N \lhd G$, $[G : H]$ finite, $H < G$, $|H|$ finite, and $[G : N]$ and $|H|$ are relatively prime, then $H < N$.

**Answer.** $N \lhd G \Rightarrow NH < G$. By the second isomorphism theorem, $NH/N \cong H/H \cap N \Rightarrow [NH : N] = [H : H \cap N]$. Assume $[G : N] = m$, $|H| = n$, $|G| = mnp$ where $(m, n) = 1$. Then $|N| = np$, $N < NH$, assume $|NH| = knp$, $NH < G \Rightarrow knp|mnp \Rightarrow k|m$. $[NH : N] = [H : H \cap N] = k \Rightarrow k|n$. So $k = 1$, $NH = N$ which means $H < N$.

**Exercise 1.5.20.** If $N \lhd G$, $|N|$ finite, $H < G$, $[G : N]$ finite, and $[G : H]$ and $|N|$ are relatively prime, then $N < H$.

**Answer.** $N \lhd G \Rightarrow NH < G$. By the second isomorphism theorem, $NH/N \cong H/H \cap N \Rightarrow [NH : N] = [H : H \cap N]$. Assume $[G : H] = m$, $|N| = n$, $|G| = mnp$ where $(m, n) = 1$. Then $|H| = np$, $H < NH$, assume $|NH| = knp$, $NH < G \Rightarrow knp|mnp \Rightarrow k|m$. $[NH : N] = [H : H \cap N] = kp \Rightarrow kp|np \Rightarrow k|n$. So $k = 1$, $NH = H$ which means $N < H$.

**Exercise 1.5.21.** If $H$ is a subgroup of $Z(p^\infty)$ and $H \neq Z(p^\infty)$, then $Z(p^\infty)/H \cong Z(p^\infty)$.

**Answer.** From **Exercise 1.3.7**(b), we know that $H$ has the form $\left\langle \frac{\bar{1}}{p^n} \right\rangle$. Take $x_i = \frac{\bar{1}}{p^{n+i}} + H$, $x_1 = \frac{\bar{1}}{p^{n+1}} + H$.

$$\sum_{m=1}^{p} x_1 = \frac{\bar{p}}{p^{n+1}} + pH = \frac{\bar{1}}{p^n} + H = H$$

$$\sum_{m=1}^{p} x_i = \frac{\bar{p}}{p^{n+i}} + pH = \frac{\bar{1}}{p^{n+i-1}} + H = x_{i-1}$$

Take $A = \{x_i | i \in \mathbf{Z}_+\}$, $\langle A \rangle \cong Z(p^\infty)$ by **Exercise 1.3.7**(e). $\forall x \in \langle A \rangle$, $x \in Z(p^\infty)/H$, so $\langle A \rangle \subset Z(p^\infty)/H$. Take $x \in Z(p^\infty)/H$, $x = y + H$ where $y = \sum\limits_{i=1}^{m} \frac{a_i}{p^{n+i}}$, $x = \sum\limits_{i=1}^{m} (\frac{a_i}{p^{n+i}} + H) \in \langle A \rangle$. Thus, $Z(p^\infty)/H \subset \langle A \rangle$, $\langle A \rangle = Z(p^\infty)/H \cong Z(p^\infty)$.

## 1.6 Symmetric, alternating, and dihedral groups

**Exercise 1.6.1.** Find four different subgroups of $S_4$ that are isomorphic to $S_3$ and nine isomorphic to $S_2$.

**Answer.** $S_4 = \{(1),(12),(13),(14),(23),(24),(34),(123),(124),(132),$
$(142),(134),(143),(234),(243),(12)(34),(13)(24),(14)(23),(1234),$
$(1243),(1324),(1342),(1423),(1432)\}$.
$A_1 = \{(1),(12),(13),(23),(123),(132)\}$;
$A_2 = \{(1),(12),(14),(24),(124),(142)\}$;
$A_3 = \{(1),(13),(14),(34),(134),(143)\}$;
$A_4 = \{(1),(23),(24),(34),(234),(243)\}$;
$A_1 \cong A_2 \cong A_3 \cong A_4$.
$B_1 = \{(1),(12)\}$; $B_2 = \{(1),(13)\}$; $B_3 = \{(1),(14)\}$; $B_4 = \{(1),(23)\}$; $B_5 = \{(1),(24)\}$; $B_6 = \{(1),(34)\}$; $B_7 = \{(1),(12)(34)\}$; $B_8 = \{(1),(13)(24)\}$; $B_9 = \{(14)(23)\}$;
$B_1 \cong B_2 \cong B_3 \cong B_4 \cong B_5 \cong B_6 \cong B_7 \cong B_8 \cong B_9$.

**Exercise 1.6.2.** (a) $S_n$ is generated by the $n-1$ transpositions (12), (13), (14), …, $(1n)$.
(b) $S_n$ is generated by the $n-1$ transpositions $(12),(23),(34),\ldots,(n-1\,n)$.

**Answer.** (a) $\forall x \in S_n$, $x$ can be written as a product of transpositions. Actually, for any transposition $(ij)$, we can obtain it by $(1i)(1j)(1i) = (ij)$. So $x \in \langle (12),(13),\ldots,(1n)\rangle$, $S_n \subset \langle (12),(13),\ldots,(1n)\rangle$.
(b) We can contruct $(1i)$ inductively since $(1i) = (1\,i-1)(i-1\,i)(1\,i-1)$. From (a), we have $\forall x \in S_n$, $x \in \langle (12),(13),\ldots,(1n)\rangle$. Thus $S_n \subset \langle (12),(13),\ldots,(1n)\rangle \subset \langle (12),(23),(34),\ldots,(n-1\,n)\rangle$.

**Exercise 1.6.3.** If $\sigma = (i_1 i_2 \cdots i_r) \in S_n$ and $\tau \in S_n$, then $\tau\sigma\tau^{-1}$ is the $r$-cycle $(\tau(i_1)\tau(i_2)\cdots\tau(i_r))$.

**Answer.** $\sigma(i_n) = i_{n+1}$ for $n = 1,2,\ldots,r-1$, $\sigma(i_r) = i_1$. Assume $\tau(i_n) = j_n$, $n = 1,2,\ldots,r-1$ and $I = \{i_n | n = 1,2,\ldots,r-1\}$, $J = \{j_n | n = 1,2,\ldots,r-1\}$. For $x \notin J$, $\tau\sigma\tau^{-1}(x) = \tau\tau^{-1}(x) = x$. For $x = j_k \in J$, $\tau^{-1}(x) = i_k$, $\sigma(\tau^{-1}(x)) = i_{k+1}$, $\tau(\sigma(\tau^{-1}(x))) = j_{k+1}$ and $\tau\sigma\tau^{-1}(j_r) = j_1$. Thus $\tau\sigma\tau^{-1} = (\tau(i_1)\tau(i_2)\cdots\tau(i_r))$.

**Exercise 1.6.4.** (a) $S_n$ is generated by $\sigma_1 = (12)$ and $\tau = (123\cdots n)$.
(b) $S_n$ is generated by (12) and $(23\cdots n)$.

**Answer.** (a) Denote $\sigma_i = \tau\sigma_{i-1}\tau^{-1}$. Applying **Exercise 1.6.3**, $\sigma_i = (i\,i+1)$. By **Exercise 1.6.2**(b), $S_n \subset \langle (12), (23), (34), \ldots, (n-1\,n)\rangle = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1}\rangle \subset \langle \tau, \sigma_1\rangle$. $S_n$ can be generated by $\tau$ and $\sigma_1$.
(b) Denote $\sigma_1 = (12)$, $\tau = (23\cdots n)$, $\sigma_i = \tau\sigma_{i-1}\tau^{-1}$. Applying **Exercise 1.6.3**,$\sigma_i = (1\,i{+}1)$. By **Exercise 1.6.2**(a), $S_n \subset \langle (12), (13), \ldots, (1n)\rangle = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1}\rangle \subset \langle \tau, \sigma_1\rangle$. $S_n$ can be generated by $\tau$ and $\sigma_1$.

**Exercise 1.6.5.** Let $\sigma, \tau \in S_n$. If $\sigma$ is even (odd), then so is $\tau\sigma\tau^{-1}$.

**Answer.** Assume $\sigma = (x_1x_2)\cdots(x_{2n-1}x_{2n})$, $\tau = (y_1y_2)\cdots(y_{2m-1}y_{2m})$. Then $\tau^{-1} = (y_{2m-1}y_{2m})\cdots(y_1y_2)$. $\sigma$ is odd (even) if an only if $n$ is odd (even). $\tau\sigma\tau^{-1}$ has $2m+n$ transpositions. We can add $(ij) = (ji) = (1)$ into some segments of $\tau\sigma\tau^{-1}$ without changing it. So $\tau\sigma\tau^{-1}$ is odd (even) if and only if $2m+n$ is odd (even). $2m+n \equiv n \mod 2$ so $\tau\sigma\tau^{-1}$ is odd (even) if and only if $\sigma$ is odd (even).

**Exercise 1.6.6.** $A_n$ is the only subgroup of $S_n$ of index 2.

**Answer.** For any subgroup $N < S_n$ and $[S_n : N] = 2$, we have $N \lhd S_n$.
Assume there exists $k$-circle $\sigma = (i_1i_2\cdots i_k) \in N$. Then for any other $k$-circle $(j_1j_2\cdots j_k)$, take $\tau = (i_ij_1)(i_2j_2)\cdots(i_kj_k)$, by **Exercise 1.6.3**, $\tau\sigma\tau^{-1} = (j_1j_2\cdots j_k) \in N$. Thus $N$ contains all the $k$-circles.
For $n \geq 5$. If there exists 3-circle in $N$, then all the 3-circles are contained in $N$, $A_n \subset N \subset S_n \Rightarrow A_n = N$.
If there exists 2-circle in $N$, then all the 2-circles are contained in $N$. Notice $(1i)(1j) = (1ij) \in N$ is a 3-circle, so $A_n = N$.
If there only contain $x$ in the form of $(a_ia_2\cdots a_{n_1})(b_1b_2\cdots b_{n_2})\cdots$ where $n_i \geq 4$ and every two circles are disjoint. Take $\tau_i : \{a_i | i = 1, 2, \ldots, n_1\} \to \{a_i | i = 1, 2, \ldots, n_1\}$. We can obtain product of two $n_1$-circles

$$x^{-1}\tau x\tau^{-1} = (a_1a_2\cdots a_{n_1})(\tau(a_1)\tau(a_2)\cdots\tau(a_n)) \in N$$

By the arbitrariness of $\tau$, take

$$(\tau(a_1)\tau(a_2)\cdots\tau(a_n)) = (a_1a_4a_5\cdots a_na_3a_2)$$

then $x^{-1}\tau x\tau^{-1} = (a_1a_3)(a_2a_4)$ is a product of 2-circles. We can take $a_1, a_2, a_3, a_4$ arbitrarily. WLOG, take $(12)(34) \in N$ and $(12)(35) \in N$, $(12)(35)(12)(34) = (345) \in N$. Then there exists 3-circle in $N$, $N = A_n$.
In conlusion, when $n \geq 5$, $S_n$ has only one normal subgroup $A_n$.
For $n = 2, 3, 4$, we can verify it by enumeration.

**Exercise 1.6.7.** Show that $N = \{(1), (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of $S_4$ contained in $A_4$ such that $S_4/N \cong S_3$ and $A_4/N \cong Z_3$.

**Answer.** Assume $\sigma = (i_1i_2)(i_3i_4) \in N$, $\forall\tau \in S_4$, $\tau(i_n) = j_n$, $J = \{j_n|n = 1, 2, 3, 4\}$. For $x \notin J$, $\tau\sigma\tau^{-1}(x) = \tau\tau^{-1}(x) = x$. For $x = j_k \in J$, $\tau^{-1}(x) = i_k$, $\sigma\tau^{-1}(x) = i_{3k-4[\frac{k}{2}]-1}$, $\tau\sigma\tau^{-1}(x) = (\tau(i_i)\tau(i_2))(\tau(i_3)\tau(i_4)) \in N$. So $N \lhd S_4$. $S_4/N = \{N, N(12), N(13), N(23), N(123), N(132)\} \cong S_3$. $A_4/N = \{N, N(123), N(132)\} \cong Z_3$.

**Exercise 1.6.8.** The group $A_4$ has no subgroup of order 6.

**Answer.** $|A_4| = 12$, assume there exists $N < A_4$, $|N| = 6$. Then $N \lhd A_4$. From **Exercise 1.6.6**, we know that all 3-circles are contained in $N$. But there're 8 3-circles in total, so $N$ can't exist.

**Exercise 1.6.9.** For $n \geq 3$ let $G_n$ be the multiplicaive group of complex matrices generated by $x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{-2\pi i/n} \end{pmatrix}$, where $i^2 = -1$. Show that $G_n \cong D_n$.

**Answer.** Take a mapping $f : G_n \to D_n$ as $f(x) = (2\,n)(3\,n-1)\cdots$, $f(y) = (123\cdots n)$. $|f(x)| = |x| = 2$, $|f(y)| = |y| = n$. $f$ is obviously a monomorphism. $\forall a \in D_n$, $a = f(x)^n f(y)^m, m = 1, 2$, then $a = f(x^ny^m)$, $f$ is a epimorphism. Thus $G_n \cong D_n$.

**Exercise 1.6.10.** Let $a$ be the generator of order $n$ of $D_n$. Show that $\langle a \rangle \lhd D_n$ and $D_n/\langle a \rangle \cong Z_2$.

**Answer.** $|\langle a \rangle| = n$, $b$ is the other generator of $D_n$, $a^n = b^2 = (1)$. $\forall k \in \mathbf{Z}$, $a^k b = ba^{-k}$ can be easily proved by induction. So $\forall x = a^m b^n \in D_n$, $x = a^{m'} b^{n'}$, here $m' \equiv m \mod 2$, $n' \equiv n \mod 2$. $D_n = \{e, a, a^2, \ldots, a^{n-1}, b, ba, \ldots, ba^{n-1}\}$. $|D_n| = 2n$. Thus, $\langle a \rangle \lhd D_n$. $D_n/\langle a \rangle = \{\langle a \rangle, \langle a \rangle b\} \cong Z_2$.

**Exercise 1.6.11.** Find all normal subgroups of $D_n$.

**Answer.** The subgroups of $\langle a \rangle$ is always normal in $D_n$. $\langle a^m \rangle < \langle a \rangle$. $\forall x \in D_n$ and $a^{km} \in \langle a^m \rangle$, $x = a^t$ or $x = ba^t$.

$$x^{-1} a^{km} x = a^{-t} a^{km} a^t = a^{km} \in \langle a^m \rangle$$

or

$$x^{-1} a^{km} x = a^{-t} b^{-1} a^{km} ba^t = a^{-t} ba^{km} ba^t = a^{-t} a^{-km} b^2 a^t = a^{-km} \in \langle a^m \rangle$$

so $\langle a^m \rangle \lhd D_n$.
Consider the subgroup $S$ which only contains $ba^i, i = 1, \ldots, n$. Since $ba^i \cdot ba^j = a^{j-i} \in S \, (i \neq j)$, so $S = \{e, ba^k\}$.
If $n$ is odd, take $x = a^{\frac{n-1}{2}} \in D_n$.

$$x^{-1} ba^k x = a^{\frac{1-n}{2}} ba^k a^{\frac{n-1}{2}} = ba^{k+n-1} \notin S$$

so $S \ntriangleleft D_n$ for all $k = 1, 2, \ldots, n$.
If $n$ is even, take $x = a^{\frac{n-2}{2}} \in D_n$, $n \geq 6$.

$$x^{-1} ba^k x = a^{\frac{2-n}{2}} ba^k a^{\frac{n-2}{2}} = ba^{k+n-2} \notin S$$

so $S \ntriangleleft D_n$ for all $k = 1, 2, \ldots, n$.
If $n = 2$, all the subgroups are normal since $|D_2| = 4$.
For subgroup $S$ contains both $ba^i$ and $a^j$. It can be written as $S = \left\langle a^d, ba^r \right\rangle$, where $d|n$, $0 \leq r \leq d{-}1$. If $\exists a^m, a^n \in S$, $(m, n) = d$, then there exist $x, y \in \mathbf{Z}$ s.t. $a^{mx+ny} = a^d \in \mathbf{Z}$. Thus, $S = \left\langle a^d, ba^r \right\rangle$.
Take $x = a^{\frac{n-w}{2}}$, then $x^{-1} ba^r x = ba^{r+n-w}$.

If $d \geq 3$, take $w \equiv n \mod 2$, $x^{-1}ba^rx \notin S$.
If $d = 2$, then $n = 2s$ and $S = \{e, a^s, ba^s, b\}$. $Sa^k = \{a^k, a^{s+k}, ba^{s-k}, ba^{-k}\}$, $k = 1, 2, \ldots, s-1$. $ba^k = ba^{-k}$ or $ba^k = ba^{s-k} \Rightarrow k = \frac{s}{2}$. So for $s = 2$, $n = 4$, $S$ is a normal subgroup of $D_4$.

**Exercise 1.6.12.** The center of the group $D_n$ is $\langle e\rangle$ if $n$ is odd and isomorphic to $Z_2$ if $n$ is even.

**Answer.** If $n$ is odd, $C$ is the center of $D_n$, $C \lhd D_n \Rightarrow C < \langle a\rangle$. Take $a^d \in C$, $x = ba^m$,

$$x^{-1}ax = a^{-m}b^{-1}a^dba^m = a^{-m}ba^dba^m = a^{-d} = a^d$$

so $d = 0$, $C = \{e\}$.
If $n$ is even, $n \geq 6$. $C$ is the center of $D_n$. $C \lhd D_n \Rightarrow C < \langle a\rangle$ or $C = \{e, ba^k\}$.
If $C = \{e, ba^k\}$, $C \cong Z_2$.
If $C < \langle a\rangle$,take $a^d \in C$, $x = ba^m$,

$$x^{-1}ax = a^{-m}b^{-1}a^dba^m = a^{-m}ba^dba^m = a^{-d} = a^d$$

so $d = \frac{n}{2}$ or $d = 0$, $C = \{a^{\frac{n}{2}}, e\} \cong Z_2$.

**Exercise 1.6.13.** For each $n \geq 3$ let $P_n$ be a regular polygon of $n$ sides (for $n = 3$, $P_n$ is an equilateral triangle; for $n = 4$, a square). A *symmetry* of $P_n$ is a bijection $P_n \to P_n$ that preserves distances and maps adjacent vertices on to adjacent vertices.

(a) The set $D_n^*$ of all symmetries of $P_n$ is a group under the binary operation of composition of functions.
(b) Every $f \in D_n^*$ is completely determined by its actions on the vertices of $P_n$. Number the vertices consecutively $1, 2, \ldots, n$; then each $f \in D_n^*$ determines a unique permutation $\sigma_f$ of $\{1, 2, \ldots, n\}$. The assignment $f \mapsto \sigma_f$ defines a monomorphism of groups $\varphi : D_n^* \to S_n$.
(c) $D_n^*$ is generated by $f$ and $g$, where $f$ is a rotation of $2\pi/n$ degrees about the center of $P_n$ and $g$ is a reflection about the "diameter" through the center and vertex 1.
(d) $\sigma_f = (123\cdots n)$ and $\sigma_g = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & n & n-1 & \cdots & 3 & 2 \end{pmatrix}$, whence $\mathrm{Im}\varphi = D_n$ and $D_n^* \cong D_n$.

**Answer.** In the following analysis, all the numbers are $\mod n$.

(a) Consider $n$ points $A_i = (\cos\frac{2\pi i}{n}, \sin\frac{2\pi i}{n})^t$, $i = 1, 2, \ldots, n$. $f$ is the transposition of $A_i \mapsto A_j$ with the consevation of $n$ regular polygon structure. So $f$ must be a bijection. $D_n^*$ is the set of $f$. By the definition, $D_n^* \subset S_n$. We prove $D_n^*$ is ta subgroup of $S_n$.

Notice that $A_{i+1} = \begin{pmatrix} \cos\frac{2\pi i}{n} & -\sin\frac{2\pi i}{n} \\ \sin\frac{2\pi i}{n} & \cos\frac{2\pi i}{n} \end{pmatrix} A_i$.

Denote $X = \begin{pmatrix} \cos\frac{2\pi i}{n} & -\sin\frac{2\pi i}{n} \\ \sin\frac{2\pi i}{n} & \cos\frac{2\pi i}{n} \end{pmatrix}$. To construct the polygon, we must have

$$f(A_{i+1}) = \begin{pmatrix} \cos\frac{2\pi i}{n} & -\sin\frac{2\pi i}{n} \\ \sin\frac{2\pi i}{n} & \cos\frac{2\pi i}{n} \end{pmatrix} f(A_i)$$

or

$$f(A_i) = \begin{pmatrix} \cos\frac{2\pi i}{n} & -\sin\frac{2\pi i}{n} \\ \sin\frac{2\pi i}{n} & \cos\frac{2\pi i}{n} \end{pmatrix} f(A_{i+1})$$

We need to verify that $\forall f_1, f_2 \in D_n^*$, $f_1 f_2^{-1} \in D_n^*$. Assume $B_i = f_2(A_i)$, $B_{i+1} = f_2(A_{i+1})$. Then $B_i = XB_{i+1}$ or $B_i = X^{-1}B_{i+1}$. Denote $B_i = A_j$, then $B_{i+1} = A_{j-1}$ or $B_{i+1} = A_{j+1}$. WLOG, assume $B_{i+1} = A_{j+1}$, then $f_1(A_j) = Xf_1(A_{j+1})$ or $f_1(A_j) = X^{-1}f_1(A_{j-1})$. So $f_1 f_2^{-1} \in D_n^*$. $D_n^*$ is a subgroup of $S_n$.

(b) Assume $A_i = f(A_1)$. If $f(A_2) = A_{i+1}$, since $f$ is a bijection, by induction, we can prove $f(A_k) = A_{k+i-1}$. $\varphi : D_n^* \to S_n$ can be defined as $\varphi : f \mapsto (1i\, 2i{-}1\, 3i{-}2 \cdots)$. If $f(A_2) = A_{i-1}$, similarly, we can also prove $f(A_k) = A_{i+1-k}$. $\varphi$ can be defined as $\varphi : f \mapsto (1i)(2\, i-1)(3\, i-2)\cdots$. This means $f$ is completely determined by $f(A_1)$ and $f(A_2)$. $D_n^*$ can be embedded into $S_n$.

(c) Denote $\alpha = \begin{pmatrix} \cos\frac{2\pi i}{n} & -\sin\frac{2\pi i}{n} \\ \sin\frac{2\pi i}{n} & \cos\frac{2\pi i}{n} \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. $f : A_i \mapsto \alpha A_i$, $g : A_i \mapsto \beta A_i$. $f$ is the rotation of $\frac{2\pi}{n}$ degrees counter-clockwisely. $g$ is the reflection about $x$-axis. Now we prove $\forall x \in D_n^*$, $x$ can be factorised into finite product of $f$ and $g$. From (b), $x$ is fully defined by $x(A_1)$ and $x(A_2)$. Assume $x(A_1) = A_i$.

If $x(A_2) = A_{i+1}$, $x(A_k) = A_{i-1+k} = \alpha^{i-1}A_k$, $k = 1, 2, \ldots, n$. So $x = f^{i-1}$.

If $x(A_2) = A_{i-2}$, $x(A_k) = A_{i+1-k} = \alpha^{i+1}A_{-k} = \alpha^{i+1}\beta A_k$. So $x = f^{i+1}g$. Thus $D_4^* \subset \langle f, g\rangle$.

(d) $\alpha^n = \beta^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We can easily verify that $|f| = n$ and $|g| = 2$. From **Exercise 1.6.9**, $\langle f, g\rangle \cong D_n$, $|\langle f, g\rangle| = |D_n| = 2n$. From (b), $x \in D_n^*$

if completely determined by $x(A_1)$ and $x(A_2)$. There are $2n$ different ways to obtain $x(A_1)$ and $x(A_2)$. So $|D_n^*| = |\langle f, g\rangle| = 2n$. $D_n^* \subset \langle f, g\rangle$, so $D_n^* = \langle f, g\rangle$. Thus, $D_n^* \cong \langle f, g\rangle \cong D_n$.

## 1.7 Categories: products, coproducts, and free objects

**Exercise 1.7.1.** A *pointed set* is a pair $(S, x)$ with $S$ a set and $x \in S$. A morphism of pointed sets $(S, x) \to (S', x')$ is a triple $(f, x, x')$, where $S \to S'$ is a function such that $f(x) = x'$. Show that pointed sets form a category.

**Answer.** Let $\mathcal{S}$ be the category and 4 objects of $\mathcal{S}$ are $(A, a)$, $(B, b)$, $(C, c)$, $(D, d)$. $f$, $g$ and $h$ are morphisms defined by $f : A \to B$, $g : B \to C$, $h : C \to D$ with $f(a) = b$, $g(b) = c$, $h(c) = d$.

$$(A, a) \xrightarrow{f} (B, b) \xrightarrow{g} (C, c) \xrightarrow{h} (D, d)$$

category $\mathcal{S}$

$$\hom(A, B) \times \hom(B, C) \to \hom(A, C)$$

because $g \circ f : A \to C$ with $g(f(a)) = g(b) = c = g \circ f(a)$. Similarly, $(h \circ g) \circ f = h \circ (g \circ f)$ with $(h \circ g) \circ f(a) = h \circ (g \circ f)(a) = d$. Take $1_B$ consist of those functions $i : B \to B$ with $i(b) = b$. Then $1_B \circ f = f$ and $g \circ 1_B = g$. So $\mathcal{S}$ is a category.

**Exercise 1.7.2.** If $f : A \to B$ is an equivalence in a category $\mathcal{C}$ and $g : B \to A$ is the morphism such that $g \circ f = 1_A$, $f \circ g = 1_B$, show that $g$ is unique.

**Answer.** Assume there exist $g$ and $g'$ satisfies the condition.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "g"] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "g'"] \end{tikzcd}$$

So $g' \circ (f \circ g) = g' \circ 1_B = g' = (g' \circ f) \circ g = 1_A \circ g = g$.

**Exercise 1.7.3.** In the category $\mathcal{G}$ of groups, show that the group $G_1 \times G_2$ together with the homomorphisms $\pi_1 : G_1 \times G_2 \to G_1$ and $\pi_2 : G_1 \times G_2 \to G_2$ is a product for $\{G_1, G_2\}$.

**Answer.** Take $\tau_1 : G_1 \to G_1 \times G_2$ as $\tau_1(g_1) = (g_1, e)$; $\tau_2 : G_2 \to G_1 \times G_2$ as $\tau_2(g_2) = (e, g_2)$; $\pi_1 : G_1 \times G_2 \to G_1$ as $\pi_1(g_1, g_2) = g_1$; $\pi_2 : G_1 \times G_2 \to G_2$ as $\pi_2(g_1, g_2) = g_2$. Then

$$\begin{tikzcd} G_1 \arrow[r, shift right, "\tau_1"'] & G_1 \times G_2 \arrow[l, shift right, "\pi_1"'] \arrow[r, shift left, "\pi_2"] & G_2 \arrow[l, shift left, "\tau_2"] \end{tikzcd}$$

For any object $B$ such that

$$\begin{tikzcd} G_1 & B \arrow[l, "\varphi_1"'] \arrow[r, "\varphi_2"] & G_2 \end{tikzcd}$$

For any $x \in B$, define $f : B \to G_1 \times G_2$ as $f(x) = (\varphi_1(x), \varphi_2(x))$. Then $\pi_1(f(x)) = \varphi_1(x)$, $\pi_1 \circ f = \varphi_1$, $\pi_2(f(x)) = \varphi_2(x)$, $\pi_2 \circ f = \varphi_2$. Thus

$$\begin{tikzcd} & B \arrow[dl, "\varphi_1"'] \arrow[d, "f"'] \arrow[dr, "\varphi_2"] & \\ G_1 \arrow[r, shift right, "\tau_1"'] & G_1 \times G_2 \arrow[l, shift right, "\pi_1"'] \arrow[r, shift left, "\pi_2"] & G_2 \arrow[l, shift left, "\tau_2"] \end{tikzcd}$$

Next we verify the uniqueness. If there exist $f$ and $f'$ satisfies the condition,

$$\pi_1(f(x)) = \pi_1(f'(x)) = \varphi_1(x)$$

$$\pi_2(f(x)) = \pi_2(f'(x)) = \varphi_2(x)$$

Thus $f(x) = f'(x)$ for all $x \in B$, so $f = f'$.

**Exercise 1.7.4.** In the category $\mathcal{A}$ of abelian groups, show that the group $A_1 \times A_2$ together with the morphisms $\tau_1 : A_1 \to A_1 \times A_2$ and $\tau_2 : A_2 \to A_1 \times A_2$ is a coproduct of $\{A_1, A_2\}$.

**Answer.** Take $\tau_1 : A_1 \to A_1 \times A_2$ as $\tau_1(a_1) = (a_1, e)$; $\tau_2 : A_2 \to A_1 \times A_2$ as $\tau_2(a_2) = (e, a_2)$; $\pi_1 : A_1 \times A_2 \to A_1$ as $\pi_1(a_1, a_2) = a_1$; $\pi_2 : A_1 \times A_2 \to A_2$ as $\pi_2(a_1, a_2) = a_2$. Then

$$\begin{tikzcd} A_1 \arrow[r, shift right, "\tau_1"'] & A_1 \times A_2 \arrow[l, shift right, "\pi_1"'] \arrow[r, shift left, "\pi_2"] & A_2 \arrow[l, shift left, "\tau_2"] \end{tikzcd}$$

For any object $B$ such that

$$A_1 \xrightarrow{\varphi_1} B \xleftarrow{\varphi_2} A_2$$

For any $(a_1, a_2) \in A_1 \times A_2$, define $f : A_1 \times A_2 \to B$ as $f(a_1, a_2) = \varphi_1(a_1)\varphi_2(a_2)$. Then $f(\tau_1(a_1)) = f(a_1, e) = \varphi_1(a_1)e = \varphi_1(a_1)$, $f \circ \tau_1 = \varphi_1$, $f(\tau_2(a_2)) = f(e, a_2) = e\varphi_2(a_2) = \varphi_2(a_2)$, $f \circ \tau_2 = \varphi_2$.

$$\begin{tikzcd}
 & B & \\
A_1 \arrow[ur, "\varphi_1"] \arrow[r, shift right, "\tau_1"'] & A_1 \times A_2 \arrow[u, "f"] \arrow[l, shift right, "\pi_1"'] \arrow[r, shift left, "\pi_2"] & A_2 \arrow[ul, "\varphi_2"'] \arrow[l, shift left, "\tau_2"]
\end{tikzcd}$$

Next we verify the uniqueness. If there exist $f$ and $f'$ satisfies the condition,

$$f(\tau_1(a_1)) = f'(\tau_1(a_1)) = f(a_1, e) = f'(a_1, e)$$

$$f(\tau_2(a_2)) = f'(\tau_2(a_2)) = f(e, a_2) = f'(e, a_2)$$

$$\begin{aligned} f(\tau_1(a_1), \tau_2(a_2)) &= f(\tau_1(a_1))f(\tau_2(a_2)) \\ =&f'(\tau_1(a_1), \tau_2(a_2)) = f'(\tau_1(a_1))f'(\tau_2(a_2)) \end{aligned}$$

so $f = f'$.

**Exercise 1.7.5.** Every family $\{A_i | i \in I\}$ in the category of sets has a coproduct.

**Answer.** We examine $\dot{\bigcup} A_i = \{(a, i) \in (\cup A_i) \times I | a \in A_i\}$ which satisfies the condition. Define the morphism $\pi_i : A_i \to \dot{\bigcup} A_i$ as $\pi_i(a) = (a, i)$. For any $B$ such that $\exists \varphi_i : A_i \to B$.

$$\begin{tikzcd}
 & & B & & \\
A_1 \arrow[urr, "\varphi_1"] & A_2 \arrow[ur, "\varphi_2"] & \cdots & \cdots & A_n \arrow[ull, "\varphi_n"']
\end{tikzcd}$$

$\varphi(a) = x \in B$. Take $\varphi(a,i) = \varphi_i(a)$ defined on the subset of $\cup A_i \times I$, we can verify that the domain of $\varphi$ is $\dot{\bigcup} A_i$. Then take $f = \varphi$, $f(\pi_i(a)) = \varphi_i(a)$, $f \circ \pi_i = \varphi_i$.
The uniqueness is obvious.

**Exercise 1.7.6.** (a) Show that in the category $\mathcal{S}_*$ of pointed sets product always exist; describe them.
(b) Show that in $\mathcal{S}_*$ every family of objects has a coproduct, describe the coproduct.

**Answer.** (a) Define $\otimes$ as an operator between points and other elements in the pointed set. $\forall a \in A_i$, $a \otimes a_i = a_1 \times a = a$. For a family of sets with their points $\{(A_i, a_i | i \in I)\}$, consider $(A_1, A_2, \cdots, A_n) = \{(a'_1, a'_2, \cdots, a'_n)\}$. Define morphisms $\pi_i(a) = (a_1, a_2, \cdots, a, \cdots, a_n)$, $\pi_i : A_i \to (A_1, A_2, \cdots, A_n)$.

$$\begin{tikzcd}
 & & (A_1, A_2, \cdots, A_n) & \\
A_1 \arrow[urr, "\pi_1"] & A_2 \arrow[ur, "\pi_2"] & \cdots & A_n \arrow[ul, "\pi_n"']
\end{tikzcd}$$

For any $B$ such that $\exists \varphi_i : A_i \to B$.

$$\begin{tikzcd}
 & & B & \\
A_1 \arrow[urr, "\varphi_1"] & A_2 \arrow[ur, "\varphi_2"] & \cdots & A_n \arrow[ul, "\varphi_n"']
\end{tikzcd}$$

Take $f : (A_1, A_2, \cdots, A_n) \to B$ as

$$f(a'_1, a'_2, \cdots, a'_n) = \varphi_1(a'_1) \otimes \varphi_2(a'_2) \otimes \cdots \otimes \varphi(a'_n)$$

Then $f \circ \pi_i(a) = f(a_1, a_2, \cdots, a, \cdots, a_n) = \varphi_1(a_1) \otimes \cdots \otimes \varphi_i(a) \otimes \cdots \otimes \varphi_n(a_n) = \varphi_i(a)$. So $f \circ \pi_i = \varphi_i$.
Next we verify the uniqueness. If there exist $f$ and $f'$ satisfies the condition. Then $\exists i \in I$ and $a \in A_i$ s.t. $f(a_1, a_2, \cdots, a, \cdots, a_n) \neq f'(a_1, a_2, \cdots, a, \cdots, a_n)$. But $f(\pi_i(a)) = f'(\pi_i(a))$, so $f = f'$.

(b) The proof is similar to **Exercise 1.7.5**.

**Exercise 1.7.7.** Let $F$ be a free object on a set $X(i : X \to F)$ in a concrete category $\mathcal{C}$. If $\mathcal{C}$ contains an object whose underlying set has at least two elements in it, then $i$ is an injective map of sets.

**Answer.** Assume $A \in \mathrm{obj}(\mathcal{C})$, $A$ has at least two elements and $X \xrightarrow{f} A$. $X \xrightarrow{i} F$ and $F$ is free on $X$, so there exists a morphism $\bar{f}$ s.t. $F \xrightarrow{\bar{f}} A$. If $|X| = 1$, $i$ must be injective. For $|X| \geq 2$. Suppose $i$ is not injective. Take $x_1, x_2 \in X$ and $i(x_1) = i(x_2) \in F$, $f(x_1) = a_1$, $f(x_2) = a_2$. $\bar{f}(i(x_1)) = \bar{f}(i(x_2)) = f(x_1) = f(x_2) = a_1 = a_2$. That means all the elements in $A$ are identical. That's contradictory to the assumption.

**Exercise 1.7.8.** Suppose $X$ is a set and $F$ is a free object on $X$ (with $i : X \to F$) in the category of groups. Prove that $i(X)$ is a set of generators for the group $F$.

**Answer.** Assume $G$ the subgroup of $F$ is the group generated by $i(X)$. Since $X \xrightarrow{i} G$ and $X \xrightarrow{i} F$, we can obtain unique morphism $\varphi$ such that $F \xrightarrow{\varphi} G$ and $\varphi \circ i = i$.
Consider morphism $1_F : F \to F$ which is the identical homomorphism. $F$ is free so $1_F$ is the unique homomorphism. Take $\subset: G \to F$ as a morphism defined as $\forall g \in G$, $\subset(g) = g$. Then

$$\begin{tikzcd} & G \arrow[dr, "\subset"] & \\ X \arrow[ur, "i"] \arrow[r, "i"] & F \arrow[u, "\varphi"] \arrow[r, "1_F"] & F \end{tikzcd}$$

$\subset \circ \varphi \circ i = 1_F \circ i = i$ so $\subset \circ \varphi = 1_F$. Thus $\subset$ is an epimorphism, $F \subset G$. So $F = G$ can be generated by $i(X)$.

## 1.8 Direct products and direct sums

**Exercise 1.8.1.** $S_3$ is not the direct product of any family of its proper subgroups. The same is true of $Z_{p^n}$($p$ prime, $n \geq 1$) and $\mathbb{Z}$.

**Answer.** We list all the subgroups of $S_3$: $\{(1),(12)\}$, $\{(1),(13)\}$, $\{(1),(23)\}$, $\{(1),(123),(132)\}$. Only $\{(1),(123),(132)\}$ is normal, so $S_3$ isn't an direct product of any family of its proper subgroups.
For $Z_{p^n}$, $Z_{p^i} \lhd Z_{p^n}$ for all $i = 1, 2, \dots, n-1$ but $Z_{p^i} \cap Z_{p^j} \neq \{e\}$. So $Z_{p^n}$ isn't an direct product of any family of its proper subgroups.
For $\mathbf{Z}$. $\forall N_1 \lhd \mathbf{Z}$, $N_2 \lhd \mathbf{Z}$, we have $N_1 = \langle a_1 \rangle$ and $N_2 = \langle a_2 \rangle$. Thus, $N_1 \cap N_2 = \langle a_1 a_2 \rangle \neq \{e\}$. So $\mathbf{Z}$ isn't an direct product of any family of its proper subgroups.

**Exercise 1.8.2.** Give an example of groups $H_i$, $K_i$ such that $H_1 \times H_2 \cong K_1 \times K_2$ and no $H_i$ is isomorphic to any $K_j$.

**Answer.** Take $H_1 \cong K_1 \times K_2$, $H_2 = \{e\}$. We verify that $H_1 \times H_2 \cong K_1 \times K_2$. There exists $f : H_1 \to K_1 \times H_2$ which is an isomorphism. There exists canonical projection $\pi_1 : H_1 \times H_2 \to H_1$ and $\pi_1$ is an epimorphism. Ker$\pi_1 = \{(e_1, e_2)\}$ thus $\pi_1$ is also a monomorphism. Therefore $\bar{f} = f \circ \pi_1$ is a well defined isomorphism. $H_1 \times H_2 \cong K_1 \times K_2$ but neither $H_1$ nor $H_2$ are isomorphic to any $K_i$, $i = 1, 2$.

**Exercise 1.8.3.** Let $G$ be and (additive) abelian group with subgroups $H$ and $K$. Show that $G \cong H \oplus K$ if and only if there are homomorphisms

$$H \underset{\tau_1}{\overset{\pi_1}{\leftrightarrows}} G \underset{\tau_2}{\overset{\pi_2}{\rightleftarrows}} K$$

such that $\pi_1\tau_1 = 1_H$, $\pi_2\tau_2 = 1_K$, $\pi_1\tau_2 = 0$ and $\pi_2\tau_1 = 0$, where 0 is the map sending every element onto the zero (identity) element, and $\tau_1\pi_1(x) + \tau_2\pi_2(x) = x$ for all $x \in G$.

**Answer.** If $G \cong H \oplus K$. Denote $f : G \to H \oplus K$ which is a isomorphism. Then there are canonical products $\pi_1'$, $\pi_2'$, $\tau_1'$, $\tau_2'$.

$$\begin{tikzcd}
H \arrow[r, shift right, "\tau_1'"'] & H \oplus K \arrow[l, shift right, "\pi_1'"'] \arrow[r, shift left, "\pi_2'"] & K \arrow[l, shift left, "\tau_2'"]
\end{tikzcd}$$

Thus

$$\begin{tikzcd}
 & G \arrow[dl, dashed, shift right, "\pi_1"'] \arrow[dr, dashed, shift left, "\pi_2"] \arrow[d, shift left, "f^{-1}"] & \\
H \arrow[ur, dashed, shift right, "\tau_1"'] \arrow[r, shift right, "\tau_1'"'] & H \oplus K \arrow[u, shift left, "f"] \arrow[l, shift right, "\pi_1'"'] \arrow[r, shift left, "\pi_2'"] & K \arrow[ul, dashed, shift left, "\tau_2"] \arrow[l, shift left, "\tau_2'"]
\end{tikzcd}$$

Take $\tau_1 = f \circ \tau_1'$, $\tau_2 = f \circ \tau_2'$, $\pi_1 = \pi_1' \circ f^{-1}$, $\pi_2 = \pi_2' \circ f^{-1}$.

$$\pi_1\tau_1 = \pi_1' f^{-1} f \tau_1' = \pi_1'\tau_1' = 1_H$$

$$\pi_2\tau_2 = \pi_2' f^{-1} f \tau_2' = \pi_2'\tau_2' = 1_K$$

$$\pi_1\tau_2 = \pi_1' f^{-1} f \tau_2' = \pi_1'\tau_2' = 0$$

$$\pi_2\tau_1 = \pi_2' f^{-1} f \tau_1' = \pi_2'\tau_1' = 0$$

$\forall x \in G$, $x = hk$ where $h \in H$ and $k \in K$.

$$\begin{aligned}
\tau_1\pi_1(x) + \tau_2\pi_2(x) &= f(\tau_1'\pi_1'(h,k)) + f(\tau_2'\pi_2(h,k)) \\
&= f(\tau_1'(h)) + f(\tau_2'(k)) \\
&= f(h,e) + f(e,k) \\
&= f(h+e, e+k) = f(h,k) \\
&= x
\end{aligned}$$

If there exist $\pi_1$, $\pi_2$, $\tau_1$, $\tau_2$ satisfies the condition. There are canonical projections $\pi_1'$, $\pi_2'$, $\tau_1'$, $\tau_2'$ between $H$ and $H \oplus K$, $K$ and $H \oplus K$.

$$\begin{tikzcd}
 & G \arrow[dl, shift right, "\pi_1"'] \arrow[dr, shift left, "\pi_2"] \arrow[d, dashed, shift left, "f^{-1}"] & \\
H \arrow[ur, shift right, "\tau_1"'] \arrow[r, shift right, "\tau_1'"'] & H \oplus K \arrow[u, dashed, shift left, "f"] \arrow[l, shift right, "\pi_1'"'] \arrow[r, shift left, "\pi_2'"] & K \arrow[ul, shift left, "\tau_2"] \arrow[l, shift left, "\tau_2'"]
\end{tikzcd}$$

For $f = \tau_1'\pi_1 + \tau_2'\pi_2$ which is a well defined homomorphism. $\forall h \in H$ and $k \in K$, $\tau_1'(h) + \tau_2'(k) = (h, k) \in H \oplus K$. Thus $f(x) = (e_1, e_2)$ if and only if $\pi_1(x) = e_1$ and $\pi_2(x) = e_2$. $\tau_1\pi_1(x) + \tau_2\pi_2(x) = \tau_1(e_1) + \tau_2(e_2) = e = x$. Thus $\mathrm{Ker} f = \{e\}$. $f$ is a monomorphism. $\forall (h, k) \in H \oplus K$, take $x = \tau_1(h) + \tau_2(k) \in G$, then

$$\begin{aligned} f(x) &= \tau_1'\pi_1\tau_1(h) + \tau_1'\pi_1\tau_2(h) + \tau_2'\pi_2\tau_1(k) + \tau_2'\pi_2\tau_1(k) \\ &= \tau_1'(h) + \tau_2'(k) = (h, k) \in H \oplus K \end{aligned}$$

$f$ is a epimorphism. Thus $G \cong H \oplus K$.

**Exercise 1.8.4.** Give an example to show that the weak direct product is not a coproduct in the category of all groups.

**Answer.** Consider $S_3$ and $S_3 \times S_3$.

$$\begin{tikzcd} & S_3 \times S_2 \\ S_3 \arrow[ur] \arrow[r] & S_3 \times S_3 \arrow[u, dashed] \end{tikzcd}$$

Since there doesn't exist homomorphism $S_3 \to S_2$, there is no homomorphism $S_3 \times S_3 \to S_3 \times S_2$.

**Exercise 1.8.5.** Let $G$, $H$ be finite cyclic groups. Then $G \times H$ is cyclic if and only if $(|G|, |H|) = 1$.

**Answer.** Assume $|G| = m$, $|H| = n$, then $G \cong Z_m$, $H \cong Z_n$ and $G \times H \cong Z_m \oplus Z_n$.
If$(|G|, |H|) = 1$. Consider $(x_1, x_2) \in Z_m \oplus Z_n$. By *Chinese Remainder Theorem*, there exists $x$ such that $a \equiv x \mod \mathrm{lcm}(m, n)$ and $a \equiv x_1 \mod m$, $a \equiv x_2 \mod n$. Thus, $a(1, 1) = (x_1, x_2)$. $Z_m \oplus Z_n < \langle(1, 1)\rangle$. $\langle(1, 1)\rangle < Z_m \oplus Z_n$ is trivial. So $Z_m \oplus Z_n = \langle(1, 1)\rangle \cong G \times H$ is cyclic.
If $G \times H$ is cyclic. Assume $l = \gcd(m, n)$ and there exist $x$ such that $x_1 \equiv x \mod m$, $x_2 \equiv x \mod n$. Take $x_1 \not\equiv x_2 \mod l$, it can be chosen properly. Consider $(x_1, x_2) \in Z_m \oplus Z_n$, $x = k_1 m + x_1 = k_2 n + x_2 \Rightarrow x_1 \equiv x_2 \mod l$. That's contradictory!

**Exercise 1.8.6.** Every finitely generated abelian group $G \neq \langle e \rangle$ in which every element (except $e$) has order $p$ ($p$ prime) is isomorphic to $Z_p \oplus Z_p \oplus \cdots \oplus Z_p$($n$ summands) for some $n \geq 1$.

**Answer.** Assume $\{a_1, a_2, \ldots, a_n\}$ generates $G$. $|a_i| = p$ for $i = 1, 2, \ldots, n$ so $\langle a_i \rangle \cong Z_p$. Now we show that $G = \prod_{i=1}^{n}{}^{w} \langle a_i \rangle \cong \sum_{i=1}^{n} Z_p$. $G = \langle a_1, a_2, \ldots, a_n \rangle$ and $\langle a_1 \rangle \lhd G$ for $i = 1, 2, \ldots, n$. If exist $\langle a_i \rangle$ s.t. $\prod_{j=1, j\neq i}^{n} \langle a_j \rangle \cap \langle a_i \rangle \neq \{e\}$. Then there exists $a_i^{s_i} = a_1^{s_1} \cdots a_{i-1}^{s_{i-1}} a_{i+1}^{s_{i+1}} \cdots a_n^{s_n}$. $(s_i, p) = 1$ so $\exists 1 \leq t_i \leq p-1$ such that $s_i t_i \equiv 1 \mod p$. So $a_i^{s_i t_i} = a_1^{s_1 t_i} \cdots a_{i-1}^{s_{i-1} t_i} a_{i+1}^{s_{i+1} t_i} \cdots a_n^{s_n t_i} = a_i$. $\{a_1, a_2, \ldots, a_n\}$ can generate $G$. That's contradictory! So $\prod_{j=1, j\neq i}^{n} \langle a_j \rangle \cap \langle a_i \rangle = \{e\}$, which means $G = \prod_{i=1}^{n}{}^{w} \langle a_i \rangle \cong \sum_{i=1}^{n} Z_p$.

**Exercise 1.8.7.** Let $H, K, N$ be nontrivial normal subgroups of a group $G$ and suppose $G = H \times K$. Prove that $N$ is in the center of $G$ or $N$ intersects one of $H, K$ nontrivially. Give examples to show that both possibilities can actually occur when $G$ is nonabelian.

**Answer.** If $N \cap H = N \cap K = \{e\}$. $G = HK$. $\forall h \in H$ and $k \in K$, since $H \cap K = \{e\}$, $hk = kh$. For any $hk \in N$, and $h_1 \in H \subset HK$, $h_1^{-1} hkh_1 = h_1^{-1} hh_1 k \in N$. Assume $h' = h_1^{-1} h_1 \in H$, $h'k \in N$. Thus $h'^{-1} k^{-1} kh = h'^{-1} h \in N$. So $h'^{-1} h = e$, $h = h'$, $h$ is in the center $C(H)$ of group $H$. Similarly, $k \in C(K)$ which is the center of $K$. Then $\forall hk \in N$ and $h_1 k_1 \in G$, $k_1^{-1} h_1^{-1} hkh_1 k_1 = h_1^{-1} hh_1 k_1^{-1} kk_1 = hk$. $N \subset N(G)$.
For $N \cup H \neq \varnothing$, the example can be trivial: $N < H$ and $N \lhd G$. There's many cyclic group satisfy the condition.
For $N \subset C(G)$. Take $G = D_4^* \times D_4^*$, $H = D_4^* \times \{I\}$, $K = \{I\} \times D_4^*$. $\{I, R^2\}$ is normal in $D_4^*$. Denote $N$ is the subgroup $\{(I, I), (R^2, R^2)\}$. We can verify that $N$ satisfies the condition.

**Exercise 1.8.8.** Corollary 8.7 is false if one of the $N_i$ is not normal.

**Answer.** Consider $N_1, N_2, \ldots, N_n$ are all finite. WLOG, assume $N_1$ is not normal. $G = \left\langle \bigcup_{i=1}^{n} N_i \cup N_1 \right\rangle$ and $N_1N_2 \cdots N_n \subset G$. Denote $A = N_2N_3 \cdots N_n$. Then $\exists a \in A$ such that $a^{-1}na = n' \notin N_1$. Thus $n'a \in G$ but $n'a \notin N_1N_2 \cdots N_n$ so $|G| > |N_1N_2 \cdots N_n| = |N_1| \times |N_2| \times \cdots \times |N_n| = |N_1 \times N_2 \times \cdots \times N_n|$.

**Exercise 1.8.9.** If a group $G$ is the (internal) direct product of its subgroups $H$, $K$, then $H \cong G/K$ and $G/H \cong K$.

**Answer.** $H \cap K = \{e\}$. $G = H \times K = HK$. Thus $HK/H \cong K/(K \cap H) = K$, $HK/K \cong H/(K \cap H) = H$.

**Exercise 1.8.10.** If $\{G_i | i \in I\}$ is a family of groups, then $\prod^w G_i$ is the internal weak product its subgroups $\{\tau_i(G_i) | i \in I\}$.

**Answer.** Take $\tau_i(g) = (e_1, e_2, \ldots, g, \ldots, e_n)$, $g \in G_i.\tau_i(G_i)$. $\tau_i(G_i)$ is normal in $\prod\limits_{i \in I} {}^w G_i$. $\tau_i(G_i) \cap \tau_j(G_j) = \{(e_1, e_2, \ldots, e_n)\}$ which is the identity element in $\prod\limits_{i \in I} {}^w G_i$. $\forall (g_1, g_2, \ldots, g_n) \in \prod\limits_{i \in I} {}^w G_i$, we have

$$(g_1, g_2, \ldots, g_n) = (g_1, e_2, \ldots, e_n)(e_1, g_2, \ldots, e_n) \cdots (e_1, e_2, \ldots, g_n)$$

Thus $\prod\limits_{i \in I} {}^w G_i \subset \left\langle \bigcup\limits_{i \in I} {}^w \tau_i(G_i) \right\rangle$ and

$$\left\langle \bigcup_{i \in I} {}^w \tau_i(G_i) \right\rangle = \tau_1(G_1)\tau_2(G_2) \cdots \tau_n(G_n) \subset \prod_{i \in I} {}^w G_i$$

Therefore $\prod\limits_{i \in I} {}^w G_i$ is the direct product of $\tau_i(G_i)$.

**Exercise 1.8.11.** Let $\{N_i | i \in I\}$ be a family of subgroups of a group $G$. Then $G$ is the internal weak product of $\{N_i | i \in I\}$if and only if:

(i) $a_i a_j = a_j a_i$ for all $i \neq j$ and $a_i \in N_i$, $a_j \in N_j$;

(ii) every nonidentity element of $G$ is uniquely a product $a_{i_1} \cdots a_{i_n}$, where $i_i, \ldots, i_n$ are distinct elements of $I$ and $e \neq a_{i_k} \in N_{i_k}$ for each $k$.

**Answer.** Trivial.

**Exercise 1.8.12.** A normal subgroup $H$ of a group $G$ is said to be a **direct factor** (**direct summand** if $G$ is additive abelian) if there exists a (normal) subgroup $K$ of $G$ such that $G = H \times K$.

(a) If $H$ is a direct factor of $K$ and $K$ is a direct factor of $G$, then $H$ is normal in $G$.

(b) If $H$ is a direct factor of $G$, then every homomorphism $H \to G$ may be extended to an endomorphism $G \to G$. However, a monomorphism $H \to G$ need not be extendible to and automorphism $G \to G$.

**Answer.** (a) $G = K \times K' = (H \times H') \times K'$. So $\forall g \in G$, $g = hh'k'$ with $h \in H$, $h' \in H'$ and $k' \in K'$. $\forall h_1 \in H$ and $g \in G$, $g^{-1}h_1g = k'^{-1}h'^{-1}h^{-1}h_1hh'k' = (h^{-1}h_1h)(h'^{-1}h')(k'^{-1}k') = h^{-1}h_1h \in H$. Thus $H \lhd G$.

(b) If $G = H \times K$. For a homomorphism $f : H \to G$, we construct a homomorphism $\bar{f} : G \to G$, $\forall g \in G$,$g$ can be uniquely written as $g = hk$ where $h \in H$, $k \in K$. Take $\tau(g) = h$ which is a homomorphism $\tau : G \to H$. We can get $\bar{f} = f \circ \tau : G \to G$ is a endomorphism but it needn't to be a automorphism.

**Exercise 1.8.13.** Let $\{G_i | i \in I\}$ be a family of groups and $J \subset I$. The map $\alpha : \prod\limits_{j \in J} G_j \to \prod\limits_{i \in I} G_i$ given by $\{a_j\} \mapsto \{b_i\}$, where $b_j = a_j$ for $j \in J$ and $b_i = e_i$(identity in $G_i$) for $i \notin J$, is a monomorphism of groups and $\prod\limits_{i \in I} G_i / \alpha(\prod\limits_{j \in J} G_j) \cong \prod\limits_{i \in I - J} G_i$.

**Answer.** Define a map $\beta : \prod\limits_{i \in I} G_i \to \prod\limits_{i \in I-J} G_i$ given by $\{a_i\} \mapsto \{b_i\}$ and for those $i \in I - J$, $\exists b_i \in \{b_i\}$ s.t. $a_i = b_i$. Thus $\beta(\{a_i\})\beta(\{a_i'\}) = \beta(\{a_ia_i'\})$, $\beta$ is a well defined homomorphism. Ker$\beta = \{\{a_i\} \in \prod\limits_{i \in I} G_i | a_i = e_i$ for $i \in I - J\} = \alpha(\prod\limits_{j \in J} G_j)$. We verify $\beta$ is a epimorphism. $\forall \{b_i\} \in \prod\limits_{i \in I-J} G_i$, take

$\{a_i\} \in \prod_{i\in I} G_i$ where $a_i = b_i$ for $i \in I - J$. Then $\beta(\{a_i\}) = \{b_i\}$. Thus $\beta$ is an isomorphism, $\mathrm{Im}\beta = \prod_{i\in I-J} G_i \cong \prod_{i\in I} G_i/\alpha(\prod_{j\in J}(G_j))$.

**Exercise 1.8.14.** For $i = 1, 2$ let $H_i \lhd G_i$ and give examples to show that each of the following statements may be false:
(a) $G_1 \cong G_2$ and $H_1 \cong H_2 \Rightarrow G_1/H_1 \cong G_2/H_2$.
(b) $G_1 \cong G_2$ and $G_1/H_1 \cong G_2/H_2 \Rightarrow H_1 \cong H_2$.
(c) $H_1 \cong H_2$ and $G_1/H_1 \cong G_2/H_2 \Rightarrow G_1 \cong G_2$.

**Answer.** (a) Take $G_1 = G_2 = Z_2 \times Z_4$, $H_1 = Z_2 \times \{\bar{0}\}$, $H_2 = \{(\bar{0}, \bar{0}), (\bar{0}, \bar{2})\}$.
(b) Take $G_1 = G_2 = Z_2 \times Z_4$, $H_1 = \{\bar{0}\} \times Z_4$, $H_2 = \{(\bar{0}, \bar{0}), (\bar{1}, \bar{0}), (\bar{0}, \bar{2}), (\bar{1}, \bar{2})\}$.
(c) Take $H_1 = \{(\bar{0}, \bar{0}), (\bar{0}, \bar{2})\}$, $H_2 = Z_2$ and $G_1 = Z_2 \times Z_4$, $G_2 = Z_2 \times K_4$.

## 1.9 Free groups, free products, generators and relations

**Exercise 1.9.1.** Every nonidentity elements in a free group $F$ has a infinite order.

**Answer.** Define the length of a word $x = a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n}$ is $n$ and denote it as $len(x)$. Assume $len(x) = n$ for some $n \in F$ and $len(1) = 0$, we prove that $len(x^m) \geq n \forall m \geq 1$.
Let $k$ be the largest integer such that $a_{n-j}^{\lambda_{n-j}} = a_n^{-\lambda_j}$ for $j = 0, 1, \ldots, k-1$. If $k > \left[\frac{n}{2}\right]$. For even $k$, $a_{\frac{n}{2}}^{\lambda_{\frac{n}{2}}} = a_{\frac{n}{2}+1}^{-(\lambda_{\frac{n}{2}+1})}$, $a_{\frac{n}{2}-1}^{\lambda_{\frac{n}{2}-1}} = a_{\frac{n}{2}+2}^{-(\lambda_{\frac{n}{2}+2})}$, $\cdots$ which means $x = a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n} = 1$. For odd $k$, $a_{\left[\frac{n}{2}\right]+1}^{\lambda_{\left[\frac{n}{2}\right]+1}} = a_{\left[\frac{n}{2}\right]+1}^{-(\lambda_{\left[\frac{n}{2}\right]+1})}$, which is contradictory to $x$ is reduced. So $k \leq \left[\frac{n}{2}\right]$.
Divide $x = x_1 x_2 x_3$ where $x_1 = a_1^{\lambda_1} \cdots a_k^{\lambda_k}$, $x_2 = a_{k+1}^{\lambda_{k+1}} \cdots a_{n-k}^{\lambda_{n-k}}$, $x_3 = a_{n-k+1}^{\lambda_{n-k+1}} \cdots a_n^{\lambda_n}$. $x_3 x_1 = 1$. So $len(x) = len(x_1) + len(x_2) + len(x_3) = n$. $x^m = x_1 x_2 x_3 x_1 x_2 x_3 \cdots x_1 x_2 x_3 = x_1 x_2^m x_3$. $len(x^m) = len(x_1) + m \cdot len(x_2) + len(x_3) \geq n$. So $\forall m \geq 1$, $x^m \neq 1$, $|x|$ is infinite.

**Exercise 1.9.2.** Show that the free group on the set $\{a\}$ is an infinite cyclic group, and hence isomorphic to $\mathbf{Z}$.

**Answer.** $F(\{a\}) = \langle a \rangle$ and thus it's a infinite cyclic group. $F(\{a\}) \cong \mathbf{Z}$.

**Exercise 1.9.3.** Let $F$ be a free group and let $N$ be the subgroup generated by the set $\{x^n | x \in F, n \text{ a fixed integer}\}$. Show that $N \lhd F$.

**Exercise 1.9.4.** Let $F$ be the free group on the set $X$, and let $Y \subset H$. If $H$ is the smallest normal subgroup of $F$ containin $Y$, then $F/H$ is a free group.

**Exercise 1.9.5.** The group defined by generators $a, b$ and relations $a^8 = b^2a^4 = ab^{-1}ab = e$ has order at most 16.

**Exercise 1.9.6.** The cyclic group of order 6 is the group defined by generators $a, b$ and relations $a^2 = b^3 = a^{-1}b^{-1}ab = e$.

**Exercise 1.9.7.** Show that the group defined by generators $a, b$ and relations $a^2 = e$, $b^3 = e$ is infinite and nonabelian.

**Exercise 1.9.8.** The group defined by generators $a, b$ and relations $a^n = e(3 \leq n \in \mathbf{N}^*)$, $b^2 = e$ and $abab = e$ is the dihedral group $D_n$.

**Exercise 1.9.9.** The group defined by the generator $b$ and $b^m = e(m \in \mathbf{N}^*)$ is the cyclic group $Z_m$.

**Exercise 1.9.10.** The operation of free product is commutative and associative: for any groups $A$, $B$, $C$, $A*B \cong B*A$ and $A*(B*C) \cong (A*B)*C$.

**Exercise 1.9.11.** If $N$ is normal subgroup of $A * B$ generated by $A$, then $(A * B)/N \cong B$.

**Exercise 1.9.12.** If $G$ and $H$ each have more than one element, then $G*H$ is an infinite group with center $\langle e \rangle$.

**Exercise 1.9.13.** A free group is a free product of infinite cyclic groups.

**Exercise 1.9.14.** If $G$ is the group defined by generators $a, b$ and relations $a^2 = e$, $b^3 = e$, then $G \cong Z_2 * Z_3$.

**Exercise 1.9.15.** If $f : G_1 \to G_2$ and $g : H_1 \to H_2$ are homomorphisms of groups, then there is a unique homomorphism $h : G_1 * H_1 \to G_2 H_2$ such that $h|G_1 = f$ and $h|H_1 = g$.

# Chapter 2

# The structure of groups

# Chapter 3

# Rings

## 3.1 Rings and homomorphisms

**Exercise 3.1.1.** (a) Let $G$ be an (additive) abelian group. Define an operation of multiplication in $G$ by $ab = 0$ (for all $a, b \in G$). Then $G$ is a ring.

(b) Let $S$ be the set of all subsets of some fixed set $U$. For $A, B \in S$, define $A + B = (A - B) \cup (B - A)$ and $AB = A \cap B$. Then $S$ is a ring. Is $S$ commutative? Does it have an identity?

**Answer.** (a) $\forall a, b \in G$, $ab = 0 \in G$, so $G$ is a monoid under multiplication, thus $G$ is a ring.

(b) $A \subset U$, $B \subset U$, so $A - B \subset U$, $B - A \subset U$. Thus $A + B = B + A = (A - B) \cup (B - A) \subset U$. Take $\varnothing$ is the identity under addition and $U - A$ as the inverse of $A$, $S$ is abelian group under the addition. $AB = A \cap B \subset U$, $AB = A \cap B = B \cap A = BA \in S$. So $S$ is a commutative ring. $\forall A \in S$, $A \cap U = AU = A$ is the identity of the ring $S$.

**Exercise 3.1.2.** Let $\{R_i | i \in I\}$ be a family of rings with identity. Make the direct sum of abelian groups $\sum\limits_{i \in I} R_i$ into a ring by defining multiplication coordinatewise. Does $\sum\limits_{i \in I} R_i$ have identity?

**Answer.** Take $1_{R_i} \in R_i$ is the identity for $i = 1, 2, \ldots, n$. $\forall (a_1, a_2, \ldots, a_n) \in \sum\limits_{i \in I} R_i$

$$\begin{aligned} &(a_1, a_2, \ldots, a_n)(1_{R_1}, 1_{R_2}, \ldots, 1_{R_n}) \\ =&(1_{R_1}, 1_{R_2}, \ldots, 1_{R_n})(a_1, a_2, \ldots, a_n) \\ =&(a_1, a_2, \ldots, a_n) \end{aligned}$$

is the identity.

**Exercise 3.1.3.** A ring $R$ such that $a^2 = a$ for all $a \in R$ is called **Boolean ring**. Prove that every Boolean ring $R$ is commutative and $a + a = 0$ for all $a \in R$.

**Answer.** $\forall a \in R$, $(a+a)^2 = a^2 + 2a + a^2 = a + 2a + a = 2a$, so $a + a = 0$. $\forall a, b \in R$, $(a+b)^2 = a^2 + b^2 + ab + ba = a + b = a + b + ba + ab$, so $ab + ba = 0 \Rightarrow ab = -ab = -ba$, $ab = ba$. Thus $R$ is commutative.

**Exercise 3.1.4.** Let $R$ be a ring and $S$ a nonempty set. Then the group $M(S, R)$ is a ring with multiplication defined as follows: the product of $f, g \in M(S, R)$ is the function $S \to R$ given by $s \mapsto f(s)g(s)$.

**Answer.** We only need to check $M(S, R)$ is a monoid under multiplication, which means $\forall f, g \in M(S, R)$, $fg \in M(S, R)$. $\forall a \in S$, $fg(a) = f(a)g(a)$. Since $f(a) \in R$, $g(a) \in R$, $f(a)g(a) \in R$, $fg : S \to G$ is a well defined function. $fg \in M(S, R)$. $M(S, R)$ is a ring.

**Exercise 3.1.5.** If $A$ is the abelian group $\mathbf{Z} \oplus \mathbf{Z}$, then $\mathrm{End}A$ is a noncommutative ring.

**Answer.** We only need to verify that $\mathrm{End}A$ is not commutative. Take $f, g \in \mathrm{End}A$, $f : (x_1, x_2) \mapsto (x_1 \mod 2, x_2 \mod 2)$, $g : (x_1, x_2) \mapsto (x_1 \mod 3, x_2 \mod 3)$. Then $gf(3,3) = (1,1)$, $fg(3,3) = (0,0)$. Thus $\mathrm{End}A$ is not commutative.

**Exercise 3.1.6.** A finite ring with more than one element and no zero divisors is a division ring.

**Answer.** For any disjoint $a, b, c \in R$, $ab \neq ac$, otherwise $a(b-c) = 0$, $b - c$ is a zero divisor. So $ax$ are different for different $x \in R$. $|\{ax | x \in R\}| = |R|$ and $\{ax | x \in R\} \subset R$. Thus $\{ax | x \in R\} = R$ which means $\exists a^{-1} \in R$ s.t. $aa^{-1} = R$. Similarly, $a$ is also left invertable and $R$ is a division ring.

**Exercise 3.1.7.** Let $R$ be a ring withe more than one element such that for each nonzero $a \in R$ there is a unique $b \in R$ such that $aba = a$. Prove:
(a) $R$ has no zero divisors.

(b) $bab = b$.
(c) $R$ has an identity.
(d) $R$ is a division ring.

**Answer.** (a) If $x$ is a zero divisor of $a$. WLOG, assume $ax = 0$, $axa \neq a$ so $b \neq x$. But $axa + aba = a(x + b)a = a$ which is contradictory to the uniqueness.
(b) $aba = a \Rightarrow abab = ab$, $a(bab - b) = 0$ and $a \neq 0$, so $bab - b =$, $bab = b$.
(c) Assume $c = ab$, $abab = ab \Rightarrow c^2 = c$. $\forall x \in R$, $xc^2 = xc \Rightarrow (xc - x)c = 0$ and $c \neq 0$, so $xc = x$ for any $x \in R$. Similarly, $cx = x$ for all $x \in R$, $c$ is the identity of $R$.
(d) $\forall a, b \in R$, $aba = a \cdot 1_R = 1_R \cdot a$. So $a(ba - 1_R) = (ab - 1_R)a = 0$, $ba = ab = 1_R$. That means $a, b$ are all units, so $R$ is a division ring.

**Exercise 3.1.8.** Let $R$ be the set of all $2 \times 2$ matrices over the complex field $\mathbf{C}$ of the form

$$\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$$

where $\bar{z}, \bar{w}$ are the complex conjugates of $z$ and $w$ respectively. Then $R$ is a division ring that is isomorphic to the division ring $K$ of real quaternions.

**Answer.** Define $f : K \to R$ with $f(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $f(i) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $f(j) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $f(k) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. Assume $z = a + bi$, $w = c + di$.

$$\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Define

$$f(\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}) = af(1) + bf(i) + cf(j) + df(k)$$

$f(xy) = f(x)f(y)$ and $f$ is a isomorphism, so $R \cong K$.

**Exercise 3.1.9.** (a) The subset $G = \{1, -1, i, -i, j, -j, k, -k\}$ of the division ring $K$ of real quaternions forms a group under multiplication.

(b) $G$ is isomorphic to the quaternion group.
(c) What is the difference between the ring $K$ and the group $\mathbf{R}(G)$($\mathbf{R}$ the field of real numbers)?

**Answer.** (a) Trivial.
(b) Define $f : G \to Q_8$ given by $f(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $f(i) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $f(j) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $f(k) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. We can verify that $f$ is a isomorphism, $G \cong Q_8$.
(c) $R(G)$ is a free abelian group while $K$ is not free on $G$.

**Exercise 3.1.10.** Let $k, n$ be integers such that $0 \leq k \leq n$ and $\binom{n}{k}$ the binomial coefficient $n!/(n-k)!k!$, where $0! = 1$ and for $n > 0$, $n! = n(n-1)(n-2)\cdots 2 \cdot 1$.
(a) $\binom{n}{k} = \binom{n}{n-k}$
(b) $\binom{n}{k} < \binom{n}{k+1}$ for $k+1 \leq n/2$.
(c) $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$ for $k < n$.
(d) $\binom{n}{k}$ is an integer.
(e) if $p$ is prime and $1 \leq k \leq p^n - 1$, then $\binom{p^n}{k}$ is divisible by $p$.

(a) $\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n!}{(n-(n-k))!(n-k)!} = \binom{n}{n-k}$.
(b) $\binom{n}{k} = \frac{n!}{(n-k)!k!}$, $\binom{n}{k+1} = \frac{n!}{(n-k-1)!(k+1)!}$, since $k+1 \leq n-k$ when $k+1 \leq \frac{n}{2}$, then $\binom{n}{k} < \binom{n}{k+1}$.
(c) $\binom{n}{k} + \binom{n}{k+1} = \frac{n!}{(n-k)!k!} + \frac{n!}{(n-k-1)!(k+1)!} = \frac{(n+1)!}{(n-k)!(k+1)!} = \binom{n+1}{k+1}$.
(d) $\binom{n}{k}$ is an integer can be easily solved by induction and (c).
(e) $\mathrm{ord}_p(p^n!) = \sum\limits_{i=1}^{\infty} \left[\frac{p^n}{p^i}\right] = \sum\limits_{i=0}^{n-1} p^i$. $\mathrm{ord}_p(k!) = \sum\limits_{i=1}^{\infty} \left[\frac{k}{p^i}\right]$, $\mathrm{ord}_p((p^n-k)!) = \sum\limits_{i=1}^{\infty} \left[\frac{p^n-k}{p^i}\right]$. $\forall i \in \mathbf{N}$, $\left[\frac{p^n-k}{p^i}\right] + \left[\frac{k}{p^i}\right] \leq \left[\frac{p^n}{p^i}\right]$, the equality holds if and only if $\frac{p^n-k}{p^i}, \frac{k}{p^i} \in \mathbf{Z}$. And $\left[\frac{p^n-k}{p^n}\right] = 0$, $\left[\frac{k}{p^n}\right] = 0$. So $\mathrm{ord}_p(\binom{p^n}{k}) = \mathrm{ord}_p(p^n!) - \mathrm{ord}_p((n-k)!) - \mathrm{ord}_p(k!) \geq 1$. $p | \binom{p^n}{k}$.

**Exercise 3.1.11.** Let $R$ be a commutative ring with identity of prime characteristic $p$. If $a, b \in R$, then $(a \pm b)^{p^n} = a^{p^n} \pm b^{p^n}$ for all integers $n \geq 0$.

**Answer.** $(a \pm b)^{p^n} = \sum\limits_{i=0}^{p^n} \binom{p^n}{i}(\pm a)^i b^{p^n-i}$. From **Exercise 3.1.10**, $p|\binom{p^n}{i}$ for all $i = 1, 2, \ldots, n-1$, so $\binom{p^n}{i} a^i b^{p^n-i} = 0$ for $i = 1, 2, \ldots, n-1$. Thus $\sum\limits_{i=0}^{p^n} \binom{p^n}{i}(\pm a)^i b^{p^n-i} = a^{p^n} \pm b^{p^n} = (a \pm b)^{p^n}$.

**Exercise 3.1.12.** An element of a ring is **nilpotent** if $a^n = 0$ for some $n$. Prove that in a commutative ring $a + b$ is nilpotent if $a$ and $b$ are. Show that this result may be false if $R$ is not commutative.

**Answer.** Assume $a^m = 0$, $b^n = 0$. For $(a+b)^{m+n} = \sum\limits_{i=1}^{m+n} \binom{m+n}{i} a^i b^{m+n-i}$. If $i \geq m$, $a^i b^{m+n-i} = 0b^{m+n-i} = 0$; if $i \leq m$, $m + n - i \geq n$ so $a^i b^{m+n-i} = a^i 0 = 0$. Thus $a^i b^{m+n-i} = 0$ for all $i = 1, 2, \ldots, m+n$. $a+b$ is also nilpotent. For the $2 \times 2$ matrix ring. $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are nilpotent, but $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is not nilpotent.

**Exercise 3.1.13.** In a ring $R$ the following conditions are equivalent.
(a) $R$ has no nonzero nilpotent elements.
(b) If $a \in R$ and $a^2 = 0$, then $a = 0$.

**Answer.** (a)$Rightarrow$(b): Trivial.
(b)$Rightarrow$(a): If $\exists a \in R$, $a^n = 0$ for some $n$ and $a \neq 0$. Assume $n = 2^m \cdot k$ and $k$ is a odd integer. Then $(a^{k \cdot 2^{m-1}})^2 = 0 \Rightarrow a^{k \cdot 2^{m-1}} = 0 \Rightarrow \cdots \Rightarrow a^k = 0$. $a^k \cdot a^{k+1} = 0$ and $2|k+1$, we can continue this step until $\frac{k+1}{2} \geq k$ which means $k = 1$. So $a = 0$.

**Exercise 3.1.14.** Let $R$ be a commutative ring with identity and prime characteristic $p$. The map $R \to R$ given by $r \mapsto r^p$ is a homomorphism of rings called the Frobenius homomorphism.

**Answer.** $\forall a, b \in R$, $pa = pb = 0$ and the map $f : r \mapsto r^p$. $f(a+b) = (a+b)^p = \sum_{i=0}^{p} \binom{p}{i} a^i b^{p-i}$. Since $p$ is a prime so $p \mid p!$ and $p \nmid i!(p-i)!$, $p \mid \binom{p}{i}$ for $i = 1, 2, \ldots, p-1$. So $f(a+b) = a^p + b^p = f(a) + f(b)$, $f(ab) = (ab)^p = a^p b^p = f(a)f(b)$, $f$ is a homomorphism of rings.

**Exercise 3.1.15.** (a) Give an example of nonzero homomorphism $f : R \to S$ of rings with the identity such that $f(1_R) \neq 1_S$.
(b) If $f : R \to S$ is an epimorphism of rings with identity, then $f(1_R) = 1_S$.
(c) If $f : R \to S$ is a homomorphism of rings with identity and $u$ is a unit in $R$ such that $f(u)$ is a unit in $S$, then $f(1_R) = 1_S$ and $f(u^{-1}) = f(u)^{-1}$.

**Answer.** (a) For $f : Z_2 \to Z_6$ defined by $f(0) = 0$, $f(1) = 3$. $f$ is a homomorphism of ring which satisfies the condition.
(b) $\forall s \in S$, $\exists r \in R$ such that $f(r) = s$, so $f(r)f(1_R) = f(1_R)f(r) = f(r) = s$, so $f(1_R) = 1_S$ is the identity of $S$.
(c) $f(u)f(u^{-1}) = f(u^{-1})f(u) = f(1_R)$. $\exists s \in S$ such that $f(u)s = sf(u) = 1_S$, $sf(u)f(u^{-1}) = sf(1_R) = f(u^{-1})$, $sf(1_R)f(u) = f(u^{-1})f(u) = f(1_R) = sf(u) = 1_S$. Thus $f(u^{-1} = s)$, $f(u^{-1}) = f(u)^{-1}$.

**Exercise 3.1.16.** Let $f : R \to S$ be a homomorphism of rings such that $f(r) \neq 0$ for some nonzero $r \in R$. If $R$ has an identity and $S$ has no zero divisors, then $S$ is a ring with identity $f(1_R)$.

**Answer.** $f(1_R)f(1_R) = f(1_R)$, so $f(1_R)(f(1_R) - 1_S) = 0 \Rightarrow f(1_R) = 1_S$.

**Exercise 3.1.17.** (a) If $R$ is a ring, then so is $R^{op}$ is defined as follows. The underlying set of $R^{op}$ is precisely $R$ and addition in $R^{op}$ coincides with addition in $R$. Multiplication in $R^{op}$, denoted $\circ$, is defined by $a \circ b = ba$, where $ba$ is the product in $R$. $R^{op}$ is called the **opposite ring** of $R$.
(b) $R$ has identity if and only if $R^{op}$ does.
(c) $R$ is a division ring if and only if $R^{op}$ is.
(d) $(R^{op})^{op} = R$.
(e) If $S$ is a ring, then $R \cong S$ if and only if $R^{op} \cong S^{op}$.

**Answer.** (a) Trivial.

(b) If $1_R$ is the identity of $R$. Take $1_{R^{op}} = 1_R$ then $\forall a \in R^{op}$, $1_{R^{op}} \circ a = a1_R = a = 1_R a = a \circ 1_{R^{op}}$. So $1_{R^{op}}$ is the identity of $R^{op}$.

(c) $\forall a \in R^{op}$, take $a^{-1} \in R$, $a^{-1} \circ a = aa^{-1} = 1_R = a^{-1}a = a \circ a^{-1}$. So $a$ is a unit, $R^{op}$ is a division ring.

(d) Denote $*$ is the multiplication in $(R^{op})^{op}$.

$$a * b = b \circ a = ab \in R$$

The multiplications are identical. The underlying set and addition of $R$ and $(R^{op})^{op}$ are identical. So $R = (R^{op})^{op}$.

(e) If $R \cong S$, there exists isomorphism $f : R \to S$. We verify that $f' R^{op} \to S^{op}$ defined by $f' = f$ is an isomorphism. $f' = f$ is obviously a bijection. $f'(a) \circ f'(b) = f(b)f(a) = f(ba) = f'(a \circ b)$. $f'$ is a well defined homomorphism, so $R^{op} \cong S^{op}$.

**Exercise 3.1.18.** Let $\mathbf{Q}$ be the field of rational numbers and $R$ any ring. If $f, g : \mathbf{Q} \to R$ are homomorphisms of rings such that $f|\mathbf{Z} = g|\mathbf{Z}$, then $f = g$.

**Answer.** $f(n) = g(n)$ for $n \in \mathbf{Z}$. $g(n)g(\frac{1}{n}) = g(1) \Rightarrow f(n)g(\frac{1}{n}) = g(1) = f(1)$, so $f(\frac{1}{n})f(n)g(\frac{1}{n}) = g(\frac{1}{n}) = f(\frac{1}{n})$ for all $n \in \mathbf{Z}$. Thus $f = g$.

## 3.2 Ideals

**Exercise 3.2.1.** The set of all nilpotent elements in a commutative ring forms an ideal.

**Answer.** Assume the set is $I$, then $\forall a, b \in I$, $a^m = b^n = 0$, $(a+b)^{m+n} = 0$ and $(ab)^{mn} = 0$ so $a+b \in I$, $ab \in I$. $I$ is a subring. $\forall x \in R$, $(xa)^m = x^m a^m = 0$, $(ax)^m = a^m x^m = 0$, so $xa \in I$ and $ax \in I$, $I$ is an ideal.

**Exercise 3.2.2.** Let $I$ be an ideal in a commutative ring $R$ and let $\mathrm{Rad}I = \{r \in R | r^n \in I \text{ for some } n\}$. Show that $\mathrm{Rad}I$ is an ideal.

**Answer.** $\mathrm{Rad}I$ is a ring since $R$ is a commutative ring. For $r \in \mathrm{Rad}I$ and $\forall x \in R$, $(xr)^n = x^n r^n \in I$ so $xr \in \mathrm{Rad}I$, $(rx)^n = r^n x^n \in I$ so $rx \in \mathrm{Rad}I$. Thus $\mathrm{Rad}I$ is an ideal.

**Exercise 3.2.3.** If $R$ is a ring and $a \in R$, then $J = \{r \in R | ra = 0\}$ is a left ideal and $K = \{r \in R | ar = 0\}$ is a right ideal in $R$.

**Answer.** $J$ is a subring of $R$. For $r \in J$ and $\forall x \in R$, $(xr)a = x(ra) = 0$ so $xr \in J$, $J$ is a left ideal. Similarly, $I$ is a right ideal.

**Exercise 3.2.4.** If $I$ is a left ideal of $R$, then $A(I) = \{r \in R | rx = 0 \text{ for every } x \in I\}$ is an ideal in $R$.

**Answer.** For any $a, b \in A(I)$, we have $ab \in A(I)$ and $a + b \in A(I)$. For $r \in A(I)$ and $\forall x \in R$, $(xr)x' = x(rx') = 0$ for every $x' \in I$, so $xr \in A(I)$. $(rx)x' = r(xx')$, $xx' \in I$ so $rxx' = 0$, $rx \in A(I)$. Thus $A(I)$ is an ideal of $R$.

**Exercise 3.2.5.** If $I$ is an ideal in a ring $R$, let $[R : I] = \{r \in R | xr \in I \text{ for every } x \in R\}$. Prove that $[R : I]$ is an ideal of $R$ which contains $I$.

**Answer.** $I$ is a subring of $R$ so $[R : I]$ is also a subring of $R$. For $r \in [R : I]$ and $x, x' \in R$, $x'xr = (x'x)r \in I$ so $xr \in [R : I]$, $x'rx = (x'r)x \in I$ so $rx \in [R : I]$. $[R : I]$ is an ideal of $R$. Since $\forall r \in I$, $xr \in I$ and $rx \in I$, $I \subset [R : I]$.

**Exercise 3.2.6.** (a) The center of the ring $S$ of all $2 \times 2$ matrices over a field $F$ consists of all matrices of the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

(b) Then center of $S$ is not an ideal in $S$.

(c) What is the center of the ring of all $n \times n$ matrices over a division ring?

**Answer.** (a) $\forall x \in M_F(2,2)$, $x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$

$$x \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} x = \begin{pmatrix} ax_1 & ax_2 \\ ax_3 & ax_4 \end{pmatrix}$$

so $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in C(M_F(2,2))$.

$\forall \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in C(M_F(2,2))$, take $\begin{pmatrix} 0 & 1_F \\ 0 & 0 \end{pmatrix} \in M_F(2,2)$

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} 1_F & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ a_3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1_F & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}$$

so $a_2 = a_3 = 0$.

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} 0 & 1_F \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_1 \\ 0 & a_3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1_F \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} a_3 & a_4 \\ 0 & 0 \end{pmatrix}$$

so $a_1 = a_4$. All the elements of $C(M_F(2,2))$ has the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

(b) For $c \in C(S)$. If $S$ is not commutative, $\forall x, x' \in R$, we need $xc \in C(S) \Rightarrow x'xc = xcx' = xx'c$, however, this may not always true.

(c) By multiplying $\begin{pmatrix} 1_F & & \\ & \ddots & \\ & & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & & & \\ & 1_F & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$, $\ldots$, $\begin{pmatrix} 0 & & \\ & \ddots & \\ & & 1_F \end{pmatrix}$,

we can have $C(M_F(2,2))$ consist of all the elements in the form of $a\begin{pmatrix} 1_F & & & \\ & 1_F & & \\ & & \ddots & \\ & & & 1_F \end{pmatrix}$.

**Exercise 3.2.7.** (a) A ring $R$ with identity is a division ring if and only if $R$ has no proper left ideals.

(b) If $S$ is a ring (possibly without identity) with no proper left ideals, then either $S^2 = 0$ or $S$ is a division ring.

**Answer.** (a) Suppose not. $I$ is an ideal in $R$. $\forall r \in I$, take $r^{-1} \in R$, then $1_R \in I$ so $I = R$ is not a proper ideal.

(b) $I = \{a \in S | Sa = 0\}$ is a left ideal since $\forall x, x' \in S$, $x'(xs) = (x'x)s = 0$, $xs \in I$. Thus $I = 0$ or $I = S$. If $I = S$, then $S^2 = 0$. If $I = 0$, we prove $S$ has no zero divisor.

For the set $I' = \{r \in S | rb = 0\}$, $I' \subset I$. $I'$ is a subring of $S$, and $I'$ is also a left ideal of $S$. So $I' = 0$, $b$ has no left zero divisors. $\forall a \in S$, $Sa$ is a left ideal of $S$. $Sa \neq 0$ so $Sa = S$. Thus, $\exists 1_S \in S$, such that $1_S a = a$. Since $s_1 - s_2$ has no left zero divisor, $as_1 = as_2 \Rightarrow s_1 = s_2$. So $aS = S$. For all $s \in S$, $\exists s'$ s.t. $s = as'$ so $\forall s \in S$, $1_S \cdot s = 1_S as' = as' = s$. $aS = S$ so $\exists 1'_S \in S$, $a1'_S = a$. Similarly, $\forall s \in S$, $s1_S = s$. Then $1_S 1'_S = 1_S = 1'_S$ so $S$ has identity. Since $Sa = aS = S$, we can have $S$ is a division ring.

**Exercise 3.2.8.** Let $R$ be a ring with identity and $S$ the ring of all $n \times n$ matrices over $R$. $J$ is an ideals of $S$ if and only if $J$ is the ring of all $n \times n$ matrices over $I$ for some ideal $I$ in $R$.

**Answer.** If $J$ is an ideal. Denote $E_{r,s}$ as the matrix which has $1_R$ as the $r$ column and $s$ row. Then $\forall A = (a_{ij})$, $E_{p,r} A E_{s,q}$ is a matrix with $a_{rs}$ in the $p$ column and $q$ row. So for $A \in J$ $(aE_{p,r})A(bE_{s,q})$ is the matrix with $aa_{rs}b$

in the $p$ column and $q$ row. $aa_{rs}b \in I$. Then because of closure we know $J$ contains all $n \times n$ matrices over $I$.
If $J$ consists of all $n \times n$ matrices over $I$, the proof is trivial.

**Exercise 3.2.9.** Let $S$ be the ring of all $n \times n$ matrices over a division ring $D$.
(a) $S$ has no proper ideals (that is, 0 is the maximal ideal).
(b) $S$ has zero divisors. Consequently, (i) $S \cong S/0$ is not a division ring and (ii) 0 is a prime ideal which does not satisfy condition (1) of Theorem 2.15.

**Answer.** (a) $J$ is an ideal of $S$ so $J$ consists of all $n \times n$ matrices over $I$ where $I$ is an ideal of $D$. From **Exercise 3.2.7**, $D$ has no proper ideal so $I = 0 \Rightarrow J = 0$.
(b) For $A = (a_{ij})$ with $a_{ri} = 0$ for $i = 1, 2 \cdots$ and other entries doesn't equals to zero, we have $E_{1r}A = 0$. $S$ has no zero divisors.

**Exercise 3.2.10.** (a) Show that $\mathbf{Z}$ is a principle ideal ring.
(b) Every homomorphic image of a principle ideal ring is also a principle ideal ring.
(c) $Z_m$ is a principle ideal ring for every $m > 0$.

**Answer.** (a) For any ideal $I$ in $\mathbf{Z}$, $I$ is a subring so $I = m\mathbf{Z}$ where $m \in \mathbf{Z}$. $m\mathbf{Z} = (m)$ is a principle ideal so $\mathbf{Z}$ is a PID.
(b) For $f : R \to S$ with $f(r) = s$ and $R$ is a principle ideal ring. Consider $f : R \to \text{Im}f \subset S$. For any ideal $J \subset \text{Im}f$, $f^{-1}(J)$ is an ideal since $\forall a \in f^{-1}(J)$ and $r \in R$, $f(ar) = f(a)f(r) \in J \Rightarrow ar \in f^{-1}(J)$. $f^{-1}(J)$ is a principle ideal, assume $f^{-1}(J) = (a)$. Then $\forall r \in R$, $ar \in (a)$, $ra \in (a)$. $f(ar) = f(a)f(r) \in J$ and $f(ra) = f(r)f(a) \in J$ since $f(a) \in J$ and $f(r) \in S$. So $(f(a)) \subset J$. $J = f((a)) = \{f(ra + as + na + \sum_{i=1}^{m} r_i a s_i) | r, s, r_i, s_i \in R, n \in \mathbf{Z}\} = \{f(r)f(a) + f(a)f(s) + nf(a) + \sum_{i=1}^{m} f(r_i)f(a_i)f(s_i) | r, s, r_i, s_i \in R, n \in \mathbf{Z}\} \subset (f(a))$. So $J = (f(a))$ is a principle ideal. The image of a principle ideal ring is also a principle ideal ring.

**Exercise 3.2.11.** If $N$ is the ideal of all nilpotent elements in a commutative ring $R$, then $R/N$ is a ring with no nonzero nilpotent elements.

**Answer.** Suppose not. $\exists r \in R$, $r \notin N$, $(r+N)^n = 0$ for some $n \in \mathbf{N}$.

$$(r+N)^n = r^n + N = N \Rightarrow r^n \in N$$

so for some $m \in \mathbf{N}$, $r^{nm} = 0 \Rightarrow r \in N$. That's contradictory!

**Exercise 3.2.12.** Let $R$ be a ring without identity and with no zero divisors. Let $S$ be the ring whose additive group is $R \times \mathbf{Z}$ as in the proof of Theorem 1.10. Let $A = \{(r,n) \in S | rx + nx = 0 \text{ for every } x \in R\}$.
(a) $A$ is an ideal in $S$.
(b) $S/A$ has an identity and contains a subring isomorphic to $R$.
(c) $S/A$ has no zero divisors.

**Answer.** (a) For $(r,n),(r',n') \in S$, $(r'+r)x + (n'+j)x = rx + nx + r'x + n'x = 0$, so $(r+r', n+n') \in A$. $(r,n)(r'n') = (rr' + nr' + n'r, nn')$, $rr'x + n'rx + nr'x + nn'x = r(r'x + n'x) + n(r'x + n'x) = 0$, so $(r,n)(r',n') \in A$. $A$ is a subring of $R \times \mathbf{Z}$. $\forall (r_1, n_1) \in R \times \mathbf{Z}$, $(r_1,n_1)(r,n) = (r_1 r + n r_1 + n_1 r, n n_1) \Rightarrow r_1 r x + n r_1 x + n_1 r x + n n_1 x = r_1(rx + nx) + n_1(rx + nx) = 0 \Rightarrow (r_1,n_1)(r,n) \in A$. $A$ is an ideal of $R \times \mathbf{Z}$.

(b) Take $0_R \in R$ and $(0_R, 1) \in S$. Then $(0_R,1) + A$ is an identity of $S/A$.

$$\forall (r,n) \in S, \quad (r,n)(0_R,1) = (0_R,1)(r,n) = (r,n)$$

(c) For any $(r,n),(s,m)$ satisfy that $(r,n)(s,m) \in A$, we prove that $(r,n) \in A$ or $(s,m) \in A$. Suppose $sx + mx \neq 0$, $r(sx+mx) + n(sx+mx) = 0 \Rightarrow (sx+mx)r(sx+mx) + n(sx+mx)^2 = 0 \Rightarrow ((sx+mx)r + n(sx+mx))(sx+mx) = 0 \Rightarrow (sx+mx)r + n(sx+mx) = 0$. For any $x \in R$, $(sx+mx)rx + n(sx+mx)x = 0 \Rightarrow (sx+mx)(rx+nx) = 0 \Rightarrow rx+nx = 0$, so $(r,n) \in A$. $S/A$ has no divisor.

**Exercise 3.2.13.** Let $f : R \to S$ be a homomorphism of rings, $I$ and ideal in $R$, and $J$ an ideal in $S$.
(a) $f^{-1}(J)$ is and ideal in $R$ that contains Ker$f$.
(b) If $f$ is an epimorphism, then $f(I)$ is an ideal in $S$. If $f$ is not surjective, $f(I)$ need not be an ideal.

**Answer.** (a) $\forall a \in f^{-1}(J)$ and $r \in R$, $f(ar) = f(a)f(r) \in J \Rightarrow ar \in J$. Similarly, $ra \in J$, $f^{-1}(J)$ is an ideal. Ker$f \subset f^{-1}(J)$ since $0_S \in J$.
(b) $\forall b \in f(I)$ and $s \in S$, $f$ is a epimorphism so $s = f(r)$, $b = f(a)$ for some $r, a \in R$. $sb = f(r)f(a) = f(ar)$, $ar \in I \Rightarrow sb \in f(I)$, similarly $bs \in f(I)$. $f(I)$ is an ideal.
If $f$ is not surjective. Take $Z[x]$ and $\mathbf{Z}$ which is a subring but not an ideal in $Z[x]$. $\mathbf{Z}$ is an ideal of itself, $f = 1_{\mathbf{Z}}$ satisfies the condition.

**Exercise 3.2.14.** If $P$ is an ideal in a not necessarily commutative ring $R$, then the following conditions are equivalent.
(a) $P$ is a prime ideal.
(b) If $r, s \in R$ are such that $rRs \subset R$, then $r \in P$ or $s \in P$.
(c) If $(r)$ and $(s)$ are principle ideals of $R$ such that $(r)(s) \subset P$, then $r \in P$ or $s \in P$.
(d) If $U$ and $V$ are right ideals in $R$ such that $UV \subset R$, then $U \subset R$ or $V \subset R$.
(e) If $U$ and $V$ are left ideals in $R$ such that $UV \subset R$, then $U \subset R$ or $V \subset R$.

**Exercise 3.2.15.** The set consisting of zero and all zero divisors in a commutative ring with identity contains at least one prime ideal.

**Answer.** Denote $S = R - Z$. $\forall a, b \in S$, we prove that $ab \in S$. Suppose $\exists (ab)c = 0$ for some $c \in R$, $a$, $b$ are not zero divisors so $abc = b(ac) = a(bc) = 0$, so $ac = 0$, $bc = 0 \Rightarrow c = 0$, so $ab$ is not a zero divisor. Thus $Z = R - S$ contains an prime ideal.

**Exercise 3.2.16.** Let $R$ be a commutative ring with identity and suppose that the ideal $A$ of $R$ is contained in a finite union of prime ideals $P_1 \cup \cdots \cup P_n$. Show that $A \subset P_i$ for some $i$.

**Answer.** Suppose not. We choose the smallest $I$ such that for all $i \in I$, $P_i \cap A \neq \varnothing$ and $A \cap P_i \not\subset \bigcup\limits_{j \neq i} P_j$ for any $i \in I$. So $\exists a_i \in (A \cap P_i) - (\bigcup\limits_{j \neq i} P_j)$, $\forall i \in I$. Take $x = a_1 + a_2 a_3 \cdots a_n$, $x \in A$ since $a_i \in A$ for all $i \in I$. And $x \notin P_i$ for $i = 2, 3, \ldots, n$ since $a_1 \notin P_i$, $i = 2, 3, \ldots, n$. $x \notin P_1$ since $P_1$ is prime and $a_2, \ldots, a_n \notin P_1$. So $x \notin \bigcup\limits_{j \neq i} P_j$, which is contradictory!

**Exercise 3.2.17.** Let $f : R \to S$ be an epimorphism of rings with kernel $K$.

(a) If $P$ is a prime ideal in $R$ that contains $K$, then $f(P)$ is a prime ideal in $S$.

(b) If $Q$ is a prime ideal in $S$, then $f^{-1}(Q)$ is a prime ideal in $R$ that contains $K$.

(c) There is a one-to-one correspondence between the set of all prime ideals in $R$ that contain $K$ and the set of all prime ideals in $S$, given by $P \mapsto f(P)$.

(d) If $I$ is an ideal in a ring $R$, then every prime ideal in $R/I$ is of the form $P/I$, where $P$ is a prime ideal in $R$ that contains $I$.

**Answer.** (a) From **Exercise 3.2.13** we know $f(P)$ is an ideal. $\forall x, y \in f(P)$, $\exists a.b \in R$, $x = f(a)$, $y = f(b)$ and $a, b \notin P$. Assume $\exists p \in P$ such that $f(ab) = f(p)$, then $f(ab - p) = 0$, $ab - p \in \mathrm{Ker} f \subset P \Rightarrow ab \in P$. That's contradictory to $a, b \notin P$ so $xy \notin f(P)$. $f(P)$ is prime.

(b) From **Exercise 3.2.13**, $f^{-1}(Q)$ is an ideal. Take $g : S \to S/Q$ and $gf : R \to S/Q$. By the Theorem of homomorphism, $R/f^{-1}(Q) \cong S/Q$ is a ring without divisor, so $f^{-1}(Q)$ is prime.

(c) From (a), (b), $f$ is a one-to-one map between prime ideals given by $P \mapsto f(P)$.

(d) Consider the homomorphism $f : R \to R/I$. For any prime ideal $P \subset R$ and $f(P)$ is an prime ideal in $R$, $\mathrm{Ker} f = I$ so for prime ideals $I \subset P \subset R$. $P$ can have one to one correspondence with $f(P) = P/I \subset R/I$. So all the prime ideals has the form $P/I$.

**Exercise 3.2.18.** An ideal $M \neq R$ in a commutative ring $R$ with identity is maximal if and only if for every $r \in R - M$, there exists $x \in R$ such that $1_R - rx \in M$.

**Answer.** If $M$ is maximal, then $M$ is prime. So $rR + M = R$, $r(R - M) + M = R$ and $r(R - M) \cap M = \varnothing$. Take $1_R \in R$ we have $x \in R - M$, $1_R - xr \in M$. If $\forall r \in R - M$, $\exists x \in R$ such that $1_R - rx \in M$. Suppose $M \subset I \subset R$ where $I$ is an ideal, $I \neq R$ so $1_R \notin R$. Take $r \in I - M \subset R - M$, then $\forall x \in R$, $rx \in I$, so $1_R - rx \notin I$ thus $1_R - rx \notin M$. That's contradictory!

**Exercise 3.2.19.** The ring $E$ of even integers contains a maximal ideal $M$ such that $E/M$ is not a field.

**Answer.** $E = 2\mathbf{Z}$ and $M$ is a maximal ideal in $E$ and for any subring of $E$ has the form $wn\mathbf{Z}$ where $n \in \mathbf{Z}$. $2n\mathbf{Z}$ is an ideal in $2\mathbf{Z}$. Take $n = 15$, $(2, 15) = 1$ so $2\mathbf{Z}/30\mathbf{Z} \cong \mathbf{Z}/15\mathbf{Z}$ which is not a field since $3 \cdot 5 = 0$ is a zero divisor.

**Exercise 3.2.20.** In the ring $\mathbf{Z}$ the following conditions on a nonzero ideal $I$ are equivalent: (i) $I$ is prime; (ii) $I$ is maximal; (iii) $I = (p)$ with $p$ prime.

**Answer.** $\mathbf{Z}$ is an integer domain so (ii)$\Rightarrow$(i).
(i)$\Rightarrow$(iii): Trivial.
(iii)$\Rightarrow$(ii): For any $n \notin (p)$, we have $p \nmid n$ thus $\exists x, y \in \mathbf{Z}$ such that $px + my = 1$. Consider an ideal $I$ and $(p) \subset I$, $n \in I$, then $1 \in I$ so $I = \mathbf{Z}$ which means $(p)$ is maximal.

**Exercise 3.2.21.** Determine all prime and maximal ideals in the ring $Z_m$.

**Answer.** $Z_m^2 = Z_m$ so every maximal ideal is prime in $Z_m$. $Z_m \cong \mathbf{Z}/m\mathbf{Z}$ via $\varphi : \bar{x} \mapsto mz + x$. From **Exercise 3.2.17**, all the prime ideals in $\mathbf{Z}/m\mathbf{Z}$ are $P/m\mathbf{Z}$, where $P$ is a prime ideal contains $m\mathbf{Z} = (m)$.
If $m$ is prime, $(m)$ is prime, too. So no such ideal exist.
If $m = p_1^{s_1} p_2^{s_2} \cdots p_n^{s_n}$ where $p_i$ are primes, then $(p_1), (p_2), \ldots, (p_n)$ are prime ideals and $f((\bar{p_i})) = (p_i)/m\mathbf{Z}$ are prime ideals. So all the prime ideals in $Z_m$ are $(\bar{p_i})$, $i, 1, 2, \ldots, n$.

**Exercise 3.2.22.** (a) If $R_1, \dots, R_n$ are rings with identity and $I$ is an ideal in $R_1 \times \cdots \times R_n$, then $I = A_1 \times \cdots \times A_m$, where each $A_i$ is an ideal in $R_i$.

(b) Show that the conclusion of (a) need not hold if the rings $R_i$ do not have identities.

**Exercise 3.2.23.** An element $e$ in a ring $R$ is said to be **idempotent** if $e^2 = e$. An element of the center of the ring $R$ is said to be central. If $e$ is a central idempotent in a ring $R$ with identity, then

(a) $1_R - e$ is a central idempotent;

(b) $eR$ and $(1_R - e)R$ are ideals in $R$ such that $R = eR \times (1_R - e)R$.

**Answer.** (a) $(1_R - e)^2 = 1_R - 2e + e^2 = 1_R - 2e + e = 1_R - e$. $\forall x \in R$, $ex = xe$ so $(1_R - e)x = x - ex = x - xe = x(1_R - e)$. $1_R - e$ is a central idempotent.

(b) $eR \cup (1_R - e)R \subset R$ so $\langle eR \cap (1_R - e)R\rangle \subset R$. $R = eR + (1_R - e)R$ so $R \subset \langle eR \cap (1_R - e)R\rangle$. So $R = \langle eR \cap (1_R - e)R\rangle$. $\langle eR\rangle = eR$ and $\langle (1_R - e)R\rangle = (1_R - e)R$ so $\langle eR\rangle \cap \langle (1_R - e)R\rangle = 0$. Thus $R = eR \times (1_R - e)R$.

**Exercise 3.2.24.** Idempotent elements $e_1, \dots, e_n$ in a ring $R$ are said to be **orthogonal** if $e_i e_j = 0$ for $i \neq j$. If $R, R_1, \dots, R_n$ are rings with identity, then the following conditions are equivalent:

(a) $R \cong R_1 \times \cdots \times R_n$.

(b) $R$ contains a set of orthogonal central idempotents $\{e_1, \dots, e_n\}$ such that $e_1 + e_2 + \cdots + e_n = 1_R$ and $e_i R \cong R$ for each $i$.

(c) $R$ is the internal direct product $R = A_1 \times \cdots \times A_n$ where each $A_i$ is an ideal of $R$ such that $A_i \cong R_i$.

**Answer.** Assume $f : R_1 \times \cdots \times R_n \to R$ is an isomorphism.
(a)$\Rightarrow$(b): Denote $\bar{e_1} = (1_R, 0, \dots, 0)$, $\bar{e_2} = (0, 1_R, \dots, 0)$, $\dots$, $\bar{e_n} = (0, 0, \dots, 1_R)$. They are orthogonal central idempotent in $S = R_1 \times \cdots \times R_n$ and $f(\bar{e_n}) = e_n$, $e_1 + e_2 + \cdots + e_n = 1_S$, $\sum\limits_{i=1}^{n} e_i S = S$.

Take $\varphi_i : (r_1, r_2, \ldots, r_i, \ldots, r_n) \mapsto r_i$. Then $\varphi_i$ is a well defined isomorphism between $e_iS$ and $R_i$. $e_iR \cong \bar{e}_iS \cong R_i$.
(b)$\Rightarrow$(c): Take $A_i = e_iR$, then $A_i \cong R_i$. We need to prove $R = e_iR \times e_2R \times \cdots \times e_nR$. $e_iR \cap (e_1R + e_2R + \cdots + e_{i-1}R + e_{i+1}R + \cdots + e_nR) = 0$ since $e_ix_i = e_1x_1 + e_2x_2 + \cdots + e_{i-1}x_{i-1} + e_{i+1}x_{i+1} + \cdots + e_nx_n \Rightarrow e_i^2x_i = 0$. $R = 1_RR = \sum_{i=1}^{n} e_iR$ so $R = e_iR \times e_2R \times \cdots \times e_nR$.
(c)$\Rightarrow$(a): Trivial.

**Exercise 3.2.25.** If $m \in \mathbf{Z}$ has a prime decomposition $m = p_1^{k_1} \cdots p_t^{k_t} (k_i > 0$; $p_i$ distinct primes), then there is an isomorphism of rings $Z_m \cong Z_{p_1^{k_1}} \times \cdots \times Z_{p_t^{k_t}}$.

**Answer.** For any $m \in \mathbf{Z}$, $\mathbf{Z}/m\mathbf{Z} \cong Z_m$. $p_1^{k_1}\mathbf{Z} \cap \cdots \cap p_t^{k_t}\mathbf{Z} = m\mathbf{Z}$. So $\exists \varphi : Z_m \mapsto Z_{p_i^{k_1}} \times \cdots \times Z_{p_t^{k_t}}$. $\forall i, j \in I$, $p_i^{k_i} \in p_i^{k_i}\mathbf{Z}$ and $p_j^{k_j} \in p_j^{k_j}\mathbf{Z}$, $\exists x, y \in \mathbf{Z}$ such that $xp_i^{k_i} + yp_j^{k_j} = 1 \in \mathbf{Z}$. So $p_i^{k_i}\mathbf{Z} + p_j^{k_j}\mathbf{Z} = \mathbf{Z}$, $\varphi$ is an isomorphism so $Z_m \cong Z_{p_1^{k_1}} \times \cdots \times Z_{p_t^{k_t}}$.

**Exercise 3.2.26.** If $R = \mathbf{Z}$, $A_1 = (6)$ and $A_2 = (4)$, then the map $\theta : R/A_1 \cap A_2 \to R_1/A_1 \times R_2/A_2$ of Corollary 2.27 is not surjective.

**Answer.** $R/(A_1 \cap A_2) = Z_{12}$, $R/A_1 = Z_6$ and $R/A_2 = Z_4$. $|Z_6 \times Z_4| = |Z_6| \times |Z_4| = 24$ but $|Z_{12}| = 12$, so $\theta$ is surjective.

## 3.3 Factorization in commutative rings

**Exercise 3.3.1.** A nonzero ideal in a principle ideal domain is maximal if and only if it is prime.

**Answer.** For PID $R$, $R^2 = R$ so every maximal ideal is prime. If $I = (p) \neq 0$ is prime in $R$, then $p$ is prime so $p$ is irreducible and $(p)$ is maximal.

**Exercise 3.3.2.** An integral domain $R$ is unique factorization domain if and only if every non zero prime ideal in $R$ contains a nonzero principle ideal that is prime.

**Answer.** Suppose $R$ is a unique factorization domain and $P \neq 0$ is a prime ideal. Let $x \in P$ be a nonzero nonunit. Then $x$ can be factored into $x = p_1 p_2 \cdots p_n$ a product of prime elements. Then $x \in P$ implies $p_i \in P$ for some $i$, so $(p_i) \subset P$.
Conversely, assume that each nonzero prime ideal of $R$ contains a principle prime ideal.

**Lemma.** *Let $R$ be a commutative ring and $S \subset R \backslash \{0\}$ a multiplicatively closed subset containing $1_R$. Let $\mathcal{I}_S$ be the set of ideals of $R$ which are disjoint from $S$. Then*
*(a) $\mathcal{I}_S$ is nonempty.*
*(b) Every element of $\mathcal{I}_S$ is contained in a maximal element oof $\mathcal{I}_S$.*
*(c) Every maximal element of $\mathcal{I}_S$ is prime.*

Here's the proof of the lemma:
(a) Trivial.
(b) Let $I \in \mathcal{I}_S$. Consider the subposet $P_I$ of $\mathcal{I}_S$ consisting of ideals which contain $I$. Since $I \in P_I$, $P_I$ is nonempty; moreover, any chain in $P_I$ has an upper bound,namely the union of all of its elements. Therefore by Zorn's lemma, $P_I$ has a maximal element of $\mathcal{I}_S$, which is clearly also a maximal element of $\mathcal{I}_S$.
(c) Let $I$ be a maximal element of $\mathcal{I}_S$; suppose that $x, y \in R$ are such that $xy \in I$. If $x$ is not in $I$, then $\langle I, x \rangle \supsetneq I$ and therefore contains an element $s_1$ of $S$, say

$$s_1 = i_1 + ax$$

Similarly, if $y$ is not in $I$, then we get an element $s_2$ of $S$ of the form

$$s_2 = i_2 + by$$

But then

$$s_1 s_2 = i_1 i_2 + (by)i_1 + (ax)i_2 + (ab)xy \in I \cap S$$

a contradiction!

A multiplicative subset $S$ is saturated if for all $x \in S$ and $y \in R$, if $y \mid x$ then $y \in S$. We define the saturation $\bar{S}$ of a multiplicatively closed subset $S$ to be the intersection of all saturated multiplicatively closed subsets containing $S$. Let $S$ be the set of units of $R$ together with all product of prime elements. One checks easily that $S$ is saturated multiplicative subset. We should show that $S = \frac{R}{\setminus}\{0\}$. Suppose then for a contradiction that there exists a nonzero nonunit $x \in R \backslash S$. Then saturation of $S$ implies that $S \cap (x) = \varnothing$, and then there exists a prime ideal $P$ contains $x$ and disjoint from $S$. But by the hypothesis, $P$ contains a prime element $p$, contradictting its disjointness from $S$.

**Exercise 3.3.3.** Let $R$ be the subring $\{a + b\sqrt{10} | a, b \in \mathbf{Z}\}$ of the field of real numbers

(a) The map $N : R \to Z$ given by $a + b\sqrt{10} \mapsto (a + b\sqrt{10})(a - b\sqrt{10}) = a^2 - 10b^2$ is such that $N(uv) = N(u)N(v)$ for all $u, v \in R$ and $N(u) = 0$ if and only if $u = 0$.

(b) $u$ is a unit in $R$ if and only if $N(u) = \pm 1$.

(c) $2, 3, 4 + \sqrt{10}$ and $4 - \sqrt{10}$ are irreducible elements of $R$.

(d) $2, 3, 4 + \sqrt{10}$ and $4 - \sqrt{10}$ are not prime elements of $R$.

**Answer.** (a) Assume $u = a_1 + b_1\sqrt{10}$, $v = a_2 + b_2\sqrt{10}$.

$$\begin{aligned} N(uv) &= N(a_1 a_2 + 10 b_1 b_2 + (a_1 b_2 + a_2 b_1)\sqrt{10}) \\ &= (a_1 a_1 + 10 b_1 b_2)^2 - 10(a_1 b_2 + a_2 b_1)^2 \\ &= a_1^2 a_2^2 + 100 b_1^2 b_2^2 - 10 a_1^2 b_2^2 - 10 a_2^2 b_1^2 \end{aligned}$$

$$N(u)N(v) = (a_1^2 - 10b_1^2)(a_2^2 - 10b_2^2) = N(uv)$$

(b) If $u$ is a unit of $R$, $N(uu^{-1}) = N(1) = N(u)N(u^{-1}) = 1$. $N(u)$ and $N(u^{-1}) \in \mathbf{Z}$ so $N(u) = \pm 1$.

(c) Suppose $4 + \sqrt{10} = (a_1 + b_1\sqrt{10})(a_2 + b_2\sqrt{10})$ where $N(a_1 + b_1\sqrt{10})$, $N(a_2 + b_2\sqrt{10}) \neq \pm 1$. $N(4 + \sqrt{10}) = 6 = N(a_1 + b_1\sqrt{10})N(a_2 + b_2\sqrt{10})$

so $N(a_1 + b_1\sqrt{10}) = \pm 2$ and $N(a_2 + b_2\sqrt{10}) = \pm 3$. WLOG, assume $N(a_1 + b_1\sqrt{10}) = 2$ and $N(a_2 + b_2\sqrt{10}) = 3$.

$$a_1^2 = 10b_1^2 + 2 \Rightarrow a_1^2 \equiv 2 \mod 10$$

$$a_2^2 = 10b_2^2 + 3 \Rightarrow a_2^2 \equiv 3 \mod 10$$

This can't be true! So $4 + \sqrt{10}$ is irreducible. Similarly, $2,3,4 - \sqrt{10}$ is irreducible.

(d) $3 \cdot 2 = (4 + \sqrt{10})(4 - \sqrt{10}) - 6$, But none of these four numbers divide another.

**Exercise 3.3.4.** Show that in the integral domain of **Exercise 3.3.3** every element can be factored into a product of irreducibles, but this factorization need not be unique.

**Answer.** Suppose $a$ can be factored into $a_1a_2\cdots a_n\cdots$ which may not be finite. We only need to prove there are finite $a_i$ are irreducible. $N(a) = N(a_1)N(a_2)\cdots N(a_n)\cdots = k \in \mathbf{Z}$. Assume $k = k_1k_2\cdots k_m$ and for irreducible $a_i$, $N(a_i) \neq \pm 1$, so there are at most $m$ $a_i$ irreducible. Thus $a$ can be factored into a product of irreducibles.

**Exercise 3.3.5.** Let $R$ be a principle ideal domain.

(a) Every proper ideal is a product $P_1P_2\cdots P_n$ of maximal ideals, which are unique ly determined up to order.

(b) An ideal $P$ in $R$ is said to be primary if $ab \in P$ and $a \notin P$ imply $b^n \in P$ for some $n$. Show that $P$ is primary if and only if for some $n$, $P = (p^n)$ where $p \in R$ is prime or $p = 0$.

(c) If $P_1, P_2, \ldots, P_n$ are primary ideals such that $P_i = (p_i^{n_i})$ and the $p_i$ are distinct primes, then $P_1P_2\cdots P_n = P_1 \cap P_2 \cap \cdots \cap P_n$.

(d) Every proper ideal in $R$ can be expressed (uniquely up to order) as the intersection of a finite number of primary ideals.

**Answer.** (a) For any ideal $(a)$, $a$ can be factored into irreducible product $a_1a_2\cdots a_n$. $(a_i)$ are maximal in $R$ and $(a) = (a_1)(a_2)\cdots(a_n)$.

(b) If $P = (p^n)$. For any $ab \in P$, $ab = p^n x$ for some $x \in R$ and $n \in \mathbf{Z}$. $R$ is a UFD so $p \mid a$ or $p \mid b$ so $b^n \in P$. Conversely, $\forall P = (k)$ we prove $k = p^t$ for some prime $p$ and $t \in \mathbf{Z}$. For any $ab = kx$, assume $a = a_1^1 \cdots a_m^{p_m}$, $b = a_1^{q_1} \cdots a_m^{q_m}$ and $k = a_1^{s_1} \cdots a_m^{s_m}$, $p_i$, $q_i$, $s_i$ are all nonnegative integers. We prove that for all but one $i$, $s_i = 0$. Take $p_i = 0$ for $i = 1, 2, \ldots, m-1$, $p_m = s_m$, $q_i = s_i$ for $i = 1, 2, \ldots, m-1$, $q_m = 0$, then $ab = k \in (k)$ but $a$, $a^n$, $b$, $b^n \notin (k)$ for all $n \in \mathbf{Z}$. So $k = a_i^{s_i}$ for some $s_i \in \mathbf{Z}$, $(k) = (a_i^{s_i})$, $a_i$ prime.

(c) $P_1P_2 \cdots P_n \subset P_1 \cap P_2 \cap \cdots \cap P_n$ is trivial.
For any $a \in P_1 \cap \cdots \cap P_n$, $p_i^{n_i}|a$, $\forall i = 1, 2, \ldots, n$. $p_i^{n_i} \neq p_j^{n_j}$ so $a = p_1^{n_1}x_1 \Rightarrow p_2^{n_2}|x_2 \Rightarrow a = p_1^{n_1}p_2^{n_2}x_2 \cdots \Rightarrow a = p_1^{n_1}p_2^{n_2}\cdots p_n^{n_n}x_n \in P_1P_2\cdots P_n$. So $P_1P_2\cdots P_n \subset P_1 \cap P_2 \cap \cdots \cap P_n$, $P_1 \cdots P_n = P_1 \cap \cdots \cap P_n$.

(d) For any ideal $(a) \subset R$, $(a) = P_1P_2 \cdots P_n$ which is the product of maximal ideals. So we can express $(a)$ as the product of $p_i' = (p_i^{s_i})$ since $n$ is finite. $(a) = P_1'P_2' \cdots P_m' = P_1' \cap P_2' \cap \cdots \cap P_m'$.

**Exercise 3.3.6.** (a) If $a$ and $n$ are integers, $n > 0$, then there exist integers $q$ and $r$ such that $a = qn + r$, where $|r| \leq n/2$.
(b) The Gaussian integers $\mathbf{Z}[i]$ form a Euclidean domain with $\varphi(a + bi) = a^2 + b^2$.

**Answer.** (a) Trivial.
(b) For $a_1 + b_1 i$, $a_2 + b_2 i \in \mathbf{Z}[i]$

$$\begin{aligned}
\varphi(a_1 + b_1 i)(a_2 + b_2 i) &= \varphi((a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)i) \\
&= (a_1a_2 - b_1b_2)^2 + (a_1b_2 + a_2b_1)^2 \\
&= (a_1a_2)^2 + (b_1b_2)^2 + (a_1b_2)^2 + (a_2b_1)^2 \\
&= (a_1^2 + b_1^2)(a_2^2 + b_2^2) \\
&= \varphi(a_1 + b_1 i)\varphi(a_2 + b_2 i)
\end{aligned}$$

For any $x \in \mathbf{Z}$, and $y = a + bi \in \mathbf{Z}[i]$, from (a) $a = q_1x + r_1$, $b = q_2x + r_2$ with $|r_1| \leq \frac{x}{2}$, $|r_2| \leq \frac{x}{2}$. Let $q = q_1 + q_2 i$, $r = r_1 + r_2 i$, then $y = qx + r$ with $r = 0$ or $\varphi(r) = r_1^2 + r_2^2 < \varphi(x)$. $\forall x = c + di \neq 0$, take $\bar{x} = c - di$, then there are $q$, $r_0 \in \mathbf{Z}[i]$ such that $y\bar{x} = qx\bar{x} + r_0$ with $r_0 = 0$ or $\varphi(r_0) < \varphi(x\bar{x})$. Let $r = y - qx$, then $y = qx + r$ and $r = 0$ or $\varphi(r) < \varphi(x)$.

**Exercise 3.3.7.** What are the units in the ring of Gaussian integers $\mathbf{Z}[i]$?

**Answer.** From **Exercise 3.3.6**, we proved that $\varphi(a+bi) = a^2+b^2$ satisfies that $\forall u, v \in \mathbf{Z}[i]$, $\varphi(uv) = \varphi(u)\varphi(v)$. So if there exist $u^{-1} = c+di$ such that $uu^{-1} = 1$, then $\varphi(u)\varphi(u^{-1}) = 1$ which means $(a^2+b^2)(c^2+d^2) = 1$. So $u = \pm 1$ or $\pm i$.

**Exercise 3.3.8.** Let $R$ be the following subring of the complex numbers: $R = \{a + b(1+\sqrt{19}i)/2 | a, b \in \mathbf{Z}\}$. The $R$ is a principle ideal domain that is not a Euclidean domain.

**Answer.** Take $\varphi(a + b(1+\sqrt{19}i)/2) = a^2 + ab + 5b^{p2}$. Denote $\tilde{R}$ as the collection of units in $R$ together with 0. An element $u \in R - \tilde{R}$ is called a universal side divisor if for every $x \in R$ there is some $z \in \tilde{R}$ such that $u$ divides $x - z$ in $R$.
Let $R$ be an integral domain that is not a field, if $R$ is a Euclidean domain then there are universal side divisors in $R$. Since $\varphi(R) \subset \mathbf{N}$ has a lower bound, we can choose $u \in R - \tilde{R}$ such that $\varphi(u)$ minimizes. Then $\forall x = qu+r$, $r = 0$ or $\varphi(r) < \varphi(u)$ so $r \in \tilde{R}$. Hence $u$ is a universal side divisor in $R$. Now we prove $R = \mathbf{Z}[(1+\sqrt{19}i)/2]$ is not a Euclidean domain by showing $R$ contains no universal side divisor. The units in $R$ are only $\pm 1$ so $\tilde{R} = \{\pm 1, 0\}$. $\forall a+b(1+\sqrt{19}i)/2 \in \mathbf{Z}[(1+\sqrt{19}i)/2] \backslash \mathbf{Z}$, $\varphi(a+b(1+\sqrt{19}i)/2) = a^2 + ab + 5b^2 \geq 5$. So the smallest nonzero value of $\varphi(x)$ is 1 and 4. Take $x = 2$ in the definition of universal side divisor, $u$ must divide 2 or 3. If $2 = ab$, then $4 = \varphi(a)\varphi(b)$ so the only divisor of 2 are $\pm 1$, $\pm 2$. Similarly the only divisor of 2 are $\pm 1$, $\pm 3$. So the value of $u$ should be $\pm 2$ or $\pm 3$. Take $x = (1+\sqrt{19}i)/2$ and it's easy to check that none of $x$, $x \pm 1$ are divisible by $\pm 2$, $\pm 3$. Thus none of these is a universal side divisor.
Next we prove $R$ is a principle ideal domain. Define $\varphi'$ to be a Dedekind-Hasse norm if $\varphi'$ is a positive norm and for every nonzero $a, b \in R$ either $a \in (b)$ or there exist $s, t \in R$ with $0 < \varphi'(sa - tb) < \varphi'(b)$.
For any principle ideal domain $R$, $R$ has a Dedekind-Hasse norm. Let $I$ be an nonzero ideal in $R$ and $b$ be a nonzero element of $I$ with $\varphi'(b)$ minimal. Suppose $a$ is any nonzero elements in $I$, so the ideal $(a, b)$ is contained in $I$. Then the Dedekind-Hasse condition on $\varphi'$ and the minimality of $b$ implies that $a \in (b)$, so $I = (b)$ is principle.
We prove $R = \mathbf{Z}[(1+\sqrt{19}i)/2]$ has a Dedekind-Hasse norm $\varphi$. Suppose $\alpha, \beta$ are nonzero elements of $R$ and $a/\beta \notin R$. We should show that there

are elements $s, t \in R$ with $0 < \varphi(s\alpha - t\beta) < \varphi(\beta)$, which is equivalent to $0 < \varphi(\frac{\alpha}{\beta}s - t) < 1$. Assume $\frac{\alpha}{\beta} = \frac{a+b\sqrt{19}i}{c} \in \mathbf{Q}[\sqrt{19}i]$ with integers $a, b, c$ having no common divisor and with $c > 1$. Since $a, b, c$ have no common divisor there are integers $x, y, z$ with $ax+by+ca = 1$. Write $ay-19bx = cq+r$ for some quatient $q$ and remainder $r$ with $|r| \leq c/2$ and let $s = y + x\sqrt{19}i$ and $t = q - z\sqrt{19}i$. Then

$$0 < \varphi(\frac{\alpha}{\beta}s - t) = \frac{(ay - 19bx - cq)^2 + 19(ax + by + cz)^2}{c^2} < \frac{1}{4} + \frac{19}{c^2}$$

so when $c \geq 5$ then condition is satisfied.
Suppose $c = 2$. Then one of $a, b$ is even and the other is odd, and then $s = 1$ and $t = \frac{(a-1)+b\sqrt{19}i}{2}$ are elements of $R$ satisfying the condition.
Suppose $c = 3$. The integer $a^2+19b^2$ is not divisible by 3. Assume $a^2+19b^2 = 3q + r$ with $r = 1$ or $r = 2$. Then $s = a - b\sqrt{19}i$ and $t = q$ satisfies the condition.
Suppose $c = 4$ so $a$ and $b$ are not both even. If one of $a, b$ is even and the other is odd, then $a^2 + 19b^2$ is odd, so we can write $a^2 + 19b^2 = 4q + r$ for some $q, r \in \mathbf{Z}$ and $0 < r < 4$. Then $s = a - b\sqrt{19}i$ and $t = q$ satisfies the condition. If $a$ and $b$ are both odd, then $a^2 + 19b^2 \equiv 4 \mod 8$, so we have $a^2 + 19b^2 = 8q + 4$ for some $q \in \mathbf{Z}$. Then $s = (a - b\sqrt{19}i)/2$ and $t = q$ are elements in $R$ satisfying the condition.

**Exercise 3.3.9.** Let $R$ be a unique factorization domain and $d$ a nonzero element of $R$. There are only a finite number of distinct principle ideals that contain the ideal $(d)$.

**Answer.** Assume $d = p_1^{s_1}p_2^{s_2}\cdots p_n^{s_n}$. For some $k$ satisfies that $(d) \subset (k)$, we have $k \mid d$. So $kx = p_1^{s_1}p_2^{s_2}\cdots p_n^{s_n}$ for $x \in \mathbf{R}$. Thus $k = p_1^{t_1}\cdots p_n^{t_n}$, where $t_i \leq s_i$, whence the choices of $k$ are finite.

**Exercise 3.3.10.** If $R$ is a unique factorization domain and $a, b \in R$ are relatively prime and $a \mid bc$, then $a \mid c$.

**Answer.** Assume $d = p_1^{s_1}p_2^{s_2}\cdots p_n^{s_n}$, $a|bc \Rightarrow ax = bc$ for some $x \in R$. $a, b$ are relatively prime so for any prime ideal $(p_i)$, $p_i \nmid b$, $c \in (p_i)$. Assume $p_ic_1 = c$, $p_ia_1 = a$, then $c_1b = a_1x$. Similarly, $c \in (p_i)$, we can continue this step so $c \in (p_i^{s_i})$. $c \in (a) = (p_1^{s_1})(p_2^{s_2})\cdots(p_n^{s_n})$.

**Exercise 3.3.11.** Let $R$ be a Euclidean ring and $a \in R$. Then $a$ is a unit in $R$ if and only if $\varphi(a) = \varphi(1_R)$.

**Answer.** If $a$ is a unit, then $\exists a^{-1} \in R$, $aa^{-1} = 1_R$. $a = a \cdot 1_R$ so $\varphi(1_R) < \varphi(a \cdot 1_R) = \varphi(a)$, $\varphi(a) \leq \varphi(aa^{-1}) = \varphi(1_R)$ so $\varphi(a) = \varphi(1_R)$.
If $\varphi(a) = \varphi(1_R)$, $\forall x \in R\backslash\{0\}$, $x = x \cdot 1_R$ so $\varphi(x) \geq \varphi(1_R)$. Assume $1_R = qa + r$, $\varphi(r) \geq \varphi(a)$ for all $r \in R\backslash\{0\}$. So $r = 0$, $1_R = qa$, $a$ is a unit.

**Exercise 3.3.12.** Every nonempty set of elements (possibly infinite) in a commutative principle ideal ring with identity has a greatest common divisor.

**Answer.** Denote $S = \{(a) | \bigcup\limits_{i \in I} (a_i) \subset (a)\}$. $S$ is nonempty since $R \in S$. For finite $I$, the conclusion is trivial. For infinite $I$. Assume $(d) = \bigcap\limits_{A \in S} A$ which is a well defined ideal. $\bigcap\limits_{i \in I} (a_i) \subset (d)$ so $(a_i) \subset (d) \Rightarrow d \mid a_i$ for all $i \in I$. And $\forall c \mid a_i$ for all $i \in I$, $(c) \subset S$ so $(d) \subset (c)$, $c \mid d$. $d$ is the greatest common divisor of $\{a_i | i \in I\}$.

**Exercise 3.3.13.** Let $R$ be a Euclidean domain with associated function $\varphi : R - \{0\} \to \mathbf{N}$. If $a, b \in R$ and $b \neq 0$, here is a method for finding the greatest common divisor of $a$ and $b$. By repeated use of Definition 3.8(ii) we have:

$$\begin{gathered}
a = q_0 b + r_1, \quad \text{with} \quad r_1 = 0 \quad \text{or} \quad \varphi(r_1) < \varphi(b); \\
b = q_1 r_1 + r_2, \quad \text{with} \quad r_2 = 0 \quad \text{or} \quad \varphi(r_2) < \varphi(1); \\
r_1 = q_2 r_2 + r_3, \quad \text{with} \quad r_3 = 0 \quad \text{or} \quad \varphi(r_3) < \varphi(2); \\
\vdots \\
r_k = q_{k+1} r_{k+1} + r_{k+2}, \quad \text{with} \quad r_{k+2} = 0 \quad \text{or} \quad \varphi(r_{k+2}) < \varphi(k+1); \\
\vdots
\end{gathered}$$

Let $r_0 = b$ and let $n$ be the least integer such that $r_{n+1} = 0$ (such an $n$ exists since the $\varphi(r_k)$ form a strictly decreasing squence of nonnegative integers). Show that $r_n$ is the greatest common divisor $a$ and $b$.

**Answer.** $r_n$exists since $\varphi(r_i)$ decreases. $r_n \mid a$ and $r_n \mid b$ is simple. We prove $(a) + (b) = (r_n)$. $r_n \mid a$, $r_n \mid b$ so $(a) \subset (r_n)$, $(b) \subset (r_n) \Rightarrow (a) + (b) \subset (r_n)$. We use induction to prove $(r_n) \subset (a) + (b)$: 1. For $i = 1$, $a = q_0 b + r \Rightarrow r_1 = a - q_0 b \in (a) + (b)$. 2. Assume for $i \leq m$, $(r_i) \subset (a) + (b)$, $r_{m-1} = q_m r_m + r_{m-1} \Rightarrow r_{m+1} = r_{m-1} - q_m r_m \in (r_m) + (r_{m-1}) \subset (a) + (b)$. So $(r_n) \subset (a) + (b)$. $r_n$ is the greatest common divisor of $a$ and $b$.

## 3.4 Rings of quotients and localization

**Exercise 3.4.1.** Determine the complete ring of quotients of the ring $Z_n$ for each $n \geq 2$.

**Answer.** For the complete multiplicative subset $S$ of $Z_n$, $S = \{\bar{x}|(x, n) = 1\}$ so the complete ring of quotient is $S^{-1}Z_n$.

**Exercise 3.4.2.** Let $S$ be a multiplicative subset of a commutative ring $R$ with identity and let $T$ be a multiplicative subset of the ring $S^{-1}R$. Let $S_* = \{r \in R | r/s \in T \text{ for some } s \in S\}$. Then $S_*$ is a multiplicative subset of $R$ and there is a ring isomorphism $S_*^{-1}R \cong T^{-1}(S^{-1}R)$.

**Answer.** For any $r_1/s_1$ , $r_2/s_2 \in T$. $r_1r_2/s_1s_2 \in T$. And there exists a monomorphism $\varphi : S_* \to T$ given by $\varphi : r \mapsto r/s$ for some $s \in S$ by the definition of $S_*$. So $\forall r_1, r_2 \in S_*$, $\exists \varphi(r_1)\varphi(r_2) = r_1r_2/s_1s_2 \in T$, thus $r_1r_2 \in S_*$. $S_*$ is a multiplicative subset.
Next we prove $S_*^{-1}R \cong T^{-1}(S^{-1}R)$. $\forall s \in S_*$ and $r \in R$, $sr \in S_*$ since if there exists some $s' \in S$, $s/s' \in T$ then $sr/s'r = s/s' \in T$. For any $(r/s)/(r'/s') \in T^{-1}(S^{-1}R)$ where $r \in R$ and $s \in S$, $r'/s' \in T$, we construct a map $\varphi : T^{-1}(S^{-1}R) \to S_*^{-1}R$ given by $\varphi : (r/s)/(r'/s') \mapsto rs'/sr'$. $\varphi$ is well defined since $rs' \in R$ and $sr' \in S_*$. Now we check $\varphi$ is an isomorphism. $\forall (r_1/s_1)/(r_1'/s_1'), (r_2/s_2)/(r_2'/s_2') \in T^{-1}(S^{-1}R)$

$$
\begin{aligned}
&\varphi((r_1/s_1)/(r_1'/s_1') + (r_2/s_2)/(r_2'/s_2')) \\
=&\varphi((((r_1/s_1)(r_2'/s_2') + (r_2/s_2)(r_1'/s_1'))/((r_1'/s_1')/(r_2'/s_{2'}))) \\
=&\varphi((r_1r_2'/s_1s_2' + r_2r_1'/s_2s_1')/(r_1'r_2'/s_1's_2')) \\
=&\varphi((((r_1r_2's_2s_1' + r_2r_1's_1s_2')/s_1s_2s_1's_2')/(r_1'r_2'/s_1's_2')) \\
=&(((r_1r_2's_2s_1' + r_2r_1's_1s_2')s_1's_2')/s_1s_2s_1's_2'r_1'r_2') \\
=&((r_1r_2's_2s_1' + r_2r_1's_1s_2')/s_1s_2r_1'r_2') \\
=&(r_1s_1')/(r_1's_1) + (r_2s_2')/(r_2's_2) \\
=&\varphi((r_1/s_1)/(r_1'/s_1/)) + \varphi((r_2/s_2)/(r_1'/s_2'))
\end{aligned}
$$

The conservation of multiplication is trivial. $\varphi$ is a homomorphism and $\varphi$ is obviously injective, so $\left|T^{-1}(S^{-1}R)\right| \leq \left|S^{-1}R\right|$.
Take $\tau : S_*^{-1}R \to T^{-1}(S^{-1}R)$ given by $\tau : r/s \mapsto (r/s')/(s/s')$. Similarly, $\tau$ is injective so $|S_*R| \leq \left|T^{-1}(S^{-1}R)\right|$. $\varphi$ is isomorphism and $S_*^{-1}R \cong T^{-1}(S^{-1}R)$.

**Exercise 3.4.3.** (a) The set $E$ of positive even integers is a multiplicative subset of $\mathbf{Z}$ such that $E^{-1}(\mathbf{Z})$ is field of rational numbers.
(b) State and prove condition(s) on a multiplicative subset of $S$ of $\mathbf{Z}$ which insure that $S^{-1}\mathbf{Z}$ is a field of rationals.

**Answer.** (a) Trivial.
(b) Assume the primes $p \in \mathbf{Z}$ forms a set $P$. For any multiplicative subset $S$ and $x \in S$ then $\{x^n | n \in \mathbf{Z}\} \subset S$. If $\forall p \in P$, $\exists x \in S$ such that $p \mid x$, we prove $S^{-1}\mathbf{Z}$ forms the field of rationals. For any $p/q \in \mathbf{Q}$, $q = q_1^{t_1} q_2^{t_2} \cdots q_n^{t_n}$ and for any $q_i$ there exists $x_1 \in S$, $x_1 = a_i q_i$. Take $x = a_1^{t_1} q_1^{t_1} \cdots a_n^{t_n} q_n^{t_n}$ and $y = a_1^{t_1} \cdots a_n^{t_n} p$. Then $y/x = p/q$, $y/x \in S^{-1}\mathbf{Z}$. So $S^{-1}\mathbf{Z}$ forms the field of rationals.
For any other multiplicative subset $S$, assume $p \in P$ and $\forall x \in S$, $p \nmid x$ then $\forall y/x \in S^{-1}\mathbf{Z}$, $yp - x \neq 0$ so $1/p \notin S^{-1}\mathbf{Z}$, $S^{-1}\mathbf{Z}$ isn't the rational field.

**Exercise 3.4.4.** If $S = \{2, 4\}$ and $R = Z_6$, then $S^{-1}R$ is isomorphic to the field $Z_3$. Consequently, the converse of Theorem 4.3(ii) is false.

**Answer.** $S^{-1}Z_6 = \{1/3, 2/3, 3/3\}$ so $S^{-1}Z_6 \cong Z_3$ is a integral domain. However, $Z_6$ has no zero divisor.

**Exercise 3.4.5.** Let $R$ be an integral domain with quotient field $F$. If $T$ is an integral domain such that $R \subset T \subset F$, then $F$ is (isomorphic to) the quotient field of $T$.

**Answer.** Consider $T_i$ which is a PID satisfying $R \subset T_i \subset F$, $T_i$ forms a category with the inclusion map as morphisms. $T_i'$ is the quotient field of $T_i$ so $R \subset T_i' \Rightarrow R \subset F \subset T_i'$ (up to isomorphic). $R \subset T_j \subset F \subset T_i'$ for all $i, j$ thus $T_i' \subset T_j'$. Similarly $T_j' \subset T_i'$ so all the $T_i'$ are universal under the inclusion map. Thus $F$ is isomorphic to the quotient field of $T$.

**Exercise 3.4.6.** Let $S$ be a multiplicative subset of an integral domain $R$ such that $0 \notin S$. If $R$ is a principle ideal domain, then so is $S^{-1}R$.

**Answer.** Actually this is true if and only if $1_R \in S$. For any ideal $J \subset S^{-1}R$, there exists ideal $I \subset R$ and $\varphi_S(I) = J$, $J = S^{-1}I = S^{-1}(a)$ for some $a \in R$. Since $1_R \in S$, $a/1_R \in S^{-1}(a)$. So $\forall s \in S$, $1_R/s$ is a unit of $S^{-1}(a)$, so $S^{-1}(a) = (a/1_R)$ is a principle ideal. Thus the multiplicative subset of $R$ is a principle ideal domain.

**Exercise 3.4.7.** Let $R_1$ and $R_2$ be integral domains with quotient fields $F_1$ and $F_2$ respectively. If $f : R_1 \to R_2$ is an isomorphism, then $f$ extends to an isomorphism $F_1 \cong F_2$.

**Answer.** For $f : R_1 \to R_2$, and the inclusion map $\subset R_2 \to F_2$, $\subset \circ f = R_1 \to F_2$ so there exists $\overline{\subset \circ f} : F_1 \to F_2$ which is a well defined homomorphism of rings. $\overline{\subset \circ f}|R_1 = f$, $\overline{\subset \circ f}$ is a monomorphism so $|F_1| \leq |F_2|$. Similarly, $|F_2| \leq |F_1|$ so $\overline{\subset \circ f}$ is an isomorphism and $F_1 \cong F_2$.

**Exercise 3.4.8.** Let $R$ be a commutative ring with identity, $I$ an ideal of $R$ and $\pi : R \to R/I$ the canonical projection.

(a) If $S$ is a multiplicative subset of $R$, then $\pi S = \pi(S)$ is a multiplicative subset of $R/I$.

(b) The mapping $\theta : S^{-1}R \to (\pi S)^{-1}(R/I)$ given by $r/s \mapsto \pi(r)/\pi(s)$ is a well-defined function.

(c) $\theta$ is a ring epimorphism with kernel $S^{-1}I$ and hence induces a ring isomorphism $S^{-1}R/S^{-1}I \cong (\pi S)^{-1}(R/I)$.

**Answer.** (a) For any $a, b \in S$, $\pi(a) = a + I$, $\pi(b) = b + I$, $\pi(a)\pi(b) = ab + I = \pi(ab) \in \pi S$, so $\pi S$ is a multiplicative subset of $R/I$.

(b) If $r_1/s_1 = r_2/s_2$ then $x(r_1s_2 - r_2s_1) = 0$ for some $x \in S$.

$$\theta(r_1/s_1) = \pi(r_1)/\pi(s_1) = (r_1 + I)/(s_1 + I)$$

$$\theta(r_2/s_2) = \pi(r_2)/\pi(s_2) = (r_2 + I)/(s_2 + I)$$

$$\begin{aligned} &(x + I)((r_1 + I)(s_2 + I) - (r_2 + I)(s_1 + I)) \\ =&(xr_1s_2 + I) - (xr_2s_1 + I) \\ =&x(r_1s_2 - r_2s_1) + I \\ =&I \end{aligned}$$

so $\theta(r_1/s_1) = \theta(r_2/s_2)$, $\theta$ is well-defined.

(c) $\pi$ is a homomorphism and so is $\theta$. $\theta$ is obviously an epimorphism and $\forall r/s \in S^{-1}I$, $\theta(r/s) = \pi(r)/\pi(s)$. $\pi(r) = I$ so $\theta(r/s) \in (\pi S)^{-1}I$, $S^{-1}I \subset \mathrm{Ker}\theta$. For any $r/s \notin S^{-1}I$, $\theta(r/s) = (r+I)/(s+I) \neq I$, so $\mathrm{Ker}\theta \subset S^{-1}I$. $\mathrm{Ker}\theta = S^{-1}I$, $S^{-1}R/\mathrm{Ker}\theta \cong \mathrm{Im}\theta \Rightarrow S^{-1}R/S^{-1}I \cong (\pi S)^{-1}(R/I)$.

**Exercise 3.4.9.** Let $S$ be a multiplicative subset of a commutative ring $R$ with identity. If $I$ is an ideal in $R$, then $S^{-1}(\mathrm{Rad}I) = \mathrm{Rad}(S^{-1}I)$.

**Answer.** $\mathrm{Rad}I = \{r | r^n \in I \text{ for some } n\}$. For any $r/s \in S^{-1}\mathrm{Rad}I$, $(r/s)^n = r^n/s^n \in S^{-1}I$ so $S^{-1}\mathrm{Rad}I \subset \mathrm{Rad}(S^{-1}I)$.
For any $a/b \in \mathrm{Rad}(S^{-1}I)$, $b \in S$ then $a^n b' - b^n a' = 0$ with $a' \in I$ and $b' \in S$. $(ab')^n = (b')^{n-1} b^n a' \in I$ so $a/b = ab'/bb' \in S^{-1}(\mathrm{Rad}I)$. Thus $S^{-1}(\mathrm{Rad}I) \subset \mathrm{Rad}(S^{-1}I)$. So $S^{-1}(\mathrm{Rad}I) = \mathrm{Rad}(S^{-1}I)$.

**Exercise 3.4.10.** Let $R$ be an integral domain and for each maximal ideal $M$, consider $R_M$ as a subring of the quotient field of $R$. Show that $\cap R_M = R$, where the intersection is taken over all maximal ideals $M$ of $R$.

**Answer.** $M$ is maximal so $1_R \in R - M$, which means $R \subset R_M$ for any $M$. So $R \subset \cap R_M$.
Denote $R'$ as the quotient field of $R$. For any $M$ maximal, $R_M \subset R'$. For any $x \in R' - R$, we prove there exists $M$ maximal and $x \notin R_M$. Take $A = \{a | ax \in R\}$, $A$ is an ideal of $R$. So $\exists A \subset M$ with $M$ maximal. If $x \in R - M$, $x = r/s$, so $xs = r \in R$, $s \in I \subset M$. That's contradictory! Thus $\cap R_M \subset R$, $R = \cap R_M$.

**Exercise 3.4.11.** Let $p$ be a prime in **Z**l then $(p)$ is a prime ideal. What can be said about the relationship of $Z_p$ and the localization $Z_{(p)}$?

**Answer.** $Z_p$ can be embedded into $\mathbf{Z}_{(p)}$ since $Z_p \subset \mathbf{Z} \subset (p)_{(p)} \subset \mathbf{Z}_{(p)}$.

**Exercise 3.4.12.** A commutative ring with identity is local if and only if for all $r, s \in R$, $r + s = 1_R$ implies $r$ or $s$ is a unit.

**Answer.** If $R$ is local, $r + s = 1_R \Rightarrow (r) + (s) = R$. $R$ has unique maximal ideal $M$ so $(r) \subset M$, $(s) \subset M$, $(r) + (s) = R \subset M$. That's contradictory! So $(r) = R$ or $(s) = R$, $r$ or $s$ is a unit.
Conversely, if there exist $M_1$, $M_2$ are maximal ideals. $M_1 + M_2 = R$ so $\exists r \in M$, $s \in M_2$ such that $r + s = 1_R$. WLOG assume $r$ is unit, $R = (r) \subset M_1$, that's contradictory! So $R$ is local.

**Exercise 3.4.13.** The ring $R$ consisting of all rational numbers with denominators not divisible by some (fixed) prime $p$ is a local ring.

**Answer.** Denote the set of primes in the question as $P$. Then $(P)$ is a prime ideal in $\mathbf{Z}$. So $S = \mathbf{Z} = (P)$ is multiplicative subset. We prove $R = \mathbf{Z}_{(P)}$. $\forall r/s \in \mathbf{Z}_{(p)}$, $r \in \mathbf{Z}$ and $s \notin (P)$ so $r/s \in R$. Thus $\mathbf{Z}_{(P)} \subset R$. Conversely, $\forall r/s \in R$, suppose $s = p_1^{t_1} p_2^{t_2} \cdots p_n^{t_n}$, $\forall p \in P$, $p \nmid s$ so $(p_i) \not\subset$ for all $i = 1, 2, \ldots, n$. Thus $(p_i) \subset S$ so $s \in S$, $r/s \in \mathbf{Z}_{(P)}$. $\mathbf{Z}_{(P)} = R$ is a lcoal ring.

**Exercise 3.4.14.** If $M$ is a maximal ideal in a commutative ring $R$ with identity and $n$ is a positive integer, then the ring $R/M^n$ has a unique prime ideal and therefore is local.

**Answer.** Consider the homomorphism $f : R \to R/M^n$. For any prime ideal $I \subset R/M^n$, $J = f^{-1}(I)$ is a prime ideal contains $M^n$. $M^n \subset P \Rightarrow M \subset P$, since $M$ is maximal, $P = M$ so the only prime ideal in $R/M^n$ is $R/M$.

**Exercise 3.4.15.** In a commutative ring $R$ with identity the following conditionns are equivalent: (i) $R$ has a unique prime ideal; (ii) every nonunit is nilpotent; (iii) $R$ has a minimal prime ideal which contains all zero divisors, and all nonunits of $R$ are zero divisors.

**Answer.** We first prove a lemma:

**Lemma.** *For an ideal $I \subset R$,* $\mathrm{Rad}I = \bigcap_{I \subset P_i} P_i$ *where $P_i$ are prime ideals.*

Proof of the lemma: $\forall a \in \mathrm{Rad}I$, $a^n \in I$ for some $n$, so $\forall I \subset P_i$ with $P_i$ prime. $a^n \in P_i \Rightarrow a \in P_i$ so $\mathrm{Rad}I \subset \bigcap_{I \subset P_i} P_i$.
Conversely $\forall a \notin \mathrm{Rad}I$, we only need to find $I \subset P_i$ and $a \notin P_i$. Take $A = \{J | a^n \in J \, \forall n \in \mathbf{N}\}$. $A$ has maximal element under $\subset$ by Zorn's lemma. Denote the maximal element as $P$. $\forall x, y \in R$ and $x \notin P$, $y \notin P$. Then $\exists m, n \in \mathbf{N}$, $a^n \in (x) + P$, $a^m \in (y) + P$, so $a^{m+n} \in (xy) + P \Rightarrow xy \notin P$. Thus $P$ is prime. That's contradictory! So $\bigcap_{I \in P_i} P_i \subset \mathrm{Rad}I$. The lemma has been proved.
(i)$\Rightarrow$(ii): $0 \in P$ where $P$ is the unique prime ideal, so $P = \{a | a^n = 0 \text{ for some } n\}$. For any nonunit $a$, $(a) \subset M = P$ so $a \in P$, there exists $n \in \mathbf{N}$ such that $a^n = 0$.
(ii)$\Rightarrow$(i): Denote $N$ as the ideal contains all the nilpotent elements. Take $\varphi : R \to R/N$. For any unit $u$, $\varphi(u)$ is also a unit. So $R/N$ is a field, $N$ is maximal in $R$. For any prime ideal $P$, $N \subset P$ from the lemma. Thus $N$ is the only prime ideal.
(ii)$\Rightarrow$(iii): Denote $N$ as the ideal contains all the nilpotent elements. All nilpotent elements are zero divisors by the definition. $N$ is prime and minimal is the direct corollary of the lemma.
(iii)$\Rightarrow$(ii): Denote $I$ as the minimal prime ideal and $N$ as the ideal contains all the nilpotent elements. Then $N \subset I$. Since all then nonunits are zero divisors, we have $N$ itself a prime ideal. So $N = I$.

**Exercise 3.4.16.** Every nonzero homomorphic image of a local ring is local.

**Answer.** Suppse $L$ is a local ring and $\varphi : L \to R$ is a ring of rings. Then $\varphi$ is an one-to-one correspondence between ideals in $L$ and ideals in $R$. For the maximal ideal $M$ in $L$, $\varphi(M) \subseteq R$, so $\varphi(M)$ contains all the proper ideals in $R$. $R$ is a local ring.

## 3.5 Rings of polynomials and formal power series

**Exercise 3.5.1.** (a) If $\varphi : R \to S$ is a homomorphism of rings, then the map $\bar{\varphi} : R[[x]] \to S[[x]]$ given by $\bar{\varphi}(\sum a_i x^i) = \sum \varphi(a_i)x^i$ is a homomorphism of rings such that $\bar{\varphi}(R[x]) \subset S[x]$.

(b) $\bar{\varphi}$ is a monomorphism if and only if $u\varphi$ is. In this case $\bar{\varphi} : R[x] \to S[x]$ is also a monomorphism.

(c) Extend the results of (a) and (b) to the polynomial rings $R[x_1, \ldots, x_n]$, $S[x_1, \ldots, x_n]$.

**Answer.** (a) It's easy to show $\bar{\varphi}(\sum a_i x^i)\bar{\varphi}(\sum b_i x^i) = \bar{\varphi}(\sum c_i)x^i$, $c_n = \sum\limits_{j=0}^{n} a_j b_{n-j}$ and $\bar{\varphi}(\sum a_i x^i) + \bar{\varphi}(\sum b_i x^i) = \bar{\varphi}(\sum(a_i + b_b)x^i)$. $\forall f(x) = \sum\limits_{i=0}^{n} a_i x^i \in R[x]$, $\bar{\varphi}(f(x)) = \bar{\varphi}(\sum\limits_{i=0}^{n} a_i x^i) = \sum\limits_{i=0}^{n} \varphi(a_i)x^i \in S[x]$. So $\bar{\varphi}(R[x]) \subset S[x]$.

(b) If $\varphi$ is monomorphism [epimorphism], it's easy to show that $\bar{\varphi}$ is also monomorphism [epimorphism].

Conversely, if $\bar{\varphi}$ is monomorphism, take $a_i \in R[[x]]$, then $\bar{\varphi}(a_i) = \varphi(a_i)$, $\varphi$ is also a monomorphism.

Similarly, $\varphi$ is epimorphism if $\bar{\varphi}$ is.

(c) It's trivial to since $R[x] \subset R[[x]]$, $S[x] \subset S[[x]]$.

**Exercise 3.5.2.** Let $\mathrm{Mat}_n R$ be the ring of $n \times n$ matrices over a ring $R$. Then for each $n \geq 1$:

(a) $(\mathrm{Mat}_n R)[x] \cong \mathrm{Mat}_n R[x]$.

(b) $(\mathrm{Mat}_n R)[[x]] \cong \mathrm{Mat}_n R[[x]]$.

**Answer.** (a) Take $x = (p_{ij}(x)) \in \mathrm{Mat}_n R[x]$, $p_{ij}(x) = \sum\limits_{k=0}^{n_{ij}} a_{ijk}x^k$. Take $n = \max\limits_{0<i,j\leq n} n_{ij}$, and for those $n \geq k > n_{ij}$, take $a_{ijk} = 0$. Denote $X_k = (a_{ijk})$, $x' = \sum\limits_{i=0}^{n} X_i x^i \in (\mathrm{Mat}_n R)[x]$. We prove $\varphi : x \mapsto x'$ is an isomorphism between rings.

For $x, x' \in \mathrm{Mat}_n R[x]$, $x = (p_{ij}(x))$, $x' = (p'_{ij}(x))$, $p_{ij}(x) = \sum\limits_{k=0}^{n_{ij}} a_{ijk}x^k$,

$p'_{ij}(x) = \sum_{k=0}^{n_{ij}} a'_{ijk}x^k$.

$$\begin{aligned}\varphi(x+x') &= \varphi(p_{ij}(x) + p'_{ij}(x)) \\ &= \begin{pmatrix} a_{110}+a'_{110} & \cdots & \\ \vdots & \ddots & \\ & & a_{nn0}+a'_{nn0} \end{pmatrix} \\ &+ \begin{pmatrix} a_{111}+a'_{111} & \cdots & \\ \vdots & \ddots & \\ & & a_{nn1}+a'_{nn1} \end{pmatrix} x + \cdots \\ &= \varphi(x) + \varphi(x')\end{aligned}$$

$$\varphi(xx') = \varphi((p_{ij}(x))(p'_{ij}(x))) = \varphi((\sum_{k=1}^{n} p_{ik}(x)p'_{kj}(x)))$$

$$\begin{aligned}\sum_{k=1}^{n} p_{ik}(x)p'_{kj}(x) &= \sum_{k=1}^{n}(\sum_{m=0}^{n_{ik}} a_{ikm}x^m)(\sum_{m=0}^{n'_{kj}} a'_{kjm}x^m) \\ &= \sum_{w}\sum_{k=1}^{n}\sum_{m=1}^{w} a_{ikm}a'_{kj(w-m)}x^w\end{aligned}$$

so

$$\begin{aligned}\varphi(xx') &= \varphi((\sum_{w}\sum_{k=1}^{n}\sum_{m=1}^{w} a_{ikm}a'_{kj(w-m)}x^w)) \\ &= \sum_{w}(\sum_{k=1}^{n}\sum_{m=1}^{w} a_{ikm}a_{kj(w-m)})x^w \\ \varphi(x)\varphi(x') &= (\sum_{w}(a_{ijw})x^w)(\sum_{w}(a'_{ijw})x^w) \\ &= \sum_{w}(\sum_{k=1}^{n}\sum_{m=1}^{w} a_{ikm}a_{kj(w-m)})x^w\end{aligned}$$

so $\varphi(xx') = \varphi(x)\varphi(x')$, $\varphi$ is a well defined homomorphism. Ker$\varphi = 0$ so $\varphi$ is a monomorphism. For any $\sum_{w}(a_{ijw})x^w \in \mathrm{Mar}_n R[x]$, $\exists(\sum_{w} a_{ijw}x^w) \in (\mathrm{Mat}_n R)[x]$ s.t. $\varphi(\sum_{w} a_{ijw}x^w = \sum_{w}(a_{ijw})x^w)$. So $\varphi$ is an epimorphism.

**Exercise 3.5.3.** Let $R$ be a ring and $G$ an infinte multiplicative cyclic group with generator denoted $x$. Is the group ring $R(G)$ isomorphic to the polynomial ring in one indeterminate over $R$?

**Answer.** $R(G)$ is not isomorphic to $R[x]$ since there's no isomorphic image of $rx^{-1} \in R(G)$ in $R[x]$.

**Exercise 3.5.4.** (a) Let $S$ be a nonempty set and let $\mathrm{N}^S$ be the set of all functions $\varphi : S \to \mathbf{N}$ such that $\varphi(s) \neq 0$ for at most a finite number of elements $s \in S$. Then $\mathbf{N}^S$ is a multiplicative abelian monoid with prooduct defined by

$$(\varphi\psi)(s) = \varphi(s) + \psi(s)\,(\varphi, \psi \in \mathbf{N}^S; s \in S)$$

The identity element in $\mathbf{N}^S$ is the zero function.

(b) For each $x \in S$ and $i \in \mathbf{N}$ let $x^i \in \mathbf{N}^S$ be defined by $x^i(x) = i$ and $x^i(s) = 0$ for $s \neq x$. If $\varphi \in \mathbf{N}^S$ and $x_1, \ldots, x_n$ are the only elements of $S$ such that $\varphi(x_i) \neq 0$, then in $\mathbf{N}^S$, $\varphi = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$, where $i_j = \varphi(x_j)$.

(c) If $R$ is a ring with identity let $R[S]$ be the set of all functions $f : \mathbf{N}^S \to R$ such that $f(\varphi) \neq 0$ for at most a finite number of $\varphi \in \mathbf{N}^S$. Then $R[S]$ is a ring with identity, where addition and multiplication are defined as follows:

$$(f + g)(\varphi) = f(\varphi) + g(\varphi)\,(f, g \in R[S]; \varphi \in \mathbf{N}^S)$$

$$(fg)(\varphi) = \sum f(\theta)g(\zeta)\,(f, g \in R[S]; \theta, \zeta, \varphi \in \mathbf{N}^S)$$

where teh sum is over all pairs $(\theta, \zeta)$ such that $\theta\zeta = \varphi$. $R[S]$ is called the ring of polynomials in $S$ over $R$.

(d) For each $\varphi = x_1^{i_1} \cdots x_n^{i_n} \in \mathbf{N}^S$ and each $r \in R$ we denote by $rx_1^{i_1} \cdots x_n^{i_n}$ the function $\mathbf{N}^S \to R$ which is $r$ at $\varphi$ and 0 elsewhere. Then every nonzero element $f$ of $R[S]$ can be written in the form $\int = \sum_{i=0}^{m} r_i x_1^{k_{i1}} x_2^{k_{i2}} \cdots x_n^{k_{in}}$ with the $r_i \in R$, $x_i \in S$ and $k_{ij} \in \mathbf{N}$ all uniquely determined.

(e) If $S$ is finite of cardinality $n$, then $R[S] \cong R[x_1, \ldots, x_n]$.

(f) State and prove an analogue of Theorem 5.5 for $R[S]$.

**Answer.** (a) $\varphi\psi = \varphi + \psi : S \to \mathbf{N}$ so $\varphi\psi \in \mathbf{N}^S$. For any $\varphi \in \mathbf{N}^S$, $\varphi 0 = 0\varphi = \varphi + 0 = 0 + \varphi = \varphi$. So $\mathbf{N}^S$ is a monoid.

(b) For any $\varphi \in \mathbf{N}^S$, $x_1, x_2, \ldots, x_n$ are the only element s.t. $\varphi(x_i) \neq 0$. We prove it has the form $\varphi = x_1^{i_1} \cdots x_n^{i_n}$. Suppose $\varphi(x_j) = i_j$. Take $\varphi_1 = \varphi - x_n$ then $x_1, x_2, \ldots, x_{n-1}$ are the only element s.t. $\varphi_1(x_i) \neq 0$. Continue this step, we can have $\varphi_{n-1} = x_i^{i_1}$ and $\varphi_n = 0$. Thus $\varphi = x_1^{i_1} + \cdots + x_n^{i_n} = x_1^{i_1} \cdots x_n^{i_n}$.

(c) $f + g(\varphi) = f(\varphi) + g(\varphi)$, $f + g : \mathbf{N}^S \to R$ and for at most finite $\varphi \in \mathbf{N}^S$, $f(\varphi) \neq 0$, so $f + g \in \mathbf{N}^S$.
$(fg)(\varphi) = \sum f(\theta)g(\zeta)$, so $fg\mathbf{N}^S \to R$. Suppose $\mathbf{N}^S_f$, $\mathbf{N}^S_g$ are the set such that $f(\mathbf{N}^S_f) = 0$, $g(\mathbf{N}^S_g) = 0$. Take $\mathbf{N}^S_{fg} = \mathbf{N}^S_f \cup \mathbf{N}^S_g$, then $\mathbf{N}^S_{fg}$ is also finite. For all $\theta, \zeta \notin \mathbf{N}^S_{fg}$, $(fg)(\varphi) = 0$. So $fg \in R[S]$.
Take the 0 element of $f$ in $R[S]$ as $0(\varphi) = 0_R$ for any $\varphi \in \mathbf{N}^S$ and the inverse element of $f$ in $R[S]$ as $f^{-1}(\varphi) = -f(\varphi)$ for any $\varphi \in \mathbf{N}^S$. Thus $R[S]$ is a ring.

(d) The proof is similar to (b).

(e) First we prove $\mathbf{N}^S \cong \mathbf{N}^n$. Assume $S = \{x_1, x_2, \ldots, x_n\}$. We can write every $\varphi \in \mathbf{N}^S$ into $x_{n_1}^{i_{n_1}} \cdots x_{n_m}^{i_{n_m}}$ and extend it to $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ by taking $i_j = 0$ if $j \neq n_1, n_2, \ldots, n_m$. Then the map $\sigma : \mathbf{N}^S \to \mathbf{N}^n$ given by $\sigma : x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \mapsto (i_1, i_2, \ldots, i_n)$ is a well defined isomorphism so $\mathbf{N}^S \cong \mathbf{N}^n$.
For any $f \in R[x_1, x_2, \ldots, x_n]$. $f$ can be expressed as
$f = \sum a_{k_1 k_2 \cdots k_n} x_1^{k_1} \cdots x_n^{k_n}$. Take $\tau : R[x_1, x_2, \ldots, x_n] \to R[S]$ given by $\tau : f \mapsto \sum a_{k_1 \cdots k_n} \sigma^{-1}(k_1, k_2, \ldots, k_n)$. It's easy to show that $\tau$ is an isomorphism.

(f) Let $R$ and $X$ be commutative rings with identity and $\varphi : R \to X$ a homomorphism of rings such that $\varphi(1_R) = 1_X$. If $x_1, x_2, \ldots, x_n \in S$, there is a unique homomorphism of rings $\bar{\varphi} : R[S] \to X$ such that $\bar{\varphi}|R = \varphi$, $|S| = n$ and $\varphi(s_i) = x_i$ for $i = 1, 2, \ldots, n$. The proof is quite simple since there exists $\tau : R[x_1, \ldots, x_n] \to R[S]$ an isomorphism.

**Exercise 3.5.5.** Let $R$ and $S$ be rings with identity, $\varphi : R \to S$ a homomorphism of rings with $\varphi(1_R) = 1_S$, and $s_1, s_2, \ldots, s_n \in S$ such that $s_i s_j = s_j s_i$ for all $i, j$ and $\varphi(r)s_i = s_i\varphi(r)$ for all $r \in R$ and all $i$. Then there is a unique homomorphism $\bar{\varphi} : R[x_1, \ldots, x_n] \to S$ such that $\bar{\varphi}|R = \varphi$ and $\bar{\varphi}(x_i) = s_i$. This property completely determines $R[x_1, \ldots, x_n]$ up to isomorphism.

**Answer.** $S' = \langle \varphi(R) \cup \{s_1, s_2, \ldots, s_n\} \rangle$ is a commutative ring. So applying Theorem 5.5. on $S'$, we can get the unique homomorphism $\bar{\varphi}$ :

$R[xt_1, x_2, \dots, x_n] \to S'$, so $\bar{\varphi} : R[x_1, \dots, x_n] \to S$ is also a homomorphism. The proof of the second statement is exactly the same as Theorem 5.5.

**Exercise 3.5.6.** (a) If $R$ is the ring of all $2 \times 2$ matrices over $\mathbf{Z}$, then for any $A \in R$,

$$(x + A)(x - A) = x^2 - A^2 \in R[x]$$

(b) There exist $C, A \in R$ such that $(C + A)(C - A) \neq C^2 - A^2$. Therefore, Corollary 5.6 is false if the rings involved are not commutative.

**Answer.** (a) For any $A \in R$, $x + A$, $x - A$, $(x + A)(x - A)$, $x^2 - A^2 \in R[x]$. $(x + A)(x - A) = x^2 + Ax - xA + A^2$. Since $Ax = xA$, $(x + A)(A + x) = x^2 - A^2$.

(b) Take $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, then $CA = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}$, $AC = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$. So $AC \neq CA$, $(C + A)(C - A) \neq C^2 - A^2$. Corollary 5.6 is false in $R$.

**Exercise 3.5.7.** If $R$ is a commutative ring with identity and $f = a_n x^n + \cdots + a_0$ is a zero divisor in $R[x]$, then there exists a nonzero $b \in R$ such that $ba_n = ba_{n-1} = \cdots = ba_0 = 0$.

**Answer.** Assume $g = b_m x^m + \cdots + b_0$ and $fg = 0$, $fg = a_n b_m x^{m+n} + (a_n b_{m-1} + a_{n-1} b_m) x^{m+n-1} + \cdots + a_0 b_0 = 0$. So for any $k = 0, 1, \dots, m + n$, $\sum\limits_{i+j=k} a_i b_j = 0$. Take $b_1' = b_n$, and then $a_n b_1' = 0$, $a_n b_{m-1} + a_{n-1} b_m = 0 \Rightarrow a_n b_{m-1} b_1' + a_{n-1} b_m b_1' = 0$. Take $b_2 = b_m b_1'$, we have $a_n b_2' = a_{n-1} b_2' = 0$. $a_n b_{m-2} + a_{n-1} b_{m-1} + a_{n-2} b_m = 0$, take $b_3' = b_m b_1'$, we have $a_m b_3' = a_{n-1} b_3' = a_n b_3' = 0$. Continue this step and we have $a_n b_n' = a_{n-1} b_n' = \cdots = a_0 b_n' = 0$. That's the $b$ we want.

**Exercise 3.5.8.** (a) The polynomial $x + 1$ is a unit in the power series ring $\mathbf{Z}[[x]]$, but is not a unit in $\mathbf{Z}[x]$.

(b) $x^2 + 3x + 2$ is irreducible in $\mathbf{Z}[[x]]$, but not in $\mathbf{Z}[x]$.

**Answer.** (a) Take $(x+1)^{-1} = 1 - x + x^2 - x^3 + \cdots \in \mathbf{Z}[[x]]$. $(1 - x + x^2 - x^3 + \cdots)(x+1) = (x+1)(1 - x + x^2 - x^3 + \cdots) = (1 - x + x^2 - x^3 + \cdots) + (x - x^2 + x^3 - \cdots) = 1$. So $x+1$ is a unit in $\mathbf{Z}[[x]]$. For any $f = \sum_{i=0}^{n} a_i x^i \in \mathbf{Z}[x]$, $(x+1)f = a_n x^{n+1} + \sum_{i=1}^{n}(a_i + a_{i-1})x^i + a_0$, $a_n \neq 0$ so $(x+1)f \neq 1$. $x+1$ is not a unit.

(b) $x^2+3x+2 = (x+2)(x+1)$ and $x+2, x+1 \in \mathbf{Z}[x]$, so $x^2+3x+2$ is not irreducible in $\mathbf{Z}[x]$. $x^2+3x+2$ itself is a unit in $\mathbf{Z}[x]$ so if $x^2+3x+2 = ab$, $a$,$b$ must be units. Thus $x^2+3x+2$ is irreducible in $\mathbf{Z}[[x]]$.

**Exercise 3.5.9.** If $F$ is a field, then $(x)$ is a maximal ideal in $F[x]$, but it is not the only maximal ideal.

**Answer.** Suppose not. $(x) \subset I \subset F[x]$ with $I \neq F[x]$. $(x)$ contains all polynomials which have zero constant term. For any $p(x) = \sum_{i=0}^{n} a_i x^i \in I$, $a_0 \neq 0$, $p(x) \notin (x)$. There exists $q(x) = \sum_{i=0}^{n} a'_i x_i$ with $a'_i = a_i$ for $i = 1, 2, \ldots, n$ and $a_0 = 0$, $q(x) \in (x) \subset I$. Thus $a_0 = p(x) - q(x) \in I$, $a_0$ is a unit so $I = F$. That's contradictory! $(x)$ is a maximal ideal.
Consider $(x+1) \subset F[x]$. $F[x]$ is a UFD since $F$ is. For any $f \in (x+1)$, $f = (x+1)g$. For any $h \in F[x]\backslash(x+1)$, $h = (x+1)k + r$, where $\deg r < \deg(x+1) = 1$. So $r$ is a unit in $F[x]$, which means $(h) + (x+1) = F[x]$. $(x+1)$ is maximal in $F[x]$.

**Exercise 3.5.10.** (a) If $F$ is a field then every nonzero element of $F[[x]]$ is of the form $x^k u$ with $u \in F[[x]]$ a unit.

(b) $F[[x]]$ is a principle ideal domain whose only ideals are $0$, $F[[x]] = (1_F) = (x^0)$ and $(x^k)$ for each $k \geq 1$.

**Answer.** (a) For any nonzero element $f$ in $F[[x]]$, $f = (a_0, a_1, \ldots)$, we can find the minimal $k$ such that $a_k \neq 0$. $f = \sum_{i=0}^{\infty} a_i x^i = x^k g$, $g = \sum_{i=0}^{\infty} a_{i+k} x^i$ which has nonzero constant term thus a unit. So $f = x^k g$.

(b) For any ideal $I \subset F[[x]]$ and $a \in I$, $a = x^k u$, $u$ a unit, we construct $\varphi : I \to \mathbf{N}$ given by $\varphi(a) = k$, $\varphi(I) \subset \mathbf{N}$, take $a \in I$ minimize $\varphi(a)$.

Assume $a = x^k u$, then $(a) = (x^k) \subset I$. For any $a' = x^{k'} u' \in I$, $k' > k$, $a' = x^k (x^{k'-k} u') \in (x^k)$. So $I \subset (x^k)$. This also shows that the only ideals are $(x^k)$ for $k \in \mathbf{N}$.

**Exercise 3.5.11.** Let $\mathcal{C}$ be the category with objects all commutative rings with identity and morphisms all ring homomorphism $f : R \to S$ such that $f(1_R) = 1_S$. Then the polynomial ring $\mathbf{Z}[x_1, \ldots, x_n]$ is a free object on the set $\{x_1, \ldots, x_n\}$ in the category $\mathcal{C}$.

**Answer.** Denote $X = \{x_1, x_2, \ldots, x_n\}$. For any object $R$ in $\mathcal{C}$, there exists a map $f : \mathbf{Z} \to R$ given by $f : n \mapsto n \cdot 1_R$ is a homomorphism of rings. If there exist $i : X \to R$ given by $i(x_i) = r_i \in R$. Applying Theorem 5.5 there exists $\bar{f} : \mathbf{Z}[x_1, x_2, \ldots, x_n] \to R$ and $\bar{f}|\mathbf{Z} = f$, $\bar{f}(x_i) = r_i$ so $\bar{f}i = f$. Thus $\mathbf{Z}[x_1, x_2, \ldots, x_n]$ is free over $X$.

## 3.6 Factorization in polynomial rings

**Exercise 3.6.1.** (a) If $D$ is an integral domain and $c$ is an irreducible element in $D$, then $D[x]$ is not a principle ideal domain.
(b) $\mathbf{Z}[x]$ is not a principle ideal domain.
(c) If $F$ is a field and $n \geq 2$, then $F[x_1, \dots, x_n]$ is not a principle ideal domain.

**Exercise 3.6.2.** If $F$ is a field and $f, g \in F[x]$ with $\deg n \geq 1$, then there exist unique polynomials $f_0, f_1, \dots, f_r \in F[x]$ such that $\deg f_i < degg$ for all $i$ and

$$f = f_0 + f_1 g + f_2 g^2 + \cdots + f_r g^r$$

**Exercise 3.6.3.** Let $f$ be a field of positive degree over an integral domain $D$.
(a) If $\text{char} D = 0$, then $f' \neq 0$.
(b) If $\text{char} D = p \neq 0$, then $f' = 0$ if and only if $f$ is a polynomial in $x^p$(that is, $f = a_0 + a_p x^p + a_{2p} x^{2p} + \cdots + a_{jp} x^{jp}$).

**Exercise 3.6.4.** If $D$ is a unique factorization domain, $a \in D$ and $f \in D[x]$, then $C(af)$ and $aC(f)$ are associates in $D$.

**Exercise 3.6.5.** Let $R$ be a commutative ring with identity and $f = \sum_{i=0}^{n} a_i x^i \in R[x]$. Then $f$ is a unit in $R[x]$ if and only if $a_0$ is a unit in $R$ and $a_1, \dots, a_n$ are nilpotent elements of $R$.

**Exercise 3.6.6.** Let $p \in \mathbf{Z}$ be prime; let $F$ be a field and let $c \in F$. Then $x^p - c$ is irreducible in $F[x]$ if and only if $x^p - c$ has no root in $F$.

**Exercise 3.6.7.** If $f = \sum a_i x^i \in \mathbf{Z}[x]$ and $p$ prime, let $\bar{f} = \sum \bar{a}_i x^i \in Z_p[x]$, where $\bar{a}$ is the image of $a$ under the canonical epimorphism $\mathbf{Z} \to Z_p$.

(a) If $f$ is monic and $\bar{f}$ is irreducible in $Z_p[x]$ for some $p$, then $f$ is irreducible in $\mathbf{Z}[x]$.

(b) Given an example to show that (a) may be false if $f$ is not monic.

(c) Extend (a) to polynomials over a unique factorization domain.

**Exercise 3.6.8.** (a) Let $c \in F$, where $F$ is a field of characteristic $p$ ($p$ prime). Then $x^p - x - c$ is irreducible in $F[x]$ if and only if $x^p - x - c$ has no root in $F$.

(b) If char$F = 0$, part (a) is false.

**Exercise 3.6.9.** Let $f = \sum\limits_{i=0} a_i x^i \in \mathbf{Z}[x]$ have degree $n$. Suppose that for some $k (0 < k < n)$ and some prime $p$: $p \nmid a_n$; $p \nmid a_k$; $p \mid a_i$ for all $0 \leq i \leq k-1$; and $p^2 \nmid a_0$. Show that $f$ has a factor $g$ of degree at least $k$ that is irreducible in $\mathbf{Z}[x]$.

**Exercise 3.6.10.** (a) Let $D$ be an integral domain and $c \in D$. Let $f(x) = \sum\limits_{i=0}^{n} a_i x^2 \in D[x]$ and $f(x-c) = \sum\limits_{i=0}^{n} a_i (x-c)^i \in D[x]$. Then $f(x)$ is irreducible in $D[x]$ if and only if $f(x-c)$ is irreducible.

(b) For each prime $p$, the **cyclotomic polynomial** $f = x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible in $\mathbf{Z}[x]$.

**Exercise 3.6.11.** If $c_0, c_1, \ldots, c_n$ are distinct elements of an integral domain $D$ and $d_0, \ldots, d_n$ are any elements of $D$, then there is at most one polynomial $f$ of degree $\leq n$ in $D[x]$ such that $f(c_i) = d_i$ for $i = 0, 1, \ldots, n$.

**Exercise 3.6.12.** *Lagrange's Interpolation Formula.* If $F$ is a field, $a_0$, $a_1$,..., $a_n$ are distinct elements of $F$ and $c_0, c_1, \ldots, c_n$ are any elements of $F$, then

$$f(x) = \sum_{i=0}^{n} \frac{(x-a_0)\cdots(x-a_{i-1})(x-a_{i+1})\cdots(x-a_n)}{(a_i-a_n)\cdots(a_i-a_{i-1})(a_i-a_{i+1})\cdots(a_i-a_n)} c_i$$

is the unique polynomial of degree $\leq n$ in $F[x]$ such that $f(a_i) = c_i$ for all $i$.

**Exercise 3.6.13.** Let $D$ be a unique factorization domain with a finite number of units and quotient field $F$. If $f \in D[x]$ has degree $n$ and $c_0, c_1, \ldots, c_n$ are $n+1$ distinct elements of $D$, then $f$ is completely determined by $f(c_0), f(c_1), \ldots, f(c_n)$ according to **Exercise 3.6.11**. Here is **Kronecker's Method** for finding all the irreducible factors of $f$ in $D[x]$.

(a) It suffices to find only those factors $g$ of degree at most $n/2$.

(b) If $g$ is a factor of $f$, then $g(c)$ is a factor of $f(c)$ for all $c \in D$.

(c) Let $m$ be the largest integer $\leq n/2$ and choose distinct elements $c_0$, $c_1$, $\ldots$, $c_m \in D$. Choose $d_0$, $d_i$, $\ldots$, $d_m \in D$ such that $d_i$ is a factor of $f(c_i)$ in $D$ for all $i$. Use **Exercise 3.6.12** to construct a polynomial $g \in F[x]$ such that $g(c_i) = d$ for all $i$; it is unique by **Exercise 3.6.11**.

(d) Check to see if the polynomial $g$ of part (c) is a factor of $f$ in $F[x]$. If not, make a new choise of $d_0, \ldots, d_m$ and repeat part (c).

(e) After a finite number of steps, all the (irreducible) factors of $f$ in $F[x]$ will have been found. If $g \in F[x]$ is such a factor (of positive degree) then choose $r \in D$ such that $rg \in D[x]$. Then $rg = C(rg)g_1$ with $g_1 \in D[x]$ primitive and irreducible in $F[x]$. By Lemma 6.13, $g_1$ is an irreducible factor of $f$ in $D[x]$. Proceed in this manner to obtain all the nonconstant irreducible factors of $f$; the constants are then easily found.

**Exercise 3.6.14.** Let $R$ be a commutative ring with identity and $c, b \in R$ with $c$ a unit.

(a) Show that the assignment $x \mapsto cx + b$ induces a unique automorphism of $R[x]$ that is the identity of $R$. What is its inverse?
(b) If $D$ is an integral domain, then show that every automorphism of $D[x]$ that is the identity on $D$ is of the type described in (a).

**Exercise 3.6.15.** If $F$ is a field, then $x$ and $y$ are relatively prime in the polynomial domain $F[x, y]$, but $F[x, y] = (1_F) \supsetneqq (x) + (y)$.

**Exercise 3.6.16.** Let $f = a_n x^n + \cdots + a_0$ be a polynomial over the field $\mathbf{R}$ of real numbers and let $\varphi = |a_n|\, x^n + \cdots + |a_0| \in \mathbf{R}[x]$.
(a) IF $|u| \leq d$, then $|f(u)| \leq \varphi(d)$.
(b) Given $a, c \in \mathbf{R}$ with $c > 0$ there exists $M \in \mathbf{R}$ such that $|f(a+h) - f(a)| \leq M\, |h|$ for all $h \in \mathbf{R}$ with $|h| \leq c$.
(c) (Intermediate Value Theorem) If $a < b$ and $f(a) < d < f(b)$, then there exists $c \in \mathbf{R}$ such that $a < c < b$ and $f(c) = d$.
(d) Every polynomial $g$ of odd degree in $\mathbf{R}[x]$ has a real root.

# Chapter 4

# Modules

## 4.1 Modules, homomorphisms and exact sequences

**Exercise 4.1.1.** If $A$ is an abelian group and $n > 0$ an integer such that $na = 0$ for all $a \in A$, then $A$ is a unitary $Z_n$-module, with the action of $Z_n$ on $A$ given by $\bar{k}a = ka$, where $k \in \mathbf{Z}$ and $k \mapsto \bar{k} \in Z_n$ under the canonical projection $\mathbf{Z} \to Z_n$.

**Answer.** $\bar{k}, \bar{l} \in Z_n$, $k, l \in \mathbf{Z}$ and $a, b \in A$, $(\bar{k} + \bar{l})a = (k + l)a = ka + la = \bar{k}a + \bar{l}a$, $\bar{k}(a+B) = k(a+b) = ka+kb = \bar{k}a+\bar{k}b$. Assume $\bar{kl} = kl \mod n = t$, $\bar{kl} = ta = k(la) = \bar{k}(\bar{l}a)$ since $kl = t + sn, s \in \mathbf{N}$. So $A$ is a $Z_n$-module.

**Exercise 4.1.2.** Let $f : A \to B$ be an $R$-module homomorphism.

(a) $f$ is a monomorphism if and only if for every pair of $R$-module homomorphisms $g, h : D \to A$ such that $fg = fh$, we have $g = h$.

(b) $f$ is an epimorphism if and only if for every pair of $R$-module homomorphisms $k, t : B \to A$ such that $kf = tf$, we have $k = t$.

**Answer.** (a) If $f$ is a monomorphism, $f(a) = f(b)$ if and only if $a = b$, so $fg(a) = fh(a)\,\forall a \in D \Rightarrow g(a) = h(a)\,\forall a \in D$, whence $g = h$.
Conversely. Take $D = \mathrm{Ker} f$ and $g : a \mapsto a \in A$, $h : a \mapsto 0 \in A$. Then $\forall a \in D$, $fg(a) = fh(a) = 0 \in B$. This means $D = \{0\}$, so $f$ is a monomorphism.

(b) If $f$ is an epimorphism. $\forall b \in B$, there is $a \in A$ such that $f(a) = b$. So $gf(a) = hf(a) \Rightarrow g(b) = f(b)\,\forall b \in B$, $g = h$.
Conversely. Take $k : b \mapsto b + \mathrm{Im} f$ and $t : b \mapsto - \in B/\mathrm{Im} f$. $\forall a \in A$, $f(a) \in \mathrm{Im} f$ so $kf(a) = \mathrm{Im} f = tf(a) \Rightarrow k = t$. So $\mathrm{Im} f = B$, $f$ is an epimorphism.

**Exercise 4.1.3.** Let $I$ be a left ideal of a ring $R$ and $A$ an $R$-module.

(a) If $S$ is a nonempty subset of $A$, then $IS = \{\sum_{i=1}^{n} r_i a_i | n \in \mathbf{N}^*; r_i \in I; a_i \in S\}$ is a submodule of $A$. Note that if $S = \{a\}$, then $IS = Ia = \{ra | r \in I\}$.

(b) If $I$ is a two-sided ideal, then $A/IA$ is an $R/I$-module with the action of $R/I$ given by $(r + I)(a + IA) = ra + IA$.

**Answer.** (a) For any $x \in IS$, $x = \sum_{i=1}^{n} r_i a_i$ so $rx = r \sum_{i=1}^{n} r_i a_i = \sum_{i=1}^{n} (rr_i) a_i \in IS$. For any $x, y \in IS$, $x = \sum_{i=1}^{n} r_i a_i$, $y = \sum_{i=1}^{n'} r_i' a_i'$. Then $x + y = x = \sum_{i=1}^{n} r_i a_i + \sum_{i=1}^{n'} r_i' a_i' \in IS$. $IS$ is a submodule of $A$.

(b) For any $r + I \in R/I$, and $a + IA = A/IA$. $(r+I)(a++IA) = ra + IA \in A/IA$ since $ra \in A$. $\forall r_1, r_2 \in R$, $a_1, a_2 \in A$.

$$\begin{aligned} ((r_1 + I) + (r_2 + I))(a + IA) &= (r_1 + r_2 + I)(a + IA) \\ &= (r_1 a + r_2 a + IA) \\ &= (r_1 a + IA) + (r_2 a + IA) \end{aligned}$$

$$\begin{aligned} (r + I)((a_1 + IA) + (a_2 + IA)) &= (r + I)(a_1 + a_2 + IA) \\ &= ra_1 + ra_2 + IA \\ &= (ra_1 + IA) + (ra_2 + IA) \end{aligned}$$

$$\begin{aligned} (r_1 + I)(r_2 + I)(a + IA) &= r_1 r_2 a + IA \\ &= r_1(r_2 a) + IA \\ &= (r_1 + I)(r_2 a + I) \end{aligned}$$

so $A/IA$ is a submodule of $R/I$.

**Exercise 4.1.4.** If $R$ has identity, then every unitary cyclic $R$-module is isomorphic to an $R$-module of the form $R/J$, where $J$ is a left ideal of $R$.

**Answer.** The cyclic unitary module generated by $a$ is $Ra$. We only need to prove $J = \{r | ra = 0 \in Ra\}$ is an left ideal of $R$. $\forall r' \in R$ and $r \in J$, $r'ra = r'(ra) = r'0 = 0 \in Ra$ so $r'r \in J$. $J$ is a left ideal of $R$. Thus $Ra \cong R/J$.

**Exercise 4.1.5.** If $R$ has identity, then a nonzero unitary $R$-module $A$ is **simple** if its only submodules are 0 and $A$.

(a) Every simple $R$-module is cyclic.

(b) If $A$ is simple every $R$-module endomorphism is either the zero map of and isomorphism.

**Answer.** (a) Trivial.

(b) For and endomorphism $f$, Im$f$ is a submodule of $A$, so $f$ is a zero map or an isomorphism.

**Exercise 4.1.6.** A finitely generated $R$-module need not to be finitely generated as an an abelian group.

**Answer.** For the polynomial ring with degree less than 3. $\mathrm{Q}_2[x]$ is finitely generated Q-module. But $\mathrm{Q} \subset \mathrm{Q}_2[x]$, $\mathrm{Q}_2[x]$ is not finitely generated abelian group since Q is not finitely generated.

**Exercise 4.1.7.** (a) If $A$ and $B$ are $R$-modules, then the set $\mathrm{Hom}_R(A, B)$ of all $R$-module homomorphisms $A \to B$ is an abelian group with $f + g$ given on $a \in A$ by $(f + g)(a) = f(a) + g(a) \in B$. The identity element is the zero map.

(b) $\mathrm{Hom}_R(A, B)$ is a ring with identity, where multiplication is composition of functions. $\mathrm{Hom}_R(A, B)$ is called the **endomorphism ring** of $A$.

(c) $A$ is a left $\mathrm{Hom}_R(A, A)$-module with $fa$ defined to be

$$f(a)(a \in A), f \in \mathrm{Hom}_R(A, A)$$

**Answer.** (a) For any $f, g \in \mathrm{Hom}_R(A, B)$, $f+g := (f+g)(a) = f(a)+g(a) \in B$ and $f+g = g+f$. Take the 0 element as the zero map and the inverse element of $f$ as $-f : a \mapsto -f(a)$. We have $\mathrm{Hom}_R(A, B)$ an abelian group.

(b) $\mathrm{Hom}_R(A, A)$ is an abelian group. $\forall f, g, h \in \mathrm{Hom}_R(A, A)$,

$$(fg)h = (f \circ h) \circ h = f \circ g \circ h = f \circ (g \circ h) = f(gh)$$

$$f \circ (g + h)(a) = f(g(a) + h(a)) = f(g(a)) + f(h(a))$$

so $f(g + h) = fg + fh$.

$$(f + g) \circ h(a) = (f + g)(h(a)) = f(h(a)) + g(h(a))$$

so $(f+g)h = fh+gh$. $\mathrm{Hom}_R(A, A)$ is a ring and the identity is $1_A$ map.

(c) $\forall a \in A$ and $f\mathrm{Hom}_R(A, A)$, $fa = f(a) \in A$. For all $a, b \in A$, $f, g \in \mathrm{Hom}_R(A, A)$,

$$(f + g)a = f(a) + g(a) = fa + ga$$

$$f(a + b) = f(a) + f(b) = fa + fb$$

$$(fg)a = f(g(a)) = f(ga)$$

so $A$ is a $\mathrm{Hom}_R(A, A)$-module.

**Exercise 4.1.8.** Prove that the obvious analogues of Theorem I.8.10 and Corollary I.8.11 are valid for $R$-modules.

**Answer.** Let $\{f_i : G_i \to H_i | i \in I\}$ be a family of homomorphisms of $R$-module. Let $f : \bigoplus_{i\in I}$ be the map $\bigoplus_{i\in I} G_i \to \bigoplus_{i\in I} H_i$ given by $\{a_i\} \mapsto \{f_i(a_i)\}$. Then $f$ is a homomorphism of $R$-modules such that $f(\bigoplus_{i\in I}) \subset \bigoplus_{i\in I} H_i$, $\mathrm{Ker}f = \bigoplus_{i\in I} \mathrm{Ker}f_i$ and $\mathrm{Im}f = \bigoplus_{i\in I} f_i$.
For any $a_i, b_i \in H_i$ with $i \in I$, $f(\{a_i\}) = \{f_i(a_i)\}$, $f(\{b_i\}) = \{f_i(b_i)\}$ and $f(\{a_ib_i\}) = \{f_i(a_ib_i)\} = \{f_i(a_i)f_i(b_i)\} = \{f(a_i)\}\{f(b_i)\} = f(\{a_i\})f(\{b_i\})$. For any $r \in R$, $f(\{ra_i\}) = \{f_i(ra_i)\} = \{rf_i(a_i)\} = r\{f_i(a_i)\} = rf(\{a_i\})$. Hence $f$ is a well defined homomorphism of $R$-modules. $\{0\} \in \bigoplus_{i\in I} H_i$ is the zero element of $\bigoplus_{i\in I} H_i$, so $\forall\{a_i\} \in \mathrm{Ker}f$, $f_i(a_i) = 0$. Thus $\mathrm{Ker}f = \bigoplus_{i\in I} \mathrm{Ker}f_i$.
The analogue of Corollary I.8.11 is the obvious corollary of the theorem above.

**Exercise 4.1.9.** If $f : A \to A$ is an $R$-module homomorphism such that $ff = f$, then

$$A = \mathrm{Ker}f \oplus \mathrm{Im}f$$

**Answer.** For the theorem of homomorphisms, $\mathrm{Im}f \cong A/\mathrm{Ker}f$. Suppose $\mathrm{Ker}f \cap \mathrm{Im}f \neq \{0\}$, $a \in \mathrm{Ker}f \cap \mathrm{Im}f$. $a = f(b)$, $f(a) = f(f(b)) = f(b) = a = 0$, that's contradictory! So $\mathrm{Ker}f \cap \mathrm{Im}f = \{0\}$. $A = \mathrm{Ker}f \oplus \mathrm{Im}f$.

**Exercise 4.1.10.** Let $A, A_1, \ldots, A_n$ be $R$-modules. Then $A \cong A_1 \oplus \cdots \oplus A_n$ if and only if for each $i = 1, 2, \ldots, n$ there is an $R$-module homomorphism $\varphi_i : A \to A$ such that $\mathrm{Im}\varphi_i \cong A_i$; $\varphi_i\varphi_j$ for $i \neq j$; and $\varphi_1 + \varphi_2 + \cdots + \varphi_n = 1_A$.

**Answer.** If $A \cong A_1 \oplus \cdots \oplus A_n$. Let $\pi_i, \tau_i$ be as in Theorem 1.14. Define $\varphi_i = \tau_i\pi_i$. Then $\varphi_i\varphi_j = 0$ for $i \neq j$ and $\sum\limits_{i=1}^{n} = 1_A$. $\mathrm{Im}\varphi_i \cong \pi_i(\tau_i(A_i)) = 1_{A_i}(A_i) = A_i$.
Conversely. If exist $\varphi_i$, $i \in I$ satisfies those conditions. $\varphi_i(\varphi_1 + \cdots + \varphi_n) = \varphi_i$, $\varphi_i\varphi_j = 0$ for $i \neq j$, so $\varphi_i\varphi_i = \varphi_i$. Let $\psi_i = \varphi_i|\mathrm{Im}\varphi_i : \mathrm{Im}\varphi_i \to A$. Then $\varphi_i\psi_i = 1_{\mathrm{Im}\varphi_i}$ since $\forall\varphi_i(a) \in \mathrm{Im}\varphi_i$, $\varphi_i\psi_i(\varphi_i(a)) = \varphi_i(a)$. $\varphi_i\psi_j = 0$ if $i \neq j$. $\sum\limits_{i=1}^{n}\psi_i\varphi_i = 1_A$ since $\sum\limits_{i=1}^{n}\varphi_i = \sum\limits_{i=1}^{n}\varphi_i\varphi_i = 1_A$. From Theorem 1.14, $A \cong \bigoplus\limits_{i=1}^{n}\mathrm{Im}\varphi_i$.

**Exercise 4.1.11.** (a) If $A$ is a module over a commutative ring $R$ and $a \in A$, then $\mathcal{O}_a = \{r \in R | ra = 0\}$ is an ideal of $R$. If $\mathcal{O}_a \neq 0$, $a$ is said to be a **torsion element** of $A$.

(b) if $R$ is an integral domain, then the set $T(A)$ of all torsion elements of $A$.($T(A)$ is called the **torsion submodule**.)

(c) Show that (b) may be false for a commutative ring $R$, which is not an integral domain.
In (d) - (f) $R$ is an integral domain.

(d) If $f : A \to B$ is an $R$-module homomorphism, then $f(T(A)) \subset T(B)$; hence the restriction $f_T$ of $f$ to $T(A)$ is an $R$-module homomorphism $T(A) \to T(B)$.

(e) If $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ is an exact sequence of $R$-module, then so is $0 \to T(A) \xrightarrow{f_T} T(B) \xrightarrow{g_T} T(C)$.

(f) If $g : B \to C$ is an $R$-module epimorphism, then $g_T : T(B) \to T(C)$ need not be an epimorphism.

**Answer.** (a) Trivial.

(b) For any $a, b \in T(A)$ and $r_1, r_2 \in R$ such that $r_1a = r_2b = 0$, $r_1r_2(a+b) = r_2(r_1a) + r_1(r_2b) = 0 \Rightarrow a + b \in T(A)$. $\forall r \in R$, $r_1ra = r(r_1a) = 0 \Rightarrow ra \in T(A)$. $T(A)$ is a submodule of $A$.

(c) Take $R = Z_6 = \{0, 1, 2, 3, 4, 5\}$. $R$ itself is an $R$-module and $2, 3 \in T(R)$, but $2 + 3 = 5 \notin T(R)$ since $5x = 0$ if and only if $x = 0$.

(d) We only need to check $\forall a \in T(A)$, $f(a) \in T(B)$. There exist $r \in R$ s.t. $ra = 0$, so $f(ra) = rf(a) = f(0) = 0$ so $f(a) \in T(B)$, $f(T(A)) \subset T(B)$.

(e) If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an exact sequence. $f(A) = \mathrm{Ker} g$, $fg_T(a) = 0 \Rightarrow g_T f_T(a) = 0$. Hence $0 \to T(A) \xrightarrow{f_T} T(B) \xrightarrow{g_T} T(C) \to 0$ is an exact sequence.

(f) $\mathbf{Z}$ itself is a $\mathbf{Z}$-module. $Z_6$ is a $\mathbf{Z}$-module as the multiplication given by $a \cdot \bar{b} = \bar{ab}$, $\mathbf{Z}$ has the torsion submodule. $\{0\}$ and $Z_6$ has the torsion submodule $\{\bar{0}, \bar{2}, \bar{3}, \bar{4}\}$ which means $f : \mathbf{Z} \to Z_6$ cannot form an epimorphism $f_T : T(\mathbf{Z}) \to T(Z_6)$.

**Exercise 4.1.12.** (The Five Lemma). Let

$$\begin{array}{ccccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\
{\scriptstyle\alpha_1}\downarrow & & {\scriptstyle\alpha_2}\downarrow & & {\scriptstyle\alpha_3}\downarrow & & {\scriptstyle\alpha_4}\downarrow & & {\scriptstyle\alpha_5}\downarrow \\
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5
\end{array}$$

be a commutative diagram of $R$-module homomorphisms, with exact rows. Prove that:

(a) $\alpha_1$ an epimorphism and $\alpha_2, \alpha_4$ monomorphisms $\Rightarrow$ $\alpha_3$ is a monomorphism;

(b) $\alpha_5$ a monomorphism and $\alpha_2, \alpha_4$ epimorphisms $\Rightarrow$ $\alpha_3$ is an epimorphism.

**Answer.** Denote all the homomorphisms as following.

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\
{\scriptstyle\alpha_1}\downarrow & & {\scriptstyle\alpha_2}\downarrow & & {\scriptstyle\alpha_3}\downarrow & & {\scriptstyle\alpha_4}\downarrow & & {\scriptstyle\alpha_5}\downarrow \\
B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5
\end{array}$$

(a) For any $a \in A_3$, $\alpha_3(a) = 0$, we need to show that $a = 0$. $g_3\alpha_3(a) = \alpha_4 f_3(a) = 0$, since $\alpha_4$ is monomorphism, $f_3(a) = 0$. So $a \in \mathrm{Ker} f_3 \Rightarrow a \in \mathrm{Im} f_2$. There's $a' \in A_2$, $f_2(a') = a$, $\alpha_3 f_2(a') = 0 = g_2\alpha_2(a')$. So $\alpha_2(a') \in \mathrm{Ker} g_2 = \mathrm{Im} g_1$. There is $b'' \in B_1$, $g_1(b'') = \alpha_2(a')$. $\alpha_1$ is epimorphism so $\exists a'' \in A_1$, $b'' = \alpha_1(a'')$, so $g_1\alpha_1(a'') = \alpha_2 f_1(a'') = \alpha_2(a')$. $\alpha_2$ is monomorphism so $f_1(a'') = a' \in \mathrm{Ker} f_2 \Rightarrow a = f_2(a') = 0$.

(b) For any $b \in B_3$, we need to show that $b \in \mathrm{Im}\alpha_3$. $g_3(b) \in B_4$, $g_3(b) = \alpha_4(a')$ for $a' \in A_4$ since $\alpha_4$ is epimorphism. $g_4\alpha_4(a') = g_4g_3(b) = 0 = \alpha_5 f_4(a')$. $f_4a' = 0$ cince $\alpha_5$ is monomorphism. So there is $a \in A_3$, $f_3(a) = a'$, $\alpha_4 f_3(a) = g_3\alpha_3(a) = \alpha_4(a') = g_3(b)$. $g_3(b - \alpha_3(a)) = 0 \Rightarrow b - \alpha_3(a) \in \mathrm{Ker}g_3 = \mathrm{Im}g_2$. There's $b' \in B_2$, $g_2(b') = -b + \alpha_3(a)$. $\alpha_2$ is epimorphism so $\exists a'' \in A_2$, $\alpha_2(a'') = b'$. Consider $\alpha_3(a - f_2(a'')) = \alpha_3(a) - \alpha_3 f_2(a'') = -g_2\alpha_2(a'') + \alpha_3(a'') = b$. Thus $b \in \mathrm{Im}\alpha_3$, whence $\alpha_3$ is epimorphism.

**Exercise 4.1.13.** (a) If $0 \to A \to B \xrightarrow{f} C \to 0$ and $0 \to C \xrightarrow{g} D \to D \to E \to 0$ are short exact sequences of modules, then the sequence $0 \to A \to B \xrightarrow{gf} D \to E \to 0$ is exact.

(b) Show that every exact sequence may be obtained by splicing together suitable short exact sequences as in (a).

**Answer.** (a) The commutative diagram

$$\begin{tikzcd}
 & & & C \arrow[dr, "g"] & & & \\
0 \arrow[r] & A \arrow[r, "k"] & B \arrow[ur, "f"] \arrow[rr, "gf"'] & & D \arrow[r, "l"] & E \arrow[r] & 0
\end{tikzcd}$$

For any $a \in A$, $k(a) \in \mathrm{Ker}f \Rightarrow fk(a) = 0$. $g$ is monomorphism so $\mathrm{Ker}g = 0$. Since $gfk(a) = 0$, $\mathrm{Im}k \subset \mathrm{Ker}gf$. $\mathrm{Ker}g = 0 \Rightarrow gf(a) = 0$ if and only if $f(a) = 0$. $\mathrm{Ker}gf \subset \mathrm{Im}k$. $\mathrm{Im}gf = \mathrm{Im}g$ since $f$ is epimorphism. So $\mathrm{Im}gf = \mathrm{Ker}l$. $0 \to A \to B \xrightarrow{gf} D \to E \to 0$ is an exact sequence.

(b) For any finite exact sequence $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n$. We can add head and tail into it and form

$$0 \to \mathrm{Coker}f_1 \to A_1 \xrightarrow{f_1}\to A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n \to \mathrm{Coim}f_{n-1} \to 0$$

For any exact sequence which has fragment

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

Consider

$$\begin{tikzcd} & & \mathrm{Im}g \arrow[dr, "\subset"] & & \\ A \arrow[r, "f"] & B \arrow[ur, "g"] \arrow[rr, "g"] & & C \arrow[r, "h"] & D \end{tikzcd}$$

$\subset$ is the inclusion map. We can split into $A \xrightarrow{f} B \xrightarrow{g} \mathrm{Im}g \to 0$ and $0 \to \mathrm{Im}g \xrightarrow{\subset} C \xrightarrow{h} D$. This provides us a way to split an exact sequence into short exact sequences.

**Exercise 4.1.14.** Show that isomorphism of short exact sequences is an equivalence relation.

**Answer.** We check isomorphism of short exact sequence is equivalence relation. $a = 0 \to A_1 \to B_1 \to C_1 \to 0$, $b = 0 \to A_2 \to B_2 \to C_2 \to 0$ and $c = 0 \to A_3 \to B_3 \to C_3 \to 0$.

1. The commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r] \arrow[d, "1_{A_1}"'] & B_1 \arrow[r] \arrow[d, "1_{B_1}"'] & C_1 \arrow[r] \arrow[d, "1_{C_1}"'] & 0 \\ 0 \arrow[r] & A_1 \arrow[r] & B_1 \arrow[r] & C_1 \arrow[r] & 0 \end{tikzcd}$$

   shows that $a \sim a$ since $1_{A_1}$, $1_{B_1}$ and $1_{C_1}$ are isomorphisms.
2. If

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r] \arrow[d, "f"'] & B_1 \arrow[r] \arrow[d, "g"'] & C_1 \arrow[r] \arrow[d, "h"'] & 0 \\ 0 \arrow[r] & A_2 \arrow[r] & B_2 \arrow[r] & C_2 \arrow[r] & 0 \end{tikzcd}$$

   then we have

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r] \arrow[d, "f^{-1}"'] & B_1 \arrow[r] \arrow[d, "g^{-1}"'] & C_1 \arrow[r] \arrow[d, "h^{-1}"'] & 0 \\ 0 \arrow[r] & A_2 \arrow[r] & B_2 \arrow[r] & C_2 \arrow[r] & 0 \end{tikzcd}$$

   is also commutative. So $a \sim b \Leftrightarrow b \sim a$.

3. If

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & 0 \\ & & {\scriptstyle f_1}\big\downarrow & & {\scriptstyle g_1}\big\downarrow & & {\scriptstyle h_1}\big\downarrow & & \\ 0 & \longrightarrow & A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & 0 \\ & & {\scriptstyle f_2}\big\downarrow & & {\scriptstyle g_2}\big\downarrow & & {\scriptstyle h_2}\big\downarrow & & \\ 0 & \longrightarrow & A_3 & \longrightarrow & B_3 & \longrightarrow & C_3 & \longrightarrow & 0 \end{array}$$

are commutative. Then

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & 0 \\ & & {\scriptstyle f_2f_1}\big\downarrow & & {\scriptstyle g_2g_1}\big\downarrow & & {\scriptstyle h_2h_1}\big\downarrow & & \\ 0 & \longrightarrow & A_3 & \longrightarrow & B_3 & \longrightarrow & C_3 & \longrightarrow & 0 \end{array}$$

is also commutative. So $a \sim b, b \sim c \Rightarrow a \sim c$.

**Exercise 4.1.15.** If $f : A \to B$ and $g : B \to A$ are $R$-module homomorphisms such that $gf = 1_A$, then $B = \text{Im}f \oplus \text{Ker}g$.

**Answer.** $gf = 1_A$ so $f$ is monomorphism and $g$ is epimorphism. So $B/\text{Ker}g \cong \text{Im}g = A \cong A/0 \cong \text{Im}f$. $\text{Ker}g \cap \text{Im}0 = \{0\}$ since $g(\text{Im}f) = A$. Thus $B = \text{Ker}g \oplus \text{Im}f$.

**Exercise 4.1.16.** Let $R$ be a ring and $R^{op}$ its opposite ring. If $A$ is a left $R$-module, then $A$ is a right $R^{op}$-module such that $ra = ar$ for all $a \in A, 4 \in R, r \in R^{op}$.

**Answer.** Trivial.

**Exercise 4.1.17.** (a) If $R$ has an identity and $A$ is an $R$-module, then there are submodules $B$ and $C$ of $A$ such that $B$ is unitary, $RC = 0$ and $A = B \oplus C$.

(b) Let $A_1$ be another $R$-module, with $A_1 = B_1 \oplus C_1$ ($B_1$ unitary, $RC = 0$), If $f : A \to A_1$ is an $R$-module homomorphism then $f(B) \subset B_1$ and $f(C) \subset C_1$.

(c) If the map $f$ of part (b) is an epimorphism, then so are $f|B : B \to B_1$ and $f|C : C \to C_1$.

**Answer.** (a) Let $B = \{1_R a | a \in A\}$, $C = \{a \in A | 1_R a = 0\}$. Then $B$ is unitary since $1_R(1_R a) = 1_R a$. $RC = 0$ since $ra = (r1_R)a = r(1_R a) = 0 \forall a \in C$. And $\forall a \in A$, $1_R(a - 1_R a) = 0 \Rightarrow a - 1_R a \in C$. So $A = \subset B \oplus C$. Obviously $B \oplus C \subset A$, $A = B \oplus C$.

(b) For any $x = b_1 + c_1 \in A_1$, $1_R x = 1_R(b_1 + c_1) = 1_R b_1$, $B_1$ is the maximal unitary submodule and $B_1$ contains all unitary elements. $f(B)$ is also unitary since $f(b) = f(1_R b) = 1_R f(b)$ for any $b \in B$. $f(B) \subset B_1$.
$C_1$ contains all elements $x \in A_1$ s.t. $Rx = 0$. Since $Rf(c) = f(Rc) = 0$ for all $c \in C$, we have $f(C) \subset C_1$.

(c) For any $b' \in B'$, we have $f(x) = b'$ since $f$ is epimorphism. Assume $x = b + c$ with $b \in B$ and $c \in C$. $f(x) = f(1_R(b + c)) = f(1_R b) = 1_R f(b) = f(b)$. So $\exists b \in B$, $f(b) = b'$. $f|B$ is epimorphism. For any $c' \in C$, we have $f(y) = c$. Assume $y = a + d$ with $a \in B$ and $d \in C$. $f(y) = f(1_R(a + d)) = f(1_R a) = 0$, so $1_R f(a) = 0 \Rightarrow a = 0$. Thus $\exists y = d \in C$, $f(d) = c'$. $f|C$ is epimorphism.

**Exercise 4.1.18.** Let $R$ be a ring without identity. Embed $R$ in a ring $S$ with identity and charateristic zero as in the proof of Theorem III.1.10. Identify $R$ with its image in $S$.

(a) Show that every element of $S$ may be uniquely expressed in the form $r1_S + n1_S (r \in R, n \in \mathbf{Z})$.

(b) If $A$ is an $R$-module and $a \in A$, show that there is a unique $R$-module homomorphism $f : S \to A$ such that $f(1_S) = a$.

**Answer.** (a) Trivial since $S = R \times \mathbf{Z}$.

(b) $S = R1_S \oplus \mathbf{Z}1_S$. Let $f(r1_S + n1_S) = ra + na$, then $f(1_S) = a$ and $f$ is a well defined homomorphism of modules. If there exists another $g$ s.t.

$g(1_S) = a$, $\forall r1_S + n1_S \in S$, $g(r1_S + n1_S) = rg(1_S) + ng(1_S) = ra + na = f(r1_S + n1_S)$. So $g = f$.

## 4.2 Free modules and vector spaces

**Exercise 4.2.1.** (a) A set of vectors $\{x_1, \cdots x_n\}$ in a vector space $V$ over a division ring $R$ is linearly dependent if and only if some $x_k$ is a linear combination of the preceding $x_i$.

(b) If $\{x_1, x_2, x_3\}$ is a linearly independent subset of $V$, then the set $\{x_1 + x_2, x_2 + x_3, x_3 + x_1\}$ is linearly independent if and only if $\mathrm{Char} R \neq 2$.

**Exercise 4.2.2.** Let $R$ be any ring (possibly without identity) and $X$ a nonempty set. In this exercise an $R$-module $F$ is called a **free module on** $X$ if $F$ is a free object on $X$ in the category of all left $R$-modules. Thus by Definition I.7.7m $F$ is the free module on $X$ if there is a function $\tau : X \to F$ such that for any left $R$-module $A$ and function $f : X \to A$ there is a unique $R$-module homomorphism $\bar{f} : F \to A$ with $\bar{f}\tau = f$.

(a) Let $\{X_i | i \in I\}$ be a collection of mutually disjoint sets and for each $i \in I$, suppose $F_i$ is a free module on $X_i$, with $\tau_i : X_i \to F_i$. Let $X = \bigcup\limits_{i \in I} X_i$ and $F = \sum\limits_{i \in I} F_i$, with $\phi_i : F_i \to F$ the canonical injection. Define $\tau : X \to F$ by $\tau(x) = \phi_i \tau_i(x)$ for $x \in X_i$. Prove that $F$ is a free module on $X$.

(b) Assume $R$ has an identity. Let the abelian group $\mathbf{Z}$ be given trivial $R$-module structure ($rm = 0$ for all $r \in R, m \in \mathbf{Z}$), so that $R \oplus \mathbf{Z}$ is an $R$-module with $r(r', m) = (rr', 0)$ for all $r, r' \in R$, $m \in \mathbf{Z}$. If $X$ is any one element set, $X = \{t\}$, let $\tau : X \to R \oplus \mathbf{Z}$ be given by $\tau(t) = (1_R, 1)$. Prove that $R \oplus \mathbf{Z}$ is a free module on $X$.

(c) If $R$ is an arbitrary ring and $X$ is any set, then there exists a free module on $X$.