Chapter 1

Mathematical Concepts and Symbols

1.1 Logical Symbols

The logical symbols are a kind of symbols using for theoretical statements.

Table 1.1: Frequently-used Logical Symbols

Symbols	Meanings
$L \Longrightarrow P$	Proposition L is contained in proposition P
$L \Longleftrightarrow P$	Proposition L is equivalent to proposition P
$\neg P$	Not P
$L \wedge P$	Proposition L and proposition P
$L \vee P$	Proposition L or proposition P

e.g.

$$((A \Longrightarrow B) \wedge (\neg B) \Longrightarrow (\neg A))$$

stands for " if A is contained in B,and B is not true,then A is not true". We also call $A \Longleftrightarrow B$ "A is the necessary and suffiecent condition of B". The typical math proposition is like "$A \Longrightarrow B$".In order to prove this proposition ,we can use the implication relationship

$$A \Longrightarrow C_1 \Longrightarrow \cdots \Longrightarrow C_n \Longrightarrow B$$

The every implication relationship in this chain is general truth or proved proposition.

Table 1.2: Truth Table

<table>
<tr><td rowspan="2">$\neg A$</td><td>A</td><td>0</td><td>1</td></tr>
<tr><td>$\neg A$</td><td>1</td><td>0</td></tr>
<tr><td rowspan="3">$A \vee B$</td><td>$A \backslash B$</td><td>0</td><td>1</td></tr>
<tr><td>0</td><td>0</td><td>1</td></tr>
<tr><td>1</td><td>1</td><td>1</td></tr>
<tr><td rowspan="3">$A \wedge B$</td><td>$A \backslash B$</td><td>0</td><td>1</td></tr>
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Question 1. $\neg(A \wedge B) \Leftrightarrow (\neg A \vee \neg B)$.

Proof. (Use the truth table)
If A is ture, B is ture, $A \wedge B$ is ture. $\neg(A \wedge B)$ is false.$\neg A$ is false,$\neg B$ is false, $(\neg A \vee \neg B)$ is false.
If A is ture, B is false, $A \wedge B$ is false. $\neg(A \wedge B)$ is true.$\neg A$ is false,$\neg B$ is true, $(\neg A \vee \neg B)$ is ture.
If A is flase, B is true, $A \wedge B$ is false. $\neg(A \wedge B)$ is true.$\neg A$ is true,$\neg B$ is false, $(\neg A \vee \neg B)$ is ture.
If A is false, B is false, $A \wedge B$ is false. $\neg(A \wedge B)$ is true.$\neg A$ is true,$\neg B$ is true, $(\neg A \vee \neg B)$ is ture.
So

$$\neg(A \wedge B) \Leftrightarrow (\neg A \vee \neg B)$$

□

Question 2. $(A \Rightarrow B) \Leftrightarrow \neg A \vee B$.

Proof. Firstly, we confirm the truth of

$$(A \Rightarrow B) \Rightarrow \neg A \vee B$$

If $(A \Rightarrow B)$ is false, then $\neg A \vee B$ is true.
If $(A \Rightarrow B)$ is ture ,then we have two posibilities. The first is A is ture, B is true, so $\neg A \vee B$ is true. The second is A is false,then B can be true or false, but $\neg A \vee B$ will be true.
Hence, $(A \Rightarrow B) \Rightarrow \neg A \vee B$.
Secondly, we prove

$$(A \Rightarrow B) \Leftarrow \neg A \vee B$$

If $\neg A \vee B$ is false, then $(A \Rightarrow B)$ is true.
If $\neg A \vee B$ is true, we have

1. $\neg A$ is true, B is false, then, A is false, $(A \Rightarrow B)$ is true.
2. $\neg A$ is false, B is true, then, A is true, $(A \Rightarrow B)$ is true.
3. $\neg A$ is true, B is true, then, A is false, $(A \Rightarrow B)$ is true.

So $(A \Rightarrow B) \Leftarrow \neg A \vee B$. □

TIPS. *1.* $\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$*;*

2. $\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$*;*

3. $(A \Rightarrow) \Leftrightarrow (\neg B \Rightarrow \neg A)$*;*

4. $(A \Rightarrow) \Leftrightarrow (\neg A \vee B)$*;*

5. $\neg(A \Rightarrow B) \Leftrightarrow A \wedge \neg B$*;*

1.2 Sets and their Operations

A set is a collection of well-defined objects.
If A is a set, we write $a \in A$ to express element a belongs to set A, the negetive proposition of which is $a \notin A$. We use the symbol $\oslash$ to denote the empty set, that is, the set with no elements.

Theorem 1.2.1 (Cantor)**.** *There is no set contains all the sets.*

Proof. We assume $P(M)$ represents M doesn't contain itself.
Consider $K = \{M|P(M)\}$ which is made of **sets** M that satisfies P. Assuming K is a set, then either $P(K)$ or $\neg P(K)$ is true.
If $P(K)$ is true, K doesn't contain itself,but because of the definition of K, K is belong to K, which means $\neg P(M)$.
If $\neg P(M)$ is ture, it's easy to find the similar conclusion.
So to the contrary, K is not a set. This reveals a set can't contain all the sets. $\square$

Theorem 1.2.1 is a typical paradox called Russell's paradox.
$\forall$and $\exists$ are logical symbols to describe

Table 1.3: Universial and Exsitential Quantifications

Symbols	Meanings
$\forall x \in A$	For all elements x in A
$\exists x \in A$	There exist at least one element x in A

To show the inclusion relation of two sets, we often use the Symbol $A \subset B$, which means set A is a **subset** of set B (All the elements in A also belong to B).We indicate that A is not a subset of B by this notaion: $A \not\subset B$.

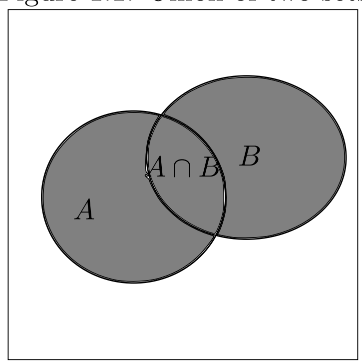
$$(A \subset B) := \forall x((x \in A) \Rightarrow (x \in B))$$

We define the equal relation between two sets, using the symbol =:

$$A = B =: (A \subset B) \wedge (B \subset A)$$

We often use this definition to prove $A = B$. Symbol $\neq$ denotes the negetive proposition of equal.
A is a **proper subset** of B, if A is a subset of B, and $A \neq B$, denoted by the symbol $\subseteq$.

Figure 1.1: Union of two sets



If A and B are sets, then their **union**, denoted by $A \cup B$,is the set of all elements that are elements of either A or B:

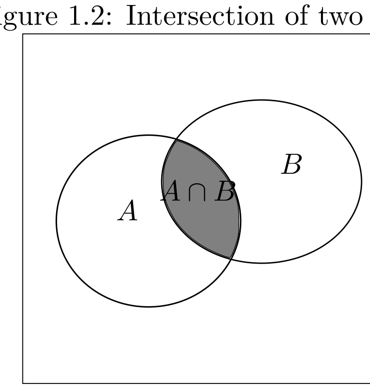
$$(A \cup B) =: \{x \in M | (x \in A) \vee (x \in B)\}$$

Clearly, $A \cup B = B \cup A$.
If A and B are sets, then their **intersection**, denoted by $A \cap B$,contains all the elements in both A and B:

$$(A \cap B) =: \{x \in M | (x \in A) \wedge (x \in B)\}$$

Also, we have $A \cap B = B \cap A$.

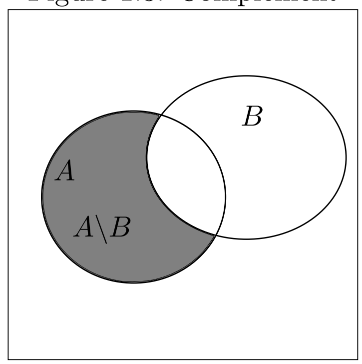
Figure 1.2: Intersection of two sets



We use the donation $A \backslash B$ to represent the set contains all the elements which belongs to A but not belong to B.

$$A \backslash B =: \{x \in M | (x \in A) \wedge (x \notin B)\}$$

Figure 1.3: Complement



For $B \subset A$, we can also denote it as the symbol $C_A B$.

Question 3 (de Morgen)**.**

$$C_M(A \cup B) = C_M A \cap C_M B$$

$$C_M(A \cap B) = C_M A \cup C_M B$$

Proof. We prove the first one.

$$\begin{aligned}&(x \in C_M(A \cup B)) \Rightarrow (x \notin (A \cup B))\\ \Rightarrow&((x \notin A) \wedge (x \notin B)) \Rightarrow ((x \in C_m A) \wedge (x \in C_M B))\\ \Rightarrow&((x \in C_m A) \cap (x \in C_M B))\end{aligned}$$

So we have proved $C_M(A \cup B) \subset C_M A \cap C_M B$. On the other hand:

$$\begin{aligned}&((x \in C_m A) \cap (x \in C_M B)) \Rightarrow ((x \in C_m A) \wedge (x \in C_M B))\\ \Rightarrow&((x \notin A) \wedge (x \notin B)) \Rightarrow (x \notin (A \cup B))\\ \Rightarrow&(x \in C_M(A \cup B))\end{aligned}$$

That's the same as $C_M(A \cup B) = C_M A \cap C_M B$. □

Question 4. $(A \subset C_M B) \Leftrightarrow (B \subset C_M A)$.

Proof.

$$\begin{aligned}&(A \subset C_M B) \Rightarrow ((x \in A) \Rightarrow ((x \notin B)) \wedge (x \in M))\\ \Rightarrow&(\neg(x \in A) \Leftarrow \neg((x \notin B))))\\ \Rightarrow&((x \in B) \Rightarrow (x \notin A)) \Rightarrow (B \subset C_M A)\end{aligned}$$

The other hand of this problem is the same. □

TIPS. *1.* $(A \subset C) \wedge (B \subset C) \Leftrightarrow ((A \cup B) \subset C)$;
2. $(C \subset A) \wedge (C \subset B) \Leftrightarrow (C \subset (A \cap B))$;
3. $C_M(C_M A) = A$;
4. $(A \subset C_M B) \Leftrightarrow (B \subset C_M A)$;
5. $(A \subset B) \Leftrightarrow (C_M A \supset C_M B)$.

Question 5. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof.

$$A \cup (B \cap C) \Leftrightarrow ((x \in A) \vee ((x \in B)) \wedge (x \in C))$$

$$((x \in A) \vee (x \in B)) \wedge ((x \in A) \vee (x \in C)) \Leftrightarrow (A \cup B) \cap (A \cup C)$$

So, we should prove:

$$((x \in A) \vee ((x \in B)) \wedge (x \in C)) \Leftrightarrow ((x \in A) \vee (x \in B)) \wedge ((x \in A) \vee (x \in C))$$

That's the same as:

$$(A \vee (B \wedge C)) \Leftrightarrow ((A \vee B) \wedge (A \vee C))$$

It's easy to prove with the help of truth table. $\square$

TIPS. *1.* $A \cup (B \cup C) = (A \cup B) \cup C =: A \cup B \cup C$;
2. $A \cap (B \cap C) = (A \cap B) \cap C =: A \cap B \cap C$;
3. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;
4. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.