Chapter 1

Groups

1.1 Semigroups, monoids and groups

Exercise 1.1.1. Give examples other than those in the text of semigroups and monoids that are not groups.

Exercise 1.1.2. Let G be a group (written additively), S a nonempty set, and $M(S, G)$ the set of all functions $f : S \to G$. Define addition in $M(S, G)$ as follows: $(f + g) : S \to G$ is given by $s \to f(s) + g(s) \in G$. Prove that $M(S, G)$ is a group, which is abelian if G is.

Exercise 1.1.3. Is it true that a semigroup which has a left identity element and in which every element has a right inverse (see Proposition 1.3) is a group?

Exercise 1.1.4. Write out a multiplication table for the group D_4^*.

Exercise 1.1.5. Prove that the symmetric group on n letters, S_n, has ordrer $n!$.

Exercise 1.1.6. Write out an addition table for $Z_2 \oplus Z_2$. $Z_2 \oplus Z_2$ is called the Klein four group.

Exercise 1.1.7. If p is prime, then the nonzero elements of Z_p form a group of order $p - 1$ under multiplication. Show that this statement is false if p is not prime.

Exercise 1.1.8.
1. The relation given by $a\ b \Leftrightarrow a-b \in \mathbf{Z}$ is a congruence relation on the additive group $\mathbf{Q}$ [see Theorem 1.5].
2. The set $\mathbf{Q}/\mathbf{Z}$ of equivalence classes is an infinite abelian group.

Exercise 1.1.9. Let p be a fixed prime. Let R_p be the set of all those rational numbers whose denominator is relatively prime to p. Let R^p be the set of rationals whose denominator is a power of $p(p^i, i > 0)$. Prove that both R_p and R^p are abelian groups under ordinary addition of rationals.

Exercise 1.1.10. Let p be a prime and let $Z(p^\infty)$ be the following subset of the group $\mathbf{Q}/\mathbf{Z}$:

$$Z(p^\infty) = \{\bar{a/b} \in \mathbf{Q}/\mathbf{Z} | a, b \in \mathbf{Z} \text{ and } b = p^i \text{ for some } i \geq 0\}$$

Show that $Z(p^\infty)$ is an infinite group under the addition operation of $\mathbf{Q}/\mathbf{Z}$.

Exercise 1.1.11. The following conditions on a group G are equivalent:
1. G is abelian,;
2. $(ab)^2 = a^2b^2$ for all $a, b \in G$;
3. $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$;
4. $(ab)^n = a^nb^n$ for all $n \in \mathbf{Z}$ and all $a, b \in G$;
5. $(ab)^n = a^nb^n$ for three consecutive integers n and all $a, b \in G$. Show that $(v) \Rightarrow (i)$ is false if 'three' is replaced by 'two'.

Exercise 1.1.12. If G is a group, $a, b \in G$ and $bab^{-1} = a^r$ for some $r \in \mathbf{N}$, then $b^jab^{-j} = a^{r^j}$ for all $j \in \mathbf{N}$.

Exercise 1.1.13. If $a^2 = e$ for all elements a of a group G, then G is abelian.

Exercise 1.1.14. If G is a finite group of even order, then G contains an element $a \neq e$ such that $a^2 = e$.

Exercise 1.1.15. Let G be a nonempty finite set with an associative binary operation such that for all $a, b, c \in G$, $ab = ac \Rightarrow b = c$ and $ba = ca \Rightarrow b = c$. Then G is a group. Show that this conclusion may be false if G is finite.

Exercise 1.1.16. Let $a_1, a_2, \ldots$ be a sequence of elements in a semigroup G. Then there exists a unique function $\Psi : \mathbf{N}^* \to G$ such that $\Psi(1) = a_1, \Psi(2) = a_1a_2, \Psi(3) = (a_1a_2)a_3$ and for $n \geq 1$, $\Psi(n+1) = (\Psi(n))a_{n+1}$. Note that $\Psi(n)$ is precisely the standard n product $\prod_{i=1}^{n} a_i$.

1.2 Homomorphisms and subgroups

Exercise 1.2.1. If $f : G \to H$ is a homomorphism of groups, then $f(e_G) = e_H$ and $f(a^{-1}) = f(a)^{-1}$ for all $a \in G$. Show by example that the first conclusion may be false if G, H are monoids that are not groups.

Exercise 1.2.2. A group G is abelian if and only if the map $G \to G$ given by $x \to x^{-1}$ is automorphism.

Exercise 1.2.3. Let Q_8 be the group(under ordinary matrix multiplication) generated by complex matrices $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, where $i^2 = -1$. Show that Q_8 is a nonabelian group of order 8. Q_8 is called the quaternion group.

Exercise 1.2.4. Let H be the group(under ordinary matrix multiplication) of real matrices generated by $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Show that H is a nonabelian group of order 8 which is not isomorphic to the quaternion group, but is isomorphic to the group D_4^*.

Exercise 1.2.5. Let S be a nonempty subset of a group G and define a relation on G by a b if and only if $ab^{-1} \in S$. Show that is an equivalence relation if and only if S is a subgroup of G.

Exercise 1.2.6. A nonempty finte subset of a group is s subgroup if and only if it is closed under the product in G.

Exercise 1.2.7. If n is a fixed integer, then $\{kn|n \in \mathbf{Z}\} \subset \mathbf{Z}$ is an additive subgroup of $\mathbf{Z}$, which is isomorphic to $\mathbf{Z}$.

Exercise 1.2.8. The set $\{\sigma \in S_n|\sigma(n) = n\}$ is a subgroup of S_n which is isomorphic to S_{n-1}.

Exercise 1.2.9. Let $f : G \to H$ be a homomorphism of groups, A a subgroup of G, and B a subgroup of H.

1. Ker$f$ and $f^{-1}(B)$ are subgroups of G.
2. $f(A)$ is a subgroup of H.

Exercise 1.2.10. List all subgroups of $Z_2 \oplus Z_2$. Is $Z_2 \oplus Z_2$ isomorphic to Z_4?

Exercise 1.2.11. If G is a subgroup, then $C = \{a \in G|ax = xa \text{ for all } x \in G\}$ is a abelian subgroup of G. C is called the center of G.

Exercise 1.2.12. The group D_4^* is not cyclic, but can be generated by two elements. The same is true of S_n(nontrivial). What is the minimal number of generators of the additive group $\mathbf{Z} \oplus \mathbf{Z}$?

Exercise 1.2.13. If $G = \langle a \rangle$ is a cyclic group and H is any group, then every homomorphism $f : G \to H$ is completely determined by the element $f(a) \in H$.

Exercise 1.2.14. The following cyclic subgroups are all isomorphic: the multiplication group $\langle i \rangle$ in $\mathbf{C}$, the additive group $\mathbf{Z_4}$ and the subgroup $\left\langle \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \right\rangle$ of S_4.

Exercise 1.2.15. Let G be a group and Aut$G$ is the set of all automorphisms of G.

1. Aut$G$ is a group with composition of functions as binary operation.
2. Aut$\mathbf{Z} \cong Z_2$ and Aut$Z_6 \cong Z_2$; Aut$Z_8 \cong Z_2 \oplus Z_2$; Aut$Z_p \cong Z_{p-1}$ (p prime).
3. What is $\mathrm{AutZ_n}$ for arbitrary $n \in \mathbf{N}^*$?

Exercise 1.2.16. For each prime p the additive subgroup $Z(p^\infty)$ of $\mathbf{Q}/\mathbf{Z}$ is generated by the set $\{\overline{1/p^n} | n \in \mathbf{N}^*\}$.

Exercise 1.2.17. Let G be an abelian group and let H, K be subgroups of G. Show that the join $H \vee K$ is the set $\{ab | a \in H, b \in K\}$. Extend this result to any finite number of subgroups of G.

Exercise 1.2.18.

1. Let G be a group and $\{H_i | i \in I\}$ a family of subgroups. State and prove a condition that will imply that $\bigcup\limits_{i \in I} H_i$ is a subgroup, that is $\bigcup\limits_{i \in I} H_i = \left\langle \bigcup\limits_{i \in I} H_i \right\rangle$.
2. Given an example of a group G and a family of subgroups $\{H_i | i \in I\}$ such that $\bigcup\limits_{i \in I} H_i \neq \left\langle \bigcup\limits_{i \in I} H_i \right\rangle$.

Exercise 1.2.19.

1. The set of all subgroups of a group G, partially ordered by set theoretic inclusion, forms a complete lattice in which the g.l.b of $\{H_i | i \in I\}$ is $\bigcap\limits_{i \in I} H_i$ and the l.u.b is $\left\langle \bigcap\limits_{i \in I} H_i \right\rangle$.
2. Exhibit the lattice of subgroups of the groups S_3, D_4^*, Z_6, Z_{27} and Z_{36}.