Chapter 1

Mathematical Concepts and Symbols

1.1 Logical Symbols

The logical symbols are a kind of symbols using for theoretical statements.

Table 1.1: Frequently-used Logical Symbols

Symbols	Meanings
$L \Longrightarrow P$	Proposition L is contained in proposition P
$L \Longleftrightarrow P$	Proposition L is equivalent to proposition P
$\neg P$	Not P
$L \wedge P$	Proposition L and proposition P
$L \vee P$	Proposition L or proposition P

e.g.

$$((A \Longrightarrow B) \wedge (\neg B) \Longrightarrow (\neg A))$$

stands for " if A is contained in B,and B is not true,then A is not true". We also call $A \Longleftrightarrow B$ "A is the necessary and suffiecent condition of B". The typical math proposition is like "$A \Longrightarrow B$".In order to prove this proposition ,we can use the implication relationship

$$A \Longrightarrow C_1 \Longrightarrow \cdots \Longrightarrow C_n \Longrightarrow B$$

The every implication relationship in this chain is general truth or proved proposition.

Table 1.2: Truth Table

<table>
<tr><td rowspan="2">$\neg A$</td><td>A</td><td>0</td><td>1</td></tr>
<tr><td>$\neg A$</td><td>1</td><td>0</td></tr>
<tr><td rowspan="3">$A \vee B$</td><td>$A \backslash B$</td><td>0</td><td>1</td></tr>
<tr><td>0</td><td>0</td><td>1</td></tr>
<tr><td>1</td><td>1</td><td>1</td></tr>
<tr><td rowspan="3">$A \wedge B$</td><td>$A \backslash B$</td><td>0</td><td>1</td></tr>
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</table>

Question 1. $\neg(A \wedge B) \Leftrightarrow (\neg A \vee \neg B)$.

Proof. (Use the truth table)
If A is ture, B is ture, $A \wedge B$ is ture. $\neg(A \wedge B)$ is false.$\neg A$ is false,$\neg B$ is false, $(\neg A \vee \neg B)$ is false.
If A is ture, B is false, $A \wedge B$ is false. $\neg(A \wedge B)$ is true.$\neg A$ is false,$\neg B$ is true, $(\neg A \vee \neg B)$ is ture.
If A is flase, B is true, $A \wedge B$ is false. $\neg(A \wedge B)$ is true.$\neg A$ is true,$\neg B$ is false, $(\neg A \vee \neg B)$ is ture.
If A is false, B is false, $A \wedge B$ is false. $\neg(A \wedge B)$ is true.$\neg A$ is true,$\neg B$ is true, $(\neg A \vee \neg B)$ is ture.
So

$$\neg(A \wedge B) \Leftrightarrow (\neg A \vee \neg B)$$

□

Question 2. $(A \Rightarrow B) \Leftrightarrow \neg A \vee B$.

Proof. Firstly, we confirm the truth of

$$(A \Rightarrow B) \Rightarrow \neg A \vee B$$

If $(A \Rightarrow B)$ is false, then $\neg A \vee B$ is true.
If $(A \Rightarrow B)$ is ture ,then we have two posibilities. The first is A is ture, B is true, so $\neg A \vee B$ is true. The second is A is false,then B can be true or false, but $\neg A \vee B$ will be true.
Hence, $(A \Rightarrow B) \Rightarrow \neg A \vee B$.
Secondly, we prove

$$(A \Rightarrow B) \Leftarrow \neg A \vee B$$

If $\neg A \vee B$ is false, then $(A \Rightarrow B)$ is true.
If $\neg A \vee B$ is true, we have

1. $\neg A$ is true, B is false, then, A is false, $(A \Rightarrow B)$ is true.
2. $\neg A$ is false, B is true, then, A is true, $(A \Rightarrow B)$ is true.
3. $\neg A$ is true, B is true, then, A is false, $(A \Rightarrow B)$ is true.

So $(A \Rightarrow B) \Leftarrow \neg A \vee B$. □

TIPS.
1. *$\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$;*
2. *$\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$;*
3. *$(A \Rightarrow) \Leftrightarrow (\neg B \Rightarrow \neg A)$;*
4. *$(A \Rightarrow) \Leftrightarrow (\neg A \vee B)$;*
5. *$\neg(A \Rightarrow B) \Leftrightarrow A \wedge \neg B$.*

1.2 Sets and their Operations

A **set** is a collection of well-defined objects.
If A is a set, we write $a \in A$ to express element a belongs to set A, the negetive proposition of which is $a \notin A$. We use the symbol $\oslash$ to denote the **empty set**, that is, the set with no elements.

Theorem 1.2.1 (Cantor)**.** *There is no set contains all the sets.*

Proof. We assume $P(M)$ represents M doesn't contain itself.
Consider $K = \{M|P(M)\}$ which is made of sets M that satisfies P. Assuming K is a set, then either $P(K)$ or $\neg P(K)$ is true.
If $P(K)$ is true, K doesn't contain itself,but because of the definition of K, K is belong to K, which means $\neg P(M)$.
If $\neg P(M)$ is ture, it's easy to find the similar conclusion.
So to the contrary, K is not a set. This reveals a set can't contain all the sets. $\square$

Theorem 1.2.1 is a typical **paradox** called Russell's paradox.
$\forall$and $\exists$ are logical symbols to describe

Table 1.3: Universial and Exsitential Quantifications

Symbols	Meanings
$\forall x \in A$	For all elements x in A
$\exists x \in A$	There exist at least one element x in A

To show the inclusion relation of two sets, we often use the Symbol $A \subset B$, which means set A is a **subset** of set B (All the elements in A also belong to B).We indicate that A is not a subset of B by this notaion: $A \not\subset B$.

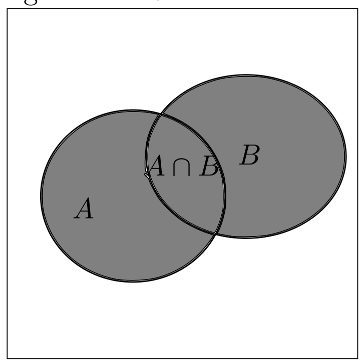
$$(A \subset B) := \forall x((x \in A) \Rightarrow (x \in B))$$

We define the equal relation between two sets, using the symbol =:

$$A = B := (A \subset B) \wedge (B \subset A)$$

We often use this definition to prove $A = B$. Symbol $\neq$ denotes the negetive proposition of equal.
A is a **proper subset** of B, if A is a subset of B, and $A \neq B$, denoted by the symbol $\subseteq$.

Figure 1.1: Union of two sets



If A and B are sets, then their **union**, denoted by $A \cup B$,is the set of all elements that are elements of either A or B:

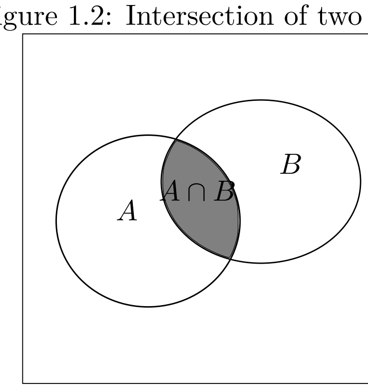
$$(A \cup B) := \{x \in M | (x \in A) \vee (x \in B)\}$$

Clearly, $A \cup B = B \cup A$.
If A and B are sets, then their **intersection**, denoted by $A \cap B$,contains all the elements in both A and B:

$$(A \cap B) := \{x \in M | (x \in A) \wedge (x \in B)\}$$

Also, we have $A \cap B = B \cap A$.

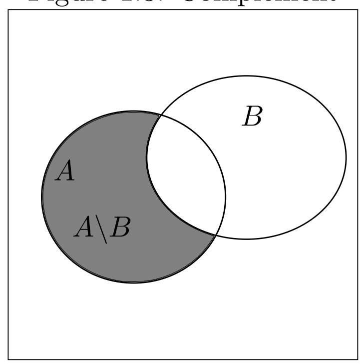
Figure 1.2: Intersection of two sets



We use the denotion $A \backslash B$ to represent the set contains all the elements which belongs to A but not belong to B, we call it the **defference set**.

$$A \backslash B := \{x \in M | (x \in A) \wedge (x \notin B)\}$$

Figure 1.3: Complement



For $B \subset A$, we can also denote it as the symbol $C_A B$.

Question 3 (de Morgen)**.**

$$C_M(A \cup B) = C_M A \cap C_M B$$

$$C_M(A \cap B) = C_M A \cup C_M B$$

Proof. We prove the first one.

$$\begin{aligned}
&(x \in C_M(A \cup B)) \Rightarrow (x \notin (A \cup B)) \\
\Rightarrow &((x \notin A) \wedge (x \notin B)) \Rightarrow ((x \in C_m A) \wedge (x \in C_M B)) \\
\Rightarrow &((x \in C_m A) \cap (x \in C_M B))
\end{aligned}$$

So we have proved $C_M(A \cup B) \subset C_M A \cap C_M B$. On the other hand:

$$\begin{aligned}
&((x \in C_m A) \cap (x \in C_M B)) \Rightarrow ((x \in C_m A) \wedge (x \in C_M B)) \\
\Rightarrow &((x \notin A) \wedge (x \notin B)) \Rightarrow (x \notin (A \cup B)) \\
\Rightarrow &(x \in C_M(A \cup B))
\end{aligned}$$

That's the same as $C_M(A \cup B) = C_M A \cap C_M B$. □

Question 4. *$(A \subset C_M B) \Leftrightarrow (B \subset C_M A)$.*

Proof.

$$\begin{aligned}
&(A \subset C_M B) \Rightarrow ((x \in A) \Rightarrow ((x \notin B)) \wedge (x \in M)) \\
\Rightarrow &(\neg(x \in A) \Leftarrow \neg((x \notin B)))) \\
\Rightarrow &((x \in B) \Rightarrow (x \notin A)) \Rightarrow (B \subset C_M A)
\end{aligned}$$

The other hand of this problem is the same. □

TIPS. *1.* $(A \subset C) \wedge (B \subset C) \Leftrightarrow ((A \cup B) \subset C)$;
2. $(C \subset A) \wedge (C \subset B) \Leftrightarrow (C \subset (A \cap B))$;
3. $C_M(C_M A) = A$;
4. $(A \subset C_M B) \Leftrightarrow (B \subset C_M A)$;
5. $(A \subset B) \Leftrightarrow (C_M A \supset C_M B)$.

Question 5. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof.

$$A \cup (B \cap C) \Leftrightarrow ((x \in A) \vee ((x \in B)) \wedge (x \in C))$$

$$((x \in A) \vee (x \in B)) \wedge ((x \in A) \vee (x \in C)) \Leftrightarrow (A \cup B) \cap (A \cup C)$$

So, we should prove:

$$((x \in A) \vee ((x \in B)) \wedge (x \in C)) \Leftrightarrow ((x \in A) \vee (x \in B)) \wedge ((x \in A) \vee (x \in C))$$

That's the same as:

$$(A \vee (B \wedge C)) \Leftrightarrow ((A \vee B) \wedge (A \vee C))$$

It's easy to prove with the help of truth table. $\square$

In this question, we also establish a relation between logical operation and sets' operation.

TIPS. *1.* $A \cup (B \cup C) = (A \cup B) \cup C := A \cup B \cup C$;
2. $A \cap (B \cap C) = (A \cap B) \cap C := A \cap B \cap C$;
3. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;
4. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

We denote two sets' **Cartesian product** as $A \times B$. $A \times B$ is a set contains ordered pairs, which means $A \times B \neq B \times A$.

$$A \times B := \{(x, y) | (x \in A) \wedge (y \in B)\}$$

Question 6. $(X \times Y) \cup (Z \times Y) = (X \cup Z) \times Y$.

Proof.

$$\begin{aligned} &(X \times Y) \cup (Z \times Y) \\ \Rightarrow &\{(x, y) | (x \in X) \wedge (y \in Y)\} \cup \{(z, y) | (z \in Z) \wedge (y \in Y)\} \\ \Rightarrow &\{(x(\text{or } z), y) | ((x \in X) \wedge (y \in Y)) \vee ((z \in Z) \wedge (y \in Y))\} \\ \Rightarrow &\{(x(\text{or } z), y) | ((x \in X) \vee (z \in Z)) \wedge (y \in Y)\} \\ \Rightarrow &(X \cup Z) \times Y \end{aligned}$$

On the other hand:

$$\begin{aligned}
&(X \cup Z) \times Y \\
\Rightarrow&\{(x(\text{or } z), y) | ((x \in X) \vee (z \in Z)) \wedge (y \in Y)\} \\
\Rightarrow&\{(x(\text{or } z), y) | ((x \in X) \wedge (y \in Y)) \vee ((z \in Z) \wedge (y \in Y))\} \\
\Rightarrow&\{(x, y) | (x \in X) \wedge (y \in Y)\} \cup \{(z, y) | (z \in Z) \wedge (y \in Y)\} \\
\Rightarrow&(X \times Y) \cup (Z \times Y)
\end{aligned}$$

In conclusion:$(X \times Y) \cup (Z \times Y) = (X \cup Z) \times Y$. □

Question 7. $(A \times B \subset X \times Y) \Leftrightarrow (A \subset X) \wedge (B \subset Y)$.

Proof.

$$(A \times B) \subset (X \times Y) \Rightarrow (((a \in A) \wedge (b \in B)) \Rightarrow ((x \in X) \wedge (y \in Y)))$$

In this formula, we will get

$$((a \in A) \Rightarrow (x \in X)) \wedge ((b \in B) \Rightarrow (y \in Y))$$

That's because $A \times B$ is a ordered pair, which means there is a consistent one-to-one match between A and X, B and Y.
The proof of the other hand is similar. □

TIPS. *1.* $(X \times Y = \oslash) \Leftrightarrow (X = \oslash) \vee (Y = \oslash)$,
if $X \times Y \neq \oslash, A \times B \neq \oslash$, we have
2. $(A \times B \subset X \times Y) \Leftrightarrow (A \subset X) \wedge (B \subset Y)$;
3. $(X \times Y) \cup (Z \times Y) = (X \cup Z) \times Y$
4. $(X \times Y) \cap (X' \times Y') = (X \cap X') \times (Y \times Y')$.

1.3 Function

A **function** or **mapping** f from X to Y is a rule, or formula, or assignment, or relation of association that assigns to each $x \in X$ a unique element $y \in Y$. Here we call X the domain of the function f,. Elements $x \in X$ are the **arguments** of the function. The elements $y \in Y$ are the **dependent varibles**(the **image** of x), denoted by the symbol $f(x)$. The set Y is made of the value of the function,which called the **range** or **codomain** of the function f.

$$f(X) := \{y \in Y | \exists x \ ((x \in X) \wedge (y = f(x))\}$$

We often use the symbol

$$f : X \longrightarrow Y, X \xrightarrow{f} Y$$

to denote the function f.
While for a subset $D \subset X$, which is filled with the image of elements in $B \in Y$, we denote it as

$$D = f^{-1}(B) := \{x \in X | f(x) \in B\}$$

We call it the **preimage** or **inverse image** of set B.Like **Figure 1.4**
Mappings $f : X \to Y$ can be divided into several types:

1. the **surjection**: $f(X) = Y$;
2. the **injection**: $\forall x_1, x_2 \in X \ (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$;
3. the **bijection**: the mapping that is both surjection and surjection.

Figure 1.4: Image and preimage