

Chapter 1

Mathematical Concepts and Symbols

1.1 Logical Symbols

The logical symbols are a kind of symbols using for theoretical statements.

Table 1.1: Frequently-used Logical Symbols

| Symbols | Meanings |
|----------------|--|
| $L \implies P$ | Proposition L is contained in proposition P |
| $L \iff P$ | Proposition L is equivalent to proposition P |
| $\neg P$ | Not P |
| $L \wedge P$ | Proposition L and proposition P |
| $L \vee P$ | Proposition L or proposition P |

e.g.

$$((A \implies B) \wedge (\neg B) \implies (\neg A))$$

stands for “ if A is contained in B ,and B is not true,then A is not true”.

We also call $A \iff B$ “ A is the necessary and suffiecent condition of B ”.

The typical math proposition is like “ $A \implies B$ ”.In order to prove this proposition ,we can use the implication relationship

$$A \implies C_1 \implies \cdots \implies C_n \implies B$$

The every implication relationship in this chain is general truth or proved proposition.

Table 1.2: Truth Table

| | | | |
|----------------|--|---|---|
| $\neg A$ | A | 0 | 1 |
| | $\neg A$ | 1 | 0 |
| $A \vee B$ | $\begin{array}{c} B \\ \diagdown \\ A \end{array}$ | 0 | 1 |
| | 0 | 0 | 1 |
| | 1 | 1 | 1 |
| $A \wedge B$ | $\begin{array}{c} B \\ \diagdown \\ A \end{array}$ | 0 | 1 |
| | 0 | 0 | 0 |
| | 1 | 0 | 1 |
| $A \implies B$ | $\begin{array}{c} B \\ \diagdown \\ A \end{array}$ | 0 | 1 |
| | 0 | 1 | 1 |
| | 1 | 0 | 1 |

Question 1. $\neg(A \wedge B) \Leftrightarrow (\neg A \vee \neg B)$.

Proof. (Use the truth table)

If A is true, B is true, $A \wedge B$ is true. $\neg(A \wedge B)$ is false. $\neg A$ is false, $\neg B$ is false, $(\neg A \vee \neg B)$ is false.

If A is true, B is false, $A \wedge B$ is false. $\neg(A \wedge B)$ is true. $\neg A$ is false, $\neg B$ is true, $(\neg A \vee \neg B)$ is true.

If A is false, B is true, $A \wedge B$ is false. $\neg(A \wedge B)$ is true. $\neg A$ is true, $\neg B$ is false, $(\neg A \vee \neg B)$ is true.

If A is false, B is false, $A \wedge B$ is false. $\neg(A \wedge B)$ is true. $\neg A$ is true, $\neg B$ is true, $(\neg A \vee \neg B)$ is true.

So

$$\neg(A \wedge B) \Leftrightarrow (\neg A \vee \neg B)$$

□

Question 2. $(A \Rightarrow B) \Leftrightarrow \neg A \vee B$.

Proof. Firstly, we confirm the truth of

$$(A \Rightarrow B) \Rightarrow \neg A \vee B$$

If $(A \Rightarrow B)$ is false, then $\neg A \vee B$ is true.

If $(A \Rightarrow B)$ is true, then we have two possibilities. The first is A is true, B is true, so $\neg A \vee B$ is true. The second is A is false, then B can be true or false, but $\neg A \vee B$ will be true.

Hence, $(A \Rightarrow B) \Rightarrow \neg A \vee B$.

Secondly, we prove

$$(A \Rightarrow B) \Leftarrow \neg A \vee B$$

If $\neg A \vee B$ is false, then $(A \Rightarrow B)$ is true.

If $\neg A \vee B$ is true, we have

1. $\neg A$ is true, B is false, then, A is false, $(A \Rightarrow B)$ is true.
2. $\neg A$ is false, B is true, then, A is true, $(A \Rightarrow B)$ is true.
3. $\neg A$ is true, B is true, then, A is false, $(A \Rightarrow B)$ is true.

So $(A \Rightarrow B) \Leftarrow \neg A \vee B$.

□

TIPS. 1. $\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$;

2. $\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$;

3. $(A \Rightarrow) \Leftrightarrow (\neg B \Rightarrow \neg A)$;

4. $(A \Rightarrow) \Leftrightarrow (\neg A \vee B)$;

5. $\neg(A \Rightarrow B) \Leftrightarrow A \wedge \neg B$.

1.2 Sets and their Operations

A set is a collection of well-defined objects.

If A is a set, we write $a \in A$ to express element a belongs to set A , the negative proposition of which is $a \notin A$. We use the symbol \emptyset to denote the empty set, that is, the set with no elements.

Theorem 1.2.1 (Cantor). *There is no set contains all the sets.*

Proof. We assume $P(M)$ represents M doesn't contain itself.

Consider $K = \{M | P(M)\}$ which is made of **sets** M that satisfies P . Assuming K is a set, then either $P(K)$ or $\neg P(K)$ is true.

If $P(K)$ is true, K doesn't contain itself, but because of the definition of K , K is belong to K , which means $\neg P(K)$.

If $\neg P(K)$ is true, it's easy to find the similar conclusion.

So to the contrary, K is not a set. This reveals a set can't contain all the sets. \square

Theorem 1.2.1 is a typical paradox called Russell's paradox.

\forall and \exists are logical symbols to describe

Table 1.3: Universal and Existential Quantifications

| Symbols | Meanings |
|-------------------|---|
| $\forall x \in A$ | For all elements x in A |
| $\exists x \in A$ | There exist at least one element x in A |

To show the inclusion relation of two sets, we often use the Symbol $A \subset B$, which means set A is a **subset** of set B (All the elements in A also belong to B). We indicate that A is not a subset of B by this notation: $A \not\subset B$.

$$(A \subset B) := \forall x((x \in A) \Rightarrow (x \in B))$$

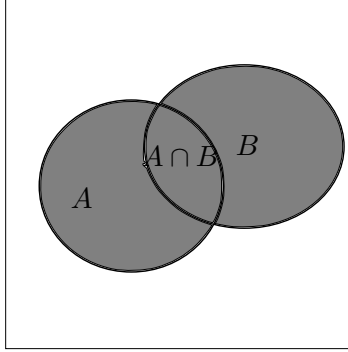
We define the equal relation between two sets, using the symbol $=$:

$$A = B := (A \subset B) \wedge (B \subset A)$$

We often use this definition to prove $A = B$. Symbol \neq denotes the negative proposition of equal.

A is a **proper subset** of B , if A is a subset of B , and $A \neq B$, denoted by the symbol \subsetneq .

Figure 1.1: Union of two sets



If A and B are sets, then their **union**, denoted by $A \cup B$, is the set of all elements that are elements of either A or B :

$$(A \cup B) =: \{x \in M \mid (x \in A) \vee (x \in B)\}$$

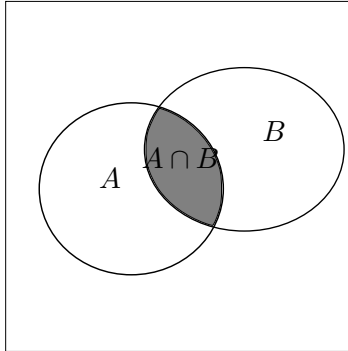
Clearly, $A \cup B = B \cup A$.

If A and B are sets, then their **intersection**, denoted by $A \cap B$, contains all the elements in both A and B :

$$(A \cap B) =: \{x \in M \mid (x \in A) \wedge (x \in B)\}$$

Also, we have $A \cap B = B \cap A$.

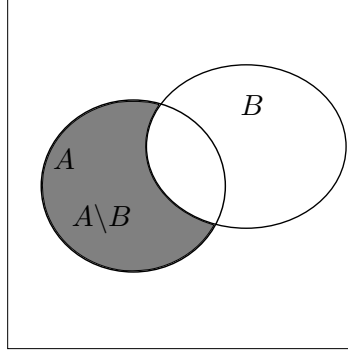
Figure 1.2: Intersection of two sets



We use the notation $A \setminus B$ to represent the set contains all the elements which belongs to A but not belong to B .

$$A \setminus B =: \{x \in M \mid (x \in A) \wedge (x \notin B)\}$$

Figure 1.3: Complement



For $B \subset A$, we can also denote it as the symbol $C_A B$.

Question 3 (de Morgan).

$$C_M(A \cup B) = C_M A \cap C_M B$$

$$C_M(A \cap B) = C_M A \cup C_M B$$

Proof. We prove the first one.

$$\begin{aligned} (x \in C_M(A \cup B)) &\Rightarrow (x \notin (A \cup B)) \\ &\Rightarrow ((x \notin A) \wedge (x \notin B)) \Rightarrow ((x \in C_M A) \wedge (x \in C_M B)) \\ &\Rightarrow ((x \in C_M A) \cap (x \in C_M B)) \end{aligned}$$

So we have proved $C_M(A \cup B) \subset C_M A \cap C_M B$. On the other hand:

$$\begin{aligned} ((x \in C_M A) \cap (x \in C_M B)) &\Rightarrow ((x \in C_M A) \wedge (x \in C_M B)) \\ &\Rightarrow ((x \notin A) \wedge (x \notin B)) \Rightarrow (x \notin (A \cup B)) \\ &\Rightarrow (x \in C_M(A \cup B)) \end{aligned}$$

That's the same as $C_M(A \cup B) = C_M A \cap C_M B$. □

Question 4. $(A \subset C_M B) \Leftrightarrow (B \subset C_M A)$.

Proof.

$$\begin{aligned} (A \subset C_M B) &\Rightarrow ((x \in A) \Rightarrow ((x \notin B) \wedge (x \in M))) \\ &\Rightarrow (\neg(x \in A) \Leftarrow \neg((x \notin B))) \\ &\Rightarrow ((x \in B) \Rightarrow (x \notin A)) \Rightarrow (B \subset C_M A) \end{aligned}$$

The other hand of this problem is the same. □

- TIPS.** 1. $(A \subset C) \wedge (B \subset C) \Leftrightarrow ((A \cup B) \subset C)$;
 2. $(C \subset A) \wedge (C \subset B) \Leftrightarrow (C \subset (A \cap B))$;
 3. $C_M(C_MA) = A$;
 4. $(A \subset C_MB) \Leftrightarrow (B \subset C_MA)$;
 5. $(A \subset B) \Leftrightarrow (C_MA \supset C_MB)$.

Question 5. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof.

$$\begin{aligned} A \cup (B \cap C) &\Leftrightarrow ((x \in A) \vee ((x \in B) \wedge (x \in C))) \\ ((x \in A) \vee (x \in B)) \wedge ((x \in A) \vee (x \in C)) &\Leftrightarrow (A \cup B) \cap (A \cup C) \end{aligned}$$

So, we should prove:

$$((x \in A) \vee ((x \in B) \wedge (x \in C))) \Leftrightarrow ((x \in A) \vee (x \in B)) \wedge ((x \in A) \vee (x \in C))$$

That's the same as:

$$(A \vee (B \wedge C)) \Leftrightarrow ((A \vee B) \wedge (A \vee C))$$

It's easy to prove with the help of truth table. □

In this question, we also establish a relation between logical operation and sets' operation.

- TIPS.** 1. $A \cup (B \cup C) = (A \cup B) \cup C =: A \cup B \cup C$;
 2. $A \cap (B \cap C) = (A \cap B) \cap C =: A \cap B \cap C$;
 3. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;
 4. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

We denote two sets' Cartesian product as $A \times B$. $A \times B$ is a set contains ordered pairs, which means $A \times B \neq B \times A$.

$$A \times B =: \{(x, y) | (x \in A) \wedge (y \in B)\}$$

Question 6. $(X \times Y) \cup (Z \times Y) = (X \cup Z) \times Y$.

Proof.

$$\begin{aligned} &(X \times Y) \cup (Z \times Y) \\ &\Rightarrow \{(x, y) | (x \in X) \wedge (y \in Y)\} \cup \{(z, y) | (z \in Z) \wedge (y \in Y)\} \\ &\Rightarrow \{(x(\text{or } z), y) | ((x \in X) \wedge (y \in Y)) \vee ((z \in Z) \wedge (y \in Y))\} \\ &\Rightarrow \{(x(\text{or } z), y) | ((x \in X) \vee (z \in Z)) \wedge (y \in Y)\} \\ &\Rightarrow (X \cup Z) \times Y \end{aligned}$$

On the other hand:

$$\begin{aligned}
& (X \cup Z) \times Y \\
& \Rightarrow \{(x(\text{or } z), y) | ((x \in X) \vee (z \in Z)) \wedge (y \in Y)\} \\
& \Rightarrow \{(x(\text{or } z), y) | ((x \in X) \wedge (y \in Y)) \vee ((z \in Z) \wedge (y \in Y))\} \\
& \Rightarrow \{(x, y) | (x \in X) \wedge (y \in Y)\} \cup \{(z, y) | (z \in Z) \wedge (y \in Y)\} \\
& \Rightarrow (X \times Y) \cup (Z \times Y)
\end{aligned}$$

In conclusion: $(X \times Y) \cup (Z \times Y) = (X \cup Z) \times Y$. □

- TIPS.**
1. $(X \times Y = \emptyset) \Leftrightarrow (X = \emptyset) \vee (Y = \emptyset)$,
if $X \times Y \neq \emptyset, A \times B \neq \emptyset$, we have
 2. $(A \times B \subset X \times Y) \Leftrightarrow (A \subset X) \wedge (B \subset Y)$;
 3. $(X \times Y) \cup (Z \times Y) = (X \cup Z) \times Y$
 4. $(X \times Y) \cap (X' \times Y') = (X \cap X') \times (Y \cap Y')$.