Chapter 1

Groups

1.1 Semigroups, monoids and groups

Exercise 1.1.1. Give examples other than those in the text of semigroups and monoids that are not groups.

Answer. Semigroup: $(\mathbf{Z}_+, +)$
Monoid: $(\mathbf{Z}_+, \times)$

Exercise 1.1.2. Let G be a group (written additively), S a nonempty set, and $M(S,G)$ the set of all functions $f : S \to G$. Define addition in $M(S,G)$ as follows: $(f+g) : S \to G$ is given by $s \to f(s) + g(s) \in G$. Prove that $M(S,G)$ is a group, which is abelian if G is.

Answer. Firstly we check $M(S,G)$ is a group

1. $f+g : s \mapsto f(s)+g(s) \in G$, so $f+g \in M(S,G)$
2. $(f+g)+h : s \mapsto (f(s)+g(s))+h(s)$, G is a group, so $s \mapsto (f(s)+g(s))+h(s) \Leftrightarrow s \mapsto f(s)+(g(s)+h(s))$, $(f+g)+h = f+(g+h)$.
3. Take the unit element as $e' : s \mapsto e$. $f+e' : s \mapsto f(s)+e'(s) = f(s)+e = f(s)$, so $f+e' = f$. Similarly, $e'+f = f$.
4. For any $f \in M(S,G)$, take $f^{-1} : s \mapsto (f(s))^{-1}$, whence $f(s) + (f(s))^{-1} = (f(s))^{-1} + f(s) = e$.

In conclusion, $M(S,G)$ is a group. If G is abelian $f+g : s \mapsto f(s)+g(s) = g(s)+f(s)$, $f+g = g+f$, so $M(S,G)$ is abelian.

Exercise 1.1.3. Is it true that a semigroup which has a left identity element and in which every element has a right inverse (see Proposition 1.3) is a group?

Answer. If e is the left identity, $\forall a \in A, ea = a$ and $\forall a \in A, \exists a^{-1} s.t. aa^{-1} = e$. We have proved that if $cc = c$, then $c = e$.

$$(a^{-1}a)(a^{-1}a) = a^{-1}(aa^{-1})a = a^{-1}(ea) = a^{-1}a \Rightarrow a^{-1}a = e$$

a^{-1} is also the left inverse. $ae = a(a^{-1}a) = (aa^{-1})a = ea = a$, e is also the right identity.

Exercise 1.1.4. Write out a multiplication table for the group D_4^*.

Answer. $D_4^* = \{R, R^2, R^3, I, T_x, T_y, T_{13}, T_{24}\}$

$$\begin{array}{c|cccccccc}
 & I & R & R^2 & R^3 & T_x & T_y & T_{13} & T_{24} \\
\hline
I & I & R & R^2 & R^3 & T_x & T_y & T_{13} & T_{24} \\
R & R & R^2 & R^3 & I & T_{13} & T_{24} & T_y & T_x \\
R^2 & R^2 & R^3 & I & R & T_y & T_x & T_{24} & T_{13} \\
R^3 & R^3 & I & R & R^2 & T_{24} & T_{13} & T_x & T_y \\
T_x & T_x & T_{24} & T_y & T_{13} & I & R^2 & R^3 & R \\
T_y & T_y & T_{13} & T_x & T_{24} & R^2 & I & R & R^3 \\
T_{13} & T_{13} & T_y & T_{24} & T_x & R^3 & R & I & R^2 \\
T_{24} & T_{24} & T_x & T_{13} & T_y & R & R^3 & R^2 & I
\end{array}$$

Exercise 1.1.5. Prove that the symmetric group on n letters, S_n, has ordrer $n!$.

Answer. For a set A whose order is n, we prove there's $n!$ different bijections by induction

1. For $n = 1$, trivial.
2. Assume $n = k$, there's $k!$ bijections. For $n = k + 1$, fix one element in A, and take $a \mapsto a$, there's k free elements, so there's $k! \cdot (k + 1)$ bijections in total.

By induction, we get the result.

Exercise 1.1.6. Write out an addition table for $Z_2 \oplus Z_2$. $Z_2 \oplus Z_2$ is called the Klein four group.

Answer. $Z_2 = \{1, 0\}$, $Z_2 \oplus Z_2 = \{(1, 1), (1, 0), (0, 1), (0, 0)\}$

$$\begin{array}{c|cccc}
 & (1,1) & (1,0) & (0,1) & (0,0) \\
\hline
(1,1) & (0,0) & (0,1) & (1,0) & (1,1) \\
(1,0) & (0,1) & (0,0) & (1,1) & (1,0) \\
(0,1) & (1,0) & (1,1) & (0,0) & (0,1) \\
(0,0) & (1,1) & (1,0) & (0,1) & (0,0)
\end{array}$$

Exercise 1.1.7. If p is prime, then the nonzero elements of Z_p form a group of order $p-1$ under multiplication. Show that this statement is false if p is not prime.

Answer. For the set $Z_p \backslash \{\bar{0}\}$

1. $Z_p \backslash \{\bar{0}\}$ is obviously associative and commutative.
2. Take $\bar{1}$ as the identity element, $\forall \bar{a} \in Z_p \backslash \{\bar{0}\}, \bar{1} \times \bar{a} = \bar{a}$.
3. We prove there is a unique element $a^{-1} \in Z_p \backslash \{\bar{0}\} s.t. aa^{-1} = \bar{1}$. Assume there exists $\bar{b}, \bar{c}$ and $\bar{a} \cdot \bar{b} = \bar{k}, \bar{a} \cdot \bar{c} = \bar{k}$, then $a(b-c) \equiv 0 \mod p$. p is a prime, so $lcm(p,a) = 1, lcm(p, b-c) = 1$, so $\bar{b} = \bar{c}$. There is at most one element s.t. $\bar{a}\bar{b} = \bar{k}$. Take $\bar{b} = \bar{1}, \bar{2}, \ldots \bar{p-1}$, $\bar{k}$ travels through $\bar{b} = \bar{1}, \bar{2}, \ldots \bar{p-1}$. There exists an element $\bar{b} \in Z_p \backslash \{\bar{0}\}, \bar{a}\bar{b} = \bar{1}$.

$Z_p \backslash \{\bar{0}\}$ is a group. If p is not a prime, the inverse element is not always unique. Take $a|p$, there's more than one inverse element in $Z_p \backslash \{\bar{0}\}$.

Exercise 1.1.8. (a) The relation given by $a \sim b \Leftrightarrow a - b \in \mathbf{Z}$ is a congruence relation on the additive group $\mathbf{Q}$ [see Theorem 1.5].
(b) The set $\mathbf{Q}/\mathbf{Z}$ of equivalence classes is an infinite abelian group.

Answer. (a) For group $(\mathbf{Q}, +)$, $a_1 \sim b_1 \Leftrightarrow a_1 - b_1 = k_1 \in \mathbf{Z}$, $a_2 \sim b_2 \Leftrightarrow a_2 - b_2 = k_2 \in \mathbf{Z}$, so $(a_1+a_2)-(b_1+b_2) = ((k_1+b_1)+(k_2+b_2))-(b_1+b_2) = k_1 + k_2 \in \mathbf{Z}$. $a \sim b$ is a congruence relation.
(b) 1 if $a+b \geq 1$, $\bar{a} + \bar{b} = \overline{a+b-1}$. If $a+b < 1$, $\bar{a} + \bar{b} = \overline{a+b}$.
2 $\mathbf{Q}/\mathbf{Z}$ is obviously associative and commutative.
3 Take the identity element as $\bar{0}$, $\bar{0} + \bar{a} = \bar{a}$.
4 If $\bar{a} \neq 0$, take $(\bar{a})^{-1} = \overline{1-a}$, then $\bar{a} + \overline{1-a} = \bar{0}$
so $\mathbf{Q}/\mathbf{Z}$ is a abelian group. (Infinite remains to be certified)

Exercise 1.1.9. Let p be a fixed prime. Let R_p be the set of all those rational numbers whose denominator is relatively prime to p. Let R^p be the set of rationals whose denominator is a power of $p(p^i, i > 0)$. Prove that both R_p and R^p are abelian groups under ordinary addition of rationals.

Answer. Trivial.

Exercise 1.1.10. Let p be a prime and let $Z(p^\infty)$ be the following subset of the group $\mathbf{Q}/\mathbf{Z}$:

$$Z(p^\infty) = \{\overline{a/b} \in \mathbf{Q}/\mathbf{Z} | a, b \in \mathbf{Z} \text{ and } b = p^i \text{ for some } i \geq 0\}$$

Show that $Z(p^\infty)$ is an infinite group under the addition operation of $\mathbf{Q}/\mathbf{Z}$.

Answer. $Z(p^\infty) = \{\overline{a/b} | a, b \in \mathbf{Z}, b = p^i, i \geq 0\}$. Take $a = \frac{\bar{a_1}}{b_1}$, $b = \frac{\bar{a_2}}{b_2}$. $b^{-1} = \frac{\overline{b_2 - a_2}}{b_2}$

$$\begin{aligned} a + b^{-1} = \frac{\bar{a_1}}{b_1} + \frac{\overline{b_2 - a_2}}{b_2} &= \frac{\bar{a_1}}{p^{s_1}} + \frac{\overline{p^{s_2} - a_2}}{p^{s_2}} \\ &= \frac{\overline{a_1 \cdot p^{s_2} + p^{s_1}(p^{s_2} - a_2)}}{p^{s_1+s_2}} \in Z(p^\infty) \end{aligned}$$

Therefore, $Z(p^\infty)$ is a subgroup of $\mathbf{Q}/\mathbf{Z}$. $\frac{1}{p^i} \in Z(p^\infty)$ for any $i \in \mathbf{Z}$, so $Z(p^\infty)$ is infinite, $\mathbf{Q}/\mathbf{Z}$ is also infinite.

Exercise 1.1.11. The following conditions on a group G are equivalent:

i G is abelian;

ii $(ab)^2 = a^2b^2$ for all $a, b \in G$;

iii $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$;

iv $(ab)^n = a^nb^n$ for all $n \in \mathbf{Z}$ and all $a, b \in G$;

v $(ab)^n = a^nb^n$ for three consecutive integers n and all $a, b \in G$. Show that v$\Rightarrow$ i is false if 'three' is replaced by 'two'.

Answer. i$\Leftrightarrow$ iii: $((ab)b^{-1})a^{-1} = (ab)(b^{-1}a^{-1}) = e$, so $(ab)^{-1} = b^{-1}a^{-1}$. If iii, $b^{-1}a^{-1} = a^{-1}b^{-1}$ for any $a, b \in G$, G is abelian. If i, G is abelian, $(ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1}$.
iv $\Rightarrow$ v, iv$\Rightarrow$ ii and i$\Rightarrow$ iv are trivial.
ii$\Rightarrow$ i:

$$(ab)(ab) = aabb \Rightarrow a^{-1}(ab)^2b^{-1} = a^{-1}aabbb^{-1} = ba = ab$$

so G is abelian.

v $\Rightarrow$ i: $a^nb^n = (ab)^n$, $a^{n-1}b^{n-1} = (ab)^{n-1}$, $a^{n+1}b^{n+1} = (ab)^{n+1}$.

$$(b^{-1})^n(a^{-1})^n = ((ab)^n)^{-1} = ((ab)^{-1})^n$$

$$((ab)^{-1})^n(ab)^{n+1} = (b^{-1})^n ab^{n+1}$$

$$((ab)^{-1})^n(ab)^{n-1} = b^{-1}a^{-1} = (b^{-1})^n a^{-1}b^{n-1}$$

$$a = (b^{-1})^n ab^n \qquad b^{-1}a^{-1}b = (b^{-1})^n a^{-1}b^n$$

So $a^{-1} = b^{-1}a^{-1}b$, which means G is abelian.
If "three" is replaced by "two": $a^nb^n = (ab)^n$, $a^{n+1}b^{n+1} = (ab)^{n+1}$.

$$(b^{-1})^n(a^{-1})^n = ((ab)^{-1})^n \qquad a = (b^{-1})^n ab^n$$

For the group $S_3 = \{(1), (12), (13), (23), (123), (132)\}$, taking any $a \in S_3$, we can check that $a^6 = (1)$. If $n = 6$, then $a = (b^{-1})^n ab^n$ for any $a, b \in S_3$. But S_3 is nonabelian.

Exercise 1.1.12. If G is a group, $a, b \in G$ and $bab^{-1} = a^r$ for some $r \in \mathbf{N}$, then $b^jab^{-j} = a^{r^j}$ for all $j \in \mathbf{N}$.

Answer. $bab^{-1} = a^r$. We prove it by induction. For $j = 1$, its always true. Assume $j = k$ the equation is correct, $b^kab^{-k} = a^{r^k}$. $ba^{r^k}b^{-1} = (a^{r^k})^{r = a^{r^{k+1}}}$. For $j = k + 1$, it's also true.

Exercise 1.1.13. If $a^2 = e$ for all elements a of a group G, then G is abelian.

Answer.

$$a^2 = e \Rightarrow a^2a^{-1} = ea^{-1} = a(aa^{-1}) = ae \Rightarrow a = a^{-1}$$

$$ab = a^{-1}b^{-1} = (ab)^{-1} = (ba)^{-1}$$

So $ab = ba \forall a, b \in G$. G is abelian.

Exercise 1.1.14. If G is a finite group of even order, then G contains an element $a \neq e$ such that $a^2 = e$.

Answer. Suppose not. $\forall a \neq e, aa \neq e \Leftrightarrow a \neq a^{-1}$. We can classify the group into some subsets. $G = \bigcup\limits_{a \neq e} \{a, a^{-1}\} \cup \{e\}$. Notice that $\{a, a^{-1}\} \cap \{b, b^{-1}\} = \varnothing$ if $a \neq b$, so $|G| = 2n + 1$, That's contradictory!

Exercise 1.1.15. Let G be a nonempty finite set with an associative binary operation such that for all $a, b, c \in G$, $ab = ac \Rightarrow b = c$ and $ba = ca \Rightarrow b = c$. Then G is a group. Show that this conclusion may be false if G is finite.

Answer. G is a semigroup. Fix $a \in G$ and take b travels through all elements in G, then ab travels through all elements in G.
There exists an element e_1 s.t. $ae_1 = a \forall a \in G$. Similarly, we can find e_2 s.t. $e_2 a = a \forall a \in G$. $e_2 e_1 = e_1 = e_2 = e$. e is the identity element of G. Easily, we can find that $\forall a \in G, \exists ! a^{-1} \in G$ s.t. $a^{-1}a = aa^{-1} = e$ because $ab = ac \Rightarrow b = c$ and $ba = ca \Rightarrow b = c$.
G is a group. If G is infinite, G may not be a group, for example: $(Z_+, \times)$.

Exercise 1.1.16. Let $a_1, a_2, \ldots$ be a sequence of elements in a semigroup G. Then there exists a unique function $\Psi : \mathbf{N}^* \to G$ such that $\Psi(1) = a_1, \Psi(2) = a_1 a_2, \Psi(3) = (a_1 a_2) a_3$ and for $n \geq 1$, $\Psi(n+1) = (\Psi(n)) a_{n+1}$. Note that $\Psi(n)$ is precisely the standard n product $\prod_{i=1}^{n} a_i$.

Answer. Applying the Recursion Theorem with $a = a_1, S = G$ and $f_n : G \to G$ given by $x \mapsto x a_{n+2}$ yields a function $\phi : \mathbf{N} \to G$. Let $\Psi = \phi\theta$, where $\theta : \mathbf{N}^* \to \mathbf{N}$ is given by $k \mapsto k - 1$.

1.2 Homomorphisms and subgroups

Exercise 1.2.1. If $f : G \to H$ is a homomorphism of groups, then $f(e_G) = e_H$ and $f(a^{-1}) = f(a)^{-1}$ for all $a \in G$. Show by example that the first conclusion may be false if G, H are monoids that are not groups.

Answer. For example, $(\mathbf{Z}_+, +)$ and $(\mathbf{N}, \times)$ are monoids. Denote $f : \mathbf{Z}_+ \to \mathbf{N}$ as $f(x) = 0 \forall x \in \mathbf{Z}_+$. f is a homomorphism satisfies those conditions.

Exercise 1.2.2. A group G is abelian if and only if the map $G \to G$ given by $x \mapsto x^{-1}$ is automorphism.

Answer. If G is abelian, $f(x) = x^{-1}$ is a monomorphism and epimorphism. $f(a)f(b) = a^{-1}b^{-1} = (ab)^{-1} = f(ab)$
If $f(x) = x^{-1}$ is a isomorphism, $f(a)f(b) = a^{-1}b^{-1} = f(ab) = (ab)^{-1} = b^{-1}a^{-1} \forall a, b \in G$, so G is abelian.

Exercise 1.2.3. Let Q_8 be the group(under ordinary matrix multiplication) generated by complex matrices $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, where $i^2 = -1$. Show that Q_8 is a nonabelian group of order 8. Q_8 is called the quaternion group.

Answer. The multiply operation is associative by the difinition. $A^4 = B^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ which is the identity element.

$$A^{-1} = A^3 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \in G \qquad B^{-1} = B^3 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \in G$$

So $\forall A^i B^j \in G$, $(A^i B^j)^{-1} \in G$. G is a group. Now we examine the order of G is 8.

$$BA = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$A^3B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix}$$

So $BA = A^3B$. Take $X = A^{s_1}B^{s_2}A^{s_3}B^{s_4}\dots A^{s_{2n-1}}B^{s_{2n}} = A^{s_1}B^{s_2-1}A^3B$ $A^{s_3-1}B^{s_4}\dots A^{s_{2n-1}}B^{s_{2n}} = \dots$ In finite steps , we can change it into $X = A^aB^b$. $A^4 = B^4 = I$, so we only consider $1 \leq a, b \leq 4$. $A^2 = B^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, we list all: $Q_8 = \{A, A^2, B, BA, AB, A^2B, AB^2, I\}$. The order of Q_8 is 8.

Exercise 1.2.4. Let H be the group(under ordinary matrix multiplication) of real matrices generated by $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Show that H is a nonabelian group of order 8 which is not isomorphic to the quaternion group, but is isomorphic to the group D_4^*.

Answer. $C^4D^2 = I, DC = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = C^3D$.Similarly, we can prove H is a nonabelian group of order 8. $H = \{C, C^2, C^3, I, D, CD, C^2D, C^3D\}$
Assume $G \cong H$ and the isomorphism is f, Let $f(D) = X$, $f(D^2) = X^2 = f(I) = I$, so $X^2 = I$. But $f^{-1}(I) = I \Rightarrow X \neq I \Rightarrow X = AB$ or $X = A^2$ or $X = B^2$.
If $X = A^2$, consider $f(C) = Y, f(C^2D) = Z$, we have $(Y, Z) = (B^2, AB)$ or $(Y, Z) = (AB, B^2)$. $f(C^2D) = f(C^2)f(D) \Leftrightarrow Z = XY$. That's contradictory!
If $X = B^2$, the proof is similar.
If $X = AB$, $(Y, Z) = (A, B)$ or $(Y, Z) = (B, A)$. That's contradictory! So f doesn't exist. G is not isomorphic to H.
Now we prove $H \cong D_4^*$. For any point $(x, y)^T$ inside the square

$$T_x = (x, -y)^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}(x, y)^T = CD(x, y)^T$$

$$T_y = (-x, y)^T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}(x, y)^T = C^3D(x, y)^T$$

$$T_{13} = (-y, x)^T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(x, y)^T = C^3(x, y)^T$$

$$T_{24} = (y, -x)^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(x, y)^T = C(x, y)^T$$

so $D_4^* = \langle T_x, T_y, T_{13}, T_{24}\rangle = H = \langle C, D\rangle$.

Exercise 1.2.5. Let S be a nonempty subset of a group G and define a relation on G by $a \sim b$ if and only if $ab^{-1} \in S$. Show that $\sim$ is an equivalence relation if and only if S is a subgroup of G.

Answer. If $\sim$ is a equivalence relation

1. $a \sim b \Rightarrow b \sim a$;
2. $a \sim a$;
3. $a \sim b, b \sim c \Rightarrow a \sim c$.

$2 \Leftrightarrow aa^{-1} = e \in S$. $1 \Rightarrow a \sim e \Rightarrow e \sim a \forall a \in S$, so $ae^{-1} = a \in S, ea^{-1} = a^{-1} \in S$. If $a, b \in S, b^{-1} \in S$, so $ae^{-1} \in S, e(b^{-1})^{-1} \in S$. By 3, $a \sim e, e \sim b^{-1} \Rightarrow a \sim b^{-1} \Rightarrow ab \in S$. S is a subgroup of G.
If S is a subgroup of G

1. $aa^{-1} \in S \Rightarrow a \sim a$;
2. $ab^{-1} \in S \Rightarrow (ab^{-1})^{-1} = ba^{-1} \in S \Rightarrow (a \sim b \Rightarrow b \sim a)$;
3. $ab^{-1} \in S, bc^{-1} \in S \Rightarrow (ab^{-1})(bc^{-1}) = ac^{-1} \in S$, which means $a \sim b, b \sim c \Rightarrow a \sim c$

In conclusion, $\sim$ is a equivalence relation.

Exercise 1.2.6. A nonempty finte subset of a group is s subgroup if and only if it is closed under the product in G.

Answer. $\Rightarrow$: Trivial.
$\Leftarrow$: S is apparently associative. $\forall a, b \in S, ab \in S$. S is a finite set, so there exists $m > n \in \mathbf{N}$ s.t. $a^m = a^n$.

Exercise 1.2.7. If n is a fixed integer, then $\{kn | n \in \mathbf{Z}\} \subset \mathbf{Z}$ is an additive subgroup of $\mathbf{Z}$, which is isomorphic to $\mathbf{Z}$.

Answer. Denote $Z^n = \{kn | k \in \mathbf{Z}\}$. We can easily check that Z^n is a subgroup of $\mathbf{Z}$. Now we build a isomorphism between Z^n and $\mathbf{Z}$. Take $f : Z^n \to \mathbf{Z}$ as $f(kn) = k$, $f^{-1}(n) = kn$. f is a bijection so Z^n and $\mathbf{Z}$ are isomorphism.

Exercise 1.2.8. The set $\{\sigma \in S_n | \sigma(n) = n\}$ is a subgroup of S_n which is isomorphic to S_{n-1}.

Answer. Denote $S_n^{(n)} = \{\sigma \in S_n | \sigma(n) = n\}$. $\forall \sigma_1, \sigma_2 \in S_n^{(n)}, \sigma_1\sigma_2(n) = \sigma_1(\sigma_2(n)) = \sigma_1(n) = n$, so $\sigma_1\sigma_2 \in S_n^{(n)}$. By the above exercise, $S_n^{(n)}$ is a subgroup of S_n. Now we build an isomorphism between $S_n^{(n)}$ and S_{n-1}. Take $f : S_{n-1} \to S_n^{(n)}$ as $f(\sigma) = \sigma'$, where $\sigma'(x) = \begin{cases} n, & x = n \\ \sigma(n), & x \neq n \end{cases}$. $\sigma' \in S_n^{(n)}$ and f is a bijection, so $S_{n-1} \cong S_n^{(n)}$.

Exercise 1.2.9. Let $f : G \to H$ be a homomorphism of groups, A a subgroup of G, and B a subgroup of H.
(a) $\mathrm{Ker} f$ and $f^{-1}(B)$ are subgroups of G.
(b) $f(A)$ is a subgroup of H.

Answer. (a) f is a homomorphism, so $f(e) = e', e \in \mathrm{Ker} f$. $\forall a \in \mathrm{Ker} f$, $f(aa^{-1}) = f(a)f(a^{-1}) = e'$, so $f(a^{-1}) = f(a)^{-1} = e'^{-1} = e'$. $\forall a, b \in \mathrm{Ker} f, f(ab^{-1}) = f(a)f(b^{-1}) = f(a)f(b)^{-1} = e' \Rightarrow ab^{-1} \in \mathrm{Ker} f$, which means $\mathrm{Ker} f$ is a subgroup of G. The proof of $f^{-1}(B)$ is a subgroup of G is similar.
(b) f is a homomorphism, $f(e) = e'$. $\forall a, b \in A, ab^{-1} \in A$, so $f(ab^{-1}) = f(a)f(b^{-1}) = f(a)f(b)^{-1} \in f(A)$, $f(A)$ is a subgroup of H.

Exercise 1.2.10. List all subgroups of $Z_2 \oplus Z_2$. Is $Z_2 \oplus Z_2$ isomorphic to Z_4?

Answer. $Z_2 \oplus Z_2$: $\{\{(1,1),(1,0),(0,1),(0,0)\},\{(1,1),(0,0)\},\{(0,0)\},$
$\{(1,0),(0,0)\},\{(0,1),(0,0)\},\{(0,1),(1,0),(0,0)\}\}$.
Z_4: $\{\{\bar{0},\bar{1},\bar{2},\bar{3}\},\{\bar{0},\bar{2}\},\{\bar{0}\}\}$.
Z_4 and $Z_2 \oplus Z_2$ are not isomorphic because they have different subgroups.

Exercise 1.2.11. If G is a subgroup, then $C = \{a \in G | ax = xa \text{ for all } x \in G\}$ is a abelian subgroup of G. C is called the center of G.

Answer. Take $a, b \in C, ab = ba$, C is commutative. $\forall a, b \in C, x \in G$, $b^{-1} \in G$, so $ab^{-1} = b^{-1}a$.

$$ax = axbb^{-1} = abxb^{-1} = baxb^{-1} = bxab^{=1} = abb^{-1}x = bab^{-1}x$$

so $b^{-1}ax = ab^{-1}x = xab^{-1}$, $ab^{-1} \in C$, C is a subgroup of G.

Exercise 1.2.12. The group D_4^* is not cyclic, but can be generated by two elements. The same is true of S_n(nontrivial). What is the minimal number of generators of the additive group $\mathbf{Z} \oplus \mathbf{Z}$?

Answer. $\mathbf{Z} \oplus \mathbf{Z} = \{(a, b) | a \in \mathbf{Z}, b \in \mathbf{Z}\} = \langle (0, 0), (1, 0), (0, 1) \rangle$. We can easily check the spanning set is the minimal.

Exercise 1.2.13. If $G = \langle a \rangle$ is a cyclic group and H is any group, then every homomorphism $f : G \to H$ is completely determined by the element $f(a) \in H$.

Answer. $\forall x \in G$, there exist $m \in \mathbf{N}$ s.t. $x = a^m$, so $f(x) = f(a^m) = f(a)^m \Rightarrow \mathrm{Im} f = \langle f(a) \rangle$. $f : a^m \mapsto f(a)^m \forall m \in \mathbf{N}$. f is completely determined by $f(a) \in H$.

Exercise 1.2.14. The following cyclic subgroups are all isomorphic: the multiplication group $\langle i \rangle$ in $\mathbf{C}$, the additive group $\mathbf{Z_4}$ and the subgroup $\left\langle \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \right\rangle$ of S_4.

Answer. $\langle i \rangle = \{i, -1, -i, 1\}$, $Z_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$,
$\langle (1234) \rangle = \{(1234), (13)(24), (1432), (1)\}$. Denote $f : \langle i \rangle \to Z_4$ as $f(i) = \bar{i}$, $g : Z_4 \to \langle (1234) \rangle$ as $g(i) = (1234)$. From the exercise above we know f and g aer homomorphisms, and they are bijections, so $\langle i \rangle \cong Z_4 \cong \langle (1234) \rangle$.

Exercise 1.2.15. Let G be a group and $\mathrm{Aut}G$ is the set of all automorphisms of G.
(a) $\mathrm{Aut}G$ is a group with composition of functions as binary operation.
(b) $\mathrm{Aut}\mathbf{Z} \cong Z_2$ and $\mathrm{Aut}Z_6 \cong Z_2$; $\mathrm{Aut}Z_8 \cong Z_2 \oplus Z_2$; $\mathrm{Aut}Z_p \cong Z_{p-1}$ (p prime).
(c) What is $\mathrm{AutZ_n}$ for arbitrary $n \in \mathbf{N}^*$?

Answer. We only prove the third question.
For $\bar{a} \in Z_n$, the order of $\bar{a}$ is $|\bar{a}| = \frac{n}{(n,a)}$. When $(n,a) = 1$, $\bar{a}$ is a generator of Z_n. Denote Euler function as $\varphi(x)$ and $Z_n^* = \{\bar{a} \in Z_n | (a,n) = 1\}$, then $|Z_n^*| = \varphi(n)$. For $\sigma \in \mathrm{Aut}Z_n$, σ is completely determined by $\sigma(\bar{1}) = \bar{a}$, and we denote σ as σ_a. For $\sigma_a, \sigma_b \in \mathrm{Aut}Z_n$, $\sigma_a(\sigma_b(\bar{1})) = \sigma_a(\bar{b}) = \bar{a}\bar{b} = \sigma_{ab}(\bar{1})$. We have proved $\mathrm{Aut}Z_n \cong Z_n^*$.
Now we give out a lemma to show the structure of Z_n^*.

Lemma. *If $n = st, (s,t) = 1$, then $Z_n^* \cong Z_s^* \oplus Z_t^*$.*

The proof of this lemma is quite simple. Consider the mappping $f^* : Z_n^* \to Z_s^* \oplus Z_t^*$ which is defined by $(x \mod n) \mapsto (x \mod s, x \mod t)$. Since for any $a, b \in Z_n^*$, $f^*(a)f^*(b) = (a \mod s, a \mod t)(b \mod s, b \mod t) = (ab \mod s, ab \mod t) = f^*(ab)$, f^* is a well defined homomorphism. For $x \in \mathrm{Ker}f^*$, $x \equiv 1 \mod s$, $x \equiv 1 \mod t$, so $x \equiv 1 \mod [s,t]$, $x \equiv 1 \mod n$, f^* is a monomorphism. Since $|f^*(Z_n^*)| = |Z_n| = \varphi(n) = \varphi(s)\varphi(t) = |Z_s^* \oplus Z_t^*|$, f^* is a epimorphism. $Z_n^* \cong Z_s^* \oplus Z_t^*$
For $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$, $Z_n^* \cong Z_{p_1^{k_1}}^* \oplus Z_{p_2^{k_2}}^* \oplus \cdots \oplus Z_{p_m^{k_m}}^*$. Now we consider the structure of $Z_{p^k}^*$.
For $p = 2$, $Z_2^* \cong Z_1$, $Z_4^* \cong Z_2$, $Z_{2^k}^* \cong Z_2 \oplus Z_{2^{k-2}}$.
For other odd prime p, $Z_{p^k}^* \cong Z_{(p-1)p^{k-1}}$.
In order to prove the result, we need the Lagrange theorem in number theory.

Lemma (Lagrange). $f(x) \in Z[n]$, $f(x) \equiv k$ *has at most n solutions when* mod p, *where p is an odd prime.*

We use induction to prove the lemma.

1. $n = 1$, the proof the trivial.
2. Assume for $n \leq m-1$ the lemma is correct, and for $n = m$, $f(x) \equiv k$ has $m+1$ solutions. $f(x) - f(x_{m+1}) = (x - x_{m+1})g(x) \equiv 0 \mod p$. Take $x = x_i, i = 1, 2, \ldots, m$, $(x_i - x_{m+1})g(x_i) \equiv 0 \mod p$, $x_i \neq x_{m+1}$, so $g(x_i) \equiv 0 \mod p$, That's contradictory to the induction assumptions!

The lemma is proved.
Come back to the original question. Firstly, we consider $k = 1$ and p is an odd prime. For any factor d of $p-1$, denote $S(d) = \{\bar{a} \in Z_p^* | \mathrm{ord}_p(a) = d\}$. $S(d)$ forms a partition of Z_p^*. If $S(d) \neq \varnothing$, there exists $\bar{a} \in S(d)$ and $a^d \equiv 1 \mod p$. By Largrange theorem, $a^d \equiv 1 \mod p$ has at most d solutions. Notice that $\{1, a, a^2, \ldots, a^{d-1}\}$ are the solutions of the equation, $a^i \not\equiv a^j \mod p$, whence $S(d) \subset \langle \bar{a} \rangle$. For $k = 1, 2, \ldots, d-1$, $\mathrm{ord}_p(\bar{a^k}) = \left|a^k\right| = \frac{d}{(d,k)} = d \Leftrightarrow (d,k) = 1$. Thus $|S(d)| = \varphi(d)$.
From $Z_p^* = \bigcup\limits_{d|p-1} S(d)$, we get

$$p - 1 = \left|Z_p^*\right| = \sum_{d|p-1} |S(d)| \leq \sum_{d|p-1} \varphi(d) = p - 1$$

If $d|p-1$, $|S(d)| = \varphi(d)$. Particularly, when $d = p-1$, $|S(p-1)| = \varphi(p-1) \neq 0$, Z_p^* has a element of order $p-1$, Z_p^* is a cyclic group.
Secondly, we consider $k \geq 2$. Take $a \in \mathbf{Z}$ and $\bar{a}$ is the class of $x \equiv a \mod p^k$. For $s \geq t$, we have a group homomorphism $f_{s,t} : Z_{p^s}^* \to Z_{p^t}^*$ which is defined by $(a \mod p^s) \mapsto (a \mod p^t)$. Since $a \equiv b \mod p^s \Rightarrow a \equiv b \mod p^t$, f is well defined. $\mathrm{Ker} f_{s,t} = \{up^t + 1 \mod p^s | u = 0, 1, \ldots, p^{s-t} - 1\}$. If $2t \geq s$, since $(up^t + 1)(vp^t + 1) \equiv uvp^{2t} + (u+v)p^t + 1 \equiv (u+v)p^t + 1 \mod p^s$, $\mathrm{Ker} f_{s,t} \cong Z_{p^{s-t}}$ is a cyclic group. There exists a isomorphism $g_{s,t} : Z_{p^s}^* / \mathrm{Ker} f_{s,t} \to Z_{p^t}^*$.

$$\{\bar{1}_{p^k}\} = \mathrm{Ker} f_{k,k} < \mathrm{Ker} f_{k,k-1} < \cdots < \mathrm{Ker} f_{k,1} < Z_{p^k}^*$$

Lemma. *Suppose $i \geq 2$, $\bar{a}_{p^k} \in \mathrm{Ker} f_{k,i}$, but $\bar{a}_{p^k} \notin \mathrm{Ker} f_{k,i+1}$, then $\bar{a}_{p^k}^p \in \mathrm{Ker} f_{k,i+1}$ and $\bar{a}_{p^k}^p \notin \mathrm{Ker} f_{k,i+2}$.*

This lemma can be proved by LTE. Here we use the language in group theory to prove it. $f_{k,i+2}(\bar{a}_{p^k}) = \bar{a}_{p^{i+2}}$, $\bar{a}_{p^{i+2}} \in f_{k,i+2}(\mathrm{Ker} f_{k,i}) = \mathrm{Ker} f_{i+2,i}$. $\mathrm{Ker} f_{i+2,i} \cong Z_{p^2}$ since $2i \geq i + 2$. $\bar{a}_{p^{i+2}} \notin f_{k,i+2}(\mathrm{Ker} f_{k,i+1}) = \mathrm{Ker} f_{i+2,i+1} \cong Z_p$. $\mathrm{Ker} f_{i+2,i+1}$ contains all the elements whose order is p in $\mathrm{Ker} f_{i+2,i}$, so $\left|\bar{a}_{p^{i+2}}\right| = p^2$. $\bar{a}_{p^{i+2}}^p \in \mathrm{Ker} f_{i+2,i+1}$, $\bar{a}_{p^{i+2}}^p \notin \mathrm{Ker} f_{i+2,i+2}$, $\bar{a}_{p^k}^p \in g_{k,i+2}^{-1}(\bar{a}_{p^{i+2}}^p) \subset g_{k,i+2}^{-1}(\mathrm{Ker} f_{i+2,i+1}) = \mathrm{Ker} f_{k,i+1}$, $\bar{a}_{p^k}^p \notin g_{k,i+2}^{-1}(\mathrm{Ker} f_{i+2,i+2}) = \mathrm{Ker} f_{k,i+2}$.
For $i = 1$, if p is an odd prime, $\mathrm{Ker} f_{3,1} = \left\langle \overline{p+1}_{p^3} \right\rangle \cong Z_{p^2}$, if $p = 2$, $\mathrm{Ker} f_{3,1} = \{\bar{1}_8, \bar{3}_8, \bar{5}_8, \bar{7}_8\} \cong Z_2 \oplus Z_2$. Thus, for $\bar{a}_{p^k} \in \mathrm{Ker} f_{k,2}, \bar{a}_{p^k} \notin \mathrm{Ker} f_{k,3}$, using the lemma above for several times, we get $\bar{a}_{p^k}^{p^{k-2}} \in \mathrm{Ker} f_{k,2}, \bar{a}_{p^k}^{p^{k-3}} \notin \mathrm{Ker} f_{k,k}$, $\left|\bar{a}_{p^k}\right| = p^{k-2}$, $\mathrm{Ker} f_{k,2} \cong Z_{p^{k-2}}$.
If p is an odd prime, we can further obtain $\mathrm{Ker} f_{k,1} \cong Z_{p^{k-1}}$.

Suppose x is a generator of Z_p^*, assume $a \in g_{k,1}^{-1}(x)$, $g_{k,1}^{-1}(x) = a\mathrm{Ker}f_{k,1}$, and $a^{p-1} \in g_{k,1}^{-1}(x^{p-1}) = g_{k,1}^{-1}(\bar{1}_p) = \mathrm{Ker}f_{k,1}$. If $a^{p-1} \notin \mathrm{Ker}f_{k,2}$, then $\left|a^{p-1}\right| = p^{k-1}$. If $a^{p-1} \in \mathrm{Ker}f_{k,2}$, $\forall h \in \mathrm{Ker}f_{k,1}, h \notin \mathrm{Ker}f_{k,2}$. Since $(ah)^{p-1} = (a^{p-1}h^p)h^{-1}$, $(ah)^{p-1} \in \mathrm{Ker}f_{k,1}, (ah)^{p-1} \notin \mathrm{Ker}f_{k,2}$, whence $\left|(ah)^{p-1}\right| = p^{k-1}$, $Z_{p^k}^* \cong Z_{(p-1)p^{k-1}}$.
If $p = 2$, $Z_{2^k}^* = \mathrm{Ker}f_{k,1} \cong Z_{2^{k-2}} \oplus Z_2$.
For Aut$\mathbf{Z}$, assume there exist $f \neq 1_G, -1_G$, $f \in \mathbf{AutZ}$. WLOG, $f(1) = x \neq \pm 1, f(-1) = y$. $f(1)+f(-1) = f(0) = x+y = 0$. Assume $af(1)+bf(-1) = f(a-b) = 1 = (a-b)x$, since $x \neq \pm 1$, there is a contradiction. Aut$\mathbf{Z} \cong Z_2$.

Exercise 1.2.16. For each prime p the additive subgroup $Z(p^\infty)$ of $\mathbf{Q}/\mathbf{Z}$ is generated by the set $\{\overline{1/p^n} | n \in \mathbf{N}^*\}$.

Answer. We prove that $\left\langle \bigcup\limits_{n=1}^{\infty} \frac{1}{p^n} \right\rangle \cong Z(p^\infty)$. $\forall x \in Z(p^\infty), x = \frac{\bar{a}}{b} = \frac{\bar{a}}{p^k}$. Expand a as $a = \sum\limits_{i=0}^{k-1} p^i a_i$, where $a_i = 1, 2, \ldots n-1$. $x = \frac{\bar{a}}{b} = \sum\limits_{i=0}^{k-1} \frac{\bar{a_i}}{p^{k-i}} = \sum\limits_{i=1}^{k} \frac{\overline{a_{k-i}}}{p^i}$. Denote $f : \left\langle \bigcup\limits_{n=1}^{\infty} \frac{1}{p^n} \right\rangle \to Z(p^\infty)$ as $f(\sum\limits_{i=1}^{n} \frac{a_i}{p^i}) = \overline{\sum\limits_{i=1}^{n} \frac{a_i}{p^i}}$. f is an isomorphism because every $x \in Z(p^\infty)$ can be written in such form.

Exercise 1.2.17. Let G be an abelian group and let H, K be subgroups of G. Show that the join $H \vee K$ is the set $\{ab | a \in H, b \in K\}$. Extend this result to any finite number of subgroups of G.

Answer. $H \vee K = \langle H \cup K \rangle, I = \{ab | a \in H, b \in K\}$. G is abelian so I is a subgroup of G. $H < I, K < I, (H \cup K) \subset I$. $\langle H \cup K \rangle \subset I \Rightarrow \langle H \cup K \rangle$.
For any $ab \in I$, $a \in H$, $b \in K$, we prove that ab is contained in any subgroup which contains $H \cup K$.
Assume $(H \cup K) \subset J$, so $a \in J, b \in J \Rightarrow ab \in J$, which means $I \subset H \vee K$.
$\langle H \cup K \rangle = I$.
G is abelian group, $H_1, H_2, \ldots H_n$ are n subgroups. $\left\langle \bigcup\limits_{i=1}^{n} H_i \right\rangle = \{\prod\limits_{i=1}^{n} h_i | h_i \in H_i, i = 1, 2, \ldots n\}$. This proposition can be proved by induction.

Exercise 1.2.18. 1. Let G be a group and $\{H_i | i \in I\}$ a family of subgroups. State and prove a condition that will imply that $\bigcup_{i\in I} H_i$ is a subgroup, that is $\bigcup_{i\in I} H_i = \left\langle \bigcup_{i\in I} H_i \right\rangle$.

2. Given an example of a group G and a family of subgroups $\{H_i | i \in I\}$ such that $\bigcup_{i\in I} H_i \neq \left\langle \bigcup_{i\in I} H_i \right\rangle$.

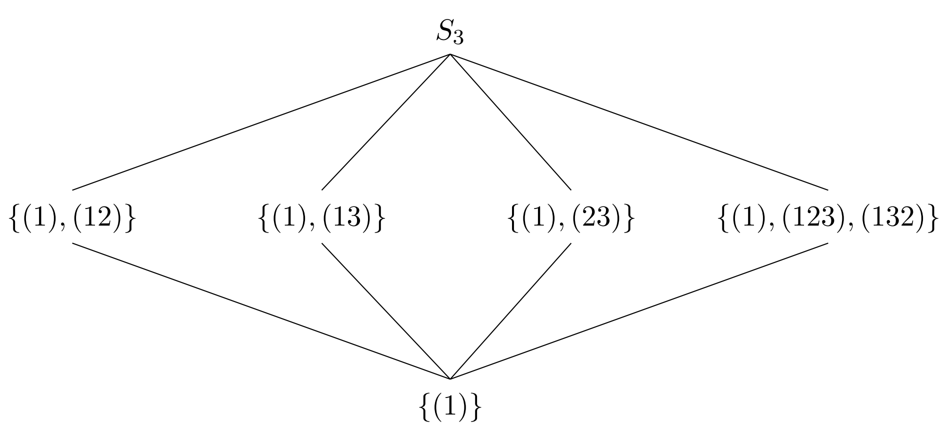
Answer. I didn't find a sufficient and necessary condition for this question, just choose one as you like:)

Exercise 1.2.19. 1. The set of all subgroups of a group G, partially ordered by set theoretic inclusion, forms a complete lattice in which the g.l.b of $\{H_i | i \in I\}$ is $\bigcap_{i\in I} H_i$ and the l.u.b is $\left\langle \bigcap_{i\in I} H_i \right\rangle$.

2. Exhibit the lattice of subgroups of the groups S_3, D_4^*, Z_6, Z_{27} and Z_{36}.

Answer. 1. The subset relation $<$ forms a partially ordered relation. By the difinition of $\left\langle \bigcup_{i\in I} H_i \right\rangle$, $\left\langle \bigcup_{i\in I} H_i \right\rangle$ is the smallest set contains $\bigcup_{i\in I} H_i$, so it's lup. For glb, we know that $\bigcap_{i\in I} H_i \subset H_i \ \forall i \in I$, and $\forall H \supset \bigcap_{i\in I} H_i$, there exists $x \in H, x \notin H_j \ j \in I$, so $\bigcap_{i\in I}$ is glb.

2. $S_3 = \{(1), (12), (13), (23), (123), (132)\}$.



The Hasse figure of the lattice of S_3

$D_4^* = \{R, R^2, R^3, I, T_x, T_y, T_{13}, T_{24}\}$.

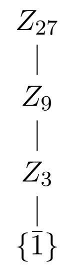
$$\begin{tikzcd}
& D_4^* \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\{I, R^2, T_x, T_y\} \arrow[dr, dash] & \{R, R^2, R^3, I\} \arrow[d, dash] & \{I, R^2, T_{13}, T_{24}\} \arrow[dl, dash] \\
& \{I, R^2\} \arrow[d, dash] & \\
& \{I\} &
\end{tikzcd}$$

The Hasse figure of the lattice of D_4^*

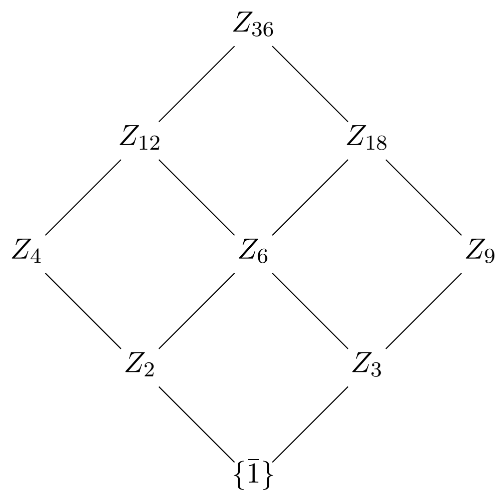
$Z_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$.

$$\begin{tikzcd}
& Z_6 \arrow[dl, dash] \arrow[dr, dash] & \\
\{\bar{0}, \bar{3}\} \arrow[dr, dash] & & \{\bar{0}, \bar{2}, \bar{4}\} \arrow[dl, dash] \\
& \{\bar{0}\} &
\end{tikzcd}$$

The Hasse figure of the lattice of Z_6



The Hasse figure of the lattice of Z_{27}



The Hasse figure of the lattice of Z_{36}

1.3 Cyclic groups

Exercise 1.3.1. Let a, b be elements of group G. Show that $|a| = \left|a^{-1}\right|$; $|ab| = |ba|$, and $|a| = \left|cac^{-1}\right|$ for all $c \in G$.

Answer. We only consider that $|a|, |b|, |c|$ are finite. Assume $a^k = e$, $(ab)^m = e$, $(ac^{-1})^n = e$, $kmn \neq 0$. $a^k \cdot (a^{-1})^k = e$, so k sialso the order of a^{-1}, $\left|a^{-1}\right| = k$. $(ab)^m = e = a(ba)^{m-1}b \Rightarrow (ba)^{m-1} = a^{-1}b^{-1}$, $(ba)^m = a^{-1}b^{-1}ba = e$. m is the order of ba. $(cac^{-1})^r = cac^{-1}cac^{-1}\cdots cac^{-1} = ca^nc^{-1} = e$, so $a^n = e$, whence $n = k$.

Exercise 1.3.2. Let G be an abelian group containing elements a and b of orders m and n respectively. Show that G contains an element whose order is the least commom multiple of m and n.

Answer. If $(m, n) = 1$, we know that $\forall a^i, i = 1, 2, \ldots, m$, $b^j, j = 1, 2, \ldots, n$, $a^ib^j \neq e$, since if $a^i = b^j$, $\left|a^i\right| = n = \left|b^{-j}\right| = \left|b^j\right| = m$. G is abelian, so $(ab)^k = a^kb^k \Rightarrow |ab| = mn = [m, n]$.
If $m|n$ or $n|m$, then a or b is the element we want. We consider $m \nmid n$ and $n \nmid m$. Factorise $n = p_1^{t_1}p_2^{t_2}\cdots p_l^{t_l}$, $m = p_1^{s_1}p_2^{s_2}\cdots p_l^{s_l}$, where $p_1, \cdots, p_l$ are primes and $t_1, \cdots, t_l, s_1, \cdots, s_l \geq 0$. We can choose a new arrangement of $p_1, \cdots, p_l$ and make $t_1 \geq s_1$, $t_2 \geq s_2$, $\ldots$, $t_i \geq s_i$, $t_{i+1} < s_{i+1}$, $\ldots$, $t_l < s_l$.

$$(m, n) = p_1^{s_1}\cdots p_i^{s_i}p_{i+1}^{t_{i+1}}\cdots p_l^{t_l}, [m, n] = p_1^{t_1}\cdots p_i^{t_i}p_{i+1}^{s_{i+1}}\cdots p_l^{s_l}$$

Take $x = a^{p_{i+1}^{s_{i+1}}\cdots p_l^{s_l}}$, $y = b^{p_1^{t_1}\cdots p_i^{t_i}}$, then $|x| = p_1^{t_1}\cdots p_i^{t_i}$, $|y| = p_{i+1}^{s_{i+1}}\cdots p_l^{s_l}$. Thus $(x, y) = 1$, the order of xy is $|x| \cdot |y| = p_1^{t_1}\cdots p_i^{t_i}p_{i+1}^{s_{i+1}}\cdots p_l^{s_l} = [m, n]$.

Exercise 1.3.3. Let G be an abelian group of order pq, with $(p, q) = 1$. Assume there exist $a, b \in G$ such that $|a| = p, |b| = q$ and show that G is cyclic.

Answer. From **Exercise 1.3.2** we know $a^ib^j \neq e$ for $i < p$, $j < q$. $|G| = pq$ for all a^ib^j and a^mb^n with $i \neq m$, $b \neq n$, $a^ib^j \neq a^mb^n$. So G can be generated by ab. G is cyclic.

Exercise 1.3.4. If $f : G \to H$ is a homomorphism, $a \in G$, and $f(a)$ has finte order in H, then $|a|$ is infinite or $|f(a)|$ divides $|a|$.

Answer. Assume $|f(a)| = n$, $|a| = m$, and $n \nmid m$. Trivially, $m \geq n$. Assume $\gcd(m, n) = k \leq n$. $a^m = e \Rightarrow f(a)^m = e' = f(a)^n$. By Bezout theorem $\exists x, y \in \mathbf{Z}$ s.t. $f(a)^{mx+ny} = f(a)^k = e'$, $k \leq n$, that's contradictory!

Exercise 1.3.5. Let G be the multiplicative group of all nonsingular 2×2 matrices with rational entries. Show that $a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has order 4 and $b = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ has order 3, but ab has infinite order. Conversely, show that the additive group $Z_2 \oplus \mathbf{Z}$ contains nonzero elements a, b of infinite order such that $a + b$ has finite order.

Answer. The verification of $|a| = 4$ and $|b| = 3$ is trivial. $ab = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\det(ab = \lambda I) = 0 \Rightarrow \lambda_1 = \lambda_2 = 1$. ab is not diagnizable. By induction, we have $(ab)^n = \begin{pmatrix} 1 & 2^{n-1} \\ 0 & 1 \end{pmatrix}$ which means (ab) has infinite order.
For $a = (\bar{0}, 1), b = (\bar{0}, -1) \in Z_2 \oplus \mathbf{Z}$, a, b have infinite order, but $a + b = (\bar{0}, 0)$ has finite order 1.

Exercise 1.3.6. If G is a cyclic group of order n and $k|n$, then G has exactly one subgroup of order k.

Answer. Assume $a^n = e$, $mk = n$, we verify that $\langle a^m \rangle$ is a subgroup of order k. $\forall x, y \in \mathbf{Z}_+$, $a^{xm} \cdot a^{-ym} = a^{(x-y)m} \in \langle a^m \rangle$, so $\langle a^m \rangle$ is a subgroup. $a^{km} = e$, $a^{sm} \neq e$ for $s < k$, so $|\langle a^m \rangle| = k$.

Exercise 1.3.7. Let p be prime and H a subgroup of $Z(p^\infty)$.
(a) Every element of $Z(p^\infty)$ has finite order p^n for some $n \geq 0$.
(b) If at least one element of H has order p^k and no element of H has order greater than p^k, then H is the cyclic subgroup generated by $\overline{1/p^k}$, whence $H \cong Z_{p^k}$.

(c) If there is no upper bound on the orders of elements of H, then $H = Z(p^\infty)$.
(d) The only proper subgroups of $Z(p^\infty)$ are the finite cyclic groups $C_n = \left\langle 1/\overline{p^n} \right\rangle$ $(n = 1, 2, \dots)$. Futhermore, $\langle 0 \rangle = C_0 < C_1 < C_2 < C_3 < \cdots$.
(e) Let $x_1, x_2, \dots$ be elements of an abelian group G such that $|x_1| = p, px_2 = x_1, px_3 = x_2, \dots, px_{n+1} = x_n, \dots$. The subgroup generated by the $x_i (i \geq 1)$ is isomorphic to $Z(p^\infty)$.

Answer. (a) $\forall x \in Z(p^\infty)$, $x = \frac{a}{p^n}$ where $a < p^n$, $p \nmid a$. p is a prime, so $\gcd(p, a) = 1$. $m \cdot a | p^n \Rightarrow m = p^n$. Thus $m \cdot \frac{a}{p^n} = e$, p^n is the smallest number satisfies it. $\frac{a}{p^n}$ has order p^n.
(b) For all $x \in Z(p^\infty)$, if x has order smaller than p^k, x must have the form $x = \frac{a}{p^i} (i \leq k)$, $(p, a) = 1$, so $x \in \left\langle \frac{1}{p^k} \right\rangle$. If not, assume $x = \frac{a}{p^i} (i > k)$, then $p^k \cdot x = \frac{a}{p^{i-k}} \neq 1$.
(c) Assume not, $H < Z(p^\infty)$, $H \neq Z(p^\infty)$. There exist $y \in H$ s.t. y has order $p^m, m \geq n$. $y = \frac{b}{p^m}$, $(p, b) = 1$, so there exists $b^{-1} \in \{1, 2, \dots, p-1\}$, $bb^{-1} \equiv 1 \mod p^m$. But $ab^{-1}p^{m-n}y = \frac{a}{p^n} = x \in H$, that's contradictory! Conversely, $H = Z(p^\infty)$.
(d) From (b), we know that if there's least upper bound p^n for elements in a subgroup S, then $S = C_n$.

$$\langle 0 \rangle = C_0 < C_1 < C_2 < C_3 < \cdots < Z(p^\infty)$$

is easy to verify.
(e) We can verify that $f : x_i \mapsto \frac{\bar{1}}{p^i}$ is a well defined isomorphism. $f(e) = f(px_1) = 1$, $f(px_{i+1}) = f(x_i) = \frac{1}{p^i} = p \cdot \frac{1}{p^{i+1}}$. f is obviously a bijection, so $H \cong Z(p^\infty)$.

Exercise 1.3.8. A group that has only a finite number of subgroups must be finite.

Answer. Suppose not. If the order of all subgroups are finite, G must be finite. So there exists a infinite subgroup $H < G$. $\forall a \in G$, if $\forall n \in \mathbf{N}$, $a^n \neq e$. then we can construct infinte subgroups $\langle a \rangle$, $\left\langle a^2 \right\rangle$, $\left\langle a^3 \right\rangle \dots$. If $\forall a \in G$, $\exists n \in \mathbf{N}$, $a^n = e$, so $\langle a \rangle$ is a proper subgroup of G, we can take $b \in G \ni \langle a \rangle$ to construct another subgroup. By induction, there are infinte subgroups in G. That's contradictory, so G must be finite.

Exercise 1.3.9. If G is an abelian group, then the set T of all elements of G with finite order is a subgroup of G.

Answer. We can easily verify that $\forall a, b \in S$, $|a| = m$, $|b| = n$ and $\left|ab^{-1}\right| \leq mn$ is finite. T is a subgroup of G.

Exercise 1.3.10. An infinite group is cyclic if and only if it is isomorphic to each of its proper subgroups.

Answer. If G is cyclic, $G \cong \mathbf{Z}$, $S < G$. For any subgroup of $\mathbf{Z}$, it has the form $\{na\}, a \in \mathbf{Z}$. We can construct a isomorphism $f : n \mapsto na$, so $S \cong \{na\} \Rightarrow G \cong S$.
If $\forall S < G$, $G \cong S$ and $|G| = |S|$ is finite. We prove there exists $S < G$ s.t. $|S| = \aleph_0$. Take $a \in G$ and $S = \{na | n \in \mathbf{Z}\}$, S is a subgroup. If there exists $ma = 0$, S must be finite, contradictory! Thus, $S \cong \mathbf{Z} \cong G$. G is a infinite cyclic group.

1.4 Cosets and counting

Exercise 1.4.1. Let G be a group and $\{H_i | i \in I\}$ a family of subgroups. Then for any $a \in G$, $(\bigcap\limits_i H_i)a = \bigcap\limits_i H_i a$.

Answer. $\bigcap\limits_i H_i$ is a subgroup of G. Take $x \in \bigcap\limits_i H_i$, $x \in H_i$, $\forall i \in I$. Then $xa \in H_i a$, $\forall i \in I$, so $xa \in \bigcap\limits_i (H_i a)$. Thus, $(\bigcap\limits_i H_i)a = \bigcap\limits_i (H_i a)$.

Exercise 1.4.2. (a) Let H be the cyclic subgroup (of order 2) of S_3 generated by $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$. Then no left cosets of H (except H itself) is also a right coset. There exists $a \in S_3$ such that $aH \cap Ha = \{a\}$.

(b) If K is the cyclic subgroup (of order 3) of S_3 generated by $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, then every left coset of K is also a right coset of K.

Answer. (a) $H = \{(12), (1)\}$. $S_3 = \{(12), (13), (23), (1), (123), (132)\}$. For $a \in H$, $aH = Ha = H$.
$a = (13)$, $aH = \{(13), (123)\}$, $Ha = \{(13), (132)\}$.
$a = (23)$, $aH = \{(23), (132)\}$, $Ha = \{(23), (123)\}$.
$a = (123)$, $aH = \{(123), (23)\}$, $Ha = \{(132), (13)\}$.
$a = (132)$, $aH = \{(132), (13)\}$, $Ha = \{(123), (23)\}$.

(b) $K = \{(123), (132), (1)\}$. For $a \in K$, $aK = Ka = K$.
$a = (12)$, $aK = Ka = \{(12), (23), (13)\}$.
$a = (13)$, $aK = Ka = \{(12), (23), (13)\}$.
$a = (23)$, $aK = Ka = \{(12), (23), (13)\}$.

Exercise 1.4.3. The following conditions on a finite group G are equivalent.
(i) $|G|$ is prime.
(ii) $G \neq \langle e \rangle$ and G has no proper subgroups.
(iii) $G \cong Z_p$ for some prime p.

Answer. (i)⇒(ii): If there exists $S < G$, $S \neq G$, then $|S| \mid |G| = p$. That's contradictory!
(ii)⇒(iii): $\forall a \in G$, take $S = \{na | n = 1, 2, \ldots, p\}$. If there exists $ma = na, (1 \leq m < n \leq p)$, $(n - m)a = 0$. So there exists subgroup S, and $|S| = n - m < p$. That's contradictory! So $S < G$, $|S| = |G| \Rightarrow S = G \cong Z_p$.

(iii)⇒(i): Trivial.

Exercise 1.4.4. Let a be an integer and p be a prime such that $p \nmid a$. Then $a^{p-1} \equiv 1 \mod p$.

Answer. $(Z_p \backslash \{\bar{0}\}, \times)$ is a group of order $p-1$. From **Exercise 1.1.7**, we know that $\forall \bar{a} \in Z_p \backslash \{\bar{0}\}$ and $b \in Z_p \backslash \{\bar{0}\}$, taking different $\bar{b}$ we will have different $\bar{a}\bar{b} \in Z_p \backslash \{\bar{0}\}$. $\bar{a}\bar{b}$ travels through all the elements in $Z_p \backslash \{\bar{0}\}$. So

$$\prod_{i=1}^{p-1} (\bar{i} \cdot \bar{a}) = \prod_{i=1}^{p-1} \bar{i}$$

By the definition of $Z_p \backslash \{\bar{0}\}$, $Z_p \backslash \{\bar{0}\}$ is commutative. So

$$(\bar{a})^{p-1} (\prod_{i=1}^{p-1} \bar{i}) = \prod_{i=1}^{p-1} \bar{i} \Rightarrow (\bar{a})^{p-1} = \bar{1}$$

Exercise 1.4.5. Prove that there are only two distinct groups of order 4 (up to isomorphism), namely Z_4 and $Z_2 \oplus Z_2$.

Answer. The only cyclic group of order 4 is Z_4. For a group G of order 4 which is not cyclic, $\forall a \in G$, $a \neq e$, if $|a| = 2$, $G \cong Z_2 \oplus Z_2$. If there exists $a \in G$, $|a| = 4$, $G \cong Z_4$. If there exists $a \in G$, $|a| = 3$, denote $a^2 = b, a^3 = e$. Then $b^2 = a^4 = a$, $\{e, a, b\} < G$, which is contradictory to the Largrange theorem.

Exercise 1.4.6. Let H, K be subgroups of a group G. Then HK is a subgroup of G if and only if $HK = KH$.

Answer. If $HK = KH$, for $a_1b_1, a_2b_2 \in HK$,

$$(a_1b_1)(a_2b_2)^{-1} = (a_1b_1)(b_2^{-1}a_2^{-1}) = (a_1b_1)(a_3b_3)$$

since $b_2^{-1}a_2^{-1} \in KH = HK$, there exists $b_2^{-1}a_2^{-1} = a_3b_3$.

$$(a_1b_1)(a_3b_3) = a_1(b_1a_3)b_3 = a_1a_4b_4b_3$$

since $b_1a_3 \in KH = HK$, there exists $b_1a_3 = a_4b_4$. $(a_1b_1)(a_2b_2)^{-1} = a_1a_4b_4b_3 = a_5b_5 \in HK$. Thus HK is a subgroup of G.
If HK is a subgroup of G, $\forall b_1a_1 \in KH$, there exists $(a_1^{-1}b_1^{-1}) \in HK$ s.t. $b_1a_1 = (a_1^{-1}b_1^{-1})^{-1} \in HK$. So $KH \subset HK$. $\forall a_1b_1 \in HK$, $(a_1b_1)^{-1} = b_1^{-1}a_1^{-1} \in HK$, so $\exists a_2b_2 \in HK$ s.t. $b_1^{-1}a_1^{-1} = a_2b_2$. $a_1b_1 = b_2^{-1}a_2^{-1} \in KH$. So $HK \subset KH$. Thus $HK = KH$.

Exercise 1.4.7. Let G be a group of order $p^k m$, with p prime and $(p, m) = 1$. Let H be a subgroup of order p^k and K a subgroup of order p^d, with $0 < d \leq k$ and $K \not\subset H$. Show that HK is not a subgroup of G.

Answer. Assume $HK < G$, $|HK| = p^k n$, $n|m$. We can get $[HK : H] = n = [K : K \cap H]$. $[K : K \cap H]\,|p^k \Rightarrow n|p^k$. That's contradictory to $(m, p^k) = 1$.

Exercise 1.4.8. If H and K are subgroups of finite index of a group G such that $[G : H]$ and $[G : K]$ are relatively prime, then $G = HK$.

Answer. Assume $[G : H] = m$, $[G : K] = n$, $(m, n) = 1$. Then $|H| = np$, $|K| = mp$. $H \cap K < H$, $H \cap K < G \Rightarrow |H \cap K|\,|p$.

$$[G : H] = m \geq [K : H \cap K] = \frac{|K|}{|H \cap K|} \geq m$$

Thus $[G : H] = [K : H \cap K] = m$, $G = HK$.

Exercise 1.4.9. If H, K and N are subgroups of a group G such that $H < N$, then $HK \cap N = H(K \cap N)$.

Answer. $\forall x = hk \in HK \cap N$, $\exists h_1^{-1} \in H$ s.t. $h_1^{-1}hk \in K \cap N$. $H < N$ so $\forall h_1^{-1} \in H, h_1^{-1}hk \in N$. Take $h_1^{-1} = h^{-1}$, $h_1^{-1}hk = k \in K$. So $HK \cap N \subset H(K \cap N)$.
$\forall x = hk \in H(K \cap N)$ where $h \in H$, $k \in K \cap N$. $hk \in HK, h, k \in N \Rightarrow hk \in N$. So $H(K \cap N) \subset HK \cap N$.
Thus, $HK \cap N = H(K \cap N)$.

Exercise 1.4.10. Let H, K, N be subgroups of a group G such that $H < K$, $H \cap N = K \cap N$, and $HN = KN$. Show that $H = K$.

Answer. Assume there exists $x \in K \backslash H$. $K \bigcup\limits_{i \in I} Ha_i$, $\forall h_i \in H$ there exists $a \in K$ s.t. $x = h_1 a$. Take $n_1 \in N$. Since $HN = KN$, $xn_1 \in HN$, there exists $h_2 \in H$, $n_2 \in N$ s.t. $xn_1 = h_2 n_2 = h_2 a n_1$. So $a = n_2 n_1^{-1} \in N$, $a \in K \cap N = H \cap N \Rightarrow a \in H, x \in H$. That's contradictory!

Exercise 1.4.11. Let G be a group of order $2n$; then G contains an element of order 2. If n is odd and G abelian, there is only one element of order 2.

Answer. The proof of the first part is exactly the same as **Exercise 1.1.14**. Assume there exists $a, b \in G$, $a^2 = b^2 = e$. We can check $H = \{e, a, b, ab\}$ is a subgroup of G. $|H| \,|\, |G| \Rightarrow 4|2n \Rightarrow 2|n$, which is contradictory to n is odd. So there's only one element a s.t. $a^2 = e$.

Exercise 1.4.12. If H and K are subgroups of a group G, then $[H \vee K : H]$ $\geq [K : H \cap K]$.

Answer. The question is a direct corollary of Proposition 4.8.

Exercise 1.4.13. If $p > q$ are primes, a group of order pq has at most one subgroup of order p.

Answer. $H \cap K < H$, $H \cap K < K$, $H \neq K \neq H \cap K$. $|H \cap K| \,|p$ and $|H \cap K| \neq q$, so $H \cap K = \{e\}$. From **Exercise 1.3.12**,

$$[H \vee K : H] \geq [K : K \cap H] = p$$

$$|H \vee K| = |H| \cdot [H \vee K : H] \geq p^2$$

But $H \vee K \in G$, $|H \vee K| \leq pq < p^2$. That's contradictory!

Exercise 1.4.14. Let G be a group and $a, b \in G$ such that (i) $|a| = 4 = |b|$; (ii) $a^2 = b^2$; (iii) $ba = a^3b = a^{-1}b$; (iv) $a \neq b$; (v) $G = \langle a, b\rangle$. Show that $|G| = 8$ and $G \cong Q_8$.

Answer. The proof is exactly the same as **Exercise 1.2.3**.

1.5 Normality, quotient groups, and homomorphisms

Exercise 1.5.1. If N is a subgroup of index 2 in a group G, then N is normal in G.

Answer. $\forall a \in G \backslash N$, $G = N \cup Na = N \cup aN$ and $N \cap Na = \varnothing$, $N \cap aN = \varnothing$. So $\forall x \in Na$, $x \in G \backslash N \Rightarrow x \in aN$, $Na \subset aN$. Similarly, $aN \subset Na$, whence $Na = aN$, $N \lhd G$.

Exercise 1.5.2. If $\{N_i | i \in I\}$ is a family of normal subgroups of a group G, then $\bigcap\limits_{i \in I} N_i$ is a normal subgroup of G.

Answer. $\bigcap\limits_{i \in I} N_i$ is a subgroup of G. $N_i (i \in I)$ are normal subgroups of G, so $\forall a \in G$, $aN_i a^{-1} = \{an_i a^{-1} | n_i \in N_i\} = N_i$. $\forall x = ana^{-1} \in a(\bigcap\limits_{i \in I} N_i)a^{-1}$, $n \in N_i \Rightarrow x \in a(\bigcap\limits_{i \in I} N_i)a^{-1} \subset \bigcap\limits_{i \in I} aN_i a^{-1} = \bigcap\limits_{i \in I} N_i$. $\bigcap\limits_{i \in I} N_i$ are normal subgroup of G.

Exercise 1.5.3. Let N be a subgroup of a group G. N is normal in G if and only if (right) congruence modulo N is a congruence relation on G.

Answer. If $N \lhd G$. $\forall a, b \in G$, $ab^{-1} \in N \Leftrightarrow a^{-1}b \in N$. If $a_1 \equiv b_1 \mod N$, $a_2 \equiv b_2 \mod N$, then $a_2 b_2^{-1} \in N$, $a_1 N = N a_1 = N b_1 \Rightarrow a_1 N b_1^{-1} = N$. So $a_1 a_2 b_1^{-1} b_2^{-1} = (a_1 a_2)(b_1 b_2)^{-1} \in N$. Similarly, $(a_1 a_2)^{-1}(b_1 b_2) \in N$. Congruence modulo N is a congruence relation.
If congruence modulo N is a congruence relation. $\forall a_1 \equiv b_1 \mod N$, $a_2 \equiv b_2 \mod N$, we will have $a_1 a_2 \equiv b_1 b_2 \mod N$. Take $n \in N$ and fix $a_2 \in G$, define $b_2 = n^{-1} a_2$. Then $\forall n \in N$, n can be expressed as $a_2 b_2^{-1}$, $a_2 \equiv b_2 \mod N$. $\forall a_1 \in G$ and $\forall b_1 \equiv a_1 \mod N$, $a_1 n b_1^{-1} = a_1 a_2 b_2^{-1} b_1^{-1} \in N$. Take $b_1 = a_1$ and n varies in N, $a_1 n a_1^{-1} \in N \Rightarrow a_1 N a_1^{-1} \subset N$. Thus $N \lhd G$.

Exercise 1.5.4. Let $\sim$ be an equivalence relation on a group G and let $N = \{a \in G | a \sim e\}$. Then $\sim$ is a congruence relation on G if and only if N is a normal subgroup of G and $\sim$ is congruence modulo N.

Answer. If $G \lhd N$ and $\sim$ is congruence modulo N. $\forall a \in G$, $aNa^{-1} \subset N$. $\forall a_1, b_1, a_2, b_2 \in G$, $a_1b_1^{-1} \in N$, $a_2b_2^{-1} \in N$. $a_1a_2(b_1b_2)^{-1} = a_1a_2b_2^{-1}b_1^{-1}$, denote $n = a_2b_2^{-1} \in N$, $a_1a_2b_2^{-1}b_1^{-1} = a_1nb_1^{-1} \in a_1Nb_1^{-1}$. $\forall n \in N$, there exists $n' = b_1^{-1}a_1, n' \in N$ s.t. $a_1n = b_1n'$. So $a_1nb_1^{-1} = b_1n'b_1^{-1} \in b_1Nb_1^{-1} \subset N$. That means $(a_1a_2)(b_1b_2)^{-1} \in N$, $a \sim b$ is a congruence relation.
If $a \sim b$ is a congruence relation. We first prove N is a subgroup of G. $\forall a \in N$, $a \sim e$, $a^{-1} \sim a^{-1} \Rightarrow e \sim a^{-1}$, so $a^{-1} \sim e$, $a^{-1} \in N$. $\forall a, b \in N$, $b^{-1} \sim e$, $a \sim e \Rightarrow ab^{-1} \in e$, thus $N < G$.
$\forall x \in G$, $xN = \{xa|a \sim e\} = \{xa|xa \sim xe\} = \{ax|ax \sim e\} = Nx$, so N is normal in G. $x \sim y \Leftrightarrow y \in xN$. $\sim$ is congruence modulo N.

Exercise 1.5.5. Let $N < S_4$ consist of all those permutations σ such that $\sigma(4) = 4$. Is N normal in S_4?

Answer. $N = \{(1), (12), (13), (23), (123), (132)\}$. Take $a = (14) \in G$, $a^{-1} = (14)$, $a^{-1}(12)a = (24) \notin N$. So N is not normal in S_4.

Exercise 1.5.6. Let $H < G$; then the set aHa^{-1} is a subgroup for each $a \in G$, and $H \cong aHa^{-1}$.

Answer. $H < G$, $aHa^{-1} = \{aha^{-1}|h \in H\}$. $\forall x, y \in aHa^{-1}$, $x = ah_1a^{-1}$, $y = ah_2a^{-1}$. $y^{-1} = ah_2^{-1}a^{-1}$, $xy = ah_1h_2^{-1}a^{-1} \in aHa^{-1}$, so $aHa^{-1} < G$.
Take $f : H \to aHa^{-1}$ as $f(h) = aha^{-1}$. If $f(h_1) = f(h_2) = ah_1a^{-1} = ah_2a^{-1}$, then $h_1 = h_2$, so f is an injection. f is a surjection because $\forall x \in aHa^{-1}$, $f(a^{-1}xa) = x$, $a^{-1}xa \in H$. In conclusion, $H \cong aHa^{-1}$.

Exercise 1.5.7. Let G be a finite group and H a subgroup of G of order n. If H is the only subgroup of G of order n, then H is normal in G.

Answer. Applying **Exercise 1.5.6**, $\forall a \in G$, $aHa^{-1} \cong H$. $\left|aHa^{-1}\right| = |H| = n \Rightarrow aHa^{-1} = H$. Whence $H \lhd G$.

Exercise 1.5.8. All subgroups of the quaternion group are normal.

Answer. $Q_8 = \{a, b, a^2, ba, ab, a^2b, ab^2, a^2b^2\}$ where $a^2 = b^2$, $a_1b = ba = a^3b$ and $|a| = |b| = 4$. There are several subgroups $\{a, a^2, ab^2, a^2b^2\}$, $\{b, a^2, a^2b, a^2b^2\}$, $\{ab, a^2b^2\}$, $\{ba, a^2b^2\}$, $\{a^2, a^2b^2\}$. From **Exercise 1.5.1**, we know the first two subgroups are normal in G. For $\{ab, a^2b^2\}$, $\{ba, a^2b^2\}$, $\{a^2, a^2b^2\}$, we can check that ab, ba, a^2 is commutative in G, that is $\forall x \in G$, $xabx^{-1} = ab$, $xbax^{-1} = ba$, $xa^2x^{-1} = a^2$. They are all normal in G.

Exercise 1.5.9. (a) If G is a group, then the center of G is a normal subgroup of G;
(b) the center of S_n is the identity subgroup for all $n > 2$.

Answer. (a) By the definition of center C, $\forall x \in G$ and $a \in C$, $ax = xa$, so $xCx^{-1} = C$. C is normal in G.
(b) $\forall x \in S_n$, x can be expressed as

$$x = (a_1a_2 \cdots a_{i_1})(a_{i_1+1}a_{i_1+2} \cdots a_{i_2}) \cdots (a_{i_{n-1}+1}a_{i_{n-1}+2} \cdots a_{i_n})$$

Those cycles $(a_1a_2 \cdots a_{i_1})$, $(a_{i_1+1}a_{i_1+2} \cdots a_{i_2})$, ..., $(a_{i_{n-1}+1}a_{i_{n-1}+2} \cdots a_{i_n})$ are all disjoint, so they are commutative.
If there exists cycles whose length is longer than 2. WLOG, assume $i_1 > 2$. Take $y = (a_1a_2)$,

$$y^{-1}xy = (a_1a_2)(a_1a_2 \cdots a_{i_1})(a_1a_2) \cdots (a_{i_{n-1}+1}a_{i_{n-1}+2} \cdots a_{i_n})$$

$(a_1a_2)(a_1a_2 \cdots a_{i_i})(a_1a_2) = (a_2a_1a_3 \cdots a_{i_1})$, so $y^{-1}xy \neq x$, $x \notin C$.
If $x = (a_1a_2)(a_3a_4) \cdots (a_{2n-1}a_{2n})$ and $n \geq 2$. Take $y = (a_1a_3)$,

$$\begin{aligned} y^{-1}xy &= (a_1a_3)(a_1a_2)(a_3a_4) \cdots (a_{2n-1}a_{2n})(a_1a_3) \\ &= (a_1a_3)(a_1a_2)(a_3a_4)(a_1a_3) \cdots (a_{2n-1}a_{2n}) \\ &= (a_1a_4)(a_2a_3) \cdots (a_{2n-1}a_{2n}) \\ &\neq x \end{aligned}$$

So $x \notin C$.
If $x = (a_1a_2)$. Take $y = (a_1a_3)$, $y^{-1}xy = (a_2a_3) \neq x$, so $x \notin C$.
In conclusion, $C = \{(1)\}$.

Exercise 1.5.10. Find subgroups H and K of D_4^* such that $H \lhd K$ and $K \lhd D_4^*$, but H is not normal in D_4^*.

Answer. $D_4^* = \{I, R, R^2, R^3, T_x, T_y, T_{13}, T_{24}\}$. Take $K = \{I, R, T_x, T_y\}$, $H = \{I, T_x\}$. We can easily verify that $H \lhd K$ and $K \lhd D_4^*$ but $K \ntriangleleft D_4^*$.

Exercise 1.5.11. If H is a cyclic subgroup of a group G and H is normal in G, then every subgroup of H is normal in G.

Answer. Assume $K < H \lhd G$, H has the generator a, and K has the generator a^n. Here we used: *Every subgroup of a cyclic group is cyclic.* This can be easily proved by the conclusion $H \cong Z_m$ for some $m \in \mathbf{Z}$. $\forall x \in G$, $h = a^s \in H$, $x^{-1}a^s x = a^t \in H$. Assume $x^{-1}ax = a^m$, then $x^{-1}a^n x = (x^{-1}ax)^n = a^{mn} = a^k$, so $n|k$, $a^k \in K$. $x^{-1}Kx \subset K$, K is normal in G.

Exercise 1.5.12. If H is a normal subgroup of a group G such that H and G/H are finitely generated, then so is G.

Answer. Assume $A = \{a_1, a_2, \ldots, a_m\}$, $B = \{b_1, b_2, \ldots, b_n\}$. $H = \langle A \rangle$, $G/H = \langle \{Hb_i | b_i \in B\} \rangle$. We prove that G can be generated by $A \cup B$. $\forall x \in G$, x is in one of the right cosets of H, $x \in Ha$. $Ha \in G/H$ so $Ha = \prod_{b_i \in B} Hb_i^{s_i} = H(\prod_{b_i \in B} b_i^{s_i})$. Thus $a^{-1}(\prod_{b_i \in B} b_i^{s_i}) = a' \in H$. H is generated by A so $xa^{-1} = \prod_{a_i \in A} a_i^{t_i}$, $a' = \prod_{a_i \in A} a_i^{-r_i}$. Then

$$x = (\prod_{a_i \in A} a_i^{t_i + r_i})(\prod_{b_i \in B} b_i^{s_i}) \in \langle A \cup B \rangle$$

Thus $G \subset \langle A \cup B \rangle$ is finitely generated.

Exercise 1.5.13. (a) Let $H \lhd G$, $K \lhd G$. Show that $H \vee K$ is normal in G.

(b) Prove that the set of all normal subgroups of G forms a complete lattice under inclusion.

Answer. (a) $\forall x \in G$, $a \in H \vee K$, we need to prove $x^{-1}ax \in H \vee K$. $a \in H \vee K$ so a can be expressed as

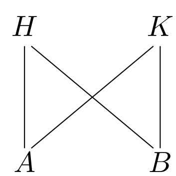
$$a = b_1^{n_1} b_2^{n_2} \cdots b_t^{n_t} \quad \text{where } b_i \in H \text{ or } b_i \in K, i = 1, 2, \ldots, t$$

so $x^{-1}ax = x^{-1}b_1^{n_1} \cdots b_t^{n_t}x = (x^{-1}b_1x)^{n_1}(x^{-1}b_2x)^{n_2} \cdots (x^{-1}b_tx)^{n_t}$.
$H \lhd G$, $K \lhd G$, so $x^{-1}b_ix \in H \vee K, i = 1, 2, \ldots, t$ and

$$x^{-1}ax = (x^{-1}b_1x)^{n_1}(x^{-1}b_2x)^{n_2} \cdots (x^{-1}b_tx)^{n_t} \in H \vee K$$

$H \vee K \lhd G$.

(b) Actually, in **Exercise 1.2.19** and (a), we have proved lub exists. Now we only consider glb. For $H \lhd G$, $K \lhd G$. If $H \cap K \lhd G$, then their glb is $H \cap K$. If not, assume there exists $A < H \cap K$, $B < H \cap K$, A, B are both normal in H and K. And there doesn't exists I s.t. $A \lhd I \lhd H$, $A \lhd I \lhd K$, $B \lhd I \lhd H$, $B \lhd I \lhd K$. Just like the figure:



But $A < H \cap K$, $B < H \cap K \Rightarrow A \vee B < H \cap K$. So $A \vee B \lhd H$, $A \vee B \lhd K$. That's contradictory! There is only one lower bound for $\{H, K\}$. Notice that $\{e\} < H \cap K$ so there exists at least one subgroup satisfies the condition. We have proved normality forms a lattice.

Exercise 1.5.14. If $N_1 \lhd G_1$, $N_2 \lhd G_2$ then $(N_1 \times N_2) \lhd (G_1 \times G_2)$ and $(G_1 \times G_2)/(N_1 \times N_2) \cong (G_1/N_1) \times (G_2/N_2)$.

Answer. Take $a \in (N_1 \times N_2)$, $a = (n_1, n_2)$ where $n_1 \in N_1$, $n_2 \in N_2$. $\forall x \in (G_1 \times G_2)$, $x = (g_1, g_2)$ where $g_1 \in G_1$, $g_2 \in G_2$. $x^{-1} = (g_1^{-1}, g_2^{-1})$, $x^{-1}ax = (g_1^{-1}n_1g_1, g_2^{-1}n_2g_2)$. $N_1 \lhd G_1$, $N_2 \lhd G_2$, so $g_1^{-1}n_1g_1 \in N_1$, $g_2^{-1}n_2g_2 \in N_2$. $x^{-1}ax \in (N_1 \times N_2)$. Thus $(N_1 \times N_2) \lhd (G_1 \times G_2)$.
Assume $G_1 = \bigcup\limits_{i \in I} N_1a_i$, $G_2 = \bigcup\limits_{j \in J} N_2b_j$. Then $G_1 \times G_2 = \bigcup\limits_{i \in I} N_1a_i \times \bigcup\limits_{j \in J} N_2b_j$. Denote $A = \{a_i | i \in I\}$, $B = \{b_j | j \in J\}$. We construct two bijections $(G_1 \times G_2)/(N_1 \times N_2) \to A \times B$ and $(G_1/N_1) \times (G_2/N_2)$.

$$f : N_1a_i \times N_2b_j \mapsto (a_i, b_j)$$

$$g : (N_1 a_i, N_2 b_j) \mapsto (a_i, b_j)$$

Take $h = g^{-1} \circ f$, f, g are bijections, so h is an isomorphism. $(G_1 \times G_2)/(N_1 \times N_2) \cong (G_1/N_1) \times (G_2/N_2)$.

Exercise 1.5.15. Let $N \lhd G$ and $K \lhd G$. If $N \cap K = \langle e \rangle$ and $N \vee K = G$, then $G/N \cong K$.

Answer. Assume $G = \bigcup_{i \in I} Na_i$, we construct $f : k \to G/N$. We prove that $\forall x, y \in K$, x, y belong to different cosets of N. Suppose not. $\exists x, y \in K$, $x, y \in Na_i$, then $xy^{-1} \in N \Rightarrow x = y$. That's contradictory! So f is a monomorphism.
$G = H \vee K$, so $G = HK$. we can write x as pq, where $p \in H$, $q \in K$. $|G/H| = [G : H] = [HK : H] = [K : K \cap H] = |K|$. f is a epimorphism.
Thus, $G/N \cong K$.

Exercise 1.5.16. If $f : G \to H$ is a homomorphism, H is abelian and N is a subgroup of G containing $\mathrm{Ker} f$, then N is normal in G.

Answer. Assume there exists $x \in G$, $x \notin N$ s.t. $f(x) \in f(N)$. $\exists n \in N$, $f(x) = f(n)$, $f(xn^{-1}) = f(x)f(n)^{-1} = e' \Rightarrow xn^{-1} \in \mathrm{Ker} f \Rightarrow x \in N$. That's contradictory! $\forall x \in G$, $n \in N$, $f(x^{-1}nx) = f(x^{-1})f(n)f(x) = f(n) \in f(N)$, so $x^{-1}nx \in N$. Thus, $N \lhd G$.

Exercise 1.5.17. (a) Consider the subgroups $\langle 6 \rangle$ and $\langle 30 \rangle$ of $\mathbf{Z}$ and show that $\langle 6 \rangle / \langle 30 \rangle \cong Z_5$.

(b) For any $k, m > 0$, $\langle k \rangle / \langle km \rangle \cong Z_m$; in particular, $\mathbf{Z} / \langle m \rangle = \langle 1 \rangle / \langle m \rangle \cong Z_m$.

Answer. (a) $\langle 6 \rangle = \{6n | n \in \mathbf{Z}\}$, $\langle 30 \rangle = \{30n | n \in \mathbf{Z}\}$. So $\langle 6 \rangle / \langle 30 \rangle = \{\langle 30 \rangle, \langle 30 \rangle + 6, \langle 30 \rangle + 12, \langle 30 \rangle + 18, \langle 30 \rangle + 24\} \cong Z_5$

(b) $\langle km \rangle \lhd \langle k \rangle$, $\langle k \rangle = \bigcup_{i \in I} (\langle km \rangle + a_i)$. For $x \in \langle k \rangle$, $x \equiv a_i \mod km$, then $x \in \langle km \rangle + a_i$. $f : \langle k \rangle / \langle km \rangle \to \{a_i | i \in I\}$ defined by $f(\langle km \rangle + a_i) = a_i$ is a bijection. We check that $g : \{a_i | i \in I\} \to Z_m$ is also a bijection. Define

$b_i \equiv \frac{a_i}{k} \mod m$, $g(a_i) = b_i$. If there exists $b_i = b_j$ for $i \neq j$, $a_i \equiv a_j \mod km$. That's contradictory! So g is an injection. g is obviously a surjection, so g is a bijection. Take $h = g \circ f : \langle k \rangle / \langle km \rangle \to Z_m$ is a isomorphism, so $\langle k \rangle / \langle km \rangle \cong Z_m$.

Exercise 1.5.18. If $f : G \to H$ is a homomorphism with kernel N and $K < G$, then prove that $f^{-1}(f(K)) = KN$. Hence $f^{-1}(f(K)) = K$ if and only if $N < K$.

Answer. Take $x \in f^{-1}(f(K))$, then there exists $k \in K$ s.t. $f(x) = f(k)$. $f(xk^{-1}) = f(x)f(k)^{-1} = e' \in f(K) \Rightarrow xk^{-1} \in \mathrm{Ker} f = N$. Thus, $x \in Nk \subset NK$, $f^{-1}(f(K)) \subset NK$.
$\forall x = nk \in NK$, where $n \in N$ and $k \in K$. $f(x) = f(n)f(k) = e'f(k) \in f(K)$, so $NK \subset f^{-1}(f(K))$.
Thus, $f^{-1}(f(K)) = NK$. Hence $f^{-1}(f(K)) = K$ if and only if $N < K$.

Exercise 1.5.19. If $N \lhd G$, $[G : H]$ finite, $H < G$, $|H|$ finite, and $[G : N]$ and $|H|$ are relatively prime, then $H < N$.

Answer. $N \lhd G \Rightarrow NH < G$. By the second isomorphism theorem, $NH/N \cong H/H \cap N \Rightarrow [NH : N] = [H : H \cap N]$. Assume $[G : N] = m$, $|H| = n$, $|G| = mnp$ where $(m, n) = 1$. Then $|N| = np$, $N < NH$, assume $|NH| = knp$, $NH < G \Rightarrow knp|mnp \Rightarrow k|m$. $[NH : N] = [H : H \cap N] = k \Rightarrow k|n$. So $k = 1$, $NH = N$ which means $H < N$.

Exercise 1.5.20. If $N \lhd G$, $|N|$ finite, $H < G$, $[G : N]$ finite, and $[G : H]$ and $|N|$ are relatively prime, then $N < H$.

Answer. $N \lhd G \Rightarrow NH < G$. By the second isomorphism theorem, $NH/N \cong H/H \cap N \Rightarrow [NH : N] = [H : H \cap N]$. Assume $[G : H] = m$, $|N| = n$, $|G| = mnp$ where $(m, n) = 1$. Then $|H| = np$, $H < NH$, assume $|NH| = knp$, $NH < G \Rightarrow knp|mnp \Rightarrow k|m$. $[NH : N] = [H : H \cap N] = kp \Rightarrow kp|np \Rightarrow k|n$. So $k = 1$, $NH = H$ which means $N < H$.

Exercise 1.5.21. If H is a subgroup of $Z(p^\infty)$ and $H \neq Z(p^\infty)$, then $Z(p^\infty)/H \cong Z(p^\infty)$.

Answer. From **Exercise 1.3.7**(b), we know that H has the form $\left\langle \frac{\bar{1}}{p^n} \right\rangle$. Take $x_i = \frac{\bar{1}}{p^{n+i}} + H$, $x_1 = \frac{\bar{1}}{p^{n+1}} + H$.

$$\sum_{m=1}^{p} x_1 = \frac{\bar{p}}{p^{n+1}} + pH = \frac{\bar{1}}{p^n} + H = H$$

$$\sum_{m=1}^{p} x_i = \frac{\bar{p}}{p^{n+i}} + pH = \frac{\bar{1}}{p^{n+i-1}} + H = x_{i-1}$$

Take $A = \{x_i | i \in \mathbf{Z}_+\}$, $\langle A \rangle \cong Z(p^\infty)$ by **Exercise 1.3.7**(e). $\forall x \in \langle A \rangle$, $x \in Z(p^\infty)/H$, so $\langle A \rangle \subset Z(p^\infty)/H$. Take $x \in Z(p^\infty)/H$, $x = y + H$ where $y = \sum\limits_{i=1}^{m} \frac{a_i}{p^{n+i}}$, $x = \sum\limits_{i=1}^{m} (\frac{a_i}{p^{n+i}} + H) \in \langle A \rangle$. Thus, $Z(p^\infty)/H \subset \langle A \rangle$, $\langle A \rangle = Z(p^\infty)/H \cong Z(p^\infty)$.

1.6 Symmetric, alternating, and dihedral groups

Exercise 1.6.1. Find four different subgroups of S_4 that are isomorphic to S_3 and nine isomorphic to S_2.

Answer. $S_4 = \{(1), (12), (13), (14), (23), (24), (34), (123), (124), (132), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23), (1234), (1243), (1324), (1342), (1423), (1432)\}$.
$A_1 = \{(1), (12), (13), (23), (123), (132)\}$;
$A_2 = \{(1), (12), (14), (24), (124), (142)\}$;
$A_3 = \{(1), (13), (14), (34), (134), (143)\}$;
$A_4 = \{(1), (23), (24), (34), (234), (243)\}$;
$A_1 \cong A_2 \cong A_3 \cong A_4$.
$B_1 = \{(1), (12)\}$; $B_2 = \{(1), (13)\}$; $B_3 = \{(1), (14)\}$; $B_4 = \{(1), (23)\}$; $B_5 = \{(1), (24)\}$; $B_6 = \{(1), (34)\}$; $B_7 = \{(1), (12)(34)\}$; $B_8 = \{(1), (13)(24)\}$; $B_9 = \{(14)(23)\}$;
$B_1 \cong B_2 \cong B_3 \cong B_4 \cong B_5 \cong B_6 \cong B_7 \cong B_8 \cong B_9$.

Exercise 1.6.2. (a) S_n is generated by the $n-1$ transpositions (12), (13), (14), ..., $(1n)$.
(b) S_n is generated by the $n-1$ transpositions $(12), (23), (34), \ldots, (n-1\,n)$.

Answer. (a) $\forall x \in S_n$, x can be written as a product of transpositions. Actually, for any transposition (ij), we can obtain it by $(1i)(1j)(1i) = (ij)$. So $x \in \langle (12), (13), \ldots, (1n)\rangle$, $S_n \subset \langle (12), (13), \ldots, (1n)\rangle$.
(b) We can contruct $(1i)$ inductively since $(1i) = (1\,i-1)(i-1\,i)(1\,i-1)$. From (a), we have $\forall x \in S_n$, $x \in \langle (12), (13), \ldots, (1n)\rangle$. Thus $S_n \subset \langle (12), (13), \ldots, (1n)\rangle \subset \langle (12), (23), (34), \ldots, (n-1\,n)\rangle$.

Exercise 1.6.3. If $\sigma = (i_1 i_2 \cdots i_r) \in S_n$ and $\tau \in S_n$, then $\tau\sigma\tau^{-1}$ is the r-cycle $(\tau(i_1)\tau(i_2)\cdots\tau(i_r))$.

Answer. $\sigma(i_n) = i_{n+1}$ for $n = 1, 2, \ldots, r-1$, $\sigma(i_r) = i_1$. Assume $\tau(i_n) = j_n$, $n = 1, 2, \ldots, r-1$ and $I = \{i_n | n = 1, 2, \ldots, r-1\}$, $J = \{j_n | n = 1, 2, \ldots, r-1\}$. For $x \notin J$, $\tau\sigma\tau^{-1}(x) = \tau\tau^{-1}(x) = x$. For $x = j_k \in J$, $\tau^{-1}(x) = i_k$, $\sigma(\tau^{-1}(x)) = i_{k+1}$, $\tau(\sigma(\tau^{-1}(x))) = j_{k+1}$ and $\tau\sigma\tau^{-1}(j_r) = j_1$. Thus $\tau\sigma\tau^{-1} = (\tau(i_1)\tau(i_2)\cdots\tau(i_r))$.

Exercise 1.6.4. (a) S_n is generated by $\sigma_1 = (12)$ and $\tau = (123\cdots n)$.
(b) S_n is generated by (12) and $(23\cdots n)$.

Answer. (a) Denote $\sigma_i = \tau\sigma_{i-1}\tau^{-1}$. Applying **Exercise 1.6.3**, $\sigma_i = (i\,i+1)$. By **Exercise 1.6.2**(b), $S_n \subset \langle (12), (23), (34), \ldots, (n-1\,n)\rangle = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1}\rangle \subset \langle \tau, \sigma_1\rangle$. S_n can be generated by τ and σ_1.
(b) Denote $\sigma_1 = (12)$, $\tau = (23\cdots n)$, $\sigma_i = \tau\sigma_{i-1}\tau^{-1}$. Applying **Exercise 1.6.3**,$\sigma_i = (1\,i{+}1)$. By **Exercise 1.6.2**(a), $S_n \subset \langle (12), (13), \ldots, (1n)\rangle = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1}\rangle \subset \langle \tau, \sigma_1\rangle$. S_n can be generated by τ and σ_1.

Exercise 1.6.5. Let $\sigma, \tau \in S_n$. If σ is even (odd), then so is $\tau\sigma\tau^{-1}$.

Answer. Assume $\sigma = (x_1x_2)\cdots(x_{2n-1}x_{2n})$, $\tau = (y_1y_2)\cdots(y_{2m-1}y_{2m})$. Then $\tau^{-1} = (y_{2m-1}y_{2m})\cdots(y_1y_2)$. σ is odd (even) if an only if n is odd (even). $\tau\sigma\tau^{-1}$ has $2m+n$ transpositions. We can add $(ij) = (ji) = (1)$ into some segments of $\tau\sigma\tau^{-1}$ without changing it. So $\tau\sigma\tau^{-1}$ is odd (even) if and only if $2m+n$ is odd (even). $2m+n \equiv n \mod 2$ so $\tau\sigma\tau^{-1}$ is odd (even) if and only if σ is odd (even).

Exercise 1.6.6. A_n is the only subgroup of S_n of index 2.

Answer. For any subgroup $N < S_n$ and $[S_n : N] = 2$, we have $N \lhd S_n$.
Assume there exists k-circle $\sigma = (i_1i_2\cdots i_k) \in N$. Then for any other k-circle $(j_1j_2\cdots j_k)$, take $\tau = (i_ij_1)(i_2j_2)\cdots(i_kj_k)$, by **Exercise 1.6.3**, $\tau\sigma\tau^{-1} = (j_1j_2\cdots j_k) \in N$. Thus N contains all the k-circles.
For $n \geq 5$. If there exists 3-circle in N, then all the 3-circles are contained in N, $A_n \subset N \subset S_n \Rightarrow A_n = N$.
If there exists 2-circle in N, then all the 2-circles are contained in N. Notice $(1i)(1j) = (1ij) \in N$ is a 3-circle, so $A_n = N$.
If there only contain x in the form of $(a_ia_2\cdots a_{n_1})(b_1b_2\cdots b_{n_2})\cdots$ where $n_i \geq 4$ and every two circles are disjoint. Take $\tau_i : \{a_i | i = 1, 2, \ldots, n_1\} \to \{a_i | i = 1, 2, \ldots, n_1\}$. We can obtain product of two n_1-circles

$$x^{-1}\tau x\tau^{-1} = (a_1a_2\cdots a_{n_1})(\tau(a_1)\tau(a_2)\cdots\tau(a_n)) \in N$$

By the arbitrariness of τ, take

$$(\tau(a_1)\tau(a_2)\cdots\tau(a_n)) = (a_1a_4a_5\cdots a_na_3a_2)$$

then $x^{-1}\tau x\tau^{-1} = (a_1a_3)(a_2a_4)$ is a product of 2-circles. We can take a_1, a_2, a_3, a_4 arbitrarily. WLOG, take $(12)(34) \in N$ and $(12)(35) \in N$, $(12)(35)(12)(34) = (345) \in N$. Then there exists 3-circle in N, $N = A_n$.
In conlusion, when $n \geq 5$, S_n has only one normal subgroup A_n.
For $n = 2, 3, 4$, we can verify it by enumeration.

Exercise 1.6.7. Show that $N = \{(1), (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of S_4 contained in A_4 such that $S_4/N \cong S_3$ and $A_4/N \cong Z_3$.

Answer. Assume $\sigma = (i_1i_2)(i_3i_4) \in N$, $\forall\tau \in S_4$, $\tau(i_n) = j_n$, $J = \{j_n|n = 1, 2, 3, 4\}$. For $x \notin J$, $\tau\sigma\tau^{-1}(x) = \tau\tau^{-1}(x) = x$. For $x = j_k \in J$, $\tau^{-1}(x) = i_k$, $\sigma\tau^{-1}(x) = i_{3k-4[\frac{k}{2}]-1}$, $\tau\sigma\tau^{-1}(x) = (\tau(i_i)\tau(i_2))(\tau(i_3)\tau(i_4)) \in N$. So $N \lhd S_4$. $S_4/N = \{N, N(12), N(13), N(23), N(123), N(132)\} \cong S_3$. $A_4/N = \{N, N(123), N(132)\} \cong Z_3$.

Exercise 1.6.8. The group A_4 has no subgroup of order 6.

Answer. $|A_4| = 12$, assume there exists $N < A_4$, $|N| = 6$. Then $N \lhd A_4$. From **Exercise 1.6.6**, we know that all 3-circles are contained in N. But there're 8 3-circles in total, so N can't exist.

Exercise 1.6.9. For $n \geq 3$ let G_n be the multiplicaive group of complex matrices generated by $x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{-2\pi i/n} \end{pmatrix}$, where $i^2 = -1$. Show that $G_n \cong D_n$.

Answer. Take a mapping $f : G_n \to D_n$ as $f(x) = (2\,n)(3\,n-1)\cdots$, $f(y) = (123\cdots n)$. $|f(x)| = |x| = 2$, $|f(y)| = |y| = n$. f is obviously a monomorphism. $\forall a \in D_n$, $a = f(x)^n f(y)^m, m = 1, 2$, then $a = f(x^n y^m)$, f is a epimorphism. Thus $G_n \cong D_n$.

Exercise 1.6.10. Let a be the generator of order n of D_n. Show that $\langle a\rangle \lhd D_n$ and $D_n/\langle a\rangle \cong Z_2$.

Answer. $|\langle a\rangle| = n$, b is the other generator of D_n, $a^n = b^2 = (1)$. $\forall k \in \mathbf{Z}$, $a^k b = ba^{-k}$ can be easily proved by induction. So $\forall x = a^m b^n \in D_n$, $x = a^{m'} b^{n'}$, here $m' \equiv m \mod 2$, $n' \equiv n \mod 2$. $D_n = \{e, a, a^2, \ldots, a^{n-1}, b, ba, \ldots, ba^{n-1}\}$. $|D_n| = 2n$. Thus, $\langle a\rangle \lhd D_n$. $D_n/\langle a\rangle = \{\langle a\rangle, \langle a\rangle b\} \cong Z_2$.

Exercise 1.6.11. Find all normal subgroups of D_n.

Answer. The subgroups of $\langle a\rangle$ is always normal in D_n. $\langle a^m\rangle < \langle a\rangle$. $\forall x \in D_n$ and $a^{km} \in \langle a^m\rangle$, $x = a^t$ or $x = ba^t$.

$$x^{-1}a^{km}x = a^{-t}a^{km}a^t = a^{km} \in \langle a^m\rangle$$

or

$$x^{-1}a^{km}x = a^{-t}b^{-1}a^{km}ba^t = a^{-t}ba^{km}ba^t = a^{-t}a^{-km}b^2a^t = a^{-km} \in \langle a^m\rangle$$

so $\langle a^m\rangle \lhd D_n$.
Consider the subgroup S which only contains $ba^i, i = 1, \ldots, n$. Since $ba^i \cdot ba^j = a^{j-i} \in S\,(i \neq j)$, so $S = \{e, ba^k\}$.
If n is odd, take $x = a^{\frac{n-1}{2}} \in D_n$.

$$x^{-1}ba^kx = a^{\frac{1-n}{2}}ba^ka^{\frac{n-1}{2}} = ba^{k+n-1} \notin S$$

so $S \ntriangleleft D_n$ for all $k = 1, 2, \ldots, n$.
If n is even, take $x = a^{\frac{n-2}{2}} \in D_n$, $n \geq 6$.

$$x^{-1}ba^kx = a^{\frac{2-n}{2}}ba^ka^{\frac{n-2}{2}} = ba^{k+n-2} \notin S$$

so $S \ntriangleleft D_n$ for all $k = 1, 2, \ldots, n$.
If $n = 2$, all the subgroups are normal since $|D_2| = 4$.
For subgroup S contains both ba^i and a^j. It can be written as $S = \left\langle a^d, ba^r\right\rangle$, where $d|n$, $0 \leq r \leq d-1$. If $\exists a^m, a^n \in S$, $(m, n) = d$, then there exist $x, y \in \mathbf{Z}$ s.t. $a^{mx+ny} = a^d \in \mathbf{Z}$. Thus, $S = \left\langle a^d, ba^r\right\rangle$.
Take $x = a^{\frac{n-w}{2}}$, then $x^{-1}ba^rx = ba^{r+n-w}$.

If $d \geq 3$, take $w \equiv n \mod 2$, $x^{-1}ba^r x \notin S$.
If $d = 2$, then $n = 2s$ and $S = \{e, a^s, ba^s, b\}$. $Sa^k = \{a^k, a^{s+k}, ba^{s-k}, ba^{-k}\}$, $k = 1, 2, \ldots, s-1$. $ba^k = ba^{-k}$ or $ba^k = ba^{s-k} \Rightarrow k = \frac{s}{2}$. So for $s = 2$, $n = 4$, S is a normal subgroup of D_4.

Exercise 1.6.12. The center of the group D_n is $\langle e \rangle$ if n is odd and isomorphic to Z_2 if n is even.

Answer. If n is odd, C is the center of D_n, $C \lhd D_n \Rightarrow C < \langle a \rangle$. Take $a^d \in C$, $x = ba^m$,

$$x^{-1}ax = a^{-m}b^{-1}a^d ba^m = a^{-m}ba^d ba^m = a^{-d} = a^d$$

so $d = 0$, $C = \{e\}$.
If n is even, $n \geq 6$. C is the center of D_n. $C \lhd D_n \Rightarrow C < \langle a \rangle$ or $C = \{e, ba^k\}$.
If $C = \{e, ba^k\}$, $C \cong Z_2$.
If $C < \langle a \rangle$,take $a^d \in C$, $x = ba^m$,

$$x^{-1}ax = a^{-m}b^{-1}a^d ba^m = a^{-m}ba^d ba^m = a^{-d} = a^d$$

so $d = \frac{n}{2}$ or $d = 0$, $C = \{a^{\frac{n}{2}}, e\} \cong Z_2$.

Exercise 1.6.13. For each $n \geq 3$ let P_n be a regular polygon of n sides (for $n = 3$, P_n is an equilateral triangle; for $n = 4$, a square). A *symmetry* of P_n is a bijection $P_n \to P_n$ that preserves distances and maps adjacent vertices on to adjacent vertices.

(a) The set D_n^* of all symmetries of P_n is a group under the binary operation of composition of functions.
(b) Every $f \in D_n^*$ is completely determined by its actions on the vertices of P_n. Number the vertices consecutively $1, 2, \ldots, n$; then each $f \in D_n^*$ determines a unique permutation σ_f of $\{1, 2, \ldots, n\}$. The assignment $f \mapsto \sigma_f$ defines a monomorphism of groups $\varphi : D_n^* \to S_n$.
(c) D_n^* is generated by f and g, where f is a rotation of $2\pi/n$ degrees about the center of P_n and g is a reflection about the "diameter" through the center and vertex 1.
(d) $\sigma_f = (123 \cdots n)$ and $\sigma_g = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & n & n-1 & \cdots & 3 & 2 \end{pmatrix}$, whence $\mathrm{Im}\varphi = D_n$ and $D_n^* \cong D_n$.

Answer. In the following analysis, all the numbers are $\mod n$.

(a) Consider n points $A_i = (\cos\frac{2\pi i}{n}, \sin\frac{2\pi i}{n})^t$, $i = 1, 2, \ldots, n$. f is the transposition of $A_i \mapsto A_j$ with the consevation of n regular polygon structure. So f must be a bijection. D_n^* is the set of f. By the definition, $D_n^* \subset S_n$. We prove D_n^* is ta subgroup of S_n.

Notice that $A_{i+1} = \begin{pmatrix} \cos\frac{2\pi i}{n} & -\sin\frac{2\pi i}{n} \\ \sin\frac{2\pi i}{n} & \cos\frac{2\pi i}{n} \end{pmatrix} A_i$.

Denote $X = \begin{pmatrix} \cos\frac{2\pi i}{n} & -\sin\frac{2\pi i}{n} \\ \sin\frac{2\pi i}{n} & \cos\frac{2\pi i}{n} \end{pmatrix}$. To construct the polygon, we must have

$$f(A_{i+1}) = \begin{pmatrix} \cos\frac{2\pi i}{n} & -\sin\frac{2\pi i}{n} \\ \sin\frac{2\pi i}{n} & \cos\frac{2\pi i}{n} \end{pmatrix} f(A_i)$$

or

$$f(A_i) = \begin{pmatrix} \cos\frac{2\pi i}{n} & -\sin\frac{2\pi i}{n} \\ \sin\frac{2\pi i}{n} & \cos\frac{2\pi i}{n} \end{pmatrix} f(A_{i+1})$$

We need to verify that $\forall f_1, f_2 \in D_n^*$, $f_1 f_2^{-1} \in D_n^*$. Assume $B_i = f_2(A_i)$, $B_{i+1} = f_2(A_{i+1})$. Then $B_i = XB_{i+1}$ or $B_i = X^{-1}B_{i+1}$. Denote $B_i = A_j$, then $B_{i+1} = A_{j-1}$ or $B_{i+1} = A_{j+1}$. WLOG, assume $B_{i+1} = A_{j+1}$, then $f_1(A_j) = Xf_1(A_{j+1})$ or $f_1(A_j) = X^{-1}f_1(A_{j-1})$. So $f_1 f_2^{-1} \in D_n^*$. D_n^* is a subgroup of S_n.

(b) Assume $A_i = f(A_1)$. If $f(A_2) = A_{i+1}$, since f is a bijection, by induction, we can prove $f(A_k) = A_{k+i-1}$. $\varphi : D_n^* \to S_n$ can be defined as $\varphi : f \mapsto (1i\, 2i{-}1\, 3i{-}2 \cdots)$. If $f(A_2) = A_{i-1}$, similarly, we can also prove $f(A_k) = A_{i+1-k}$. φ can be defined as $\varphi : f \mapsto (1i)(2\,i-1)(3\,i-2)\cdots$. This means f is completely determined by $f(A_1)$ and $f(A_2)$. D_n^* can be embedded into S_n.

(c) Denote $\alpha = \begin{pmatrix} \cos\frac{2\pi i}{n} & -\sin\frac{2\pi i}{n} \\ \sin\frac{2\pi i}{n} & \cos\frac{2\pi i}{n} \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. $f : A_i \mapsto \alpha A_i$, $g : A_i \mapsto \beta A_i$. f is the rotation of $\frac{2\pi}{n}$ degrees counter-clockwisely. g is the reflection about x-axis. Now we prove $\forall x \in D_n^*$, x can be factorised into finite product of f and g. From (b), x is fully defined by $x(A_1)$ and $x(A_2)$. Assume $x(A_1) = A_i$.

If $x(A_2) = A_{i+1}$, $x(A_k) = A_{i-1+k} = \alpha^{i-1}A_k$, $k = 1, 2, \ldots, n$. So $x = f^{i-1}$.

If $x(A_2) = A_{i-2}$, $x(A_k) = A_{i+1-k} = \alpha^{i+1}A_{-k} = \alpha^{i+1}\beta A_k$. So $x = f^{i+1}g$. Thus $D_4^* \subset \langle f, g \rangle$.

(d) $\alpha^n = \beta^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We can easily verify that $|f| = n$ and $|g| = 2$. From **Exercise 1.6.9**, $\langle f, g \rangle \cong D_n$, $|\langle f, g \rangle| = |D_n| = 2n$. From (b), $x \in D_n^*$

if completely determined by $x(A_1)$ and $x(A_2)$. There are $2n$ different ways to obtain $x(A_1)$ and $x(A_2)$. So $|D_n^*| = |\langle f, g\rangle| = 2n$. $D_n^* \subset \langle f, g\rangle$, so $D_n^* = \langle f, g\rangle$. Thus, $D_n^* \cong \langle f, g\rangle \cong D_n$.

1.7 Categories: products, coproducts, and free objects

Exercise 1.7.1. A *pointed set* is a pair (S, x) with S a set and $x \in S$. A morphism of pointed sets $(S, x) \to (S', x')$ is a triple (f, x, x'), where $S \to S'$ is a function such that $f(x) = x'$. Show that pointed sets form a category.

Answer. Let $\mathcal{S}$ be the category and 4 objects of $\mathcal{S}$ are (A, a), (B, b), (C, c), (D, d). f, g and h are morphisms defined by $f : A \to B$, $g : B \to C$, $h : C \to D$ with $f(a) = b$, $g(b) = c$, $h(c) = d$.

$$(A, a) \xrightarrow{f} (B, b) \xrightarrow{g} (C, c) \xrightarrow{h} (D, d)$$

category $\mathcal{S}$

$$\hom(A, B) \times \hom(B, C) \to \hom(A, C)$$

because $g \circ f : A \to C$ with $g(f(a)) = g(b) = c = g \circ f(a)$. Similarly, $(h \circ g) \circ f = h \circ (g \circ f)$ with $(h \circ g) \circ f(a) = h \circ (g \circ f)(a) = d$. Take 1_B consist of those functions $i : B \to B$ with $i(b) = b$. Then $1_B \circ f = f$ and $g \circ 1_B = g$. So $\mathcal{S}$ is a category.

Exercise 1.7.2. If $f : A \to B$ is an equivalence in a category $\mathcal{C}$ and $g : B \to A$ is the morphism such that $g \circ f = 1_A$, $f \circ g = 1_B$, show that g is unique.

Answer. Assume there exist g and g' satisfies the condition.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "g"] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "g'"] \end{tikzcd}$$

So $g' \circ (f \circ g) = g' \circ 1_B = g' = (g' \circ f) \circ g = 1_A \circ g = g$.

Exercise 1.7.3. In the category $\mathcal{G}$ of groups, show that the group $G_1 \times G_2$ together with the homomorphisms $\pi_1 : G_1 \times G_2 \to G_1$ and $\pi_2 : G_1 \times G_2 \to G_2$ is a product for $\{G_1, G_2\}$.

Answer. Take $\tau_1 : G_1 \to G_1 \times G_2$ as $\tau_1(g_1) = (g_1, e)$; $\tau_2 : G_2 \to G_1 \times G_2$ as $\tau_2(g_2) = (e, g_2)$; $\pi_1 : G_1 \times G_2 \to G_1$ as $\pi_1(g_1, g_2) = g_1$; $\pi_2 : G_1 \times G_2 \to G_2$ as $\pi_2(g_1, g_2) = g_2$. Then

$$\begin{tikzcd} G_1 \arrow[r, shift right, "\tau_1"'] & G_1 \times G_2 \arrow[l, shift right, "\pi_1"'] \arrow[r, shift left, "\pi_2"] & G_2 \arrow[l, shift left, "\tau_2"] \end{tikzcd}$$

For any object B such that

$$\begin{tikzcd} G_1 & B \arrow[l, "\varphi_1"'] \arrow[r, "\varphi_2"] & G_2 \end{tikzcd}$$

For any $x \in B$, define $f : B \to G_1 \times G_2$ as $f(x) = (\varphi_1(x), \varphi_2(x))$. Then $\pi_1(f(x)) = \varphi_1(x)$, $\pi_1 \circ f = \varphi_1$, $\pi_2(f(x)) = \varphi_2(x)$, $\pi_2 \circ f = \varphi_2$. Thus

$$\begin{tikzcd} & B \arrow[dl, "\varphi_1"'] \arrow[d, "f"'] \arrow[dr, "\varphi_2"] & \\ G_1 \arrow[r, shift right, "\tau_1"'] & G_1 \times G_2 \arrow[l, shift right, "\pi_1"'] \arrow[r, shift left, "\pi_2"] & G_2 \arrow[l, shift left, "\tau_2"] \end{tikzcd}$$

Next we verify the uniqueness. If there exist f and f' satisfies the condition,

$$\pi_1(f(x)) = \pi_1(f'(x)) = \varphi_1(x)$$

$$\pi_2(f(x)) = \pi_2(f'(x)) = \varphi_2(x)$$

Thus $f(x) = f'(x)$ for all $x \in B$, so $f = f'$.

Exercise 1.7.4. In the category $\mathcal{A}$ of abelian groups, show that the group $A_1 \times A_2$ together with the morphisms $\tau_1 : A_1 \to A_1 \times A_2$ and $\tau_2 : A_2 \to A_1 \times A_2$ is a coproduct of $\{A_1, A_2\}$.

Answer. Take $\tau_1 : A_1 \to A_1 \times A_2$ as $\tau_1(a_1) = (a_1, e)$; $\tau_2 : A_2 \to A_1 \times A_2$ as $\tau_2(a_2) = (e, a_2)$; $\pi_1 : A_1 \times A_2 \to A_1$ as $\pi_1(a_1, a_2) = a_1$; $\pi_2 : A_1 \times A_2 \to A_2$ as $\pi_2(a_1, a_2) = a_2$. Then

$$\begin{tikzcd} A_1 \arrow[r, shift right, "\tau_1"'] & A_1 \times A_2 \arrow[l, shift right, "\pi_1"'] \arrow[r, shift left, "\pi_2"] & A_2 \arrow[l, shift left, "\tau_2"] \end{tikzcd}$$

For any object B such that

$$A_1 \xrightarrow{\varphi_1} B \xleftarrow{\varphi_2} A_2$$

For any $(a_1, a_2) \in A_1 \times A_2$, define $f : A_1 \times A_2 \to B$ as $f(a_1, a_2) = \varphi_1(a_1)\varphi_2(a_2)$. Then $f(\tau_1(a_1)) = f(a_1, e) = \varphi_1(a_1)e = \varphi_1(a_1)$, $f \circ \tau_1 = \varphi_1$, $f(\tau_2(a_2)) = f(e, a_2) = e\varphi_2(a_2) = \varphi_2(a_2)$, $f \circ \tau_2 = \varphi_2$.

$$\begin{tikzcd}
 & B & \\
A_1 \arrow[ur, "\varphi_1"] \arrow[r, shift right, "\tau_1"'] & A_1 \times A_2 \arrow[u, "f"] \arrow[l, shift right, "\pi_1"'] \arrow[r, shift left, "\pi_2"] & A_2 \arrow[ul, "\varphi_2"'] \arrow[l, shift left, "\tau_2"]
\end{tikzcd}$$

Next we verify the uniqueness. If there exist f and f' satisfies the condition,

$$f(\tau_1(a_1)) = f'(\tau_1(a_1)) = f(a_1, e) = f'(a_1, e)$$

$$f(\tau_2(a_2)) = f'(\tau_2(a_2)) = f(e, a_2) = f'(e, a_2)$$

$$\begin{aligned} f(\tau_1(a_1), \tau_2(a_2)) &= f(\tau_1(a_1))f(\tau_2(a_2)) \\ =f'(\tau_1(a_1), \tau_2(a_2)) &= f'(\tau_1(a_1))f'(\tau_2(a_2)) \end{aligned}$$

so $f = f'$.

Exercise 1.7.5. Every family $\{A_i | i \in I\}$ in the category of sets has a coproduct.

Answer. We examine $\dot{\bigcup} A_i = \{(a, i) \in (\cup A_i) \times I | a \in A_i\}$ which satisfies the condition. Define the morphism $\pi_i : A_i \to \dot{\bigcup} A_i$ as $\pi_i(a) = (a, i)$. For any B such that $\exists \varphi_i : A_i \to B$.

$$\begin{tikzcd}
 & & B & & \\
A_1 \arrow[urr, "\varphi_1"] & A_2 \arrow[ur, "\varphi_2"] & \cdots & \cdots & A_n \arrow[ull, "\varphi_n"']
\end{tikzcd}$$

$\varphi(a) = x \in B$. Take $\varphi(a, i) = \varphi_i(a)$ defined on the subset of $\cup A_i \times I$, we can verify that the domain of φ is $\dot{\bigcup} A_i$. Then take $f = \varphi$, $f(\pi_i(a)) = \varphi_i(a)$, $f \circ \pi_i = \varphi_i$.
The uniqueness is obvious.

Exercise 1.7.6. (a) Show that in the category $\mathcal{S}_*$ of pointed sets product always exist; describe them.

(b) Show that in $\mathcal{S}_*$ every family of objects has a coproduct, describe the coproduct.

Answer. (a) Define $\otimes$ as an operator between points and other elements in the pointed set. $\forall a \in A_i$, $a \otimes a_i = a_1 \times a = a$. For a family of sets with their points $\{(A_i, a_i | i \in I)\}$, consider $(A_1, A_2, \cdots, A_n) = \{(a'_1, a'_2, \cdots, a'_n)\}$. Define morphisms $\pi_i(a) = (a_1, a_2, \cdots, a, \cdots, a_n)$, $\pi_i : A_i \to (A_1, A_2, \cdots, A_n)$.

$$\begin{tikzcd}
 & & (A_1, A_2, \cdots, A_n) & \\
A_1 \arrow[urr, "\pi_1"] & A_2 \arrow[ur, "\pi_2"] & \cdots & A_n \arrow[ul, "\pi_n"']
\end{tikzcd}$$

For any B such that $\exists \varphi_i : A_i \to B$.

$$\begin{tikzcd}
 & & B & \\
A_1 \arrow[urr, "\varphi_1"] & A_2 \arrow[ur, "\varphi_2"] & \cdots & A_n \arrow[ul, "\varphi_n"']
\end{tikzcd}$$

Take $f : (A_1, A_2, \cdots, A_n) \to B$ as

$$f(a'_1, a'_2, \cdots, a'_n) = \varphi_1(a'_1) \otimes \varphi_2(a'_2) \otimes \cdots \otimes \varphi(a'_n)$$

Then $f \circ \pi_i(a) = f(a_1, a_2, \cdots, a, \cdots, a_n) = \varphi_1(a_1) \otimes \cdots \otimes \varphi_i(a) \otimes \cdots \otimes \varphi_n(a_n) = \varphi_i(a)$. So $f \circ \pi_i = \varphi_i$.
Next we verify the uniqueness. If there exist f and f' satisfies the condition. Then $\exists i \in I$ and $a \in A_i$ s.t. $f(a_1, a_2, \cdots, a, \cdots, a_n) \neq f'(a_1, a_2, \cdots, a, \cdots, a_n)$. But $f(\pi_i(a)) = f'(\pi_i(a))$, so $f = f'$.

(b) The proof is similar to **Exercise 1.7.5**.

Exercise 1.7.7. Let F be a free object on a set $X(i : X \to F)$ in a concrete category $\mathcal{C}$. If $\mathcal{C}$ contains an object whose underlying set has at least two elements in it, then i is an injective map of sets.

Answer. Assume $A \in \text{obj}(\mathcal{C})$, A has at least two elements and $X \xrightarrow{f} A$. $X \xrightarrow{i} F$ and F is free on X, so there exists a morphism $\bar{f}$ s.t. $F \xrightarrow{\bar{f}} A$. If $|X| = 1$, i must be injective. For $|X| \geq 2$. Suppose i is not injective. Take $x_1, x_2 \in X$ and $i(x_1) = i(x_2) \in F$, $f(x_1) = a_1$, $f(x_2) = a_2$. $\bar{f}(i(x_1)) = \bar{f}(i(x_2)) = f(x_1) = f(x_2) = a_1 = a_2$. That means all the elements in A are identical. That's contradictory to the assumption.

Exercise 1.7.8. Suppose X is a set and F is a free object on X (with $i : X \to F$) in the category of groups. Prove that $i(X)$ is a set of generators for the group F.

Answer. Assume G the subgroup of F is the group generated by $i(X)$. Since $X \xrightarrow{i} G$ and $X \xrightarrow{i} F$, we can obtain unique morphism φ such that $F \xrightarrow{\varphi} G$ and $\varphi \circ i = i$.
Consider morphism $1_F : F \to F$ which is the identical homomorphism. F is free so 1_F is the unique homomorphism. Take $\subset : G \to F$ as a morphism defined as $\forall g \in G$, $\subset (g) = g$. Then

$$\begin{tikzcd}
& G \arrow[dr, "\subset"] & \\
X \arrow[ur, "i"] \arrow[r, "i"] & F \arrow[u, "\varphi"] \arrow[r, "1_F"] & F
\end{tikzcd}$$

$\subset \circ \varphi \circ i = 1_F \circ i = i$ so $\subset \circ \varphi = 1_F$. Thus $\subset$ is an epimorphism, $F \subset G$. So $F = G$ can be generated by $i(X)$.

1.8 Direct products and direct sums

Exercise 1.8.1. S_3 is not the direct product of any family of its proper subgroups. The same is true of $Z_{p^n}(p$ prime, $n \geq 1)$ and $\mathbb{Z}$.

Answer. We list all the subgroups of S_3: $\{(1),(12)\}$, $\{(1),(13)\}$, $\{(1),(23)\}$, $\{(1),(123),(132)\}$. Only $\{(1),(123),(132)\}$ is normal, so S_3 isn't an direct product of any family of its proper subgroups.
For Z_{p^n}, $Z_{p^i} \lhd Z_{p^n}$ for all $i = 1, 2, \ldots, n-1$ but $Z_{p^i} \cap Z_{p^j} \neq \{e\}$. So Z_{p^n} isn't an direct product of any family of its proper subgroups.
For $\mathbf{Z}$. $\forall N_1 \lhd \mathbf{Z}$, $N_2 \lhd \mathbf{Z}$, we have $N_1 = \langle a_1 \rangle$ and $N_2 = \langle a_2 \rangle$. Thus, $N_1 \cap N_2 = \langle a_1 a_2 \rangle \neq \{e\}$. So $\mathbf{Z}$ isn't an direct product of any family of its proper subgroups.

Exercise 1.8.2. Give an example of groups H_i, K_i such that $H_1 \times H_2 \cong K_1 \times K_2$ and no H_i is isomorphic to any K_j.

Answer. Take $H_1 \cong K_1 \times K_2$, $H_2 = \{e\}$. We verify that $H_1 \times H_2 \cong K_1 \times K_2$. There exists $f : H_1 \to K_1 \times H_2$ which is an isomorphism. There exists canonical projection $\pi_1 : H_1 \times H_2 \to H_1$ and π_1 is an epimorphism. Ker$\pi_1 = \{(e_1, e_2)\}$ thus π_1 is also a monomorphism. Therefore $\bar{f} = f \circ \pi_1$ is a well defined isomorphism. $H_1 \times H_2 \cong K_1 \times K_2$ but neither H_1 nor H_2 are isomorphic to any K_i, $i = 1, 2$.

Exercise 1.8.3. Let G be and (additive) abelian group with subgroups H and K. Show that $G \cong H \oplus K$ if and only if there are homomorphisms

$$\begin{tikzcd} H \arrow[r, shift left, "\tau_1"'] & G \arrow[l, shift left, "\pi_1"'] \arrow[r, shift left, "\pi_2"] & K \arrow[l, shift left, "\tau_2"] \end{tikzcd}$$

such that $\pi_1\tau_1 = 1_H$, $\pi_2\tau_2 = 1_K$, $\pi_1\tau_2 = 0$ and $\pi_2\tau_1 = 0$, where 0 is the map sending every element onto the zero (identity) element, and $\tau_1\pi_1(x) + \tau_2\pi_2(x) = x$ for all $x \in G$.

Answer. If $G \cong H \oplus K$. Denote $f : G \to H \oplus K$ which is a isomorphism. Then there are canonical products π_1', π_2', τ_1', τ_2'.

$$\begin{tikzcd} H \arrow[r, shift right, "\tau_1'"'] & H \oplus K \arrow[l, shift right, "\pi_1'"'] \arrow[r, shift left, "\pi_2'"] & K \arrow[l, shift left, "\tau_2'"] \end{tikzcd}$$

Thus

$$\begin{tikzcd} & G \arrow[dl, dashed, shift right, "\pi_1"'] \arrow[d, shift left, "f^{-1}"] \arrow[dr, dashed, shift left, "\pi_2"] & \\ H \arrow[ur, dashed, shift right, "\tau_1"'] \arrow[r, shift right, "\tau_1'"'] & H \oplus K \arrow[u, shift left, "f"] \arrow[l, shift right, "\pi_1'"'] \arrow[r, shift left, "\pi_2'"] & K \arrow[ul, dashed, shift left, "\tau_2"] \arrow[l, shift left, "\tau_2'"] \end{tikzcd}$$

Take $\tau_1 = f \circ \tau_1'$, $\tau_2 = f \circ \tau_2'$, $\pi_1 = \pi_1' \circ f^{-1}$, $\pi_2 = \pi_2' \circ f^{-1}$.

$$\pi_1\tau_1 = \pi_1' f^{-1} f \tau_1' = \pi_1'\tau_1' = 1_H$$

$$\pi_2\tau_2 = \pi_2' f^{-1} f \tau_2' = \pi_2'\tau_2' = 1_K$$

$$\pi_1\tau_2 = \pi_1' f^{-1} f \tau_2' = \pi_1'\tau_2' = 0$$

$$\pi_2\tau_1 = \pi_2' f^{-1} f \tau_1' = \pi_2'\tau_1' = 0$$

$\forall x \in G$, $x = hk$ where $h \in H$ and $k \in K$.

$$\begin{aligned} \tau_1\pi_1(x) + \tau_2\pi_2(x) &= f(\tau_1'\pi_1'(h,k)) + f(\tau_2'\pi_2(h,k)) \\ &= f(\tau_1'(h)) + f(\tau_2'(k)) \\ &= f(h,e) + f(e,k) \\ &= f(h+e, e+k) = f(h,k) \\ &= x \end{aligned}$$

If there exist π_1, π_2, τ_1, τ_2 satisfies the condition. There are canonical projections π_1', π_2', τ_1', τ_2' between H and $H \oplus K$, K and $H \oplus K$.

$$\begin{tikzcd} & G \arrow[dl, shift right, "\pi_1"'] \arrow[d, dashed, shift left, "f^{-1}"] \arrow[dr, shift left, "\pi_2"] & \\ H \arrow[ur, shift right, "\tau_1"'] \arrow[r, shift right, "\tau_1'"'] & H \oplus K \arrow[u, dashed, shift left, "f"] \arrow[l, shift right, "\pi_1'"'] \arrow[r, shift left, "\pi_2'"] & K \arrow[ul, shift left, "\tau_2"] \arrow[l, shift left, "\tau_2'"] \end{tikzcd}$$

For $f = \tau_1'\pi_1 + \tau_2'\pi_2$ which is a well defined homomorphism. $\forall h \in H$ and $k \in K$, $\tau_1'(h) + \tau_2'(k) = (h,k) \in H \oplus K$. Thus $f(x) = (e_1, e_2)$ if and only if $\pi_1(x) = e_1$ and $\pi_2(x) = e_2$. $\tau_1\pi_1(x) + \tau_2\pi_2(x) = \tau_1(e_1) + \tau_2(e_2) = e = x$. Thus $\mathrm{Ker} f = \{e\}$. f is a monomorphism. $\forall (h,k) \in H \oplus K$, take $x = \tau_1(h) + \tau_2(k) \in G$, then

$$\begin{aligned} f(x) &= \tau_1'\pi_1\tau_1(h) + \tau_1'\pi_1\tau_2(h) + \tau_2'\pi_2\tau_1(k) + \tau_2'\pi_2\tau_1(k) \\ &= \tau_1'(h) + \tau_2'(k) = (h,k) \in H \oplus K \end{aligned}$$

f is a epimorphism. Thus $G \cong H \oplus K$.

Exercise 1.8.4. Give an example to show that the weak direct product is not a coproduct in the category of all groups.

Answer. Consider S_3 and $S_3 \times S_3$.

$$\begin{tikzcd} & S_3 \times S_2 \\ S_3 \arrow[ur] \arrow[r] & S_3 \times S_3 \arrow[u, dashed] \end{tikzcd}$$

Since there doesn't exist homomorphism $S_3 \to S_2$, there is no homomorphism $S_3 \times S_3 \to S_3 \times S_2$.

Exercise 1.8.5. Let G, H be finite cyclic groups. Then $G \times H$ is cyclic if and only if $(|G|, |H|) = 1$.

Answer. Assume $|G| = m$, $|H| = n$, then $G \cong Z_m$, $H \cong Z_n$ and $G \times H \cong Z_m \oplus Z_n$.
If$(|G|, |H|) = 1$. Consider $(x_1, x_2) \in Z_m \oplus Z_n$. By *Chinese Remainder Theorem*, there exists x such that $a \equiv x \mod \mathrm{lcm}(m,n)$ and $a \equiv x_1 \mod m$, $a \equiv x_2 \mod n$. Thus, $a(1,1) = (x_1, x_2)$. $Z_m \oplus Z_n < \langle(1,1)\rangle$. $\langle(1,1)\rangle < Z_m \oplus Z_n$ is trivial. So $Z_m \oplus Z_n = \langle(1,1)\rangle \cong G \times H$ is cyclic.
If $G \times H$ is cyclic. Assume $l = \gcd(m,n)$ and there exist x such that $x_1 \equiv x \mod m$, $x_2 \equiv x \mod n$. Take $x_1 \not\equiv x_2 \mod l$, it can be chosen properly. Consider $(x_1, x_2) \in Z_m \oplus Z_n$, $x = k_1 m + x_1 = k_2 n + x_2 \Rightarrow x_1 \equiv x_2 \mod l$. That's contradictory!

Exercise 1.8.6. Every finitely generated abelian group $G \neq \langle e \rangle$ in which every element (except e) has order p (p prime) is isomorphic to $Z_p \oplus Z_p \oplus \cdots \oplus Z_p$($n$ summands) for some $n \geq 1$.

Answer. Assume $\{a_1, a_2, \ldots, a_n\}$ generates G. $|a_i| = p$ for $i = 1, 2, \ldots, n$ so $\langle a_i \rangle \cong Z_p$. Now we show that $G = \prod_{i=1}^{n} {}^w \langle a_i \rangle \cong \sum_{i=1}^{n} Z_p$. $G = \langle a_1, a_2, \ldots, a_n \rangle$ and $\langle a_1 \rangle \lhd G$ for $i = 1, 2, \ldots, n$. If exist $\langle a_i \rangle$ s.t. $\prod_{j=1, j\neq i}^{n} \langle a_j \rangle \cap \langle a_i \rangle \neq \{e\}$. Then there exists $a_i^{s_i} = a_1^{s_1} \cdots a_{i-1}^{s_{i-1}} a_{i+1}^{s_{i+1}} \cdots a_n^{s_n}$. $(s_i, p) = 1$ so $\exists 1 \leq t_i \leq p-1$ such that $s_i t_i \equiv 1 \mod p$. So $a_i^{s_i t_i} = a_1^{s_1 t_i} \cdots a_{i-1}^{s_{i-1} t_i} a_{i+1}^{s_{i+1} t_i} \cdots a_n^{s_n t_i} = a_i$. $\{a_1, a_2, \ldots, a_n\}$ can generate G. That's contradictory! So $\prod_{j=1, j\neq i}^{n} \langle a_j \rangle \cap \langle a_i \rangle = \{e\}$, which means $G = \prod_{i=1}^{n} {}^w \langle a_i \rangle \cong \sum_{i=1}^{n} Z_p$.

Exercise 1.8.7. Let H, K, N be nontrivial normal subgroups of a group G and suppose $G = H \times K$. Prove that N is in the center of G or N intersects one of H, K nontrivially. Give examples to show that both possibilities can actually occur when G is nonabelian.

Answer. If $N \cap H = N \cap K = \{e\}$. $G = HK$. $\forall h \in H$ and $k \in K$, since $H \cap K = \{e\}$, $hk = kh$. For any $hk \in N$, and $h_1 \in H \subset HK$, $h_1^{-1} hkh_1 = h_1^{-1} hh_1 k \in N$. Assume $h' = h_1^{-1} h_1 \in H$, $h'k \in N$. Thus $h'^{-1} k^{-1} kh = h'^{-1} h \in N$. So $h'^{-1} h = e$, $h = h'$, h is in the center $C(H)$ of group H. Similarly, $k \in C(K)$ which is the center of K. Then $\forall hk \in N$ and $h_1 k_1 \in G$, $k_1^{-1} h_1^{-1} hkh_1 k_1 = h_1^{-1} hh_1 k_1^{-1} kk_1 = hk$. $N \subset N(G)$.
For $N \cup H \neq \varnothing$, the example can be trivial: $N < H$ and $N \lhd G$. There's many cyclic group satisfy the condition.
For $N \subset C(G)$. Take $G = D_4^* \times D_4^*$, $H = D_4^* \times \{I\}$, $K = \{I\} \times D_4^*$. $\{I, R^2\}$ is normal in D_4^*. Denote N is the subgroup $\{(I, I), (R^2, R^2)\}$. We can verify that N satisfies the condition.

Exercise 1.8.8. Corollary 8.7 is false if one of the N_i is not normal.

Answer. Consider $N_1, N_2, \ldots, N_n$ are all finite. WLOG, assume N_1 is not normal. $G = \left\langle \bigcup_{i=1}^{n} N_i \cup N_1 \right\rangle$ and $N_1 N_2 \cdots N_n \subset G$. Denote $A = N_2 N_3 \cdots N_n$. Then $\exists a \in A$ such that $a^{-1}na = n' \notin N_1$. Thus $n'a \in G$ but $n'a \notin N_1 N_2 \cdots N_n$ so $|G| > |N_1 N_2 \cdots N_n| = |N_1| \times |N_2| \times \cdots \times |N_n| = |N_1 \times N_2 \times \cdots \times N_n|$.

Exercise 1.8.9. If a group G is the (internal) direct product of its subgroups H, K, then $H \cong G/K$ and $G/H \cong K$.

Answer. $H \cap K = \{e\}$. $G = H \times K = HK$. Thus $HK/H \cong K/(K \cap H) = K$, $HK/K \cong H/(K \cap H) = H$.

Exercise 1.8.10. If $\{G_i | i \in I\}$ is a family of groups, then $\prod^w G_i$ is the internal weak product its subgroups $\{\tau_i(G_i) | i \in I\}$.

Answer. Take $\tau_i(g) = (e_1, e_2, \ldots, g, \ldots, e_n)$, $g \in G_i.\tau_i(G_i)$. $\tau_i(G_i)$ is normal in $\prod\limits_{i \in I}{}^w G_i$. $\tau_i(G_i) \cap \tau_j(G_j) = \{(e_1, e_2, \ldots, e_n)\}$ which is the identity element in $\prod\limits_{i \in I}{}^w G_i$. $\forall (g_1, g_2, \ldots, g_n) \in \prod\limits_{i \in I}{}^w G_i$, we have

$$(g_1, g_2, \ldots, g_n) = (g_1, e_2, \ldots, e_n)(e_1, g_2, \ldots, e_n) \cdots (e_1, e_2, \ldots, g_n)$$

Thus $\prod\limits_{i \in I}{}^w G_i \subset \left\langle \bigcup\limits_{i \in I}{}^w \tau_i(G_i) \right\rangle$ and

$$\left\langle \bigcup_{i \in I}{}^w \tau_i(G_i) \right\rangle = \tau_1(G_1)\tau_2(G_2) \cdots \tau_n(G_n) \subset \prod_{i \in I}{}^w G_i$$

Therefore $\prod\limits_{i \in I}{}^w G_i$ is the direct product of $\tau_i(G_i)$.

Exercise 1.8.11. Let $\{N_i | i \in I\}$ be a family of subgroups of a group G. Then G is the internal weak product of $\{N_i | i \in I\}$if and only if:

(i) $a_i a_j = a_j a_i$ for all $i \neq j$ and $a_i \in N_i$, $a_j \in N_j$;

(ii) every nonidentity element of G is uniquely a product $a_{i_1} \cdots a_{i_n}$, where $i_i, \ldots, i_n$ are distinct elements of I and $e \neq a_{i_k} \in N_{i_k}$ for each k.

Answer. Trivial.

Exercise 1.8.12. A normal subgroup H of a group G is said to be a **direct factor** (**direct summand** if G is additive abelian) if there exists a (normal) subgroup K of G such that $G = H \times K$.

(a) If H is a direct factor of K and K is a direct factor of G, then H is normal in G.

(b) If H is a direct factor of G, then every homomorphism $H \to G$ may be extended to an endomorphism $G \to G$. However, a monomorphism $H \to G$ need not be extendible to and automorphism $G \to G$.

Answer. (a) $G = K \times K' = (H \times H') \times K'$. So $\forall g \in G$, $g = hh'k'$ with $h \in H$, $h' \in H'$ and $k' \in K'$. $\forall h_1 \in H$ and $g \in G$, $g^{-1}h_1g = k'^{-1}h'^{-1}h^{-1}h_1hh'k' = (h^{-1}h_1h)(h'^{-1}h')(k'^{-1}k') = h^{-1}h_1h \in H$. Thus $H \lhd G$.

(b) If $G = H \times K$. For a homomorphism $f : H \to G$, we construct a homomorphism $\bar{f} : G \to G$, $\forall g \in G$,g can be uniquely written as $g = hk$ where $h \in H$, $k \in K$. Take $\tau(g) = h$ which is a homomorphism $\tau : G \to H$. We can get $\bar{f} = f \circ \tau : G \to G$ is a endomorphism but it needn't to be a automorphism.

Exercise 1.8.13. Let $\{G_i | i \in I\}$ be a family of groups and $J \subset I$. The map $\alpha : \prod\limits_{j \in J} G_j \to \prod\limits_{i \in I} G_i$ given by $\{a_j\} \mapsto \{b_i\}$, where $b_j = a_j$ for $j \in J$ and $b_i = e_i$(identity in G_i) for $i \notin J$, is a monomorphism of groups and $\prod\limits_{i \in I} G_i / \alpha(\prod\limits_{j \in J} G_j) \cong \prod\limits_{i \in I - J} G_i$.

Answer. Define a map $\beta : \prod\limits_{i \in I} G_i \to \prod\limits_{i \in I - J} G_i$ given by $\{a_i\} \mapsto \{b_i\}$ and for those $i \in I - J$, $\exists b_i \in \{b_i\}$ s.t. $a_i = b_i$. Thus $\beta(\{a_i\})\beta(\{a_i'\}) = \beta(\{a_ia_i'\})$, β is a well defined homomorphism. Ker$\beta = \{\{a_i\} \in \prod\limits_{i \in I} G_i | a_i = e_i \text{ for } i \in I - J\} = \alpha(\prod\limits_{j \in J} G_j)$. We verify β is a epimorphism. $\forall \{b_i\} \in \prod\limits_{i \in I - J} G_i$, take

$\{a_i\} \in \prod_{i\in I} G_i$ where $a_i = b_i$ for $i \in I - J$. Then $\beta(\{a_i\}) = \{b_i\}$. Thus β is an isomorphism, $\mathrm{Im}\beta = \prod_{i\in I-J} G_i \cong \prod_{i\in I} G_i/\alpha(\prod_{j\in J}(G_j))$.

Exercise 1.8.14. For $i = 1, 2$ let $H_i \lhd G_i$ and give examples to show that each of the following statements may be false:
(a) $G_1 \cong G_2$ and $H_1 \cong H_2 \Rightarrow G_1/H_1 \cong G_2/H_2$.
(b) $G_1 \cong G_2$ and $G_1/H_1 \cong G_2/H_2 \Rightarrow H_1 \cong H_2$.
(c) $H_1 \cong H_2$ and $G_1/H_1 \cong G_2/H_2 \Rightarrow G_1 \cong G_2$.

Answer. (a) Take $G_1 = G_2 = Z_2 \times Z_4$, $H_1 = Z_2 \times \{\bar{0}\}$, $H_2 = \{(\bar{0}, \bar{0}), (\bar{0}, \bar{2})\}$.
(b) Take $G_1 = G_2 = Z_2 \times Z_4$, $H_1 = \{\bar{0}\} \times Z_4$, $H_2 = \{(\bar{0}, \bar{0}), (\bar{1}, \bar{0}), (\bar{0}, \bar{2}), (\bar{1}, \bar{2})\}$.
(c) Take $H_1 = \{(\bar{0}, \bar{0}), (\bar{0}, \bar{2})\}$, $H_2 = Z_2$ and $G_1 = Z_2 \times Z_4$, $G_2 = Z_2 \times K_4$.

1.9 Free groups, free products, generators and relations

Exercise 1.9.1. Every nonidentity elements in a free group F has a infinite order.

Answer. Define the length of a word $x = a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n}$ is n and denote it as $len(x)$. Assume $len(x) = n$ for some $n \in F$ and $len(1) = 0$, we prove that $len(x^m) \geq n \forall m \geq 1$.
Let k be the largest integer such that $a_{n-j}^{\lambda_{n-j}} = a_n^{-\lambda_j}$ for $j = 0, 1, \ldots, k-1$. If $k > \left[\frac{n}{2}\right]$. For even k, $a_{\frac{n}{2}}^{\lambda_{\frac{n}{2}}} = a_{\frac{n}{2}+1}^{-(\lambda_{\frac{n}{2}+1})}$, $a_{\frac{n}{2}-1}^{\lambda_{\frac{n}{2}-1}} = a_{\frac{n}{2}+2}^{-(\lambda_{\frac{n}{2}+2})}$, $\cdots$ which means $x = a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n} = 1$. For odd k, $a_{\left[\frac{n}{2}\right]+1}^{\lambda_{\left[\frac{n}{2}\right]+1}} = a_{\left[\frac{n}{2}\right]+1}^{-(\lambda_{\left[\frac{n}{2}\right]+1})}$, which is contradictory to x is reduced. So $k \leq \left[\frac{n}{2}\right]$.
Divide $x = x_1 x_2 x_3$ where $x_1 = a_1^{\lambda_1} \cdots a_k^{\lambda_k}$, $x_2 = a_{k+1}^{\lambda_{k+1}} \cdots a_{n-k}^{\lambda_{n-k}}$, $x_3 = a_{n-k+1}^{\lambda_{n-k+1}} \cdots a_n^{\lambda_n}$. $x_3 x_1 = 1$. So $len(x) = len(x_1) + len(x_2) + len(x_3) = n$. $x^m = x_1 x_2 x_3 x_1 x_2 x_3 \cdots x_1 x_2 x_3 = x_1 x_2^m x_3$. $len(x^m) = len(x_1) + m \cdot len(x_2) + len(x_3) \geq n$. So $\forall m \geq 1$, $x^m \neq 1$, $|x|$ is infinite.

Exercise 1.9.2. Show that the free group on the set $\{a\}$ is an infinite cyclic group, and hence isomorphic to $\mathbf{Z}$.

Answer. $F(\{a\}) = \langle a \rangle$ and thus it's a infinite cyclic group. $F(\{a\}) \cong \mathbf{Z}$.

Exercise 1.9.3. Let F be a free group and let N be the subgroup generated by the set $\{x^n | x \in F, n \text{ a fixed integer}\}$. Show that $N \lhd F$.

Exercise 1.9.4. Let F be the free group on the set X, and let $Y \subset H$. If H is the smallest normal subgroup of F containin Y, then F/H is a free group.

Exercise 1.9.5. The group defined by generators a, b and relations $a^8 = b^2a^4 = ab^{-1}ab = e$ has order at most 16.

Exercise 1.9.6. The cyclic group of order 6 is the group defined by generators a, b and relations $a^2 = b^3 = a^{-1}b^{-1}ab = e$.

Exercise 1.9.7. Show that the group defined by generators a, b and relations $a^2 = e$, $b^3 = e$ is infinite and nonabelian.

Exercise 1.9.8. The group defined by generators a, b and relations $a^n = e(3 \le n \in \mathbf{N}^*)$, $b^2 = e$ and $abab = e$ is the dihedral group D_n.

Exercise 1.9.9. The group defined by the generator b and $b^m = e(m \in \mathbf{N}^*)$ is the cyclic group Z_m.

Exercise 1.9.10. The operation of free product is commutative and associative: for any groups A, B, C, $A*B \cong B*A$ and $A*(B*C) \cong (A*B)*C$.

Exercise 1.9.11. If N is normal subgroup of $A * B$ generated by A, then $(A * B)/N \cong B$.

Exercise 1.9.12. If G and H each have more than one element, then $G*H$ is an infinite group with center $\langle e \rangle$.

Exercise 1.9.13. A free group is a free product of infinite cyclic groups.

Exercise 1.9.14. If G is the group defined by generators a, b and relations $a^2 = e$, $b^3 = e$, then $G \cong Z_2 * Z_3$.

Exercise 1.9.15. If $f : G_1 \to G_2$ and $g : H_1 \to H_2$ are homomorphisms of groups, then there is a unique homomorphism $h : G_1 * H_1 \to G_2 H_2$ such that $h|G_1 = f$ and $h|H_1 = g$.

Chapter 2

The structure of groups

Chapter 3

Rings

3.1 Rings and homomorphisms

Exercise 3.1.1. (a) Let G be an (additive) abelian group. Define an operation of multiplication in G by $ab = 0$ (for all $a, b \in G$). Then G is a ring.

(b) Let S be the set of all subsets of some fixed set U. For $A, B \in S$, define $A + B = (A - B) \cup (B - A)$ and $AB = A \cap B$. Then S is a ring. Is S commutative? Does it have an identity?

Answer. (a) $\forall a, b \in G$, $ab = 0 \in G$, so G is a monoid under multiplication, thus G is a ring.

(b) $A \subset U$, $B \subset U$, so $A - B \subset U$, $B - A \subset U$. Thus $A + B = B + A = (A - B) \cup (B - A) \subset U$. Take $\varnothing$ is the identity under addition and $U - A$ as the inverse of A, S is abelian group under the addition. $AB = A \cap B \subset U$, $AB = A \cap B = B \cap A = BA \in S$. So S is a commutative ring. $\forall A \in S$, $A \cap U = AU = A$ is the identity of the ring S.

Exercise 3.1.2. Let $\{R_i | i \in I\}$ be a family of rings with identity. Make the direct sum of abelian groups $\sum\limits_{i \in I} R_i$ into a ring by defining multiplication coordinatewise. Does $\sum\limits_{i \in I} R_i$ have identity?

Answer. Take $1_{R_i} \in R_i$ is the identity for $i = 1, 2, \dots, n$. $\forall (a_1, a_2, \dots, a_n) \in \sum\limits_{i \in I} R_i$

$$\begin{aligned}
&(a_1, a_2, \dots, a_n)(1_{R_1}, 1_{R_2}, \dots, 1_{R_n}) \\
=&(1_{R_1}, 1_{R_2}, \dots, 1_{R_n})(a_1, a_2, \dots, a_n) \\
=&(a_1, a_2, \dots, a_n)
\end{aligned}$$

is the identity.

Exercise 3.1.3. A ring R such that $a^2 = a$ for all $a \in R$ is called **Boolean ring**. Prove that every Boolean ring R is commutative and $a + a = 0$ for all $a \in R$.

Answer. $\forall a \in R$, $(a+a)^2 = a^2 + 2a + a^2 = a + 2a + a = 2a$, so $a + a = 0$. $\forall a, b \in R$, $(a+b)^2 = a^2 + b^2 + ab + ba = a + b = a + b + ba + ab$, so $ab + ba = 0 \Rightarrow ab = -ab = -ba$, $ab = ba$. Thus R is commutative.

Exercise 3.1.4. Let R be a ring and S a nonempty set. Then the group $M(S, R)$ is a ring with multiplication defined as follows: the product of $f, g \in M(S, R)$ is the function $S \to R$ given by $s \mapsto f(s)g(s)$.

Answer. We only need to check $M(S, R)$ is a monoid under multiplication, which means $\forall f, g \in M(S, R)$, $fg \in M(S, R)$. $\forall a \in S$, $fg(a) = f(a)g(a)$. Since $f(a) \in R$, $g(a) \in R$, $f(a)g(a) \in R$, $fg : S \to G$ is a well defined function. $fg \in M(S, R)$. $M(S, R)$ is a ring.

Exercise 3.1.5. If A is the abelian group $\mathbf{Z} \oplus \mathbf{Z}$, then End$A$ is a noncommutative ring.

Answer. We only need to verify that End$A$ is not commutative. Take $f, g \in \text{End}A$, $f : (x_1, x_2) \mapsto (x_1 \mod 2, x_2 \mod 2)$, $g : (x_1, x_2) \mapsto (x_1 \mod 3, x_2 \mod 3)$. Then $gf(3,3) = (1,1)$, $fg(3,3) = (0,0)$. Thus End$A$ is not commutative.

Exercise 3.1.6. A finite ring with more than one element and no zero divisors is a division ring.

Answer. For any disjoint $a, b, c \in R$, $ab \neq ac$, otherwise $a(b-c) = 0$, $b - c$ is a zero divisor. So ax are different for different $x \in R$. $|\{ax|x \in R\}| = |R|$ and $\{ax|x \in R\} \subset R$. Thus $\{ax|x \in R\} = R$ which means $\exists a^{-1} \in R$ s.t. $aa^{-1} = R$. Similarly, a is also left invertable and R is a division ring.

Exercise 3.1.7. Let R be a ring withe more than one element such that for each nonzero $a \in R$ there is a unique $b \in R$ such that $aba = a$. Prove:
(a) R has no zero divisors.

(b) $bab = b$.
(c) R has an identity.
(d) R is a division ring.

Answer. (a) If x is a zero divisor of a. WLOG, assume $ax = 0$, $axa \neq a$ so $b \neq x$. But $axa + aba = a(x+b)a = a$ which is contradictory to the uniqueness.
(b) $aba = a \Rightarrow abab = ab$, $a(bab - b) = 0$ and $a \neq 0$, so $bab - b =$, $bab = b$.
(c) Assume $c = ab$, $abab = ab \Rightarrow c^2 = c$. $\forall x \in R$, $xc^2 = xc \Rightarrow (xc - x)c = 0$ and $c \neq 0$, so $xc = x$ for any $x \in R$. Similarly, $cx = x$ for all $x \in R$, c is the identity of R.
(d) $\forall a, b \in R$, $aba = a \cdot 1_R = 1_R \cdot a$. So $a(ba - 1_R) = (ab - 1_R)a = 0$, $ba = ab = 1_R$. That means a, b are all units, so R is a division ring.

Exercise 3.1.8. Let R be the set of all 2×2 matrices over the complex field $\mathbf{C}$ of the form

$$\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$$

where $\bar{z}, \bar{w}$ are the complex conjugates of z and w respectively. Then R is a division ring that is isomorphic to the division ring K of real quaternions.

Answer. Define $f : K \to R$ with $f(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $f(i) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $f(j) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $f(k) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. Assume $z = a + bi$, $w = c + di$.

$$\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Define

$$f(\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}) = af(1) + bf(i) + cf(j) + df(k)$$

$f(xy) = f(x)f(y)$ and f is a isomorphism, so $R \cong K$.

Exercise 3.1.9. (a) The subset $G = \{1, -1, i, -i, j, -j, k, -k\}$ of the division ring K of real quaternions forms a group under multiplication.

(b) G is isomorphic to the quaternion group.
(c) What is the difference between the ring K and the group $\mathbf{R}(G)$($\mathbf{R}$ the field of real numbers)?

Answer. (a) Trivial.

(b) Define $f : G \to Q_8$ given by $f(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $f(i) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $f(j) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $f(k) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. We can verify that f is a isomorphism, $G \cong Q_8$.

(c) $R(G)$ is a free abelian group while K is not free on G.

Exercise 3.1.10. Let k, n be integers such that $0 \le k \le n$ and $\binom{n}{k}$ the binomial coefficient $n!/(n-k)!k!$, where $0! = 1$ and for $n > 0$, $n! = n(n-1)(n-2)\cdots 2 \cdot 1$.

(a) $\binom{n}{k} = \binom{n}{n-k}$
(b) $\binom{n}{k} < \binom{n}{k+1}$ for $k+1 \le n/2$.
(c) $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$ for $k < n$.
(d) $\binom{n}{k}$ is an integer.
(e) if p is prime and $1 \le k \le p^n - 1$, then $\binom{p^n}{k}$ is divisible by p.

(a) $\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n!}{(n-(n-k))!(n-k)!} = \binom{n}{n-k}$.

(b) $\binom{n}{k} = \frac{n!}{(n-k)!k!}$, $\binom{n}{k+1} = \frac{n!}{(n-k-1)!(k+1)!}$, since $k+1 \le n-k$ when $k+1 \le \frac{n}{2}$, then $\binom{n}{k} < \binom{n}{k+1}$.

(c) $\binom{n}{k} + \binom{n}{k+1} = \frac{n!}{(n-k)!k!} + \frac{n!}{(n-k-1)!(k+1)!} = \frac{(n+1)!}{(n-k)!(k+1)!} = \binom{n+1}{k+1}$.

(d) $\binom{n}{k}$ is an integer can be easily solved by induction and (c).

(e) $\text{ord}_p(p^n!) = \sum_{i=1}^{\infty} \left[\frac{p^n}{p^i}\right] = \sum_{i=0}^{n-1} p^i$. $\text{ord}_p(k!) = \sum_{i=1}^{\infty} \left[\frac{k}{p^i}\right]$, $\text{ord}_p((p^n-k)!) = \sum_{i=1}^{\infty} \left[\frac{p^n-k}{p^i}\right]$. $\forall i \in \mathbf{N}$, $\left[\frac{p^n-k}{p^i}\right] + \left[\frac{k}{p^i}\right] \le \left[\frac{p^n}{p^i}\right]$, the equality holds if and only if $\frac{p^n-k}{p^i}, \frac{k}{p^i} \in \mathbf{Z}$. And $\left[\frac{p^n-k}{p^n}\right] = 0$, $\left[\frac{k}{p^n}\right] = 0$. So $\text{ord}_p(\binom{p^n}{k}) = \text{ord}_p(p^n!) - \text{ord}_p((n-k)!) - \text{ord}_p(k!) \ge 1$. $p | \binom{p^n}{k}$.

Exercise 3.1.11. Let R be a commutative ring with identity of prime characteristic p. If $a, b \in R$, then $(a \pm b)^{p^n} = a^{p^n} \pm b^{p^n}$ for all integers $n \ge 0$.

Answer. $(a \pm b)^{p^n} = \sum_{i=0}^{p^n} \binom{p^n}{i} (\pm a)^i b^{p^n-i}$. From **Exercise 3.1.10**, $p | \binom{p^n}{i}$ for all $i = 1, 2, \ldots, n-1$, so $\binom{p^n}{i} a^i b^{p^n-i} = 0$ for $i = 1, 2, \ldots, n-1$. Thus $\sum_{i=0}^{p^n} \binom{p^n}{i} (\pm a)^i b^{p^n-i} = a^{p^n} \pm b^{p^n} = (a \pm b)^{p^n}$.

Exercise 3.1.12. An element of a ring is **nilpotent** if $a^n = 0$ for some n. Prove that in a commutative ring $a + b$ is nilpotent if a and b are. Show that this result may be false if R is not commutative.

Answer. Assume $a^m = 0$, $b^n = 0$. For $(a+b)^{m+n} = \sum_{i=1}^{m+n} \binom{m+n}{i} a^i b^{m+n-i}$. If $i \geq m$, $a^i b^{m+n-i} = 0 b^{m+n-i} = 0$; if $i \leq m$, $m + n - i \geq n$ so $a^i b^{m+n-i} = a^i 0 = 0$. Thus $a^i b^{m+n-i} = 0$ for all $i = 1, 2, \ldots, m+n$. $a+b$ is also nilpotent. For the 2×2 matrix ring. $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are nilpotent, but $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is not nilpotent.

Exercise 3.1.13. In a ring R the following conditions are equivalent.
(a) R has no nonzero nilpotent elements.
(b) If $a \in R$ and $a^2 = 0$, then $a = 0$.

Answer. (a)$Rightarrow$(b): Trivial.
(b)$Rightarrow$(a): If $\exists a \in R$, $a^n = 0$ for some n and $a \neq 0$. Assume $n = 2^m \cdot k$ and k is a odd integer. Then $(a^{k \cdot 2^{m-1}})^2 = 0 \Rightarrow a^{k \cdot 2^{m-1}} = 0 \Rightarrow \cdots \Rightarrow a^k = 0$. $a^k \cdot a^{k+1} = 0$ and $2 | k+1$, we can continue this step until $\frac{k+1}{2} \geq k$ which means $k = 1$. So $a = 0$.

Exercise 3.1.14. Let R be a commutative ring with identity and prime characteristic p. The map $R \to R$ given by $r \mapsto r^p$ is a homomorphism of rings called the Frobenius homomorphism.

Answer. $\forall a, b \in R$, $pa = pb = 0$ and the map $f : r \mapsto r^p$. $f(a+b) = (a+b)^p = \sum_{i=0}^{p} \binom{p}{i} a^i b^{p-i}$. Since p is a prime so $p \mid p!$ and $p \nmid i!(p-i)!$, $p \mid \binom{p}{i}$ for $i = 1, 2, \ldots, p-1$. So $f(a+b) = a^p + b^p = f(a) + f(b)$, $f(ab) = (ab)^p = a^p b^p = f(a)f(b)$, f is a homomorphism of rings.

Exercise 3.1.15. (a) Give an example of nonzero homomorphism $f : R \to S$ of rings with the identity such that $f(1_R) \neq 1_S$.

(b) If $f : R \to S$ is an epimorphism of rings with identity, then $f(1_R) = 1_S$.

(c) If $f : R \to S$ is a homomorphism of rings with identity and u is a unit in R such that $f(u)$ is a unit in S, then $f(1_R) = 1_S$ and $f(u^{-1}) = f(u)^{-1}$.

Answer. (a) For $f : Z_2 \to Z_6$ defined by $f(0) = 0$, $f(1) = 3$. f is a homomorphism of ring which satisfies the condition.

(b) $\forall s \in S$, $\exists r \in R$ such that $f(r) = s$, so $f(r)f(1_R) = f(1_R)f(r) = f(r) = s$, so $f(1_R) = 1_S$ is the identity of S.

(c) $f(u)f(u^{-1}) = f(u^{-1})f(u) = f(1_R)$. $\exists s \in S$ such that $f(u)s = sf(u) = 1_S$, $sf(u)f(u^{-1}) = sf(1_R) = f(u^{-1})$, $sf(1_R)f(u) = f(u^{-1})f(u) = f(1_R) = sf(u) = 1_S$. Thus $f(u^{-1} = s)$, $f(u^{-1}) = f(u)^{-1}$.

Exercise 3.1.16. Let $f : R \to S$ be a homomorphism of rings such that $f(r) \neq 0$ for some nonzero $r \in R$. If R has an identity and S has no zero divisors, then S is a ring with identity $f(1_R)$.

Answer. $f(1_R)f(1_R) = f(1_R)$, so $f(1_R)(f(1_R) - 1_S) = 0 \Rightarrow f(1_R) = 1_S$.

Exercise 3.1.17. (a) If R is a ring, then so is R^{op} is defined as follows. The underlying set of R^{op} is precisely R and addition in R^{op} coincides with addition in R. Multiplication in R^{op}, denoted $\circ$, is defined by $a \circ b = ba$, where ba is the product in R. R^{op} is called the **opposite ring** of R.

(b) R has identity if and only if R^{op} does.

(c) R is a division ring if and only if R^{op} is.

(d) $(R^{op})^{op} = R$.

(e) If S is a ring, then $R \cong S$ if and only if $R^{op} \cong S^{op}$.

Answer. (a) Trivial.

(b) If 1_R is the identity of R. Take $1_{R^{op}} = 1_R$ then $\forall a \in R^{op}$, $1_{R^{op}} \circ a = a1_R = a = 1_R a = a \circ 1_{R^{op}}$. So $1_{R^{op}}$ is the identity of R^{op}.

(c) $\forall a \in R^{op}$, take $a^{-1} \in R$, $a^{-1} \circ a = aa^{-1} = 1_R = a^{-1}a = a \circ a^{-1}$. So a is a unit, R^{op} is a division ring.

(d) Denote $*$ is the multiplication in $(R^{op})^{op}$.

$$a * b = b \circ a = ab \in R$$

The multiplications are identical. The underlying set and addition of R and $(R^{op})^{op}$ are identical. So $R = (R^{op})^{op}$.

(e) If $R \cong S$, there exists isomorphism $f : R \to S$. We verify that $f' R^{op} \to S^{op}$ defined by $f' = f$ is an isomorphism. $f' = f$ is obviously a bijection. $f'(a) \circ f'(b) = f(b)f(a) = f(ba) = f'(a \circ b)$. f' is a well defined homomorphism, so $R^{op} \cong S^{op}$.

Exercise 3.1.18. Let $\mathbf{Q}$ be the field of rational numbers and R any ring. If $f, g : \mathbf{Q} \to R$ are homomorphisms of rings such that $f|\mathbf{Z} = g|\mathbf{Z}$, then $f = g$.

Answer. $f(n) = g(n)$ for $n \in \mathbf{Z}$. $g(n)g(\frac{1}{n}) = g(1) \Rightarrow f(n)g(\frac{1}{n}) = g(1) = f(1)$, so $f(\frac{1}{n})f(n)g(\frac{1}{n}) = g(\frac{1}{n}) = f(\frac{1}{n})$ for all $n \in \mathbf{Z}$. Thus $f = g$.

3.2 Ideals

Exercise 3.2.1. The set of all nilpotent elements in a commutative ring forms an ideal.

Answer. Assume the set is I, then $\forall a, b \in I$, $a^m = b^n = 0$, $(a+b)^{m+n} = 0$ and $(ab)^{mn} = 0$ so $a+b \in I$, $ab \in I$. I is a subring. $\forall x \in R$, $(xa)^m = x^m a^m = 0$, $(ax)^m = a^m x^m = 0$, so $xa \in I$ and $ax \in I$, I is an ideal.

Exercise 3.2.2. Let I be an ideal in a commutative ring R and let $\text{Rad}I = \{r \in R | r^n \in I \text{ for some } n\}$. Show that $\text{Rad}I$ is an ideal.

Answer. $\text{Rad}I$ is a ring since R is a commutative ring. For $r \in \text{Rad}I$ and $\forall x \in R$, $(xr)^n = x^n r^n \in I$ so $xr \in \text{Rad}I$, $(rx)^n = r^n x^n \in I$ so $rx \in \text{Rad}I$. Thus $\text{Rad}I$ is an ideal.

Exercise 3.2.3. If R is a ring and $a \in R$, then $J = \{r \in R | ra = 0\}$ is a left ideal and $K = \{r \in R | ar = 0\}$ is a right ideal in R.

Answer. J is a subring of R. For $r \in J$ and $\forall x \in R$, $(xr)a = x(ra) = 0$ so $xr \in J$, J is a left ideal. Similarly, I is a right ideal.

Exercise 3.2.4. If I is a left ideal of R, then $A(I) = \{r \in R | rx = 0 \text{ for every } x \in I\}$ is an ideal in R.

Answer. For any $a, b \in A(I)$, we have $ab \in A(I)$ and $a+b \in A(I)$. For $r \in A(I)$ and $\forall x \in R$, $(xr)x' = x(rx') = 0$ for every $x' \in I$, so $xr \in A(I)$. $(rx)x' = r(xx')$, $xx' \in I$ so $rxx' = 0$, $rx \in A(I)$. Thus $A(I)$ is an ideal of R.

Exercise 3.2.5. If I is an ideal in a ring R, let $[R : I] = \{r \in R | xr \in I \text{ for every } x \in R\}$. Prove that $[R : I]$ is an ideal of R which contains I.

Answer. I is a subring of R so $[R:I]$ is also a subring of R. For $r \in [R:I]$ and $x, x' \in R$, $x'xr = (x'x)r \in I$ so $xr \in [R:I]$, $x'rx = (x'r)x \in I$ so $rx \in [R:I]$. $[R:I]$ is an ideal of R. Since $\forall r \in I$, $xr \in I$ and $rx \in I$, $I \subset [R:I]$.

Exercise 3.2.6. (a) The center of the ring S of all 2×2 matrices over a field F consists of all matrices of the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

(b) Then center of S is not an ideal in S.

(c) What is the center of the ring of all $n \times n$ matrices over a division ring?

Answer. (a) $\forall x \in M_F(2,2)$, $x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$

$$x \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} x = \begin{pmatrix} ax_1 & ax_2 \\ ax_3 & ax_4 \end{pmatrix}$$

so $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in C(M_F(2,2))$.

$\forall \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in C(M_F(2,2))$, take $\begin{pmatrix} 0 & 1_F \\ 0 & 0 \end{pmatrix} \in M_F(2,2)$

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} 1_F & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ a_3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1_F & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}$$

so $a_2 = a_3 = 0$.

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} 0 & 1_F \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_1 \\ 0 & a_3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1_F \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} a_3 & a_4 \\ 0 & 0 \end{pmatrix}$$

so $a_1 = a_4$. All the elements of $C(M_F(2,2))$ has the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

(b) For $c \in C(S)$. If S is not commutative, $\forall x, x' \in R$, we need $xc \in C(S) \Rightarrow x'xc = xcx' = xx'c$, however, this may not always true.

(c) By multiplying $\begin{pmatrix} 1_F & & \\ & \ddots & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & \\ & 1_F & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 1_F \end{pmatrix}$,

we can have $C(M_F(2,2))$ consist of all the elements in the form of $a\begin{pmatrix} 1_F & & & \\ & 1_F & & \\ & & \ddots & \\ & & & 1_F \end{pmatrix}$.

Exercise 3.2.7. (a) A ring R with identity is a division ring if and only if R has no proper left ideals.

(b) If S is a ring (possibly without identity) with no proper left ideals, then either $S^2 = 0$ or S is a division ring.

Answer. (a) Suppose not. I is an ideal in R. $\forall r \in I$, take $r^{-1} \in R$, then $1_R \in I$ so $I = R$ is not a proper ideal.

(b) $I = \{a \in S | Sa = 0\}$ is a left ideal since $\forall x, x' \in S$, $x'(xs) = (x'x)s = 0$, $xs \in I$. Thus $I = 0$ or $I = S$. If $I = S$, then $S^2 = 0$. If $I = 0$, we prove S has no zero divisor.

For the set $I' = \{r \in S | rb = 0\}$, $I' \subset I$. I' is a subring of S, and I' is also a left ideal of S. So $I' = 0$, b has no left zero divisors. $\forall a \in S$, Sa is a left ideal of S. $Sa \neq 0$ so $Sa = S$. Thus, $\exists 1_S \in S$, such that $1_S a = a$. Since $s_1 - s_2$ has no left zero divisor, $as_1 = as_2 \Rightarrow s_1 = s_2$. So $aS = S$. For all $s \in S$, $\exists s'$ s.t. $s = as'$ so $\forall s \in S$, $1_S \cdot s = 1_S as' = as' = s$. $aS = S$ so $\exists 1'_S \in S$, $a1'_S = a$. Similarly, $\forall s \in S$, $s1_S = s$. Then $1_S 1'_S = 1_S = 1'_S$ so S has identity. Since $Sa = aS = S$, we can have S is a division ring.

Exercise 3.2.8. Let R be a ring with identity and S the ring of all $n \times n$ matrices over R. J is an ideals of S if and only if J is the ring of all $n \times n$ matrices over I for some ideal I in R.

Answer. If J is an ideal. Denote $E_{r,s}$ as the matrix which has 1_R as the r column and s row. Then $\forall A = (a_{ij})$, $E_{p,r} A E_{s,q}$ is a matrix with a_{rs} in the p column and q row. So for $A \in J$ $(aE_{p,r})A(bE_{s,q})$ is the matrix with $aa_{rs}b$

in the p column and q row. $aa_{rs}b \in I$. Then because of closure we know J contains all $n \times n$ matrices over I.
If J consists of all $n \times n$ matrices over I, the proof is trivial.

Exercise 3.2.9. Let S be the ring of all $n \times n$ matrices over a division ring D.
(a) S has no proper ideals (that is, 0 is the maximal ideal).
(b) S has zero divisors. Consequently, (i) $S \cong S/0$ is not a division ring and (ii) 0 is a prime ideal which does not satisfy condition (1) of Theorem 2.15.

Answer. (a) J is an ideal of S so J consists of all $n \times n$ matrices over I where I is an ideal of D. From **Exercise 3.2.7**, D has no proper ideal so $I = 0 \Rightarrow J = 0$.
(b) For $A = (a_{ij})$ with $a_{ri} = 0$ for $i = 1, 2 \cdots$ and other entries doesn't equals to zero, we have $E_{1r}A = 0$. S has no zero divisors.

Exercise 3.2.10. (a) Show that $\mathbf{Z}$ is a principle ideal ring.
(b) Every homomorphic image of a principle ideal ring is also a principle ideal ring.
(c) Z_m is a principle ideal ring for every $m > 0$.

Answer. (a) For any ideal I in $\mathbf{Z}$, I is a subring so $I = m\mathbf{Z}$ where $m \in \mathbf{Z}$. $m\mathbf{Z} = (m)$ is a principle ideal so $\mathbf{Z}$ is a PID.
(b) For $f : R \to S$ with $f(r) = s$ and R is a principle ideal ring. Consider $f : R \to \mathrm{Im} f \subset S$. For any ideal $J \subset \mathrm{Im} f$, $f^{-1}(J)$ is an ideal since $\forall a \in f^{-1}(J)$ and $r \in R$, $f(ar) = f(a)f(r) \in J \Rightarrow ar \in f^{-1}(J)$. $f^{-1}(J)$ is a principle ideal, assume $f^{-1}(J) = (a)$. Then $\forall r \in R$, $ar \in (a)$, $ra \in (a)$. $f(ar) = f(a)f(r) \in J$ and $f(ra) = f(r)f(a) \in J$ since $f(a) \in J$ and $f(r) \in S$. So $(f(a)) \subset J$. $J = f((a)) = \{f(ra + as + na + \sum\limits_{i=1}^{m} r_i a s_i) | r, s, r_i, s_i \in R, n \in \mathbf{Z}\} = \{f(r)f(a) + f(a)f(s) + nf(a) + \sum\limits_{i=1}^{m} f(r_i)f(a_i)f(s_i) | r, s, r_i, s_i \in R, n \in \mathbf{Z}\} \subset (f(a))$. So $J = (f(a))$ is a principle ideal. The image of a principle ideal ring is also a principle ideal ring.

Exercise 3.2.11. If N is the ideal of all nilpotent elements in a commutative ring R, then R/N is a ring with no nonzero nilpotent elements.

Answer. Suppose not. $\exists r \in R$, $r \notin N$, $(r+N)^n = 0$ for some $n \in \mathbf{N}$.

$$(r+N)^n = r^n + N = N \Rightarrow r^n \in N$$

so for some $m \in \mathbf{N}$, $r^{nm} = 0 \Rightarrow r \in N$. That's contradictory!

Exercise 3.2.12. Let R be a ring without identity and with no zero divisors. Let S be the ring whose additive group is $R \times \mathbf{Z}$ as in the proof of Theorem 1.10. Let $A = \{(r,n) \in S | rx + nx = 0 \text{ for every } x \in R\}$.
(a) A is an ideal in S.
(b) S/A has an identity and contains a subring isomorphic to R.
(c) S/A has no zero divisors.

Answer. (a) For $(r,n),(r',n') \in S$, $(r'+r)x + (n'+j)x = rx + nx + r'x + n'x = 0$, so $(r+r', n+n') \in A$. $(r,n)(r'n') = (rr' + nr' + n'r, nn')$, $rr'x + n'rx + nr'x + nn'x = r(r'x + n'x) + n(r'x + n'x) = 0$, so $(r,n)(r',n') \in A$. A is a subring of $R \times \mathbf{Z}$. $\forall (r_1, n_1) \in R \times \mathbf{Z}$, $(r_1,n_1)(r,n) = (r_1 r + n r_1 + n_1 r, n n_1) \Rightarrow r_1 r x + n r_1 x + n_1 r x + n n_1 x = r_1(rx + nx) + n_1(rx + nx) = 0 \Rightarrow (r_1,n_1)(r,n) \in A$. A is an ideal of $R \times \mathbf{Z}$.

(b) Take $0_R \in R$ and $(0_R, 1) \in S$. Then $(0_R, 1) + A$ is an identity of S/A.

$$\forall (r,n) \in S, \quad (r,n)(0_R,1) = (0_R,1)(r,n) = (r,n)$$

(c) For any $(r,n),(s,m)$ satisfy that $(r,n)(s,m) \in A$, we prove that $(r,n) \in A$ or $(s,m) \in A$. Suppose $sx + mx \neq 0$, $r(sx+mx) + n(sx+mx) = 0 \Rightarrow (sx+mx)r(sx+mx) + n(sx+mx)^2 = 0 \Rightarrow ((sx+mx)r + n(sx+mx))(sx+mx) = 0 \Rightarrow (sx+mx)r + n(sx+mx) = 0$. For any $x \in R$, $(sx+mx)rx + n(sx+mx)x = 0 \Rightarrow (sx+mx)(rx+nx) = 0 \Rightarrow rx+nx = 0$, so $(r,n) \in A$. S/A has no divisor.

Exercise 3.2.13. Let $f : R \to S$ be a homomorphism of rings, I and ideal in R, and J an ideal in S.
(a) $f^{-1}(J)$ is and ideal in R that contains Ker$f$.
(b) If f is an epimorphism, then $f(I)$ is an ideal in S. If f is not surjective, $f(I)$ need not be an ideal.

Answer. (a) $\forall a \in f^{-1}(J)$ and $r \in R$, $f(ar) = f(a)f(r) \in J \Rightarrow ar \in J$. Similarly, $ra \in J$, $f^{-1}(J)$ is an ideal. Ker$f \subset f^{-1}(J)$ since $0_S \in J$.
(b) $\forall b \in f(I)$ and $s \in S$, f is a epimorphism so $s = f(r)$, $b = f(a)$ for some $r, a \in R$. $sb = f(r)f(a) = f(ar)$, $ar \in I \Rightarrow sb \in f(I)$, similarly $bs \in f(I)$. $f(I)$ is an ideal.
If f is not surjective. Take $Z[x]$ and $\mathbf{Z}$ which is a subring but not an ideal in $Z[x]$. $\mathbf{Z}$ is an ideal of itself, $f = 1_{\mathbf{Z}}$ satisfies the condition.

Exercise 3.2.14. If P is an ideal in a not necessarily commutative ring R, then the following conditions are equivalent.
(a) P is a prime ideal.
(b) If $r, s \in R$ are such that $rRs \subset R$, then $r \in P$ or $s \in P$.
(c) If (r) and (s) are principle ideals of R such that $(r)(s) \subset P$, then $r \in P$ or $s \in P$.
(d) If U and V are right ideals in R such that $UV \subset R$, then $U \subset R$ or $V \subset R$.
(e) If U and V are left ideals in R such that $UV \subset R$, then $U \subset R$ or $V \subset R$.

Exercise 3.2.15. The set consisting of zero and all zero divisors in a commutative ring with identity contains at least one prime ideal.

Answer. Denote $S = R - Z$. $\forall a, b \in S$, we prove that $ab \in S$. Suppose $\exists (ab)c = 0$ for some $c \in R$, a, b are not zero divisors so $abc = b(ac) = a(bc) = 0$, so $ac = 0$, $bc = 0 \Rightarrow c = 0$, so ab is not a zero divisor. Thus $Z = R - S$ contains an prime ideal.

Exercise 3.2.16. Let R be a commutative ring with identity and suppose that the ideal A of R is contained in a finite union of prime ideals $P_1 \cup \cdots \cup P_n$. Show that $A \subset P_i$ for some i.

Answer. Suppose not. We choose the smallest I such that for all $i \in I$, $P_i \cap A \neq \varnothing$ and $A \cap P_i \not\subset \bigcup_{j \neq i} P_j$ for any $i \in I$. So $\exists a_i \in (A \cap P_i) - (\bigcup_{j \neq i} P_j)$, $\forall i \in I$. Take $x = a_1 + a_2 a_3 \cdots a_n$, $x \in A$ since $a_i \in A$ for all $i \in I$. And $x \notin P_i$ for $i = 2, 3, \ldots, n$ since $a_1 \notin P_i$, $i = 2, 3, \ldots, n$. $x \notin P_1$ since P_1 is prime and $a_2, \ldots, a_n \notin P_1$. So $x \notin \bigcup_{j \neq i} P_j$, which is contradictory!

Exercise 3.2.17. Let $f : R \to S$ be an epimorphism of rings with kernel K.

(a) If P is a prime ideal in R that contains K, then $f(P)$ is a prime ideal in S.

(b) If Q is a prime ideal in S, then $f^{-1}(Q)$ is a prime ideal in R that contains K.

(c) There is a one-to-one correspondence between the set of all prime ideals in R that contain K and the set of all prime ideals in S, given by $P \mapsto f(P)$.

(d) If I is an ideal in a ring R, then every prime ideal in R/I is of the form P/I, where P is a prime ideal in R that contains I.

Answer. (a) From **Exercise 3.2.13** we know $f(P)$ is an ideal. $\forall x, y \in f(P)$, $\exists a.b \in R$, $x = f(a)$, $y = f(b)$ and $a, b \notin P$. Assume $\exists p \in P$ such that $f(ab) = f(p)$, then $f(ab - p) = 0$, $ab - p \in \text{Ker} f \subset P \Rightarrow ab \in P$. That's contradictory to $a, b \notin P$ so $xy \notin f(P)$. $f(P)$ is prime.

(b) From **Exercise 3.2.13**, $f^{-1}(Q)$ is an ideal. Take $g : S \to S/Q$ and $gf : R \to S/Q$. By the Theorem of homomorphism, $R/f^{-1}(Q) \cong S/Q$ is a ring without divisor, so $f^{-1}(Q)$ is prime.

(c) From (a), (b), f is a one-to-one map between prime ideals given by $P \mapsto f(P)$.

(d) Consider the homomorphism $f : R \to R/I$. For any prime ideal $P \subset R$ and $f(P)$ is an prime ideal in R, $\text{Ker} f = I$ so for prime ideals $I \subset P \subset R$. P can have one to one correspondence with $f(P) = P/I \subset R/I$. So all the prime ideals has the form P/I.

Exercise 3.2.18. An ideal $M \neq R$ in a commutative ring R with identity is maximal if and only if for every $r \in R - M$, there exists $x \in R$ such that $1_R - rx \in M$.

Answer. If M is maximal, then M is prime. So $rR + M = R$, $r(R-M) + M = R$ and $r(R-M) \cap M = \varnothing$. Take $1_R \in R$ we have $x \in R - M$, $1_R - xr \in M$. If $\forall r \in R - M$, $\exists x \in R$ such that $1_R - rx \in M$. Suppose $M \subset I \subset R$ where I is an ideal, $I \neq R$ so $1_R \notin R$. Take $r \in I - M \subset R - M$, then $\forall x \in R$, $rx \in I$, so $1_R - rx \notin I$ thus $1_R - rx \notin M$. That's contradictory!

Exercise 3.2.19. The ring E of even integers contains a maximal ideal M such that E/M is not a field.

Answer. $E = 2\mathbf{Z}$ and M is a maximal ideal in E and for any subring of E has the form $wn\mathbf{Z}$ where $n \in \mathbf{Z}$. $2n\mathbf{Z}$ is an ideal in $2\mathbf{Z}$. Take $n = 15$, $(2, 15) = 1$ so $2\mathbf{Z}/30\mathbf{Z} \cong \mathbf{Z}/15\mathbf{Z}$ which is not a field since $3 \cdot 5 = 0$ is a zero divisor.

Exercise 3.2.20. In the ring $\mathbf{Z}$ the following conditions on a nonzero ideal I are equivalent: (i) I is prime; (ii) I is maximal; (iii) $I = (p)$ with p prime.

Answer. $\mathbf{Z}$ is an integer domain so (ii)$\Rightarrow$(i).
(i)$\Rightarrow$(iii): Trivial.
(iii)$\Rightarrow$(ii): For any $n \notin (p)$, we have $p \nmid n$ thus $\exists x, y \in \mathbf{Z}$ such that $px + my = 1$. Consider an ideal I and $(p) \subset I$, $n \in I$, then $1 \in I$ so $I = \mathbf{Z}$ which means (p) is maximal.

Exercise 3.2.21. Determine all prime and maximal ideals in the ring Z_m.

Answer. $Z_m^2 = Z_m$ so every maximal ideal is prime in Z_m. $Z_m \cong \mathbf{Z}/m\mathbf{Z}$ via $\varphi : \bar{x} \mapsto mz + x$. From **Exercise 3.2.17**, all the prime ideals in $\mathbf{Z}/m\mathbf{Z}$ are $P/m\mathbf{Z}$, where P is a prime ideal contains $m\mathbf{Z} = (m)$.
If m is prime, (m) is prime, too. So no such ideal exist.
If $m = p_1^{s_1} p_2^{s_2} \cdots p_n^{s_n}$ where p_i are primes, then $(p_1), (p_2), \ldots, (p_n)$ are prime ideals and $f((\bar{p_i})) = (p_i)/m\mathbf{Z}$ are prime ideals. So all the prime ideals in Z_m are $(\bar{p_i})$, $i, 1, 2, \ldots, n$.

Exercise 3.2.22. (a) If $R_1, \dots, R_n$ are rings with identity and I is an ideal in $R_1 \times \cdots \times R_n$, then $I = A_1 \times \cdots \times A_m$, where each A_i is an ideal in R_i.

(b) Show that the conclusion of (a) need not hold if the rings R_i do not have identities.

Exercise 3.2.23. An element e in a ring R is said to be **idempotent** if $e^2 = e$. An element of the center of the ring R is said to be central. If e is a central idempotent in a ring R with identity, then

(a) $1_R - e$ is a central idempotent;

(b) eR and $(1_R - e)R$ are ideals in R such that $R = eR \times (1_R - e)R$.

Answer. (a) $(1_R - e)^2 = 1_R - 2e + e^2 = 1_R - 2e + e = 1_R - e$. $\forall x \in R$, $ex = xe$ so $(1_R - e)x = x - ex = x - xe = x(1_R - e)$. $1_R - e$ is a central idempotent.

(b) $eR \cup (1_R - e)R \subset R$ so $\langle eR \cap (1_R - e)R \rangle \subset R$. $R = eR + (1_R - e)R$ so $R \subset \langle eR \cap (1_R - e)R \rangle$. So $R = \langle eR \cap (1_R - e)R \rangle$. $\langle eR \rangle = eR$ and $\langle (1_R - e)R \rangle = (1_R - e)R$ so $\langle eR \rangle \cap \langle (1_R - e)R \rangle = 0$. Thus $R = eR \times (1_R - e)R$.

Exercise 3.2.24. Idempotent elements $e_1, \dots, e_n$ in a ring R are said to be **orthogonal** if $e_i e_j = 0$ for $i \neq j$. If $R, R_1, \dots, R_n$ are rings with identity, then the following conditions are equivalent:

(a) $R \cong R_1 \times \cdots \times R_n$.

(b) R contains a set of orthogonal central idempotents $\{e_1, \dots, e_n\}$ such that $e_1 + e_2 + \cdots + e_n = 1_R$ and $e_i R \cong R$ for each i.

(c) R is the internal direct product $R = A_1 \times \cdots \times A_n$ where each A_i is an ideal of R such that $A_i \cong R_i$.

Answer. Assume $f : R_1 \times \cdots \times R_n \to R$ is an isomorphism.

(a)$\Rightarrow$(b): Denote $\bar{e_1} = (1_R, 0, \dots, 0)$, $\bar{e_2} = (0, 1_R, \dots, 0)$, $\dots$, $\bar{e_n} = (0, 0, \dots,$ $1_R)$. They are orthogonal central idempotent in $S = R_1 \times \cdots \times R_n$ and $f(\bar{e_n}) = e_n$, $e_1 + e_2 + \cdots + e_n = 1_S$, $\sum\limits_{i=1}^{n} e_i S = S$.

Take $\varphi_i : (r_1, r_2, \ldots, r_i, \ldots, r_n) \mapsto r_i$. Then φ_i is a well defined isomorphism between $e_i S$ and R_i. $e_i R \cong \bar{e}_i S \cong R_i$.
(b)$\Rightarrow$(c): Take $A_i = e_i R$, then $A_i \cong R_i$. We need to prove $R = e_i R \times e_2 R \times \cdots \times e_n R$. $e_i R \cap (e_1 R + e_2 R + \cdots + e_{i-1} R + e_{i+1} R + \cdots + e_n R) = 0$ since $e_i x_i = e_1 x_1 + e_2 x_2 + \cdots + e_{i-1} x_{i-1} + e_{i+1} x_{i+1} + \cdots + e_n x_n \Rightarrow e_i^2 x_i = 0$. $R = 1_R R = \sum\limits_{i=1}^{n} e_i R$ so $R = e_i R \times e_2 R \times \cdots \times e_n R$.
(c)$\Rightarrow$(a): Trivial.

Exercise 3.2.25. If $m \in \mathbf{Z}$ has a prime decomposition $m = p_1^{k_1} \cdots p_t^{k_t} (k_i > 0$; p_i distinct primes), then there is an isomorphism of rings $Z_m \cong Z_{p_1^{k_1}} \times \cdots \times Z_{p_t^{k_t}}$.

Answer. For any $m \in \mathbf{Z}$, $\mathbf{Z}/m\mathbf{Z} \cong Z_m$. $p_1^{k_1}\mathbf{Z} \cap \cdots \cap p_t^{k_t}\mathbf{Z} = m\mathbf{Z}$. So $\exists \varphi : Z_m \mapsto Z_{p_i^{k_1}} \times \cdots \times Z_{p_t^{k_t}}$. $\forall i, j \in I$, $p_i^{k_i} \in p_i^{k_i}\mathbf{Z}$ and $p_j^{k_j} \in p_j^{k_j}\mathbf{Z}$, $\exists x, y \in \mathbf{Z}$ such that $x p_i^{k_i} + y p_j^{k_j} = 1 \in \mathbf{Z}$. So $p_i^{k_i}\mathbf{Z} + p_j^{k_j}\mathbf{Z} = \mathbf{Z}$, φ is an isomorphism so $Z_m \cong Z_{p_1^{k_1}} \times \cdots \times Z_{p_t^{k_t}}$.

Exercise 3.2.26. If $R = \mathbf{Z}$, $A_1 = (6)$ and $A_2 = (4)$, then the map $\theta : R/A_1 \cap A_2 \to R_1/A_1 \times R_2/A_2$ of Corollary 2.27 is not surjective.

Answer. $R/(A_1 \cap A_2) = Z_{12}$, $R/A_1 = Z_6$ and $R/A_2 = Z_4$. $|Z_6 \times Z_4| = |Z_6| \times |Z_4| = 24$ but $|Z_{12}| = 12$, so θ is surjective.

3.3 Factorization in commutative rings

Exercise 3.3.1. A nonzero ideal in a principle ideal domain is maximal if and only if it is prime.

Answer. For PID R, $R^2 = R$ so every maximal ideal is prime. If $I = (p) \neq 0$ is prime in R, then p is prime so p is irreducible and (p) is maximal.

Exercise 3.3.2. An integral domain R is unique factorization domain if and only if every non zero prime ideal in R contains a nonzero principle ideal that is prime.

Answer. Suppose R is a unique factorization domain and $P \neq 0$ is a prime ideal. Let $x \in P$ be a nonzero nonunit. Then x can be factored into $x = p_1 p_2 \cdots p_n$ a product of prime elements. Then $x \in P$ implies $p_i \in P$ for some i, so $(p_i) \subset P$.
Conversely, assume that each nonzero prime ideal of R contains a principle prime ideal.

Lemma. *Let R be a commutative ring and $S \subset R \backslash \{0\}$ a multiplicatively closed subset containing 1_R. Let $\mathcal{I}_S$ be the set of ideals of R which are disjoint from S. Then*
(a) $\mathcal{I}_S$ is nonempty.
(b) Every element of $\mathcal{I}_S$ is contained in a maximal element oof $\mathcal{I}_S$.
(c) Every maximal element of $\mathcal{I}_S$ is prime.

Here's the proof of the lemma:
(a) Trivial.
(b) Let $I \in \mathcal{I}_S$. Consider the subposet P_I of $\mathcal{I}_S$ consisting of ideals which contain I. Since $I \in P_I$, P_I is nonempty; moreover, any chain in P_I has an upper bound,namely the union of all of its elements. Therefore by Zorn's lemma, P_I has a maximal element of $\mathcal{I}_S$, which is clearly also a maximal element of $\mathcal{I}_S$.
(c) Let I be a maximal element of $\mathcal{I}_S$; suppose that $x, y \in R$ are such that $xy \in I$. If x is not in I, then $\langle I, x\rangle \supsetneq I$ and therefore contains an element s_1 of S, say

$$s_1 = i_1 + ax$$

Similarly, if y is not in I, then we get an element s_2 of S of the form

$$s_2 = i_2 + by$$

But then

$$s_1s_2 = i_1i_2 + (by)i_1 + (ax)i_2 + (ab)xy \in I \cap S$$

a contradiction!
A multiplicative subset S is saturated if for all $x \in S$ and $y \in R$, if $y \mid x$ then $y \in S$. We define the saturation $\bar{S}$ of a multiplicatively closed subset S to be the intersection of all saturated multiplicatively closed subsets containing S. Let S be the set of units of R together with all product of prime elements. One checks easily that S is saturated multiplicative subset. We should show that $S = \frac{R}{\setminus}\{0\}$. Suppose then for a contradiction that there exists a nonzero nonunit $x \in R\setminus S$. Then saturation of S implies that $S \cap (x) = \varnothing$, and then there exists a prime ideal P contains x and disjoint from S. But by the hypothesis, P contains a prime element p, contradictting its disjointness from S.

Exercise 3.3.3. Let R be the subring $\{a + b\sqrt{10} | a, b \in \mathbf{Z}\}$ of the field of real numbers
(a) The map $N : R \to Z$ given by $a + b\sqrt{10} \mapsto (a + b\sqrt{10})(a - b\sqrt{10}) = a^2 - 10b^2$ is such that $N(uv) = N(u)N(v)$ for all $u, v \in R$ and $N(u) = 0$ if and only if $u = 0$.
(b) u is a unit in R if and only if $N(u) = \pm 1$.
(c) $2, 3, 4 + \sqrt{10}$ and $4 - \sqrt{10}$ are irreducible elements of R.
(d) $2, 3, 4 + \sqrt{10}$ and $4 - \sqrt{10}$ are not prime elements of R.

Answer. (a) Assume $u = a_1 + b_1\sqrt{10}$, $v = a_2 + b_2\sqrt{10}$.

$$\begin{aligned} N(uv) &= N(a_1a_2 + 10b_1b_2 + (a_1b_2 + a_2b_1)\sqrt{10}) \\ &= (a_1a_1 + 10b_1b_2)^2 - 10(a_1b_2 + a_2b_1)^2 \\ &= a_1^2a_2^2 + 100b_1^2b_2^2 - 10a_1^2b_2^2 - 10a_2^2b_1^2 \end{aligned}$$

$$N(u)N(v) = (a_1^2 - 10b_1^2)(a_2^2 - 10b_2^2) = N(uv)$$

(b) If u is a unit of R, $N(uu^{-1}) = N(1) = N(u)N(u^{-1}) = 1$. $N(u)$ and $N(u^{-1}) \in \mathbf{Z}$ so $N(u) = \pm 1$.
(c) Suppose $4 + \sqrt{10} = (a_1 + b_1\sqrt{10})(a_2 + b_2\sqrt{10})$ where $N(a_1 + b_1\sqrt{10})$, $N(a_2 + b_2\sqrt{10}) \neq \pm 1$. $N(4 + \sqrt{10}) = 6 = N(a_1 + b_1\sqrt{10})N(a_2 + b_2\sqrt{10})$

so $N(a_1 + b_1\sqrt{10}) = \pm 2$ and $N(a_2 + b_2\sqrt{10}) = \pm 3$. WLOG, assume $N(a_1 + b_1\sqrt{10}) = 2$ and $N(a_2 + b_2\sqrt{10}) = 3$.

$$a_1^2 = 10b_1^2 + 2 \Rightarrow a_1^2 \equiv 2 \mod 10$$

$$a_2^2 = 10b_2^2 + 3 \Rightarrow a_2^2 \equiv 3 \mod 10$$

This can't be true! So $4 + \sqrt{10}$ is irreducible. Similarly, 2,3,$4 - \sqrt{10}$ is irreducible.

(d) $3 \cdot 2 = (4 + \sqrt{10})(4 - \sqrt{10}) - 6$, But none of these four numbers divide another.

Exercise 3.3.4. Show that in the integral domain of **Exercise 3.3.3** every element can be factored into a product of irreducibles, but this factorization need not be unique.

Answer. Suppose a can be factored into $a_1 a_2 \cdots a_n \cdots$ which may not be finite. We only need to prove there are finite a_i are irreducible. $N(a) = N(a_1)N(a_2)\cdots N(a_n)\cdots = k \in \mathbf{Z}$. Assume $k = k_1 k_2 \cdots k_m$ and for irreducible a_i, $N(a_i) \neq \pm 1$, so there are at most m a_i irreducible. Thus a can be factored into a product of irreducibles.

Exercise 3.3.5. Let R be a principle ideal domain.

(a) Every proper ideal is a product $P_1 P_2 \cdots P_n$ of maximal ideals, which are unique ly determined up to order.

(b) An ideal P in R is said to be primary if $ab \in P$ and $a \notin P$ imply $b^n \in P$ for some n. Show that P is primary if and only if for some n, $P = (p^n)$ where $p \in R$ is prime or $p = 0$.

(c) If $P_1, P_2, \ldots, P_n$ are primary ideals such that $P_i = (p_i^{n_i})$ and the p_i are distinct primes, then $P_1 P_2 \cdots P_n = P_1 \cap P_2 \cap \cdots \cap P_n$.

(d) Every proper ideal in R can be expressed (uniquely up to order) as the intersection of a finite number of primary ideals.

Answer. (a) For any ideal (a), a can be factored into irreducible product $a_1 a_2 \cdots a_n$. (a_i) are maximal in R and $(a) = (a_1)(a_2)\cdots(a_n)$.

(b) If $P = (p^n)$. For any $ab \in P$, $ab = p^n x$ for some $x \in R$ and $n \in \mathbf{Z}$. R is a UFD so $p \mid a$ or $p \mid b$ so $b^n \in P$. Conversely, $\forall P = (k)$ we prove $k = p^t$ for some prime p and $t \in \mathbf{Z}$. For any $ab = kx$, assume $a = a_1^1 \cdots a_m^{p_m}$, $b = a_1^{q_1} \cdots a_m^{q_m}$ and $k = a_1^{s_1} \cdots a_m^{s_m}$, p_i, q_i, s_i are all nonnegative integers. We prove that for all but one i, $s_i = 0$. Take $p_i = 0$ for $i = 1, 2, \ldots, m-1$, $p_m = s_m$, $q_i = s_i$ for $i = 1, 2, \ldots, m-1$, $q_m = 0$, then $ab = k \in (k)$ but a, a^n, b, $b^n \notin (k)$ for all $n \in \mathbf{Z}$. So $k = a_i^{s_i}$ for some $s_i \in \mathbf{Z}$, $(k) = (a_i^{s_i})$, a_i prime.

(c) $P_1 P_2 \cdots P_n \subset P_1 \cap P_2 \cap \cdots \cap P_n$ is trivial.
For any $a \in P_1 \cap \cdots \cap P_n$, $p_i^{n_i} | a$, $\forall i = 1, 2, \ldots, n$. $p_i^{n_i} \neq p_j^{n_j}$ so $a = p_1^{n_1} x_1 \Rightarrow p_2^{n_2} | x_2 \Rightarrow a = p_1^{n_1} p_2^{n_2} x_2 \cdots \Rightarrow a = p_1^{n_1} p_2^{n_2} \cdots p_n^{n_n} x_n \in P_1 P_2 \cdots P_n$. So $P_1 P_2 \cdots P_n \subset P_1 \cap P_2 \cap \cdots \cap P_n$, $P_1 \cdots P_n = P_1 \cap \cdots \cap P_n$.

(d) For any ideal $(a) \subset R$, $(a) = P_1 P_2 \cdots P_n$ which is the product of maximal ideals. So we can express (a) as the product of $p_i' = (p_i^{s_i})$ since n is finite. $(a) = P_1' P_2' \cdots P_m' = P_1' \cap P_2' \cap \cdots \cap P_m'$.

Exercise 3.3.6. (a) If a and n are integers, $n > 0$, then there exist integers q and r such that $a = qn + r$, where $|r| \leq n/2$.

(b) The Gaussian integers $\mathbf{Z}[i]$ form a Euclidean domain with $\varphi(a + bi) = a^2 + b^2$.

Answer. (a) Trivial.

(b) For $a_1 + b_1 i$, $a_2 + b_2 i \in \mathbf{Z}[i]$

$$\begin{aligned}
\varphi(a_1 + b_1 i)(a_2 + b_2 i) &= \varphi((a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i) \\
&= (a_1 a_2 - b_1 b_2)^2 + (a_1 b_2 + a_2 b_1)^2 \\
&= (a_1 a_2)^2 + (b_1 b_2)^2 + (a_1 b_2)^2 + (a_2 b_1)^2 \\
&= (a_1^2 + b_1^2)(a_2^2 + b_2^2) \\
&= \varphi(a_1 + b_1 i)\varphi(a_2 + b_2 i)
\end{aligned}$$

For any $x \in \mathbf{Z}$, and $y = a + bi \in \mathbf{Z}[i]$, from (a) $a = q_1 x + r_1$, $b = q_2 x + r_2$ with $|r_1| \leq \frac{x}{2}$, $|r_2| \leq \frac{x}{2}$. Let $q = q_1 + q_2 i$, $r = r_1 + r_2 i$, then $y = qx + r$ with $r = 0$ or $\varphi(r) = r_1^2 + r_2^2 < \varphi(x)$. $\forall x = c + di \neq 0$, take $\bar{x} = c - di$, then there are q, $r_0 \in \mathbf{Z}[i]$ such that $y\bar{x} = qx\bar{x} + r_0$ with $r_0 = 0$ or $\varphi(r_0) < \varphi(x\bar{x})$. Let $r = y - qx$, then $y = qx + r$ and $r = 0$ or $\varphi(r) < \varphi(x)$.

Exercise 3.3.7. What are the units in the ring of Gaussian integers $\mathbf{Z}[i]$?

Answer. From **Exercise 3.3.6**, we proved that $\varphi(a+bi) = a^2+b^2$ satisfies that $\forall u, v \in \mathbf{Z}[i]$, $\varphi(uv) = \varphi(u)\varphi(v)$. So if there exist $u^{-1} = c + di$ such that $uu^{-1} = 1$, then $\varphi(u)\varphi(u^{-1}) = 1$ which means $(a^2 + b^2)(c^2 + d^2) = 1$. So $u = \pm 1$ or $\pm i$.

Exercise 3.3.8. Let R be the following subring of the complex numbers: $R = \{a + b(1 + \sqrt{19}i)/2 | a, b \in \mathbf{Z}\}$. The R is a principle ideal domain that is not a Euclidean domain.

Answer. Take $\varphi(a + b(1 + \sqrt{19}i)/2) = a^2 + ab + 5b^{p2}$. Denote $\tilde{R}$ as the collection of units in R together with 0. An element $u \in R - \tilde{R}$ is called a universal side divisor if for every $x \in R$ there is some $z \in \tilde{R}$ such that u divides $x - z$ in R.
Let R be an integral domain that is not a field, if R is a Euclidean domain then there are universal side divisors in R. Since $\varphi(R) \subset \mathbf{N}$ has a lower bound, we can choose $u \in R-\tilde{R}$ such that $\varphi(u)$ minimizes. Then $\forall x = qu+r$, $r = 0$ or $\varphi(r) < \varphi(u)$ so $r \in \tilde{R}$. Hence u is a universal side divisor in R. Now we prove $R = \mathbf{Z}[(1 + \sqrt{19}i)/2]$ is not a Euclidean domain by showing R contains no universal side divisor. The units in R are only ± 1 so $\tilde{R} = \{\pm 1, 0\}$. $\forall a+b(1+\sqrt{19}i)/2 \in \mathbf{Z}[(1+\sqrt{19}i)/2]\backslash\mathbf{Z}$, $\varphi(a+b(1+\sqrt{19}i)/2) = a^2 + ab + 5b^2 \geq 5$. So the smallest nonzero value of $\varphi(x)$ is 1 and 4. Take $x = 2$ in the definition of universal side divisor, u must divide 2 or 3. If $2 = ab$, then $4 = \varphi(a)\varphi(b)$ so the only divisor of 2 are ± 1, ± 2. Similarly the only divisor of 2 are ± 1, ± 3. So the value of u should be ± 2 or ± 3. Take $x = (1 + \sqrt{19}i)/2$ and it's easy to check that none of x, $x \pm 1$ are divisible by ± 2, ± 3. Thus none of these is a universal side divisor.
Next we prove R is a principle ideal domain. Define φ' to be a Dedekind-Hasse norm if φ' is a positive norm and for every nonzero $a, b \in R$ either $a \in (b)$ or there exist $s, t \in R$ with $0 < \varphi'(sa - tb) < \varphi'(b)$.
For any principle ideal domain R, R has a Dedekind-Hasse norm. Let I be an nonzero ideal in R and b be a nonzero element of I with $\varphi'(b)$ minimal. Suppose a is any nonzero elements in I, so the ideal (a, b) is contained in I. Then the Dedekind-Hasse condition on φ' and the minimality of b implies that $a \in (b)$, so $I = (b)$ is principle.
We prove $R = \mathbf{Z}[(1 + \sqrt{19}i)/2]$ has a Dedekind-Hasse norm φ. Suppose α, β are nonzero elements of R and $a/\beta \notin R$. We should show that there

are elements $s, t \in R$ with $0 < \varphi(s\alpha - t\beta) < \varphi(\beta)$, which is equivalent to $0 < \varphi(\frac{\alpha}{\beta}s - t) < 1$. Assume $\frac{\alpha}{\beta} = \frac{a+b\sqrt{19}i}{c} \in \mathbf{Q}[\sqrt{19}i]$ with integers a, b, c having no common divisor and with $c > 1$. Since a, b, c have no common divisor there are integers x, y, z with $ax+by+ca = 1$. Write $ay-19bx = cq+r$ for some quatient q and remainder r with $|r| \leq c/2$ and let $s = y + x\sqrt{19}i$ and $t = q - z\sqrt{19}i$. Then

$$0 < \varphi(\frac{\alpha}{\beta}s - t) = \frac{(ay - 19bx - cq)^2 + 19(ax + by + cz)^2}{c^2} < \frac{1}{4} + \frac{19}{c^2}$$

so when $c \geq 5$ then condition is satisfied.
Suppose $c = 2$. Then one of a, b is even and the other is odd, and then $s = 1$ and $t = \frac{(a-1)+b\sqrt{19}i}{2}$ are elements of R satisfying the condition.
Suppose $c = 3$. The integer a^2+19b^2 is not divisible by 3. Assume $a^2+19b^2 = 3q + r$ with $r = 1$ or $r = 2$. Then $s = a - b\sqrt{19}i$ and $t = q$ satisfies the condition.
Suppose $c = 4$ so a and b are not both even. If one of a, b is even and the other is odd, then $a^2 + 19b^2$ is odd, so we can write $a^2 + 19b^2 = 4q + r$ for some $q, r \in \mathbf{Z}$ and $0 < r < 4$. Then $s = a - b\sqrt{19}i$ and $t = q$ satisfies the condition. If a and b are both odd, then $a^2 + 19b^2 \equiv 4 \mod 8$, so we have $a^2 + 19b^2 = 8q + 4$ for some $q \in \mathbf{Z}$. Then $s = (a - b\sqrt{19}i)/2$ and $t = q$ are elements in R satisfying the condition.

Exercise 3.3.9. Let R be a unique factorization domain and d a nonzero element of R. There are only a finite number of distinct principle ideals that contain the ideal (d).

Answer. Assume $d = p_1^{s_1} p_2^{s_2} \cdots p_n^{s_n}$. For some k satisfies that $(d) \subset (k)$, we have $k \mid d$. So $kx = p_1^{s_1} p_2^{s_2} \cdots p_n^{s_n}$ for $x \in \mathbf{R}$. Thus $k = p_1^{t_1} \cdots p_n^{t_n}$, where $t_i \leq s_i$, whence the choices of k are finite.

Exercise 3.3.10. If R is a unique factorization domain and $a, b \in R$ are relatively prime and $a \mid bc$, then $a \mid c$.

Answer. Assume $d = p_1^{s_1} p_2^{s_2} \cdots p_n^{s_n}$, $a|bc \Rightarrow ax = bc$ for some $x \in R$. a, b are relatively prime so for any prime ideal (p_i), $p_i \nmid b$, $c \in (p_i)$. Assume $p_i c_1 = c$, $p_i a_1 = a$, then $c_1 b = a_1 x$. Similarly, $c \in (p_i)$, we can continue this step so $c \in (p_i^{s_i})$. $c \in (a) = (p_1^{s_1})(p_2^{s_2}) \cdots (p_n^{s_n})$.

Exercise 3.3.11. Let R be a Euclidean ring and $a \in R$. Then a is a unit in R if and only if $\varphi(a) = \varphi(1_R)$.

Answer. If a is a unit, then $\exists a^{-1} \in R$, $aa^{-1} = 1_R$. $a = a \cdot 1_R$ so $\varphi(1_R) < \varphi(a \cdot 1_R) = \varphi(a)$, $\varphi(a) \leq \varphi(aa^{-1}) = \varphi(1_R)$ so $\varphi(a) = \varphi(1_R)$.
If $\varphi(a) = \varphi(1_R)$, $\forall x \in R\backslash\{0\}$, $x = x \cdot 1_R$ so $\varphi(x) \geq \varphi(1_R)$. Assume $1_R = qa + r$, $\varphi(r) \geq \varphi(a)$ for all $r \in R\backslash\{0\}$. So $r = 0$, $1_R = qa$, a is a unit.

Exercise 3.3.12. Every nonempty set of elements (possibly infinite) in a commutative principle ideal ring with identity has a greatest common divisor.

Answer. Denote $S = \{(a) | \bigcup_{i \in I}(a_i) \subset (a)\}$. S is nonempty since $R \in S$. For finite I, the conclusion is trivial. For infinite I. Assume $(d) = \bigcap_{A \in S} A$ which is a well defined ideal. $\bigcap_{i \in I}(a_i) \subset (d)$ so $(a_i) \subset (d) \Rightarrow d \mid a_i$ for all $i \in I$. And $\forall c \mid a_i$ for all $i \in I$, $(c) \subset S$ so $(d) \subset (c)$, $c \mid d$. d is the greatest common divisor of $\{a_i | i \in I\}$.

Exercise 3.3.13. Let R be a Euclidean domain with associated function $\varphi : R - \{0\} \to \mathbf{N}$. If $a, b \in R$ and $b \neq 0$, here is a method for finding the greatest common divisor of a and b. By repeated use of Definition 3.8(ii) we have:

$$\begin{gathered}
a = q_0 b + r_1, \quad \text{with} \quad r_1 = 0 \quad \text{or} \quad \varphi(r_1) < \varphi(b); \\
b = q_1 r_1 + r_2, \quad \text{with} \quad r_2 = 0 \quad \text{or} \quad \varphi(r_2) < \varphi(1); \\
r_1 = q_2 r_2 + r_3, \quad \text{with} \quad r_3 = 0 \quad \text{or} \quad \varphi(r_3) < \varphi(2); \\
\vdots \\
r_k = q_{k+1} r_{k+1} + r_{k+2}, \quad \text{with} \quad r_{k+2} = 0 \quad \text{or} \quad \varphi(r_{k+2}) < \varphi(k+1); \\
\vdots
\end{gathered}$$

Let $r_0 = b$ and let n be the least integer such that $r_{n+1} = 0$ (such an n exists since the $\varphi(r_k)$ form a strictly decreasing squence of nonnegative integers). Show that r_n is the greatest common divisor a and b.

Answer. $r_n$exists since $\varphi(r_i)$ decreases. $r_n \mid a$ and $r_n \mid b$ is simple. We prove $(a)+(b)=(r_n)$. $r_n \mid a$, $r_n \mid b$ so $(a) \subset (r_n)$, $(b) \subset (r_n) \Rightarrow (a)+(b) \subset (r_n)$. We use induction to prove $(r_n) \subset (a)+(b)$: 1. For $i=1$, $a=q_0b+r \Rightarrow r_1=a-q_0b \in (a)+(b)$. 2. Assume for $i \leq m$, $(r_i) \subset (a)+(b)$, $r_{m-1}=q_mr_m+r_{m-1} \Rightarrow r_{m+1}=r_{m-1}-q_mr_m \in (r_m)+(r_{m-1}) \subset (a)+(b)$. So $(r_n) \subset (a)+(b)$. r_n is the greatest common divisor of a and b.

3.4 Rings of quotients and localization

Exercise 3.4.1. Determine the complete ring of quotients of the ring Z_n for each $n \geq 2$.

Answer. For the complete multiplicative subset S of Z_n, $S = \{\bar{x}|(x,n) = 1\}$ so the complete ring of quotient is $S^{-1}Z_n$.

Exercise 3.4.2. Let S be a multiplicative subset of a commutative ring R with identity and let T be a multiplicative subset of the ring $S^{-1}R$. Let $S_* = \{r \in R | r/s \in T \text{ for some } s \in S\}$. Then S_* is a multiplicative subset of R and there is a ring isomorphism $S_*^{-1}R \cong T^{-1}(S^{-1}R)$.

Answer. For any r_1/s_1 , $r_2/s_2 \in T$. $r_1r_2/s_1s_2 \in T$. And there exists a monomorphism $\varphi : S_* \to T$ given by $\varphi : r \mapsto r/s$ for some $s \in S$ by the definition of S_*. So $\forall r_1, r_2 \in S_*$, $\exists \varphi(r_1)\varphi(r_2) = r_1r_2/s_1s_2 \in T$, thus $r_1r_2 \in S_*$. S_* is a multiplicative subset.
Next we prove $S_*^{-1}R \cong T^{-1}(S^{-1}R)$. $\forall s \in S_*$ and $r \in R$, $sr \in S_*$ since if there exists some $s' \in S$, $s/s' \in T$ then $sr/s'r = s/s' \in T$. For any $(r/s)/(r'/s') \in T^{-1}(S^{-1}R)$ where $r \in R$ and $s \in S$, $r'/s' \in T$, we construct a map $\varphi : T^{-1}(S^{-1}R) \to S_*^{-1}R$ given by $\varphi : (r/s)/(r'/s') \mapsto rs'/sr'$. φ is well defined since $rs' \in R$ and $sr' \in S_*$. Now we check φ is an isomorphism. $\forall (r_1/s_1)/(r_1'/s_1'), (r_2/s_2)/(r_2'/s_2') \in T^{-1}(S^{-1}R)$

$$
\begin{aligned}
&\varphi((r_1/s_1)/(r_1'/s_1') + (r_2/s_2)/(r_2'/s_2')) \\
=&\varphi(((r_1/s_1)(r_2'/s_2') + (r_2/s_2)(r_1'/s_1'))/((r_1'/s_1')/(r_2'/s_{2'}))) \\
=&\varphi((r_1r_2'/s_1s_2' + r_2r_1'/s_2s_1')/(r_1'r_2'/s_1's_2')) \\
=&\varphi(((r_1r_2's_2s_1' + r_2r_1's_1s_2')/s_1s_2s_1's_2')/(r_1'r_2'/s_1's_2')) \\
=&(((r_1r_2's_2s_1' + r_2r_1's_1s_2')s_1's_2')/s_1s_2s_1's_2'r_1'r_2') \\
=&((r_1r_2's_2s_1' + r_2r_1's_1s_2')/s_1s_2r_1'r_2') \\
=&(r_1s_1')/(r_1's_1) + (r_2s_2')/(r_2's_2) \\
=&\varphi((r_1/s_1)/(r_1'/s_1/)) + \varphi((r_2/s_2)/(r_1'/s_2'))
\end{aligned}
$$

The conservation of multiplication is trivial. φ is a homomorphism and φ is obviously injective, so $\left|T^{-1}(S^{-1}R)\right| \leq \left|S^{-1}R\right|$.
Take $\tau : S_*^{-1}R \to T^{-1}(S^{-1}R)$ given by $\tau : r/s \mapsto (r/s')/(s/s')$. Similarly, τ is injective so $|S_*R| \leq \left|T^{-1}(S^{-1}R)\right|$. φ is isomorphism and $S_*^{-1}R \cong T^{-1}(S^{-1}R)$.

Exercise 3.4.3. (a) The set E of positive even integers is a multiplicative subset of $\mathbf{Z}$ such that $E^{-1}(\mathbf{Z})$ is field of rational numbers.

(b) State and prove condition(s) on a multiplicative subset of S of $\mathbf{Z}$ which insure that $S^{-1}\mathbf{Z}$ is a field of rationals.

Answer. (a) Trivial.

(b) Assume the primes $p \in \mathbf{Z}$ forms a set P. For any multiplicative subset S and $x \in S$ then $\{x^n | n \in \mathbf{Z}\} \subset S$. If $\forall p \in P$, $\exists x \in S$ such that $p \mid x$, we prove $S^{-1}\mathbf{Z}$ forms the field of rationals. For any $p/q \in \mathbf{Q}$, $q = q_1^{t_1} q_2^{t_2} \cdots q_n^{t_n}$ and for any q_i there exists $x_1 \in S$, $x_1 = a_i q_i$. Take $x = a_1^{t_1} q_1^{t_1} \cdots a_n^{t_n} q_n^{t_n}$ and $y = a_1^{t_1} \cdots a_n^{t_n} p$. Then $y/x = p/q$, $y/x \in S^{-1}\mathbf{Z}$. So $S^{-1}\mathbf{Z}$ forms the field of rationals.

For any other multiplicative subset S, assume $p \in P$ and $\forall x \in S$, $p \nmid x$ then $\forall y/x \in S^{-1}\mathbf{Z}$, $yp - x \neq 0$ so $1/p \notin S^{-1}\mathbf{Z}$, $S^{-1}\mathbf{Z}$ isn't the rational field.

Exercise 3.4.4. If $S = \{2, 4\}$ and $R = Z_6$, then $S^{-1}R$ is isomorphic to the field Z_3. Consequently, the converse of Theorem 4.3(ii) is false.

Answer. $S^{-1}Z_6 = \{1/3, 2/3, 3/3\}$ so $S^{-1}Z_6 \cong Z_3$ is a integral domain. However, Z_6 has no zero divisor.

Exercise 3.4.5. Let R be an integral domain with quotient field F. If T is an integral domain such that $R \subset T \subset F$, then F is (isomorphic to) the quotient field of T.

Answer. Consider T_i which is a PID satisfying $R \subset T_i \subset F$, T_i forms a category with the inclusion map as morphisms. T_i' is the quotient field of T_i so $R \subset T_i' \Rightarrow R \subset F \subset T_i'$ (up to isomorphic). $R \subset T_j \subset F \subset T_i'$ for all i, j thus $T_i' \subset T_j'$. Similarly $T_j' \subset T_i'$ so all the T_i' are universal under the inclusion map. Thus F is isomorphic to the quotient field of T.

Exercise 3.4.6. Let S be a multiplicative subset of an integral domain R such that $0 \notin S$. If R is a principle ideal domain, then so is $S^{-1}R$.

Answer. Actually this is true if and only if $1_R \in S$. For any ideal $J \subset S^{-1}R$, there exists ideal $I \subset R$ and $\varphi_S(I) = J$, $J = S^{-1}I = S^{-1}(a)$ for some $a \in R$. Since $1_R \in S$, $a/1_R \in S^{-1}(a)$. So $\forall s \in S$, $1_R/s$ is a unit of $S^{-1}(a)$, so $S^{-1}(a) = (a/1_R)$ is a principle ideal. Thus the multiplicative subset of R is a principle ideal domain.

Exercise 3.4.7. Let R_1 and R_2 be integral domains with quotient fields F_1 and F_2 respectively. If $f : R_1 \to R_2$ is an isomorphism, then f extends to an isomorphism $F_1 \cong F_2$.

Answer. For $f : R_1 \to R_2$, and the inclusion map $\subset R_2 \to F_2$, $\subset \circ f = R_1 \to F_2$ so there exists $\overline{\subset \circ f} : F_1 \to F_2$ which is a well defined homomorphism of rings. $\overline{\subset \circ f}|R_1 = f$, $\overline{\subset \circ f}$ is a monomorphism so $|F_1| \leq |F_2|$. Similarly, $|F_2| \leq |F_1|$ so $\overline{\subset \circ f}$ is an isomorphism and $F_1 \cong F_2$.

Exercise 3.4.8. Let R be a commutative ring with identity, I an ideal of R and $\pi : R \to R/I$ the canonical projection.

(a) If S is a multiplicative subset of R, then $\pi S = \pi(S)$ is a multiplicative subset of R/I.

(b) The mapping $\theta : S^{-1}R \to (\pi S)^{-1}(R/I)$ given by $r/s \mapsto \pi(r)/\pi(s)$ is a well-defined function.

(c) θ is a ring epimorphism with kernel $S^{-1}I$ and hence induces a ring isomorphism $S^{-1}R/S^{-1}I \cong (\pi S)^{-1}(R/I)$.

Answer. (a) For any $a, b \in S$, $\pi(a) = a + I$, $\pi(b) = b + I$, $\pi(a)\pi(b) = ab + I = \pi(ab) \in \pi S$, so πS is a multiplicative subset of R/I.

(b) If $r_1/s_1 = r_2/s_2$ then $x(r_1s_2 - r_2s_1) = 0$ for some $x \in S$.

$$\theta(r_1/s_1) = \pi(r_1)/\pi(s_1) = (r_1 + I)/(s_1 + I)$$

$$\theta(r_2/s_2) = \pi(r_2)/\pi(s_2) = (r_2 + I)/(s_2 + I)$$

$$\begin{aligned}
&(x + I)((r_1 + I)(s_2 + I) - (r_2 + I)(s_1 + I)) \\
=&(xr_1s_2 + I) - (xr_2s_1 + I) \\
=&x(r_1s_2 - r_2s_1) + I \\
=&I
\end{aligned}$$

so $\theta(r_1/s_1) = \theta(r_2/s_2)$, θ is well-defined.

(c) π is a homomorphism and so is θ. θ is obviously an epimorphism and $\forall r/s \in S^{-1}I$, $\theta(r/s) = \pi(r)/\pi(s)$. $\pi(r) = I$ so $\theta(r/s) \in (\pi S)^{-1}I$, $S^{-1}I \subset \mathrm{Ker}\theta$. For any $r/s \notin S^{-1}I$, $\theta(r/s) = (r+I)/(s+I) \neq I$, so $\mathrm{Ker}\theta \subset S^{-1}I$. $\mathrm{Ker}\theta = S^{-1}I$, $S^{-1}R/\mathrm{Ker}\theta \cong \mathrm{Im}\theta \Rightarrow S^{-1}R/S^{-1}I \cong (\pi S)^{-1}(R/I)$.

Exercise 3.4.9. Let S be a multiplicative subset of a commutative ring R with identity. If I is an ideal in R, then $S^{-1}(\mathrm{Rad}I) = \mathrm{Rad}(S^{-1}I)$.

Answer. $\mathrm{Rad}I = \{r | r^n \in I \text{ for some } n\}$. For any $r/s \in S^{-1}\mathrm{Rad}I$, $(r/s)^n = r^n/s^n \in S^{-1}I$ so $S^{-1}\mathrm{Rad}I \subset \mathrm{Rad}(S^{-1}I)$.
For any $a/b \in \mathrm{Rad}(S^{-1}I)$, $b \in S$ then $a^n b' - b^n a' = 0$ with $a' \in I$ and $b' \in S$. $(ab')^n = (b')^{n-1} b^n a' \in I$ so $a/b = ab'/bb' \in S^{-1}(\mathrm{Rad}I)$. Thus $S^{-1}(\mathrm{Rad}I) \subset \mathrm{Rad}(S^{-1}I)$. So $S^{-1}(\mathrm{Rad}I) = \mathrm{Rad}(S^{-1}I)$.

Exercise 3.4.10. Let R be an integral domain and for each maximal ideal M, consider R_M as a subring of the quotient field of R. Show that $\cap R_M = R$, where the intersection is taken over all maximal ideals M of R.

Answer. M is maximal so $1_R \in R - M$, which means $R \subset R_M$ for any M. So $R \subset \cap R_M$.
Denote R' as the quotient field of R. For any M maximal, $R_M \subset R'$. For any $x \in R' - R$, we prove there exists M maximal and $x \notin R_M$. Take $A = \{a | ax \in R\}$, A is an ideal of R. So $\exists A \subset M$ with M maximal. If $x \in R - M$, $x = r/s$, so $xs = r \in R$, $s \in I \subset M$. That's contradictory! Thus $\cap R_M \subset R$, $R = \cap R_M$.

Exercise 3.4.11. Let p be a prime in **Z**l then (p) is a prime ideal. What can be said about the relationship of Z_p and the localization $Z_{(p)}$?

Answer. Z_p can be embedded into $\mathbf{Z}_{(p)}$ since $Z_p \subset \mathbf{Z} \subset (p)_{(p)} \subset \mathbf{Z}_{(p)}$.

Exercise 3.4.12. A commutative ring with identity is local if and only if for all $r, s \in R$, $r + s = 1_R$ implies r or s is a unit.

Answer. If R is local, $r + s = 1_R \Rightarrow (r) + (s) = R$. R has unique maximal ideal M so $(r) \subset M$, $(s) \subset M$, $(r) + (s) = R \subset M$. That's contradictory! So $(r) = R$ or $(s) = R$, r or s is a unit.
Conversely, if there exist M_1, M_2 are maximal ideals. $M_1 + M_2 = R$ so $\exists r \in M$, $s \in M_2$ such that $r + s = 1_R$. WLOG assume r is unit, $R = (r) \subset M_1$, that's contradictory! So R is local.

Exercise 3.4.13. The ring R consisting of all rational numbers with denominators not divisible by some (fixed) prime p is a local ring.

Answer. Denote the set of primes in the question as P. Then (P) is a prime ideal in $\mathbf{Z}$. So $S = \mathbf{Z} = (P)$ is multiplicative subset. We prove $R = \mathbf{Z}_{(P)}$. $\forall r/s \in \mathbf{Z}_{(p)}$, $r \in \mathbf{Z}$ and $s \notin (P)$ so $r/s \in R$. Thus $\mathbf{Z}_{(P)} \subset R$. Conversely, $\forall r/s \in R$, suppose $s = p_1^{t_1} p_2^{t_2} \cdots p_n^{t_n}$, $\forall p \in P$, $p \nmid s$ so $(p_i) \not\subset$ for all $i = 1, 2, \ldots, n$. Thus $(p_i) \subset S$ so $s \in S$, $r/s \in \mathbf{Z}_{(P)}$. $\mathbf{Z}_{(P)} = R$ is a lcoal ring.

Exercise 3.4.14. If M is a maximal ideal in a commutative ring R with identity and n is a positive integer, then the ring R/M^n has a unique prime ideal and therefore is local.

Answer. Consider the homomorphism $f : R \to R/M^n$. For any prime ideal $I \subset R/M^n$, $J = f^{-1}(I)$ is a prime ideal contains M^n. $M^n \subset P \Rightarrow M \subset P$, since M is maximal, $P = M$ so the only prime ideal in R/M^n is R/M.

Exercise 3.4.15. In a commutative ring R with identity the following conditionns are equivalent: (i) R has a unique prime ideal; (ii) every nonunit is nilpotent; (iii) R has a minimal prime ideal which contains all zero divisors, and all nonunits of R are zero divisors.

Answer. We first prove a lemma:

Lemma. *For an ideal $I \subset R$,* $\mathrm{Rad}I = \bigcap\limits_{I \subset P_i} P_i$ *where P_i are prime ideals.*

Proof of the lemma: $\forall a \in \mathrm{Rad}I$, $a^n \in I$ for some n, so $\forall I \subset P_i$ with P_i prime. $a^n \in P_i \Rightarrow a \in P_i$ so $\mathrm{Rad}I \subset \bigcap\limits_{I \subset P_i} P_i$.
Conversely $\forall a \notin \mathrm{Rad}I$, we only need to find $I \subset P_i$ and $a \notin P_i$. Take $A = \{J | a^n \in J \, \forall n \in \mathbf{N}\}$. A has maximal element under $\subset$ by Zorn's lemma. Denote the maximal element as P. $\forall x, y \in R$ and $x \notin P$, $y \notin P$. Then $\exists m, n \in \mathbf{N}$, $a^n \in (x) + P$, $a^m \in (y) + P$, so $a^{m+n} \in (xy) + P \Rightarrow xy \notin P$. Thus P is prime. That's contradictory! So $\bigcap\limits_{I \in P_i} P_i \subset \mathrm{Rad}I$. The lemma has been proved.
(i)$\Rightarrow$(ii): $0 \in P$ where P is the unique prime ideal, so $P = \{a | a^n = 0 \text{ for some } n\}$. For any nonunit a, $(a) \subset M = P$ so $a \in P$, there exists $n \in \mathbf{N}$ such that $a^n = 0$.
(ii)$\Rightarrow$(i): Denote N as the ideal contains all the nilpotent elements. Take $\varphi : R \to R/N$. For any unit u, $\varphi(u)$ is also a unit. So R/N is a field, N is maximal in R. For any prime ideal P, $N \subset P$ from the lemma. Thus N is the only prime ideal.
(ii)$\Rightarrow$(iii): Denote N as the ideal contains all the nilpotent elements. All nilpotent elements are zero divisors by the definition. N is prime and minimal is the direct corollary of the lemma.
(iii)$\Rightarrow$(ii): Denote I as the minimal prime ideal and N as the ideal contains all the nilpotent elements. Then $N \subset I$. Since all then nonunits are zero divisors, we have N itself a prime ideal. So $N = I$.

Exercise 3.4.16. Every nonzero homomorphic image of a local ring is local.

Answer. Suppse L is a local ring and $\varphi : L \to R$ is a ring of rings. Then φ is an one-to-one correspondence between ideals in L and ideals in R. For the maximal ideal M in L, $\varphi(M) \subseteq R$, so $\varphi(M)$ contains all the proper ideals in R. R is a local ring.

3.5 Rings of polynomials and formal power series

Exercise 3.5.1. (a) If $\varphi : R \to S$ is a homomorphism of rings, then the map $\bar{\varphi} : R[[x]] \to S[[x]]$ given by $\bar{\varphi}(\sum a_i x^i) = \sum \varphi(a_i)x^i$ is a homomorphism of rings such that $\bar{\varphi}(R[x]) \subset S[x]$.
(b) $\bar{\varphi}$ is a monomorphism if and only if $u\varphi$ is. In this case $\bar{\varphi} : R[x] \to S[x]$ is also a monomorphism.
(c) Extend the results of (a) and (b) to the polynomial rings $R[x_1, \dots, x_n]$, $S[x_1, \dots, x_n]$.

Answer. (a) It's easy to show $\bar{\varphi}(\sum a_i x^i)\bar{\varphi}(\sum b_i x^i) = \bar{\varphi}(\sum c_i)x^i$, $c_n = \sum_{j=0}^{n} a_j b_{n-j}$ and $\bar{\varphi}(\sum a_i x^i) + \bar{\varphi}(\sum b_i x^i) = \bar{\varphi}(\sum (a_i + b_b)x^i)$. $\forall f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$, $\bar{\varphi}(f(x)) = \bar{\varphi}(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n} \varphi(a_i)x^i \in S[x]$. So $\bar{\varphi}(R[x]) \subset S[x]$.
(b) If φ is monomorphism [epimorphism], it's easy to show that $\bar{\varphi}$ is also monomorphism [epimorphism].
Conversely, if $\bar{\varphi}$ is monomorphism, take $a_i \in R[[x]]$, then $\bar{\varphi}(a_i) = \varphi(a_i)$, φ is also a monomorphism.
Similarly, φ is epimorphism if $\bar{\varphi}$ is.
(c) It's trivial to since $R[x] \subset R[[x]]$, $S[x] \subset S[[x]]$.

Exercise 3.5.2. Let $\mathrm{Mat}_n R$ be the ring of $n \times n$ matrices over a ring R. Then for each $n \geq 1$:
(a) $(\mathrm{Mat}_n R)[x] \cong \mathrm{Mat}_n R[x]$.
(b) $(\mathrm{Mat}_n R)[[x]] \cong \mathrm{Mat}_n R[[x]]$.

Answer. (a) Take $x = (p_{ij}(x)) \in \mathrm{Mat}_n R[x]$, $p_{ij}(x) = \sum_{k=0}^{n_{ij}} a_{ijk}x^k$. Take $n = \max_{0<i,j\leq n} n_{ij}$, and for those $n \geq k > n_{ij}$, take $a_{ijk} = 0$. Denote $X_k = (a_{ijk})$, $x' = \sum_{i=0}^{n} X_i x^i \in (\mathrm{Mat}_n R)[x]$. We prove $\varphi : x \mapsto x'$ is an isomorphism between rings.
For $x, x' \in \mathrm{Mat}_n R[x]$, $x = (p_{ij}(x))$, $x' = (p'_{ij}(x))$, $p_{ij}(x) = \sum_{k=0}^{n_{ij}} a_{ijk}x^k$,

$p'_{ij}(x) = \sum\limits_{k=0}^{n_{ij}} a'_{ijk}x^k$.

$$\begin{aligned}
\varphi(x+x') &= \varphi(p_{ij}(x) + p'_{ij}(x)) \\
&= \begin{pmatrix} a_{110}+a'_{110} & \cdots & \\ \vdots & \ddots & \\ & & a_{nn0}+a'_{nn0} \end{pmatrix} \\
&+ \begin{pmatrix} a_{111}+a'_{111} & \cdots & \\ \vdots & \ddots & \\ & & a_{nn1}+a'_{nn1} \end{pmatrix} x + \cdots \\
&= \varphi(x) + \varphi(x')
\end{aligned}$$

$$\varphi(xx') = \varphi((p_{ij}(x))(p'_{ij}(x))) = \varphi((\sum_{k=1}^{n} p_{ik}(x)p'_{kj}(x)))$$

$$\begin{aligned}
\sum_{k=1}^{n} p_{ik}(x)p'_{kj}(x) &= \sum_{k=1}^{n}(\sum_{m=0}^{n_{ik}} a_{ikm}x^m)(\sum_{m=0}^{n'_{kj}} a'_{kjm}x^m) \\
&= \sum_{w}\sum_{k=1}^{n}\sum_{m=1}^{w} a_{ikm}a'_{kj(w-m)}x^w
\end{aligned}$$

so

$$\begin{aligned}
\varphi(xx') &= \varphi((\sum_{w}\sum_{k=1}^{n}\sum_{m=1}^{w} a_{ikm}a'_{kj(w-m)}x^w)) \\
&= \sum_{w}(\sum_{k=1}^{n}\sum_{m=1}^{w} a_{ikm}a_{kj(w-m)})x^w \\
\varphi(x)\varphi(x') &= (\sum_{w}(a_{ijw})x^w)(\sum_{w}(a'_{ijw})x^w) \\
&= \sum_{w}(\sum_{k=1}^{n}\sum_{m=1}^{w} a_{ikm}a_{kj(w-m)})x^w
\end{aligned}$$

so $\varphi(xx') = \varphi(x)\varphi(x')$, φ is a well defined homomorphism. Ker$\varphi = 0$ so φ is a monomorphism. For any $\sum\limits_{w}(a_{ijw})x^w \in \mathrm{Mar}_n R[x]$, $\exists(\sum\limits_{w} a_{ijw}x^w) \in (\mathrm{Mat}_n R)[x]$ s.t. $\varphi(\sum\limits_{w} a_{ijw}x^w = \sum\limits_{w}(a_{ijw})x^w)$. So φ is an epimorphism.

Exercise 3.5.3. Let R be a ring and G an infinte multiplicative cyclic group with generator denoted x. Is the group ring $R(G)$ isomorphic to the polynomial ring in one indeterminate over R?

Answer. $R(G)$ is not isomorphic to $R[x]$ since there's no isomorphic image of $rx^{-1} \in R(G)$ in $R[x]$.

Exercise 3.5.4. (a) Let S be a nonempty set and let N^S be the set of all functions $\varphi : S \to \mathbf{N}$ such that $\varphi(s) \neq 0$ for at most a finite number of elements $s \in S$. Then $\mathbf{N}^S$ is a multiplicative abelian monoid with prooduct defined by

$$(\varphi\psi)(s) = \varphi(s) + \psi(s)\,(\varphi, \psi \in \mathbf{N}^S; s \in S)$$

The identity element in $\mathbf{N}^S$ is the zero function.

(b) For each $x \in S$ and $i \in \mathbf{N}$ let $x^i \in \mathbf{N}^S$ be defined by $x^i(x) = i$ and $x^i(s) = 0$ for $s \neq x$. If $\varphi \in \mathbf{N}^S$ and $x_1, \dots, x_n$ are the only elements of S such that $\varphi(x_i) \neq 0$, then in $\mathbf{N}^S$, $\varphi = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$, where $i_j = \varphi(x_j)$.

(c) If R is a ring with identity let $R[S]$ be the set of all functions $f : \mathbf{N}^S \to R$ such that $f(\varphi) \neq 0$ for at most a finite number of $\varphi \in \mathbf{N}^S$. Then $R[S]$ is a ring with identity, where addition and multiplication are defined as follows:

$$(f + g)(\varphi) = f(\varphi) + g(\varphi)\,(f, g \in R[S]; \varphi \in \mathbf{N}^S)$$

$$(fg)(\varphi) = \sum f(\theta)g(\zeta)\,(f, g \in R[S]; \theta, \zeta, \varphi \in \mathbf{N}^S)$$

where teh sum is over all pairs (θ, ζ) such that $\theta\zeta = \varphi$. $R[S]$ is called the ring of polynomials in S over R.

(d) For each $\varphi = x_1^{i_1} \cdots x_n^{i_n} \in \mathbf{N}^S$ and each $r \in R$ we denote by $rx_1^{i_1} \cdots x_n^{i_n}$ the function $\mathbf{N}^S \to R$ which is r at φ and 0 elsewhere. Then every nonzero element f of $R[S]$ can be written in the form $\int = \sum_{i=0}^{m} r_i x_1^{k_{i1}} x_2^{k_{i2}} \cdots x_n^{k_{in}}$ with the $r_i \in R$, $x_i \in S$ and $k_{ij} \in \mathbf{N}$ all uniquely determined.

(e) If S is finite of cardinality n, then $R[S] \cong R[x_1, \dots, x_n]$.

(f) State and prove an analogue of Theorem 5.5 for $R[S]$.

Answer. (a) $\varphi\psi = \varphi + \psi : S \to \mathbf{N}$ so $\varphi\psi \in \mathbf{N}^S$. For any $\varphi \in \mathbf{N}^S$, $\varphi 0 = 0\varphi = \varphi + 0 = 0 + \varphi = \varphi$. So $\mathbf{N}^S$ is a monoid.

(b) For any $\varphi \in \mathbf{N}^S$, $x_1, x_2, \ldots, x_n$ are the only element s.t. $\varphi(x_i) \neq 0$. We prove it has the form $\varphi = x_1^{i_1} \cdots x_n^{i_n}$. Suppose $\varphi(x_j) = i_j$. Take $\varphi_1 = \varphi - x_n$ then $x_1, x_2, \ldots, x_{n-1}$ are the only element s.t. $\varphi_1(x_i) \neq 0$. Continue this step, we can have $\varphi_{n-1} = x_i^{i_1}$ and $\varphi_n = 0$. Thus $\varphi = x_1^{i_1} + \cdots + x_n^{i_n} = x_1^{i_1} \cdots x_n^{i_n}$.

(c) $f + g(\varphi) = f(\varphi) + g(\varphi)$, $f + g : \mathbf{N}^S \to R$ and for at most finite $\varphi \in \mathbf{N}^S$, $f(\varphi) \neq 0$, so $f + g \in \mathbf{N}^S$.
$(fg)(\varphi) = \sum f(\theta)g(\zeta)$, so $fg\mathbf{N}^S \to R$. Suppose $\mathbf{N}_f^S$, $\mathbf{N}_g^S$ are the set such that $f(\mathbf{N}_f^S) = 0$, $g(\mathbf{N}_g^S) = 0$. Take $\mathbf{N}_{fg}^S = \mathbf{N}_f^S \cup \mathbf{N}_g^S$, then $\mathbf{N}_{fg}^S$ is also finite. For all $\theta, \zeta \notin \mathbf{N}_{fg}^S$, $(fg)(\varphi) = 0$. So $fg \in R[S]$.
Take the 0 element of f in $R[S]$ as $0(\varphi) = 0_R$ for any $\varphi \in \mathbf{N}^S$ and the inverse element of f in $R[S]$ as $f^{-1}(\varphi) = -f(\varphi)$ for any $\varphi \in \mathbf{N}^S$. Thus $R[S]$ is a ring.

(d) The proof is similar to (b).

(e) First we prove $\mathbf{N}^S \cong \mathbf{N}^n$. Assume $S = \{x_1, x_2, \ldots, x_n\}$. We can write every $\varphi \in \mathbf{N}^S$ into $x_{n_1}^{i_{n_1}} \cdots x_{n_m}^{i_{n_m}}$ and extend it to $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ by taking $i_j = 0$ if $j \neq n_1, n_2, \ldots, n_m$. Then the map $\sigma : \mathbf{N}^S \to \mathbf{N}^n$ given by $\sigma : x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \mapsto (i_1, i_2, \ldots, i_n)$ is a well defined isomorphism so $\mathbf{N}^S \cong \mathbf{N}^n$.
For any $f \in R[x_1, x_2, \ldots, x_n]$. f can be expressed as $f = \sum a_{k_1 k_2 \cdots k_n} x_1^{k_1} \cdots x_n^{k_n}$. Take $\tau : R[x_1, x_2, \ldots, x_n] \to R[S]$ given by $\tau : f \mapsto \sum a_{k_1 \cdots k_n} \sigma^{-1}(k_1, k_2, \ldots, k_n)$. It's easy to show that τ is an isomorphism.

(f) Let R and X be commutative rings with identity and $\varphi : R \to X$ a homomorphism of rings such that $\varphi(1_R) = 1_X$. If $x_1, x_2, \ldots, x_n \in S$, there is a unique homomorphism of rings $\bar{\varphi} : R[S] \to X$ such that $\bar{\varphi}|R = \varphi$, $|S| = n$ and $\varphi(s_i) = x_i$ for $i = 1, 2, \ldots, n$. The proof is quite simple since there exists $\tau : R[x_1, \ldots, x_n] \to R[S]$ an isomorphism.

Exercise 3.5.5. Let R and S be rings with identity, $\varphi : R \to S$ a homomorphism of rings with $\varphi(1_R) = 1_S$, and $s_1, s_2, \ldots, s_n \in S$ such that $s_i s_j = s_j s_i$ for all i, j and $\varphi(r)s_i = s_i\varphi(r)$ for all $r \in R$ and all i. Then there is a unique homomorphism $\bar{\varphi} : R[x_1, \ldots, x_n] \to S$ such that $\bar{\varphi}|R = \varphi$ and $\varphi(\bar{x}_i) = s_i$. This property completely determines $R[x_1, \ldots, x_n]$ up to isomorphism.

Answer. $S' = \langle \varphi(R) \cup \{s_1, s_2, \ldots, s_n\} \rangle$ is a commutative ring. So applying Theorem 5.5. on S', we can get the unique homomorphism $\bar{\varphi}$:

$R[xt_1, x_2, \ldots, x_n] \to S'$, so $\bar{\varphi} : R[x_1, \ldots, x_n] \to S$ is also a homomorphism. The proof of the second statement is exactly the same as Theorem 5.5.

Exercise 3.5.6. (a) If R is the ring of all 2×2 matrices over $\mathbf{Z}$, then for any $A \in R$,

$$(x + A)(x - A) = x^2 - A^2 \in R[x]$$

(b) There exist $C, A \in R$ such that $(C + A)(C - A) \neq C^2 - A^2$. Therefore, Corollary 5.6 is false if the rings involved are not commutative.

Answer. (a) For any $A \in R$, $x + A$, $x - A$, $(x+A)(x-A)$, $x^2 - A^2 \in R[x]$. $(x+A)(x-A) = x^2 + Ax - xA + A^2$. Since $Ax = xA$, $(x+A)(A+x) = x^2 - A^2$.

(b) Take $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, then $CA = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}$, $AC = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$. So $AC \neq CA$, $(C + A)(C - A) \neq C^2 - A^2$. Corollary 5.6 is false in R.

Exercise 3.5.7. If R is a commutative ring with identity and $f = a_n x^n + \cdots + a_0$ is a zero divisor in $R[x]$, then there exists a nonzero $b \in R$ such that $ba_n = ba_{n-1} = \cdots = ba_0 = 0$.

Answer. Assume $g = b_m x^m + \cdots + b_0$ and $fg = 0$, $fg = a_n b_m x^{m+n} + (a_n b_{m-1} + a_{n-1} b_m) x^{m+n-1} + \cdots + a_0 b_0 = 0$. So for any $k = 0, 1, \ldots, m+n$, $\sum\limits_{i+j=k} a_i b_j = 0$. Take $b_1' = b_n$, and then $a_n b_1' = 0$, $a_n b_{m-1} + a_{n-1} b_m = 0 \Rightarrow a_n b_{m-1} b_1' + a_{n-1} b_m b_1' = 0$. Take $b_2 = b_m b_1'$, we have $a_n b_2' = a_{n-1} b_2' = 0$. $a_n b_{m-2} + a_{n-1} b_{m-1} + a_{n-2} b_m = 0$, take $b_3' = b_m b_1'$, we have $a_m b_3' = a_{n-1} b_3' = a_n b_3' = 0$. Continue this step and we have $a_n b_n' = a_{n-1} b_n' = \cdots = a_0 b_n' = 0$. That's the b we want.

Exercise 3.5.8. (a) The polynomial $x + 1$ is a unit in the power series ring $\mathbf{Z}[[x]]$, but is not a unit in $\mathbf{Z}[x]$.

(b) $x^2 + 3x + 2$ is irreducible in $\mathbf{Z}[[x]]$, but not in $\mathbf{Z}[x]$.

Answer. (a) Take $(x+1)^{-1} = 1 - x + x^2 - x^3 + \cdots \in \mathbf{Z}[[x]]$. $(1 - x + x^2 - x^3 + \cdots)(x+1) = (x+1)(1 - x + x^2 - x^3 + \cdots) = (1 - x + x^2 - x^3 + \cdots) + (x - x^2 + x^3 - \cdots) = 1$. So $x+1$ is a unit in $\mathbf{Z}[[x]]$. For any $f = \sum_{i=0}^{n} a_i x^i \in \mathbf{Z}[x]$, $(x+1)f = a_n x^{n+1} + \sum_{i=1}^{n}(a_i + a_{i-1})x^i + a_0$, $a_n \neq 0$ so $(x+1)f \neq 1$. $x+1$ is not a unit.

(b) $x^2+3x+2 = (x+2)(x+1)$ and $x+2, x+1 \in \mathbf{Z}[x]$, so x^2+3x+2 is not irreducible in $\mathbf{Z}[x]$. x^2+3x+2 itself is a unit in $\mathbf{Z}[x]$ so if $x^2+3x+2 = ab$, a,b must be units. Thus x^2+3x+2 is irreducible in $\mathbf{Z}[[x]]$.

Exercise 3.5.9. If F is a field, then (x) is a maximal ideal in $F[x]$, but it is not the only maximal ideal.

Answer. Suppose not. $(x) \subset I \subset F[x]$ with $I \neq F[x]$. (x) contains all polynomials which have zero constant term. For any $p(x) = \sum_{i=0}^{n} a_i x^i \in I$, $a_0 \neq 0$, $p(x) \notin (x)$. There exists $q(x) = \sum_{i=0}^{n} a_i' x_i$ with $a_i' = a_i$ for $i = 1, 2, \ldots, n$ and $a_0 = 0$, $q(x) \in (x) \subset I$. Thus $a_0 = p(x) - q(x) \in I$, a_0 is a unit so $I = F$. That's contradictory! (x) is a maximal ideal.
Consider $(x+1) \subset F[x]$. $F[x]$ is a UFD since F is. For any $f \in (x+1)$, $f = (x+1)g$. For any $h \in F[x] \backslash (x+1)$, $h = (x+1)k + r$, where $\deg r < \deg(x+1) = 1$. So r is a unit in $F[x]$, which means $(h) + (x+1) = F[x]$. $(x+1)$ is maximal in $F[x]$.

Exercise 3.5.10. (a) If F is a field then every nonzero element of $F[[x]]$ is of the form $x^k u$ with $u \in F[[x]]$ a unit.

(b) $F[[x]]$ is a principle ideal domain whose only ideals are 0, $F[[x]] = (1_F) = (x^0)$ and (x^k) for each $k \geq 1$.

Answer. (a) For any nonzero element f in $F[[x]]$, $f = (a_0, a_1, \ldots)$, we can find the minimal k such that $a_k \neq 0$. $f = \sum_{i=0}^{\infty} a_i x^i = x^k g$, $g = \sum_{i=0}^{\infty} a_{i+k} x^i$ which has nonzero constant term thus a unit. So $f = x^k g$.

(b) For any ideal $I \subset F[[x]]$ and $a \in I$, $a = x^k u$, u a unit, we construct $\varphi : I \to \mathbf{N}$ given by $\varphi(a) = k$, $\varphi(I) \subset \mathbf{N}$, take $a \in I$ minimize $\varphi(a)$.

Assume $a = x^k u$, then $(a) = (x^k) \subset I$. For any $a' = x^{k'} u' \in I$, $k' > k$, $a' = x^k(x^{k'-k}u') \in (x^k)$. So $I \subset (x^k)$. This also shows that the only ideals are (x^k) for $k \in \mathbf{N}$.

Exercise 3.5.11. Let $\mathcal{C}$ be the category with objects all commutative rings with identity and morphisms all ring homomorphism $f : R \to S$ such that $f(1_R) = 1_S$. Then the polynomial ring $\mathbf{Z}[x_1, \dots, x_n]$ is a free object on the set $\{x_1, \dots, x_n\}$ in the category $\mathcal{C}$.

Answer. Denote $X = \{x_1, x_2, \dots, x_n\}$. For any object R in $\mathcal{C}$, there exists a map $f : \mathbf{Z} \to R$ given by $f : n \mapsto n \cdot 1_R$ is a homomorphism of rings. If there exist $i : X \to R$ given by $i(x_i) = r_i \in R$. Applying Theorem 5.5 there exists $\bar{f} : \mathbf{Z}[x_1, x_2, \dots, x_n] \to R$ and $\bar{f}|\mathbf{Z} = f$, $\bar{f}(x_i) = r_i$ so $\bar{f}i = f$. Thus $\mathbf{Z}[x_1, x_2, \dots, x_n]$ is free over X.