Chapter 1

Groups

1.1 Semigroups, monoids and groups

Exercise 1.1.1. Give examples other than those in the text of semigroups and monoids that are not groups.

Exercise 1.1.2. Let G be a group (written additively), S a nonempty set, and $M(S, G)$ the set of all functions $f : S \to G$. Define addition in $M(S, G)$ as follows: $(f + g) : S \to G$ is given by $s \to f(s) + g(s) \in G$. Prove that $M(S, G)$ is a group, which is abelian if G is.

Exercise 1.1.3. Is it true that a semigroup which has a left identity element and in which every element has a right inverse (see Proposition 1.3) is a group?

Exercise 1.1.4. Write ou a multiplication table for the group D_4^*.

Exercise 1.1.5. Prove that the symmetric group on n letters, S_n, has ordrer $n!$.

Exercise 1.1.6. Write out an addition table for $Z_2 \oplus Z_2$. $Z_2 \oplus Z_2$ is called the Klein four group.

Exercise 1.1.7. If p is prime, then the nonzero elements of Z_p form a group of order $p - 1$ under multiplication. [Hint: $\bar{a} \neq 0 \Rightarrow (a, p) = 1$; use Introduction, Theorem 6.5.] Show that this statement is false if p is not prime.

Exercise 1.1.8. 1. The relation given by $a\ b \Leftrightarrow a - b \in \mathbf{Z}$ is a congruence relation on the additive group $\mathbf{Q}$ [see Theorem 1.5].

2. The set $\mathbf{Q}/\mathbf{Z}$ of equivalence classes is an infinite abelian group.

Exercise 1.1.9. Let p be a fixed prime. Let R_p be the set of all those rational numbers whose de nominator is relatively prime to p. Let R^p be the set of rationals whose de nominator is a power of $p(p^i, i > 0)$. Prove that both R_p and R^p are abelian groups under ordinary addition of rationals.

Exercise 1.1.10. Let p be a prime and let $Z(p^\infty)$ be the following subset of the group $\mathbf{Q}/\mathbf{Z}$:

$$Z(p^\infty) = \{\overline{a/b} \in \mathbf{Q}/\mathbf{Z} | a, b \in \mathbf{Z} \text{ and } b = p^i \text{ for some } i \geq 0\}$$

Show that $Z(p^\infty)$ is an infinite group under the addition operation of $\mathbf{Q}/\mathbf{Z}$.

Exercise 1.1.11. The following conditions on a group G are equivalent:

1. G is abelian,;
2. $(ab)^2 = a^2b^2$ for all $a, b \in G$;
3. $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$;
4. $(ab)^n = a^n b^n$ for all $n \in \mathbf{Z}$ and all $a, b \in G$;
5. $(ab)^n = a^n b^n$ for three consecutive integers n and all $a, b \in G$. Show that $(v) \Rightarrow (i)$ is false if ‘three’ is replaced by ‘two’.

Exercise 1.1.12. If G is a group, $a, b \in G$ and $bab^{-1} = a^r$ for some $r \in \mathbf{N}$, then $b^j ab^{-j} = a^{r^j}$ for all $j \in \mathbf{n}$.

Exercise 1.1.13. If $a^2 = e$ for all elements a of a group G, then G is abelian.

Exercise 1.1.14. If G is a finite group of even order, then G contains an element $a \neq e$ such that $a^2 = e$.

Exercise 1.1.15. Let G be a nonempty finite set with an associative binary operation such that for all $a, b, c \in G$, $ab = ac \Rightarrow b = c$ and $ba = ca \Rightarrow b = c$. Then G is a group. Show that this conclusion may be false if G is finite.

Exercise 1.1.16. Let $a_1, a_2, \dots$ be a sequence of elements in a semigroup G. Then there exists a unique function $\Psi : \mathbf{N}^* \to G$ such that $\Psi(1) = a_1, \Psi(2) = a_1 a_2, \Psi(3) = (a_1 a_2) a_3$ and for $n \geq 1$, $\Psi(n+1) = (\Psi(n)) a_{n+1}$. Note that $\Psi(n)$ is precisely the standard n product $\prod_{i=1}^{n} a_i$.

1.2 Homomorphisms and subgroups

Exercise 1.2.1. If $f : G \to H$ is a homomorphism of groups, then $f(e_G) = e_H$ and $f(a^{-1}) = f(a)^{-1}$ for all $a \in G$. Show by example that the first conclusion may be false if G, H are monoids that are not groups.