Boundedness of μ-Semistable Sheaves

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Abstract

In this expository report we follow ideas in [8] and [9] to prove the boundedness of torsion-free semistable sheaves of fixed Hilbert polynomial in arbitrary characteristic case.

Contents

1 Introduction

The existence of moduli space is one of the oldest and hottest topics in algebraic geometry. We are interested in whether there exists a space that "parametrizes" desired geometric objects and, if it exists, whether it's proper or not. The moduli space $\mathfrak{M}$ that parametrizes coherent sheaves over algebraic varieties X encodes a lot of geometric information about X. However, such $\mathfrak{M}$ does not always exist. Sometimes the family we want to parametrize is too "big" to make $\mathfrak{M}$ into a scheme, so we may focus on a family $\mathfrak{F}$ with specific property. In our case, we want to know some information about the moduli space of torsion-free semistable sheaves with fixed Hilbert polynomial. To construct such a moduli space, the first and the most fundamental problem is the boundedness of the family $\mathfrak{F}$, i.e., the moduli space, if it exists, whether it is of finite type over the base field.

We will establish the bounded of torsion-free semistable sheaves with an algebraically closed base field with characteristic zero following steps in [8]. We will begin with an introduction of μ-semistable sheaves and a complete proof of Kleiman's criterion, which is an important criterion for boundedness. The second result is the Grauert-Mülich Theorem. It describes the behavior of semistable sheaf $\mathcal{F}$ under restriction on general $\mathcal{F}$-regular sequences. Then, we will discuss semistability of tensor product of sheaves and use it to prove the Le Potier-Simpson Theorem. We will follow [9] and [3] to give the Bogomolov theorems in positive characteristic situation. This result also leads to an estimation of strongly semistable sheaves restricted on a hypersurface, which gives out the boundedness in positive characteristic cases.

2 Preliminaries

In this section, we set up the preliminaries for the semistable sheaves and boundedness of sheaves. For the part of semistabilities, we refer [8] for details and proofs. We will provide a helpful criterion Theorem 17.

Let X be a projective variety over some algebraically closed field k. Fix an ample line sheaf $\mathcal{O}_X(1)$ and the corresponding ample divisor class H. For any coherent sheaf $\mathcal{F}$, the dimension of $\mathcal{F}$ is the dimension of $\operatorname{Supp}\mathcal{F}$, which is exactly the degree of Hilbert polynomial $P(\mathcal{F})(m)$ with respect to H. $P(\mathcal{F})(m)$ can be expressed into the form $\sum_{i=0}^{\dim\mathcal{F}} \alpha_i(\mathcal{F})\frac{m^i}{i!}$ with rational coefficients.

Definition 1. Let $\mathcal{F}$ be a coherent sheaf of dimension $n = \dim(X)$, the *degree* of $\mathcal{F}$ is defined by

$$\deg(\mathcal{F}) := \alpha_{n-1}(\mathcal{F}) - \operatorname{rk}(\mathcal{F}) \cdot \alpha_{n-1}(\mathcal{O}_X)$$

and its *slope* by

$$\mu(\mathcal{F}) := \frac{\deg(\mathcal{F})}{\operatorname{rk}(\mathcal{F})},$$

and the *slope of Hilbert polynomial* by

$$\hat{\mu}(\mathcal{F}) := \frac{\alpha_{n-1}(\mathcal{F})}{\alpha_n(\mathcal{F})}.$$

Clearly, the relation of $\mu(\mathcal{F})$ and $\hat{\mu(\mathcal{F})}$ can be given by $\mu(\mathcal{F}) = \alpha_n(\mathcal{O}_X)\hat{\mu(\mathcal{F})} - \alpha_{n-1}(\mathcal{O}_X)$. For smooth X, Hirzebruch-Riemann-Roch Theorem shows that $\deg\mathcal{F} = c_1(\mathcal{F}) \cdot H^{n-1}$, which is an integer.

Definition 2. Let $\mathcal{F}$ be a coherent sheaf of dimension $n = \dim(X)$. $\mathcal{F}$ is called *μ-semistable* (or just *semistable*) if any subsheaf $\mathcal{E}$ has dimension less than n, then $\mathcal{E}$ has dimension less than $n-1$, and for all subsheaf $\mathcal{G} \subset \mathcal{F}$ with $0 < \operatorname{rk}\mathcal{G} < \operatorname{rk}\mathcal{F}$, we have $\mu(\mathcal{G}) \leq \mu(\mathcal{F})$. $\mathcal{F}$ is called μ-stable if the above inequality is strict.

For torsion-free sheaf $\mathcal{F}$, the definition is equivalent to say: $\mathcal{F}$ is (semi)stable if for all subsheaf $\mathcal{G}$ of $\mathcal{F}$ with $0 < \mathrm{rk}(\mathcal{G}) < \mathrm{rk}(\mathcal{F})$, $\mu(\mathcal{G}) < \mu(\mathcal{F})$ (resp. $\mu(\mathcal{G}) \leq \mu(\mathcal{F})$). We will mainly focus on torsion-free semistable sheaves. An equivalent definition can be stated in terms of quotient sheaves:

There's another stability called "Gieseker stability". The stability involves pure sheaf:

Definition 3. A coherent sheaf $\mathcal{F}$ is *pure* of dimension n if $\dim \mathcal{F} = n$ and all nontrivial coherent subsheaf $\mathcal{E}$ of $\mathcal{F}$ has dimension n.

Clearly, a sheaf $\mathcal{F}$ is pure of dimension $\dim X$ is equivalent to $\mathcal{F}$ is torsion-free. Gieseker stability encodes more information than μ-stability. The coherent sheaf $\mathcal{F}$ is (semi)stable if $\mathcal{F}$ is pure and for all proper subsheaf $\mathcal{E}$, on has $\frac{P(\mathcal{E})(m)}{\alpha_{\dim \mathcal{E}}(\mathcal{E})}(\leq)\frac{P(\mathcal{F})(m)}{\alpha_{\dim \mathcal{F}}(\mathcal{F})}$.

Lemma 4. *Let $\mathcal{F}$ be a coherent sheaf. Then $\mathcal{F}$ is semistable if and only if for all quotient sheaves $\mathcal{E}$ of $\mathcal{F}$, $\mu(\mathcal{E}) \geq \mu(\mathcal{F})$.*

Proof. See [8] section 1.2. □

The following lemma is useful in constructing Harder-Narasimhan filtration:

Lemma 5. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be semistable sheaves with $\mu(\mathcal{F}_1) > \mu(\mathcal{F}_2)$, then* $\mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2) = 0$.

Proof. Loc. cit. □

The following are some basic facts about semistability.

Remark 6.
1. A line sheaf $\mathcal{L}$ is stable.
2. Let $\mathcal{F}$ be a semistable sheaf and $\mathcal{L}$ a line sheaf, then $\mathcal{F} \otimes \mathcal{L}$ is semistable.
3. Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of sheaves. Then
   (a) If $\mathcal{F}$ is semistable and either $\mu(\mathcal{F}) = \mu(\mathcal{F}')$ or $\mu(\mathcal{F}) = \mu(\mathcal{F}'')$, then $\mathcal{F}'$ and $\mathcal{F}''$ are semistable.
   (b) If $\mathcal{F}'$ and $\mathcal{F}''$ are semistable and $\mu(\mathcal{F}') = \mu(\mathcal{F}'')$, then $\mathcal{F}$ is semistable.

A torsion-free sheaf $\mathcal{F}$ may not be semistable, but $\mathcal{F}$ always admits a Harder-Narasimhan filtration whose factors are semistable. Harder-Narasimhan filtration is a very important tool in the study of semistable sheaves.

Definition 7. Let $\mathcal{F}$ be a coherent sheaf on X. A *Harder-Narasimhan filtration* is an increasing filtration

$$0 = HN_0(\mathcal{F}) \subset HN_1(\mathcal{F}) \subset \cdots \subset HN_l(\mathcal{F}) = \mathcal{F},$$

such that the factors $gr_i^{HN} = HN_i(\mathcal{F})/HN_{i-1}(\mathcal{F})$ for $i = 1, \ldots, l$, are semistable sheaves with slope μ_i, satisfying

$$\mu_{max} := \mu_1 > \mu_2 > \cdots > \mu_l =: \mu_{min}.$$

We call gr_1^{HN} the maximal destabilizing sheaf and gr_l^{HN} the minimal destabilizing quotient. One can generalize Lemma 5 to torsion-free sheaves using the notation above:

Lemma 8. *Let $\mathcal{F}$ and $\mathcal{G}$ be torsion-free sheaves and $\mu_{min}(\mathcal{F}) > \mu_{max}(\mathcal{G})$, then* $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$.

Theorem 9. *Every torsion-free sheaf $\mathcal{F}$ has a unique Harder-Narasimhan filtration. Moreover, all factors of the Harder-Narasimhan filtration is torsion-free.*

Proof. The idea is to find a subsheaf $\mathcal{E}$ of $\mathcal{F}$ such that for all subsheaf $\mathcal{G} \subset \mathcal{F}$, one has $\mu(\mathcal{E}) \geq \mu(\mathcal{G})$, and moreover, $\mu(\mathcal{E}) = \mu(\mathcal{G})$ only if $\mathcal{G} \subset \mathcal{E}$. Then we have $\mathcal{E}$ unique and semistable. For more details, see [8] section 1.3. □

Harder-Narasimhan filtration can be done on a family of sheaves parametrized on some integral scheme. For the proof we again refer to [8] section 2.3. We explain some notation that will be used in the articles here:

For $f : X \to S$ a morphism of finite type noetherian schemes an $g : T \to S$, the notation X_T will be used for the fibre product $T \times_S X$, and $g_X : X_T \to X$, $f_T : X_T \to T$ are the natural projections. For $s \in S$ and coherent sheaf $\mathcal{F}$ on X, X_s denotes the fibre $f^{-1}(s) = \operatorname{Spec} k(s) \times_S X$ and $\mathcal{F}_s = \mathcal{F}|_{X_s}$.

Theorem 10. *Let S be a finite type integral scheme over k. Let $f : X \to S$ bea projective morphism and $\mathcal{O}_X(1)$ be an f-ample line sheaf on X. Let $\mathcal{F}$ be a flat family of coherent sheaves on the fibre of closed point on S. There is a projective birational morphism $g : T \to S$ of integral k-scheme and a filtration*

$$0 = HN_0(\mathcal{F}) \subset HN_1(\mathcal{F}) \subset \cdots \subset HN_l(\mathcal{F}) = g_X^* \mathcal{F}$$

such that the following holds:

1. *The factors $HN_i(\mathcal{F})/HN_{i-1}(\mathcal{F})$ are T-flat for all $i = 1, \ldots, l$;*
2. *There is a dense open subscheme $U \subset T$ such that $HN_i(\mathcal{F})_t = g_X^* HN_i(\mathcal{F}_{g(t)})$ for all $t \in U$. Here $HN_i(\mathcal{F}_{(g(t))})$ is the Harder-Narasimhan filtration of $\mathcal{F}_{g(t)}$.*

Remark 11.
1. $\mu_{min}(\mathcal{F} \oplus \mathcal{E}) = \min\{\mu_{min}(\mathcal{F}), \mu_{min}(\mathcal{E})\}$;
2. $\mu_{max}(\mathcal{F} \oplus \mathcal{E}) = \max\{\mu_{max}(\mathcal{F}), \mu_{max}(\mathcal{E})\}$.
3. $\mu(\mathcal{F} \otimes \mathcal{E}) = \mu(\mathcal{F}) + \mu(\mathcal{E})$.
4. $\mu_{min}(\mathcal{F} \otimes \mathcal{E}) \leq \mu_{min}(\mathcal{F}) + \mu_{min}(\mathcal{E})$, $\mu_{max}(\mathcal{F} \otimes \mathcal{E}) \geq \mu_{max}(\mathcal{F}) + \mu_{max}(\mathcal{E})$.
5. For exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$, $\mu_{min}(F) \geq \mu_{min}(\mathcal{F}'')$ and $\mu_{max}(\mathcal{F}) \geq \mu_{max}(\mathcal{F}')$.

To construct a moduli space of sheaves, we need to make sure that our family of sheaves is not too big to parametrize. We will prove this is true for semistable sheaves on k variety with $\operatorname{char} k = 0$ in this report. We introduce the idea of boundedness and several criteria of boundedness next. The notation of Castelnuovo-Mumford regularity is needed.

Definition 12. Let $m \in \mathbb{Z}$ and $\mathcal{F}$ coherent sheaf on X. $\mathcal{F}$ is *m-regular* if $H^i(X, \mathcal{F}(m-i)) = 0$ for all $i > 0$.

Lemma 13. *If $\mathcal{F}$ is m-regular, then the following facts hold:*

1. *$\mathcal{F}$ is m'-regular for all $m' \geq m$.*

2. *$\mathcal{F}(m)$ is generated by global sections.*
3. *The natural evaluation homomorphism $H^0(X, \mathcal{F}(m)) \otimes H^0(X, \mathcal{O}(n)) \to H^0(X, \mathcal{F}(m+n))$ are surjective for all $n \geq 0$.*

Proof. Follow the idea in [13]. Without losing of generality, we can assume $X = \mathbb{P}^d$. k is algebraically closed so we can find a hyperplane section H which does not contain any of the associated points of $\mathcal{F}$. Then the sequence

$$0 \to \mathcal{F}(n-1) \to \mathcal{F}(n) \to \mathcal{F}_H(n) \to 0$$

is exact.

For 1, the long exact sequence gives $H^i(X, \mathcal{F}(n-i-1)) \to H^i(X, \mathcal{F}(n-i)) \to H^i(H, \mathcal{F}_H(n-i))$. Proceeding a induction on n, we may assume $H^i(X, \mathcal{F}(n-i-1)) = 0$. Similarly, by induction on the dimension d, we may assume $H^i(H, \mathcal{F}_H(n)) = 0$. So $H^i(X, \mathcal{F}(n-i)) = 0$. This shows $\mathcal{F}$ is m'-regular for $m' \geq m$.

For 3, we only prove the case $n = 1$; the general cases are similar. We use induction on d. Consider the commutative diagram

$$\begin{tikzcd}
 & H^0(X, \mathcal{F}(m)) \otimes H^0(X, \mathcal{O}_X(1)) \arrow[r, "\sigma"] \arrow[d, "\mu"] & H^0(H, \mathcal{F}_H(m)) \arrow[d, "\tau"] \\
H^0(X, \mathcal{F}(m)) \arrow[r, "\alpha"] & H^0(X, \mathcal{F}(m+1)) \arrow[r, "\nu"] & H^0(H, \mathcal{F}_H(n+1))
\end{tikzcd}$$

The top morphism is the tensor product of two surjective morphisms so it's surjective. By inductive hypothesis, τ is also surjective, so $\nu \circ \mu$ is surjective and $H^0(X, \mathcal{F}(m+1)) = \operatorname{im} \mu + \ker \nu$. The bottom row is exact, we also have $H^0(X, \mathcal{F}(m+1)) = \operatorname{im} \mu + \operatorname{im} \alpha$. Note that α is given by tensoring the local section defining H, so $\operatorname{im} \alpha \subset \operatorname{im} \mu$. So $H^0(X, \mathcal{F}(m+1)) = \operatorname{im} \mu$.

For 2, take sufficiently large n, $\mathcal{F}(n)$ is generated by global sections. Note $H^0(X, \mathcal{F}(m)) \otimes H^0(X, \mathcal{O}(n-m)) \to H^0(X, \mathcal{F}(n))$ is surjective, $\mathcal{F}(n)$ is generated by $H^0(X, \mathcal{F}(m)) \otimes H^0(X, \mathcal{O}(n-m))$. At any point $p \in X$, the local case of above surjection shows that $\mathcal{F}(m)_p = \mathcal{F}(n)_p$ is generated by $H^0(X, \mathcal{F}(m))$. So $\mathcal{F}(m)$ is generated by $H^0(X, \mathcal{F}(m))$. $\square$

Thanks to 1 in Lemma 13, we can define the regularity of a coherent sheaf $\mathcal{F}$ to be $\operatorname{reg}(\mathcal{F}) := \inf\{m \in \mathbb{Z} | \mathcal{F} \text{ is } m\text{-regular}\}$. Now we give the definition of boundedness for a family of sheaves.

Definition 14. A family $\mathfrak{F}$ of isomorphism class of coherent sheaves on X is *bounded* if there is a k-scheme S of finite type and a coherent sheaf $\mathcal{F}$ on $X \times S$, such that $\mathfrak{F}$ is the subset of $\{\mathcal{F}_s | s \text{ is closed point in } S\}$.

Lemma 15. *The following property of a family of sheaves $\{\mathcal{F}_i\}_{i \in I}$ are equivalent:*

1. *The family is bounded.*
2. *The set of Hilbert polynomial $\{P(\mathcal{F}_i)(m)\}_{i \in I}$ is finite and there is a uniform bound $\operatorname{reg}(\mathcal{F}_i) \leq \rho$ for all $i \in I$.*

3. *The set of Hilbert polynomial $\{P(\mathcal{F}_i)(m)\}_{i\in I}$ is finite and there is a coherent sheaf $\mathcal{F}$ such that all $\mathcal{F}_i$ admit a surjection $\mathcal{F} \to \mathcal{F}_i$ for all $i \in I$.*

Proof. 1⇒2: The finiteness of Hilbert polynomial is from the flatten stratification lemma (c.f. [8] section 2.1). For the regularity part, note that S is quasicompact and we may reduce the problem to the case that S is affine. There is a $m > 0$ such that $H^i(X \times S, \mathcal{F}(n)) = 0$ for all $i > 0$ and $n > m$. On the fibre $H^i(X, \mathcal{F}|_{\operatorname{Spec} k(s)\times X}(m + d - i)) = H^i(X \times S, \mathcal{F}) \times k(s) = 0$, here $d = \dim X$. Thus $\operatorname{reg}(\mathcal{F}_i) \leq m + d$.

2⇒3: Lemma 13 shows $\mathcal{F}_i(\rho)$ are generated by global sections, so there's surjections $\mathcal{O}(-\rho)^m \to \mathcal{F}_i$ with $m \geq \max\{P(\mathcal{F}_i)(\rho)\}$. We will need the finiteness of the set $\{P(\mathcal{F}_i)(m)\}$ here.

3⇒1: There're only finitely many Hilbert polynomials. Let $S = \coprod \operatorname{Quot}_{P(\mathcal{F}_i)}$ be the disjoint union of the Quot scheme corresponding to those finitely many Hilbert polynomials. Then 1 is immediate from the definition of Quot scheme. □

The following proposition allows us to estimate the regularity of a coherent sheaf $\mathcal{F}$ in terms of Hilbert polynomial and the number of global sections of the restriction of $\mathcal{F}$ to the regular sequence of hyperplane sections.

Proposition 16. *There're universal polynomials $P_i \in \mathbb{Q}[T_0, \dots, T_i]$ such that the following holds: Let $\mathcal{F}$ be a coherent sheaf of dimension $\dim(\mathcal{F}) \leq d$ and let $H_1, \dots, H_d$ be a regular sequence of hyperplane sections. If $\chi(\mathcal{F}|_{\cap_{j\leq i} H_j}) = a_i$ and $h^0(\mathcal{F}|_{\cap_{j\leq i} H_j}) \leq b_i$, then*

$$\operatorname{reg}(\mathcal{F}) \leq P_d(a_0 - b_0, \dots, a_d - b_d).$$

Proof. It suffices to show for the case $X = \mathbb{P}^d$ and $\dim \mathcal{F} = d$. By the argument of Lemma 1.2.1 in [8], the Hilbert polynomial $P(\mathcal{F})(m)$ can be written into $\sum_{i=0}^{d} a_i \binom{m+i-1}{i}$. The proof proceeds by induction on the dimension of the sheaf.

The base case is clear: for zero dimension sheaf, P_0 can be taken as any polynomial.

Let $d \geq 1$, take any hyperplane section H which does not meet any associated points of $\mathcal{F}$, we have the exact sequence

$$0 \to \mathcal{F}(m-1) \to \mathcal{F}(m) \to \mathcal{F}_H(m) \to 0$$

and the long exact sequence

$$\cdots \to H^i(X, \mathcal{F}(m-1)) \to H^i(X, \mathcal{F}(m)) \to H^i(H, \mathcal{F}_H(m)) \to H^{i+1}(X, \mathcal{F}(m-1)) \to \cdots$$

By induction hypothesis, $\mathcal{F}|_H$ is $n = P_{d-1}(a_1 - b_1, \dots, a_d - b_d)$-regular. For $m \geq n-1$, the long exact sequence and Lemma 13 shows that $H^i(X, \mathcal{F}(m)) \cong H^i(X, \mathcal{F}(m-1))$ for all $i \geq 2$, $m \geq n-2$. For sufficiently large m, the cohomologies vanishes so all $H^i(X, \mathcal{F}(m)) = 0$ for $i \geq 2$, $m \geq n-2$. We also get a surjection

$$\nu : H^1(X, \mathcal{F}(m-1)) \to H^1(X, \mathcal{F}(m)),$$

the function $h^1(\mathcal{F}(m))$ is decreasing in m.

ν becomes an isomorphism if and only if the homomorphism $H^0(X, \mathcal{F}(m)) \to H^0(H, \mathcal{F}_H(m))$ is surjective. Use the same diagram as in the proof of Lemma 13, we can conclude that if $H^0(X, \mathcal{F}(m)) \to H^0(H, \mathcal{F}_H(m))$ is surjective, then $H^0(X, \mathcal{F}(m+1)) \to H^0(H, \mathcal{F}_H(m+1))$ is surjective. Once $h^1(\mathcal{F}(m)) = h^1(\mathcal{F}(m+1))$, the value never decreases anymore. So $h^1(\mathcal{F}(m))$ strictly decrease to 0. For $m \geq n + h^1(\mathcal{F}(n)) + 1$, $H^1(X, \mathcal{F}(m-1)) = 0$.

Now we estimate the upper bound for $h^1(\mathcal{F}(n))$ by a polynomial in $a_i - b_i$.

$$h^1(\mathcal{F}(n)) = h^0(\mathcal{F}(n)) - \chi(\mathcal{F}(n)) = h^0(\mathcal{F}(n)) - \sum_{i=0}^{d} a_i \binom{n+i-1}{i}$$

and

$$h^0(\mathcal{F}(n)) \leq \sum_{i=0}^{d} b_i \binom{n+i-1}{i}$$

can be done inductively:

$$h^0(\mathcal{F}(n)) \leq h^0(\mathcal{F}(n-1)) + h^0(\mathcal{F}_H(n))$$

and $h^0(\mathcal{F}_H(n)) \leq \sum_{i=1}^{d} b_i \binom{n+i-2}{i-1}$ implies $h^0(\mathcal{F}(n)) \leq \sum_{i=0}^{d} b_i \binom{n+i-1}{i}$. So $h^1(\mathcal{F}(n)) \leq \sum_{i=0}^{d} (a_i - b_i) \binom{n+i-1}{i}$, where $n = P_{d-1}(a_1 - b_1, \ldots, a_d - b_d)$. We may take $P_d(a_0 - b_0, \ldots, a_d - b_d) = n + \sum_{i=0}^{d} (a_i - b_i) \binom{n+i-1}{i}$. $\square$

Combining Lemma 13 and Proposition 16, we get the important criterion for boundedness.

Theorem 17 (Kleiman Criterion). *Let $\{\mathcal{F}_i\}$ be a family of coherent sheaf on X with the same Hilbert polynomial P. Then this family is bounded if and only if there are constants C_i, $i = 0, \ldots, d = \deg(P)$, such that for every $\mathcal{F}_i$ there exists an $\mathcal{F}_i$ regular sequence of hyperplane sections $H_1, \ldots, H_d$ such that $h^0(\mathcal{F}|_{\cap j \leq i}) \leq C_i$.*

Example 18. Let X be a smooth projective curve over algebraically closed field k. If $\{\mathcal{F}_i\}$ is a family of coherent sheaves with $h^0(\mathcal{F}_i)$ and $\operatorname{rk}(\mathcal{F}_i)$, then $\{\mathcal{F}_i\}$ is bounded. This is because $h^0(\mathcal{F}_i|_H) = \operatorname{rk}(\mathcal{F}) \cdot \deg X$ is bounded.

When X is a smooth projective curve over algebraically closed field k, the semistability of sheaves behaves well in arbitrary characteristic.

Theorem 19. *$\{\mathcal{F}_i\}$ is a family of semistable sheaves with fixed Hilbert polynomial, then $\{\mathcal{F}_i\}$ is bounded.*

Proof. We show that there is an integer m such that $\mathcal{F}_i$ is m-regular for all i. Assume the Hilbert polynomial is given by $P(n) = n \cdot \operatorname{rk}(\mathcal{F}_i) \cdot \deg(\mathcal{O}(1)) + \deg(\mathcal{F}_i) + \operatorname{rk}(\mathcal{F}_i)(1-g)$, so all the sheaves have the same rank and degree. By Serre duality, $H^1(X, \mathcal{F}_i(m-1)) = \operatorname{Hom}(\mathcal{F}_i(m-1), \omega)\check{}$. Note $\mathcal{F}_i(m-1)$ and ω are all semistable. Set

$$m = \left\lceil \frac{2\operatorname{rk}(\mathcal{F}_i) g(X) - 2\operatorname{rk}(\mathcal{F}_i) - \deg(\mathcal{F}_i)}{\deg(\mathcal{O}(1))} + 1 \right\rceil$$

and using Lemma 5, $\operatorname{Hom}(\mathcal{F}_i(m-1), \omega) = 0$ for all i. Thus $\{F_i\}$ is bound. $\square$

Dimension 2 case is much more complicated since there is no good estimation for $H^1(X, \mathcal{F})$. We will show that the family of semistable sheaves with rank two on a smooth surface is bounded. Let X be a smooth projective surface and $\mathcal{O}(1)$ be a very ample line sheaf. Let $\mathfrak{F}(P)$ be the family consisting of rank two torsion-free semistable sheaf with fixed Hilbert polynomial P.

Lemma 20 ([15])**.** *There are integer n_1 and n_2 such that for any $\mathcal{F} \in \mathfrak{F}(P)$, there is a subsheaf $\mathcal{E}$ of $\mathcal{F}$ with $n_1 \leq \deg(\mathcal{E}) \leq n_2$.*

Proof. First $\mathcal{F}$ is semistable so one can set $n_2 = \frac{\deg(\mathcal{F})}{2}$, then for all rank 1 subsheaf $\mathcal{E}$ we have $\deg(\mathcal{E}) \leq \frac{\deg(\mathcal{F})}{2}$.

We can choose $n_1 > 0$ such that $P(\mathcal{F})(n) > 0$ for all $\mathcal{F} \in \mathfrak{F}(P)$. We may also assume $\deg(\mathcal{F}) \geq -2n_1 + 2\deg(\omega)$ so $\deg(\check{\mathcal{F}}(-m) \otimes \omega) < 0$. Note that any global section will induce a morphism $\mathcal{O}_X \to \check{\mathcal{F}}(-m) \otimes \omega$, which contradicts the fact $\check{\mathcal{F}}(-m) \otimes \omega$ is semistable with negative degree. So

$$H^0(X, \check{\mathcal{F}}(-m) \otimes \omega) = \operatorname{Hom}(\mathcal{F}(m), \omega) = H^2(X, \mathcal{F}(m))\check{} = 0.$$

Thus $H^0(X, \mathcal{F}(m)) \neq 0$ for all $\mathcal{F}$ and a global section induces the morphism $\mathcal{O}_X \to \mathcal{F}(m)$. Let the image be $\mathcal{G}$, then $\deg(\mathcal{G}(n_1)) \geq 0$ and $\deg(\mathcal{G}) \geq -n_1$. □

Theorem 21. *$\mathfrak{F}(P)$ is bounded.*

Proof. We will use Theorem 17 to prove this theorem. From Lemma 15 and the definition of Castelnuovo-Mumford regularity, if the family $\{\mathcal{F}(m) | \mathcal{F} \in \mathfrak{F}(P)\}$ is bounded, then $\mathfrak{F}(P)$ is bounded. So we may let m small enough and assume $\deg(\mathcal{F}) < 0$ for all $\mathcal{F} \in \mathfrak{F}(P)$. Since any nonzero global section of $\mathcal{F}$ will induce a morphism $\mathcal{O}_X \to \mathcal{F}$, which will contradict to the assumption $\mathcal{F}$ is semistable. Thus $H^0(X, \mathcal{F}) = 0$ for all $\mathcal{F}$.

For general hyperplane H_1, H_2 in the linear system $|\mathcal{O}(1)|$, $h^0(\mathcal{F}|_{H_1 \cap H_2}) = 2\deg X$ is bounded.

Now we show $h^0(\mathcal{F}|_H)$ is bounded for general hyperplane H. According to Lemma 20, there are constants n_1, n_2 and a subsheaf $\mathcal{E} \subset \mathcal{F}$ such that $n_1 \leq \deg(\mathcal{E}) \leq n_2$. We may assume $X \cap H$ is smooth curve and $\mathcal{E}|_H$, $(\mathcal{F}/\mathcal{E})|_H$ are locally free. Let g be the genus of $X \cap H$ for general H. Let $n_3 = \deg(\mathcal{F}) - n_1$, $n_4 = \deg(\mathcal{F}) - n_2$ and $n = \max\{0, 2g - n_1, 2g - n_4\}$. Then $n_4 \leq \deg(\mathcal{F}/\mathcal{E}) \leq n_3$. Since $\deg(\mathcal{E}|_H) > 2g - 2$ and $\deg((\mathcal{F}/\mathcal{E})|_H) > 2g - 2$, $h^1(\mathcal{E}|_H) = 0$ and $h^1((\mathcal{F}/\mathcal{E})|_H) = 0$. By Riemann-Roch theorem,

$$h^0(\mathcal{E}|_H(n)) \leq n + 1 - g + n_2,$$

$$h^0((\mathcal{F}/\mathcal{E})|_H(n)) \leq n + 1 - g + \deg(\mathcal{F}) - n_1.$$

So

$$\begin{aligned} h^0(\mathcal{F}|_H) &\leq h^0(\mathcal{E}|_H) + h^0((\mathcal{F}/\mathcal{E})|_H) \\ &\leq h^0(\mathcal{E}|_H(n)) + h^0((\mathcal{F}/\mathcal{E})|_H(n)) \\ &\leq 2 - 2g + 2n + n_2 - n_1 \end{aligned}$$

is bounded. □

3 Grauert-Mülich Theorem

To make use of Kleiman criterion, we first need to understand the behavior of semistable sheaves under the restriction to some hypersurface sections. Although for general hypersurface H, the restriction of semistable sheaf $\mathcal{F}$ may not be semistable, $\mathcal{F}|_H$ cannot be so 'far' from semistable. The first thing we need to prove is the Grauert-Mülich theorem. We begin with some setup on incidence structures.

Let k be an algebraically closed field of characteristic zero. Let X be a normal projective variety over k of $\dim X \geq 2$ and fix a very ample sheaf $\mathcal{O}(1)$ on X. Denote the linear system $|\mathcal{O}(a)|$ by Π_a and let $Z_a = \{(D,x) \in \Pi_a \times X | x \in D\}$ be the incidence variety. We also allow the incidence structures on different linear systems: Let $\Pi = \Pi_{a_1} \times \cdots \times \Pi_{a_l}$ and $Z = Z_{a_1} \times_X \cdots \times_X Z_{a_l}$. Then there are natural projection $p : Z \to \Pi$ and $q : Z \to X$.

Let $V_a = H^0(X, \mathcal{O}(a))$ and $\mathcal{K}$ be the kernel of the natural evaluation map $V_a \otimes \mathcal{O}_X \to \mathcal{O}_X(a)$. Then $Z_a = \mathbb{P}(\check{\mathcal{K}})$ and there's a natural closed immersion $Z_a \to \mathbb{P}(\check{V}) \times X$. q is the bundle morphism and therefore open. We can compute the relation tangent bundle by Euler sequence:

$$0 \to \mathcal{O}_{Z_a} \to q^*\mathcal{K} \otimes p^*\mathcal{O}(1) \to \mathcal{T}_{Z_a/X} \to 0$$

Z parametrizes the intersection of element in Π_{a_i} in such a way: For $s = (s_1, s_2, \ldots, s_l)$ be a closed point in Π. Each s_i corresponds to a divisor D_i. Then the fibre $p^{-1}(s)$ can be identified by q with the scheme-theoretic intersection $D_1 \cap D_2 \cap \cdots \cap D_l \subset X$. The relative tangent bundle of Z can be similarly computed as $\mathcal{T}_{Z/X} = p_1^*\mathcal{T}_{Z_{a_1}/X} \oplus \cdots \oplus p_l^*\mathcal{T}_{Z_{a_l}/X}$, where $p_i : Z \to Z_{a_i}$ are the natural projections. For a coherent sheaf $\mathcal{F}$ on X, let $\mathcal{E} = q^*\mathcal{F}$, from the construction we have $\mathcal{E}_s = \mathcal{F}|_{Z_s}$.

Lemma 22. *Let $\mathcal{F}$ be a torsion-free coherent sheaf on X and $\mathcal{E} \cong q^*\mathcal{F}$. Then*

1. *There is a nonempty open subset $S' \subset \Pi$ such that the morphism $p_{S'} : Z_{S'} \to S'$ is flat and for all $s \in S'$, the fibre $Z_s = p^{-1}(s)$ is a normal irreducible complete intersection of codimension l in X.*
2. *There is a nonempty dense open subset $S \subset S'$ such that the family $\mathcal{E}_S = q^*\mathcal{F}|_{Z_S}$ is flat over S and for all $s \in S$, the fibre $\mathcal{E}_s$ is torsion-free.*

Proof. 1. The flatness is a generic condition. The remaining part follows from Bertini theorem, see [7] section 2.8.

2. The flatness is the same as above. We reduce to the case of X smooth first. Let $f : X \to \mathbb{P}^n$ be the closed immersion defining $\mathcal{O}(1)$. We can regard $\mathcal{F}$ as a pure sheaf of dimension $\dim X$ supported on X. Let S'' be the open subset which contains point s that parametrizes regular sequence for $\mathcal{F}$ and $\mathcal{E}xt^i(\mathcal{F}, \omega_X)$, $\forall i \geq 0$. Then clearly S'' is not empty. Let $S = S'' \cap S$. On each $Z_s = p^{-1}(s)$, the torsion-freeness is from the following lemma:

   Lemma 23 ([8]). *Let X be a smooth projective variety over k. For a coherent sheaf $\mathcal{F}$ of codimension c, we say $\mathcal{F}$ satisfies Serre condition $S_{k,c}$ if $\operatorname{depth} \mathcal{F}_x \geq \min\{k, \dim \mathcal{O}_{X,x} - c\}$ for all $x \in \operatorname{Supp}(\mathcal{F})$. Then*

   (a) $\mathcal{F}$ is pure if and only if $\mathcal{F}$ satisfies $S_{1,c}$.

(b) *Let H be a hypersurface defined by some ample line sheaf $\mathcal{L}$. If H is a $\mathcal{F}$ regular section and $\mathcal{F}$ satisfies $S_{k,c}$, then $\mathcal{F}|_H$ satisfies $S_{k-1,c+1}$.*

Use the lemma and induction, one can easily see $\mathcal{F}|_{Z_s}$ satisfies $S_{1,n-\dim X+l}$, which means $\mathcal{F}|_{Z_s}$ is pure of dimension $\dim X - l$ and thus torsion-free on Z_s.

□

Now we can apply Theorem 10 to the family $\mathcal{E}_S$ and get a Harder-Narasimhan filtration

$$0 = \mathcal{E}_0 \subset \cdots \subset \mathcal{E}_j = \mathcal{E}_S$$

We may shrink to a smaller open dense set S such that all the factors are flat. S is irreducible and thus connected, so $\mu((\mathcal{E}_i/\mathcal{E}_{i-1})_s)$ is a constant for each $s \in S$. We may define $\mu_i = \mu((\mathcal{E}_i/\mathcal{E}_{i-1})_s)$. Then

$$\mu_1 > \mu_2 > \cdots > \mu_{j-1}.$$

Define the number of gap by

$$\delta\mu = \begin{cases} 0 & \text{if } \mathcal{E}_s \text{ is semistable} \\ \max\{\mu_i - \mu_{i+1}\} & \text{otherwise} \end{cases}$$

The Grauert-Mülich theorem gives us an upper bound $\delta\mu$ for sufficient general $s \in S$.

Theorem 24 (Grauert-Mülich)**.** *Let $\mathcal{F}$ be a semistable torsion-free sheaf. Then there is a nonempty open dense subset S of Π such that for all $s \in S$, the following inequality holds:*

$$\delta\mu(\mathcal{F}|_{Z_s}) \leq \max\{a_i\} \cdot \prod_{i=1}^{l} a_i \cdot \deg X.$$

Proof. The case $\mathcal{E}_s$ is semistable is trivial. Assume $\delta\mu > 0$ and $\delta\mu = \mu_i - \mu_{i+1}$ for specific i and let $\mathcal{E}' = \mathcal{E}_i$, $\mathcal{E}'' = \mathcal{E}/\mathcal{E}'$. For all $s \in S$, $\mathcal{E}'_s$ and $\mathcal{E}''_s$ are torsion-free and from the uniqueness of Harder-Narasimhan filtration, $\mu_{min}(\mathcal{E}'_s) = \mu_i$ and $\mu_{max}(\mathcal{E}''_s) = \mu_{i+1}$.

Since torsion-free sheaves are locally free on an open subset, we may let Z_0 be the maximal open subset of Z_S such that $\mathcal{E}|_{Z_0}$ and $\mathcal{E}''|_{Z_0}$ are locally free. Let their ranks be r and r'', respectively. The surjection $\mathcal{E}_{Z_0} \to \mathcal{E}''_{Z_0}$ gives a morphism $\varphi : Z_0 \to \mathrm{Grass}(\mathcal{F}, r'')$, and $\mathcal{E}|_{Z_0} \to \mathcal{E}|_{Z_0}$ is the pullback of $\mathcal{F} \to \mathcal{U}$. Let X_0 be the image of Z_0 in X, since $q : Z \to X$ is the bundle morphism, X_0 is open. Note $\mathcal{F}$ is torsion-free, for any $s \in S$ the complement of $Z_0 \cap Z_s$ in Z_s has codimension larger than 1, the codimension of complement of X_0 in X is also larger than 1.

Let

$$D\varphi : \mathcal{T}_{Z/X}|_{Z_0} \to \varphi^* \mathcal{T}_{\mathrm{Grass}(\mathcal{F},r'')/X}$$

be the relative differential morphism related to φ. Since $\mathcal{T}_{\mathrm{Grass}(\mathcal{F},r'')/X} = \mathcal{H}om(\ker(\mathcal{F} \to \mathcal{U}), \mathcal{U})$, we can identify $\varphi^* \mathcal{T}_{\mathrm{Grass}(\mathcal{F},r'')/X}$ as $\mathcal{H}om(\varphi^* \ker(\mathcal{F} \to \mathcal{U}), \varphi^* \mathcal{U}) = \mathcal{H}om(\mathcal{E}|'_{Z_0}, \mathcal{E}|''_{Z_0})$. Thus $D\varphi$ corresponds to

$$\Phi : (\mathcal{E}' \otimes \mathcal{T}_{Z/X})|_{Z_0} \to \mathcal{E}|''_{Z_0}$$

via the isomorphism $\mathrm{Hom}((\mathcal{E}'\otimes\mathcal{T}_{Z/X})|_{Z_0},\mathcal{E}|''_{Z_0})\cong\mathrm{Hom}(\mathcal{T}_{Z/X}|_{Z_0},\mathcal{H}om(\mathcal{E}|'_{Z_0},\mathcal{E}|''_{Z_0}))$.

Next we want to show $\Phi_s\neq 0$ for general $s\in S$. Suppose on the contrary. We may shrink S smaller if necessary to make $\Phi=0$. Since q is faithfully flat, according to Theorem 57, $\mathcal{E}|_{X_0}$ is also locally free. Restricting φ to Z_0, we have the following diagram:

$$\begin{tikzcd} Z_0 \arrow[r, "\varphi"] \arrow[dr, "q_0"'] & \mathrm{Grass}(\mathcal{F}|_{X_0}, r'') \arrow[d] \\ & X_0 \end{tikzcd}$$

q_0 is a smooth morphism with connected fibres. If $\Phi=0$, then $D\varphi=0$ and in characteristic zero case, this will imply φ is constant on fibre of q_0. Then by rigidity lemma, there is a morphism $\rho: X_0\to\mathrm{Grass}(\mathcal{F}|_{X_0},r'')$. From the universal property of $\mathrm{Grass}(\mathcal{F}|_{X_0},r'')$, there's a quotient $\mathcal{F}|_{X_0}\to\mathcal{F}''$ of rank r''. Moreover, $\mathcal{F}|''_{X_0\cap Z_s}\cong\mathcal{E}''_s|_{Z_0\cap Z_s}$ for general $s\in S$. Since $\mathcal{F}|_{X_0}$ is support on codimension ≥ 2 sets in X, any extension $\mathcal{F}'''$ of $\mathcal{F}''$ to X satisfies $\mu(\mathcal{F}''')=\mu(\mathcal{F}'')$. By our assumption, $\mathcal{E}''_s$ is a destabilizing quotient of $\mathcal{E}_s$, this means $\mathcal{F}'''$ is destabilizing quotient of $\mathcal{F}$. This contradicts the assumption that $\mathcal{F}$ is semistable.

A nontrivial Φ_s defines a morphism $\mathcal{E}'_s\otimes\mathcal{T}_{Z/X}|_{Z_s}$ to $\mathcal{E}''_s$ in the quotient category $\mathrm{Coh}_{n-l,n-l-1}(Z_s)$. Then by Lemma 5, we have the inequality

$$\mu_{min}(\mathcal{E}'_s\otimes\mathcal{T}_{Z/X}|_{Z_s})\leq\mu_{max}(\mathcal{E}''_s).$$

Consider the Koszul complex associated with the evaluation map $e: V_a\otimes\mathcal{O}_X\to\mathcal{O}(a)$. Taking the last terms, we get a surjection $\wedge^2V_a\otimes\mathcal{O}_X(-a)\to\ker(e)=\mathcal{K}$ and hence a surjection

$$\wedge^2V_a\otimes q^*\mathcal{O}_X(-a)\otimes p^*\mathcal{O}(1)\to q^*\mathcal{K}\otimes p^*\mathcal{O}(1)\to\mathcal{T}_{Z_a/X}.$$

Using $\mathcal{T}_{Z/X}=\bigoplus\limits_i p_i^*\mathcal{T}_{Z_{a_i}/X}$, we have a surjection

$$(\bigoplus_i\wedge^2V_{a_i}\otimes\mathcal{O}(-a_i))|_{Z_s}\to\mathcal{T}_{Z/X}|_{Z_s}.$$

Therefore,

$$\begin{aligned}\mu_{min}(\mathcal{E}'_s\otimes\mathcal{T}_{Z/X}|_{Z_s})&\geq\mu_{min}(\bigoplus_i\wedge^2V_{a_i}\otimes\mathcal{O}(-a_i)\otimes\mathcal{E}'|_{Z_s})\\&=\min_i\{\mu_{min}(\mathcal{O}_{Z_s(-a_i)}\otimes\mathcal{E}'_s)\}\\&=\mu_{min}(\mathcal{E}'_s)-\max\{a_i\}\cdot\deg(Z_s).\end{aligned}$$

In conclusion,

$$\begin{aligned}\delta\mu=\mu_i-\mu_{i+1}&=\mu_{min}(\mathcal{F}'_s)-\mu_{max}(\mathcal{F}''_s)\\&\leq\max\{a_i\}\cdot\deg(Z_s)\\&=\max\{a_i\}\cdot\prod a_i\cdot\deg(X).\end{aligned}$$

$\square$

Corollary 25. *Let $\mathcal{F}$ be a torsion-free semistable sheaf of rank r on X. Let Y be the intersection of $s < \dim X$ general hyperplanes in $|\mathcal{O}_X(1)|$. Then*

$$\mu_{min}(\mathcal{F}|_Y) \geq \mu(\mathcal{F}) - \frac{r-1}{2}\deg(X)$$

and

$$\mu_{max}(\mathcal{F}|_Y) \leq \mu(\mathcal{F}) + \frac{r-1}{2}\deg(X).$$

Proof. We only show the inequality for μ_{max}, μ_{min} is similar. If $\mathcal{F}|_Y$ is semistable then there's nothing to prove. Let $\mu_1, \ldots, \mu_j$ and $r_1, \ldots, r_j$ be the slopes and ranks of the factors of the Harder-Narasimhan filtration of $\mathcal{F}|_Y$. Note all the r_i are positive integers since the factors are torsion-free. By Theorem 24, we have

$$\mu_i - \mu_{i+1} \leq \deg X.$$

Take the sum from 1 to i we get $\mu_i \geq \mu_1 - (i-1)\deg X$. So

$$\begin{aligned}
\mu(\mathcal{F}) &= \sum_{i=1}^{j} \frac{r_i}{r}\mu_i \\
&\geq \mu_1 - \frac{\deg X}{r}\sum_{i=1}^{j}(i-1)r_i \\
&\geq \mu_1 - \frac{\deg X}{r}\sum_{i=1}^{r}(i-1) \\
&= \mu_{max}(\mathcal{F}|_Y) - \deg X \cdot \frac{r-1}{2}.
\end{aligned}$$

□

4 Semistability and Tensor Products

One thing we need to prove in this section is tensor products preserve semistability. The proof involves Theorem 24 and ampleness of positive degree sheaves, which becomes true only in the characteristic zero case. We need to first figure out the behavior of semistable sheaves under pullback and pushforward via finite morphisms.

Let $f : Y \to X$ be a finite morphism of normal projective varieties over k of dimension n. Fix an ample line sheaf $\mathcal{O}_X(1)$, then $\mathcal{O}_Y(1) = f^*\mathcal{O}_X(1)$ is also ample. f is affine morphism so all the higher direct image vanishes, and $H^i(X, f_*\mathcal{F}) = H^i(Y, \mathcal{F})$. In particular, the Hilbert polynomial $P(\mathcal{F})(m) = P(f_*\mathcal{F})(m)$ and therefore f_* preserves the dimension of sheaves.

Let $\mathcal{A}$ be the sheaf of algebras $f_*\mathcal{O}_Y$, then $\mathcal{A}$ is a torsion-free coherent sheaf of rank d. f_* gives an equivalence between the category of coherent sheaves on Y and the category of coherent sheaves on X with $\mathcal{A}$-module structure. $\mathcal{A}$ is torsion-free and thus locally free in codimension 1, which means f is flat in codimension 1. f^* is exact functor from the quotient category $\mathrm{Coh}_{n,n-1}(X)$ to $\mathrm{Coh}_{n,n-1}(Y)$.

For a pure sheaf $\mathcal{F}$ of dimension m, suppose there's a n-dimensional quotient $\mathcal{G}$ of $f_*\mathcal{F}$. $\mathcal{G}$ admits a natural $\mathcal{A}$-module structure, so there's a coherent $\mathcal{O}_Y$-module $\mathcal{G}'$, which is a m-dimensional quotient of $\mathcal{F}$. This contradicts the assumption that $\mathcal{F}$ is pure. Therefore f_* preserves the purity of sheaves.

Assume $\mathcal{F}$ is pure of dimension n which is torsion-free in codimension 1 in $\mathrm{Coh}(X)$. Since $\deg(f^*\mathcal{F}) = d\deg(\mathcal{F})$ and $\mathrm{rk}(f^*\mathcal{F}) = \mathrm{rk}(\mathcal{F})$, $\mu(f^*\mathcal{F}) = d\mu(\mathcal{F})$.

Assume $\mathcal{G}$ is pure of dimension n which is torsion-free in codimension 1 in $\mathrm{Coh}(Y)$. Then $c_1(\mathcal{O}_Y(1))^n = d{\cdot}c_1(\mathcal{O}_X(1))^n$ and $\deg Y = d\deg X$. Since $P(\mathcal{G})(m) = P(f_*\mathcal{G})(m)$, we have $\mathrm{rk}(f_*\mathcal{G}) = d\,\mathrm{rk}(\mathcal{G})$. Note that

$$\deg(\mathcal{A}) = \alpha_{d-1}(\mathcal{O}_Y) - \mathrm{rk}(\mathcal{A})\alpha_{d-1}(\mathcal{O}_X) = \alpha_{d-1}(\mathcal{O}_Y) - d\cdot\alpha_{d-1}\mathcal{O}_X,$$

$$\deg(\mathcal{G}) = \alpha_{d-1}(\mathcal{G}) - \mathrm{rk}(\mathcal{G})\alpha_{d-1}(\mathcal{O}_Y) = \alpha_{d-1}(f_*\mathcal{G}) - \mathrm{rk}(f_*\mathcal{G})\alpha_{d-1}(\mathcal{O}_X) - \mathrm{rk}(\mathcal{G})\deg(\mathcal{A}).$$

We have $\mu(\mathcal{G}) = d(\mu(f_*\mathcal{G}) - \mu(\mathcal{A}))$.

In characteristic zero case, we have the following lemma:

Lemma 26. *Let $\mathcal{F}$ be a n-dimensional coherent sheaf on X. Then $\mathcal{F}$ is semistable if and only if $f^*\mathcal{F}$ is semistable.*

Proof. $\mathcal{F}$ is torsion-free in codimension 1 if and only if $f^*\mathcal{F}$ is torsion-free, so we are allowed to work in the category $\mathrm{Coh}_{n,n-1}(Y)$.

We first show the if direction: Suppose there is subsheaf $\mathcal{E}$ of $\mathcal{F}$ such that $\mu(\mathcal{E}) > \mu(\mathcal{F})$, then $\mu(f^*\mathcal{E}) > \mu(\mathcal{F})$. This leads to a contradiction.

Then we show the only if direction: Let K be the splitting field of the function field $K(Y)$ over $K(X)$. Let Z be the normalization of Y in K, then we have finite morphisms $Z \to Y \to X$. By pulling back $f^*\mathcal{F}$ to Z, we may consider the finite morphism $g : Z \to X$. Note that $K(Z)$ is Galois over $K(X)$, $Z \to X$ is a Galois cover with Galois group G. Suppose $g^*\mathcal{F}$ is not semistable and it's maximal destabilizing subsheaf $\mathcal{E}$. Since $\mathcal{E}$ is unique, it's invariant under the action of G. By Theorem 60, there's a coherent subsheaf $\mathcal{F}' \subset \mathcal{F}$ such that $f^*\mathcal{F}'$ isomorphic to $\mathcal{E}$ in codimension 1. Then $\mu(\mathcal{F}') > \mu(\mathcal{F})$ and we have a contradiction. ☐

We will omit the proof of the next lemma here and refer to [12].

Lemma 27. *Let X be a projective normal variety over algebraically closed field k and let $\mathcal{O}_X(1)$ be a very ample line sheaf. For integer d there exist a projective normal variety X' with very ample line sheaf $\mathcal{O}_{X'}(1)$ and a finite morphism $f : X' \to X$ such that $f^*\mathcal{O}_X(1) \cong \mathcal{O}_{X'}(d)$. Moreover, if X is smooth, X' can be chosen to be smooth.*

Proof. See Corollary 1.15.1 in [12]. ☐

Using the above lemmas and results in Appendix B, we are able to prove the theorem:

Theorem 28. *Let X be a normal projective variety over an algebraically closed field of characteristic zero. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are semistable sheaves, then $\mathcal{F}_1 \otimes \mathcal{F}_2$ is semistable.*

Proof. Suppose on the contrary. Let $\mathcal{E}$ be a torsion-free destabilizing quotient of $\mathcal{F}_1 \otimes \mathcal{F}_2$. We first reduce to the case that $\mu(\mathcal{F}_1)+\mu(\mathcal{F}_2)-\mu(\mathcal{E})$ is large enough case. Using Lemma 27, there's a finite morphism $f : X' \to X$ such that $f^*\mathcal{O}_X(1) \cong \mathcal{O}_{X'}(d)$ for large d. $\mu(f^*\mathcal{E})$ are defined with respect to $\mathcal{O}_{X'}(1)$. $f^*\mathcal{F}_1$ and $f^*\mathcal{F}_2$ are also semistable by Lemma 26, and $f^*\mathcal{E}$ is a torsion-free destabilizing quotient of $f^*(\mathcal{F}_1 \otimes \mathcal{F}_2)$. Take

$$d > \left\lceil \frac{\deg X \cdot (\mathrm{rk}(\mathcal{F}_1) + \mathrm{rk}(\mathcal{F}_2) + 2)}{2(\mu(\mathcal{F}_1) + \mu(\mathcal{F}_2) - \mu(\mathcal{E}))} \right\rceil,$$

we have

$$\begin{aligned}
\frac{\mu(f^*\mathcal{F}_1) + \mu(f^*\mathcal{F}_2) - \mu(f^*\mathcal{E})}{\deg X'} &= d \cdot \frac{\mu(\mathcal{F}_1) + \mu(\mathcal{F}_2) - \mu(\mathcal{E})}{\deg X} \\
&> \frac{\mathrm{rk}(\mathcal{F}_1) + \mathrm{rk}(\mathcal{F}_2) + 2}{2} \\
&= \frac{\mathrm{rk}(f^*\mathcal{F}_1) + \mathrm{rk}(f^*\mathcal{F}_2) - 2}{2}.
\end{aligned}$$

So we may assume

$$\mu(\mathcal{F}_1) + \mu(\mathcal{F}_2) - \mu(\mathcal{E}) > \frac{\deg X \cdot (\mathrm{rk}(\mathcal{F}_1) + \mathrm{rk}(\mathcal{F}_2) + 2)}{2}.$$

According to Bertini theorem, the complete intersection for general $\dim X - 1$ hyperplanes forms a smooth curve C. Applying Corollary 25, the Harder-Narasimhan factors satisfy

$$\mu(gr_j^{HN}(\mathcal{F}_i|_C)) \geq \mu(\mathcal{F}_i) - \deg X \cdot \frac{\mathrm{rk}(\mathcal{F}_i) - 1}{2}$$

for each $i = 1, 2$. Let

$$n_i = \left\lceil \frac{\mu(\mathcal{F}_i)}{\deg X} - \frac{\mathrm{rk}(\mathcal{F}_i) - 1}{2} \right\rceil - 1.$$

Then

$$\mu(gr_j^{HN}\mathcal{F}_i(-n_i)|_C) \geq \mu(\mathcal{F}_i) - \deg X \cdot (n_i + \frac{\mathrm{rk}(\mathcal{F}_i) - 1}{2}) > 0.$$

Then $gr_j^{HN}(\mathcal{F}_i(-n_i)|_C)$ is a semistable torsion-free sheaf on C and therefore a semistable vector bundle by 62. $\mu(gr_j^{HN}(\mathcal{F}_i(-n_i)|_C)) > 0$ so by Theorem 68 $gr_j^{HN}(\mathcal{F}_i(-n_i)|_C)$ is ample. According to Corollary 67, $gr_i^{HN}(\mathcal{F}_1(-n_1)|_C) \otimes gr_j^{HN}(\mathcal{F}_2(-n_2)|_C)$ is ample. Therefore by Proposition 65 $(\mathcal{F}_1 \otimes \mathcal{F}_2)(-n_1 - n_2)|_C$ is also ample. $\mathcal{E}(-n_1 - n_2)|_C$ as a quotient of ample vector bundle is also ample.

Note that

$$\begin{aligned}
\mu(\mathcal{E}(-n_1 - n_2)) &= \mu(\mathcal{E}) - (n_1 + n_2) \deg X \\
&< \mu(\mathcal{F}_1) + \mu(\mathcal{F}_2) - \deg X \cdot (n_1 + n_2 + \frac{\mathrm{rk}(\mathcal{F}_1) + \mathrm{rk}(\mathcal{F}_2) + 2}{2}) \\
&= \mu(\mathcal{F}_1) - \deg X \cdot (\frac{\mathrm{rk}(\mathcal{F}_1) - 1}{2} + n_1 + 1) \\
&+ \mu(\mathcal{F}_2) - \deg X \cdot (\frac{\mathrm{rk}(\mathcal{F}_2) - 1}{2} + n_2 + 1) \\
&\leq 0.
\end{aligned}$$

So $\deg(\mathcal{E}(-n_1 - n_2)|_C) = \deg(\mathcal{E}(-n_1 - n_2)) < 0$, which contardicts the fact $\mathcal{E}(-n_1 - n_2)|_C$ is ample. □

Corollary 29. *Let $\mathcal{F}$ and $\mathcal{G}$ be torsion-free coherent sheaves on a normal projective variety X. Then*

1. $\mu_{min}(\mathcal{F} \otimes \mathcal{G}) = \mu_{min}(\mathcal{F}) + \mu_{max}(\mathcal{G})$.
2. $\mu_{max}(\mathcal{F} \otimes \mathcal{G}) = \mu_{max}(\mathcal{F}) + \mu_{max}(\mathcal{G})$.

Proof. We only prove for μ_{max}. One direction is easy: As in Remark 11, pick the maximal destabilizing sheaf $\mathcal{F}'$ and $\mathcal{G}'$ of $\mathcal{F}$, $\mathcal{G}$ respectively. Then $\mathcal{F}' \otimes \mathcal{G}'$ destablizes $\mathcal{F} \otimes \mathcal{G}$ and $\mu(\mathcal{F}') + \mu(\mathcal{G}') = \mu_{max}(\mathcal{F}) + \mu_{max}(\mathcal{G})$. So $\mu_{max}(\mathcal{F} \otimes \mathcal{G}) \geq \mu_{max}(\mathcal{F}) + \mu_{max}(\mathcal{G})$.

For the other direction, $\mathcal{F}$ and $\mathcal{G}$ are locally free in codimension 1, we may work in the cateogory $\mathrm{Coh}_{\dim X, \dim X-1}(X)$. First, we prove for the case that $\mathcal{G}$ is semistable. Let $gr_i^{HN}(\mathcal{F}) = HN_i(\mathcal{F})/HN_{i-1}(\mathcal{F})$ be the Harder-Narasimhan filtration of $\mathcal{F}$. Then

$$0 = HN_0\mathcal{F} \otimes \mathcal{G} \subset \cdots \subset HN_n(\mathcal{F}) \otimes \mathcal{G} = \mathcal{F} \otimes \mathcal{G}$$

gives a filtration of $\mathcal{F} \otimes \mathcal{G}$ with semistable factors and strictly decreasing slope. Thus the uniqueness of Harder-Narasimhan filtration shows that $gr_i^{HN}(\mathcal{F}) \otimes \mathcal{G}$ gives a Harder-Narasimhan filtration of $\mathcal{F} \otimes \mathcal{G}$. $HN_0(\mathcal{F}) \otimes \mathcal{G}$ is the maximal destabilizing sheaf of $\mathcal{F} \otimes \mathcal{G}$ and $\mu_{max}(\mathcal{F} \otimes \mathcal{G}) = \mu_{max}(\mathcal{F}) + \mu_{max}(\mathcal{G})$.

For general $\mathcal{G}$, we prove $\mu_{max}(gr_i^{HN}(\mathcal{F}) \otimes \mathcal{G}) = \mu_{max}(\mathcal{F}) + \mu_{max}(\mathcal{G})$ by induction on i. The base case is from our discussion above. For $i > 1$, consider the exact sequence

$$0 \to HN_{i-1}(\mathcal{F}) \otimes \mathcal{G} \to HN_i(\mathcal{F}) \otimes \mathcal{G} \to gr_i^{HN}(\mathcal{F}) \otimes \mathcal{G} \to 0$$

in the category $\mathrm{Coh}_{\dim X, \dim X-1}$. Then by Remark 6,

$$\mu_{max}(HN_{i-1}(\mathcal{F}) \otimes \mathcal{G}) \leq \mu_{max}(HN_i(\mathcal{F}) \otimes \mathcal{G}).$$

Let $\mathcal{E}$ be the maximal destabilizing sheaf of $HN_i(\mathcal{F}) \otimes \mathcal{G}$. Let $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'/\mathcal{E} \to 0$ be the induced exact sequence by setting $\mathcal{E}' = \mathcal{E} \cap HN_{i-1}(\mathcal{F}) \otimes \mathcal{G}$, then $\mathcal{E}'/\mathcal{E}$ is a subsheaf of $gr_i^{HN}(\mathcal{F}) \otimes \mathcal{G}$. So we have

$$\begin{aligned}\mu_{max}(HN_i(\mathcal{F} \otimes \mathcal{G})) &= \mu(\mathcal{E}) \leq \max\{\mu(\mathcal{E}), \mu(\mathcal{E}'/\mathcal{E})\} \\ &\leq \max\{\mu_{max}(HN_{i-1}(\mathcal{F}) \otimes \mathcal{G}), \mu_{max}(gr_i^{HN}(\mathcal{F}) \otimes \mathcal{G})\} \\ &= \mu_{max}(HN_{i-1}(\mathcal{F}) \otimes \mathcal{G})\end{aligned}$$

Thus $\mu_{max}(HN_i(\mathcal{F}) \otimes \mathcal{G}) = \mu_{max}(\mathcal{F}) + \mu_{max}(\mathcal{G})$. □

Corollary 30. *Let $\mathcal{F}$ be a torsion-free semistable sheaf on X. Then all exterior products $\wedge^n \mathcal{F}$, all symmetric products $S^n \mathcal{F}$ and $\mathcal{H}om(\mathcal{F}, \mathcal{F})$ are semistable.*

Proof. Let $\dim X = d$. Note that $\wedge^n \mathcal{F}$ and $S^n \mathcal{F}$ are all direct summands of $\mathcal{F}^{\otimes n}$. One can easily calculate that $\mu(\mathcal{F}^{\otimes n}) = \mu(\wedge^n \mathcal{F}) = \mu(S^n \mathcal{F})$, so according to Remark 6, they are semistable.

We show that $\check{\mathcal{F}}$ is semistable. Suppose there is a destabilizing sheaf $\mathcal{E}$ of $\check{\mathcal{F}}$. We may assume $\check{\mathcal{F}}/\mathcal{E}$ is torsion-free, otherwise we may replace $\mathcal{E}$ by its saturation in $\check{\mathcal{F}}$. Therefore $\check{\mathcal{F}}/\mathcal{E}$ is locally free in codimension 1. Consider the exact sequence

$$0 \to \mathcal{E} \to \check{\mathcal{F}} \to \check{\mathcal{F}}/\mathcal{E} \to 0$$

in $\mathrm{Coh}_{d,d-1}(X)$, it's dual

$$0 \to (\check{\mathcal{F}}/\mathcal{E})\check{} \to \mathcal{F} \to \check{\mathcal{E}} \to 0$$

in $\mathrm{Coh}_{d,d-1}(X)$ is still exact. Then $\mu(\mathcal{E}) > \mu(\check{\mathcal{F}})$ implies $\mu(\mathcal{F}) > \mu(\check{\mathcal{E}})$, which contradict the assumption $\mathcal{F}$ is semistable. In codimension 1 $\mathcal{F}$ is locally free so $\mathcal{H}om(\mathcal{F}, \mathcal{F})$ is isomorphic to $\mathcal{F} \otimes \check{\mathcal{F}}$ in $\mathrm{Coh}_{d,d-1}(X)$. Since $\mathcal{F}$ and $\check{\mathcal{F}}$ are semistable, $\mathcal{H}om(\mathcal{F}, \mathcal{F})$ is semistable. □

5 Boundedness in Zero Characteristic

In this section, we will use Theorem 17 to show the boundedness of semistable sheaves with fixed Hilbert polynomial. The base field k is assumed to be algebraically closed with characteristic zero. Let $[x]_+ = \max\{x, 0\}$ for any real number x.

We first deal with the case that X is normal.

Lemma 31. *Let X be a normal projective variety of dimension d. Let $\mathcal{F}$ be a torsion-free sheaf on X. Then for any $\mathcal{F}$-regular sequence of hyperplane sections $H_1, \ldots, H_d$ and $X_v = H_1 \cap \cdots \cap H_{d-v}$ the following inequality holds for all $v = 1, \ldots, d$:*

$$\frac{h^0(X_v, \mathcal{F}|_{X_v})}{\mathrm{rk}(\mathcal{F}) \cdot \deg X} \leq \frac{1}{v!} \left[\frac{\mu_{max}(\mathcal{F}|_{X_1})}{\deg X} + v \right]_+^v .$$

Proof. The proof proceeds by induction on v. For the base case, let $gr_i^{HN}(\mathcal{F}|_{X_1})$, $i = 1, \ldots, l$ be the Harder-Narasimhan filtration of $\mathcal{F}|_{X_1}$. Taking the global sections one has

$$h^0(X_1, \mathcal{F}|_{X_1}) \leq \sum_{i=1}^{l} h^0(X_1, gr_i^{HN}(\mathcal{F}|_{X_1})).$$

We may assume $\mathcal{F}|_{X_1}$ is semistable and therefore $\mu_{max}(\mathcal{F}|_{X_1}) = \mu(\mathcal{F}|_{X_1})$. For any $n \geq 0$, we can take $\mathcal{F}|_{X_1}$-regular section H in $\mathcal{O}_{X_1}(n)$ and an exact sequence

$$0 \to \mathcal{F}|_{X_1}(-n) \to \mathcal{F}|_{X_1} \to \mathcal{F}|_{X_1 \cap H} \to 0$$

So

$$h^0(X_1, \mathcal{F}|_{X_1}) \leq h^0(X_1, \mathcal{F}|_{X_1}) + n \cdot \mathrm{rk}(\mathcal{F}) \cdot \deg X$$

For $n \geq \left\lceil \frac{\mu(\mathcal{F}|_{X_1})}{\deg X} \right\rceil + 1$, we have $h^0(X_1, \mathcal{F}|_{X_1}(-n)) = \hom(\mathcal{O}_{X_1}(n), \mathcal{F}|_{X_1}) = 0$. Thus, we get our desired upper bound.

Assume the inequality for $v-1$ with $v \geq 2$. Consider the exact sequence

$$0 \to \mathcal{F}|_{X_v}(-k-1) \to \mathcal{F}|_{X_v}(-k) \to \mathcal{F}|_{X_{v-1}}(-k) \to 0$$

for $k = 0, 1, 2, \ldots$, inductively we have

$$\begin{aligned} h^0(X_v, \mathcal{F}|_{X_v}) &\leq h^0(X_v, \mathcal{F}|_{X_v}(-n)) + \sum_{k=0}^{n-1} h^0(X_{v-1}, \mathcal{F}|_{X_{v-1}}(-k)) \\ &\leq \sum_{k=0}^{\infty} h^0(X_{v-1}, \mathcal{F}|_{X_{v-1}}(-k)). \end{aligned}$$

Similarly, $h^0(X_{v-1}, \mathcal{F}|_{X_{v-1}}(-k))$ vanishes for $k > \mu_{max}(\mathcal{F}|_{X_{v-1}})$ so the summation is actually finite. Using the induction hypothesis and replacing the summation by integral, we have

$$\frac{h^0(X_v, \mathcal{F}|_{X_v})}{\mathrm{rk}(\mathcal{F}) \cdot \deg X} \leq \frac{1}{(v-1)!} \int_{-1}^{\infty} \left[\frac{\mu_{max}(\mathcal{F}_{X_1})}{\deg X} + v - 1 - t \right]_+^{v-1} dt.$$

By simple calculus we have the right hand side of the lemma. $\square$

Using the above lemma and Corollary 25, we immediately have

Corollary 32. *Let X be a normal projective variety of dimension d and $\mathcal{F}$ be a torsion-free sheaf on X. For general hyperplanes $H_1, \ldots, H_d$ in $|\mathcal{O}_X(1)|$, the following inequality holds for all $v = 1, \ldots, d$:*

$$\frac{h^0(X_v, \mathcal{F}|_{X_v})}{\mathrm{rk}(\mathcal{F}) \cdot \deg X} \leq \frac{1}{v!} \left[\frac{\mu_{max}(\mathcal{F})}{\deg X} + \frac{\mathrm{rk}(\mathcal{F}) - 1}{2} + v \right]_+^v.$$

Corollary 32 gives a uniform bound for torsion-free semistable sheaves with fixed Hilbert polynomial (note $\mu_{max}(\mathcal{F}) = \mu(\mathcal{F})$) on normal projective variety. Combining it with Theorem 17, we get the boundedness for the specific case.

Theorem 33 (Le Potier-Simpson)**.** *Let X be a d-dimensional projective variety and $\mathcal{F}$ be a torsion-free sheaf on X. Let $r(\mathcal{F}) = \alpha_d(\mathcal{F})$ be the multiplicity of $\mathcal{F}$. Then there is a $\mathcal{F}$-regular sequence of hyperplane sections $H_1, \ldots, H_d$ and $X_v = H_1 \cap \cdots \cap H_{d-v}$ such that the following inequality holds for all $v = 1, \ldots, d$:*

$$\frac{h^0(X_v, \mathcal{F}|_{X_v})}{r(\mathcal{F})} \leq \frac{1}{v!} \left[\hat{\mu}_{max}(\mathcal{F}) + r(\mathcal{F})^2 + \frac{r(\mathcal{F}) + d}{2} - 1 \right]_+^v$$

Proof. Let $i : X \to \mathbb{P}^N$ be the closed immersion corresponding to the very ample sheaf $\mathcal{O}_X(1)$. Consider a $N-d-1$-dimensional linear subspace $L \subset \mathbb{P}^N$ which does not intersect X. Let $\pi : \mathbb{P}^N - L \to Y \cong \mathbb{P}^d$ be the projection with center L, π is a finite morphism. Denote $\pi_* \mathcal{O}_X$ by $\mathcal{A}$. $\pi_* \mathcal{F}$ is also torsion-free and $r(\mathcal{F}) = \alpha_d(\pi_* \mathcal{F}) = \mathrm{rk}(\pi_* \mathcal{F})$. We also have

$$\hat{\mu}(\mathcal{F}) = \hat{\mu}(\pi_* \mathcal{F}) = \frac{\mu(\pi_* \mathcal{F})}{\alpha_d(\mathcal{O}_Y)} + \hat{\mu}(\mathcal{O}_Y) = \mu(\pi_* \mathcal{F}) + \frac{d+1}{2}.$$

A $\pi_*\mathcal{F}$-regular sequence H'_i in $Y \cong \mathbb{P}^d$ naturally induces an $\mathcal{F}$-regular sequence H_i in X. Denote $Y_v = H'_1 \cap \cdots \cap H'_v$, then $\pi_*(\mathcal{F}|_{X_v}) = (\pi_*\mathcal{F})|_{Y_v}$. Apply Corollary 32 to $\pi_*\mathcal{F}$, there is an inequality for $v = 1, \dots, d$:

$$\frac{h^0(X_v, \mathcal{F}|_{X_v})}{\operatorname{rk}(\pi_*\mathcal{F})} \leq \frac{1}{v!}\left[\mu_{max}(\pi_*\mathcal{F}) + \frac{\operatorname{rk}(\pi_*\mathcal{F}) - 1}{2} + v\right]_+^v.$$

We need to estimate $\mu_{max}(\pi_*\mathcal{F})$ by $\hat{\mu}_{max}(\pi_*\mathcal{F})$. First we show that $\mu_{min}(\mathcal{A}) \geq -r(\mathcal{F})^2$: Clearly $\mathcal{A}$ is a torsion-free sheaf. The injection $\mathcal{A} \to \mathcal{H}om(\pi_*\mathcal{F}, \pi_*\mathcal{F})$ shows that $\operatorname{rk}(\mathcal{A}) \leq \operatorname{rk}(\pi_*\mathcal{F})^2 = r(\mathcal{F})^2$. Let $\mathcal{W} = \mathcal{O}_Y(-1)^{N-d}$, then X is a closed subscheme of the vector bundle $\pi : \mathbb{P}^N - L \cong \operatorname{Spec} S^*W \to Y$, so there is a surjection $\varphi : S^*\mathcal{W} \to \mathcal{A}$. Consider the filtration of $\mathcal{A}$ by ascending $\mathcal{O}_Y$-modules

$$F_i\mathcal{A} = \varphi(\mathcal{O}_Y \oplus \mathcal{W} \oplus \cdots \oplus S^i\mathcal{W})$$

Since $\mathcal{A}$ is coherent, only finitely many factors $gr_i^F(\mathcal{A})$ are not zero. Since $\mathcal{W} \otimes gr_i^F(\mathcal{A}) \to gr_{i+1}^F(\mathcal{A})$ is surjective, once $gr_i^F(\mathcal{A})$ has torsion, all $gr_j^F(\mathcal{A})$ has torsion for $j \geq i$. If $gr_i^F(\mathcal{A})$ has no torsion then $i \leq \operatorname{rk}(\mathcal{A})$, so the cokernel of $\varphi : \mathcal{O}_Y \oplus \mathcal{W} \oplus \cdots \oplus S^{\operatorname{rk}(\mathcal{A})}\mathcal{W} \to \mathcal{A}$ has torsion. Hence

$$\mu_{min}(\mathcal{A}) \geq \mu_{min}(S^{\operatorname{rk}(\mathcal{A})}\mathcal{W}) = \mu(S^{\operatorname{rk}(\mathcal{A})\mathcal{W}}) = \operatorname{rk}(\mathcal{A})\mu(\mathcal{W}) = -\operatorname{rk}(\mathcal{A})$$

and

$$\mu_{min}(\mathcal{A}) \geq -r(\mathcal{F})^2.$$

Let $\mathcal{E}$ be the maximal destabilizing sheaf of $\pi_*\mathcal{F}$ and $\mathcal{E}'$ be its image under the multiplication morphism $\mathcal{A} \otimes \mathcal{E} \to \mathcal{A} \otimes \pi_*\mathcal{F} \to \pi_*\mathcal{E}$. $\mathcal{E}'$ is the $\mathcal{A}$-submodule of $\pi_*\mathcal{F}$ generated by $\mathcal{E}$, and $\mathcal{E}' \cong \pi_*\mathcal{E}''$ for some $\mathcal{O}_X$-submodule $\mathcal{E}''$ of $\mathcal{F}$. So

$$\hat{\mu}_{max}(\mathcal{F}) \geq \hat{\mu}(\mathcal{E}'') = \hat{\mu}(\mathcal{E}') = \mu(\mathcal{E}') + \hat{\mu}(\mathcal{O}_Y),$$

by Lemma 5,

$$\mu(\mathcal{E}') + \hat{\mu}(\mathcal{O}_Y) \geq \mu_{min}(\mathcal{A} \otimes \mathcal{E}) + \frac{d+1}{2}$$

by Corollary 29,

$$\begin{aligned}\mu_{min}(\mathcal{A} \otimes \mathcal{E}) + \frac{d+1}{2} &= \mu(\mathcal{E}) + \mu_{min}(\mathcal{A}) + \frac{d+1}{2} \\ &\geq \mu_{max}(\pi_*\mathcal{F}) - r(\mathcal{F})^2 + \frac{d+1}{2}.\end{aligned}$$

Therefore we have

$$\mu_{max}(\pi_*\mathcal{F}) + \frac{\operatorname{rk}(\pi_*\mathcal{F}) - 1}{2} + v \leq \hat{\mu}_{max}(\mathcal{F}) + r(\mathcal{F})^2 + \frac{r(\mathcal{F}) - 1}{2} + \frac{d-1}{2}.$$

□

Theorem 34. *Let X be a projective variety over k. The family of semistable sheaves on X with fixed Hilbert polynomial is bounded.*

Proof. Immediate corollary from Theorem 17 and Theorem 33. □

6 Semistability in Positive Characteristic

In high dimensional, the traditional approach for boundedness won't work when the base field is of positive characteristic. Theorem 24, Lemma 26 and Theorem 68 strongly depend on the assumption of characteristic. The problem in the positive characteristic case is completely solved by Langer in his recent work [9].

In fact, Gieseker provided a counter-example for Lemma 26 in positive characteristic in [2]: Assume the base field has characteristic p. Let $F : X \to X$ be the absolute Frobenius morphism, i.e. the morphism which is an identity on topological space and sends $x \mapsto x^p$ on $\mathcal{O}_X$.

Example 35 ([2])**.** Let p be any positive integer and $g \geq 2$, then there is a curve X of genus g over a field of characteristic p, and a semistable bundle $\mathcal{E}$ of rank 2 on X, such that $F^*\mathcal{E}$ is not semistable.

The argument using ampleness does not work in positive characteristic cases either. To make the result similar to Lemma 28, Langer introduced *strongly semistable* in [9].

Definition 36. A coherent sheaf $\mathcal{F}$ in characteristic p is *strongly semistable* if $(F^n)^*\mathcal{F}$ is semistable for all $n \geq 0$.

For a strongly semistable sheaf $\mathcal{F}$, the Frobenius pullback $F^*\mathcal{F}$ is also strongly semistable. However, for a general morphism $f : Y \to X$, $f^*\mathcal{F}$ is not necessary to be strongly semistable. We have the following lemma

Lemma 37. *Let $\mathcal{F}$ be a strongly semistable sheaf on a curve C and $f : C' \to C$ be a surjective morphism between curves. Then $f^*\mathcal{F}$ is semistable.*

Proof. Consider $f : C' \to C$ factor through smooth C'' such that $K(C'')$ is the separable closure of $K(C)$ in $K(C')$. Then $g : C' \to C''$ gives an purely inseparable extension and thus a Frobenius morphism (c.f. [7] Chapter 4 Theorem 2.5). Then by Lemma 26 $f^*\mathcal{F}$ is semistable. $\square$

In the curve case, we can have a general criteria for semistability using nefness:

Theorem 38. *Let $\mathcal{F}$ be a torsion-free sheaf of rank r on curve C. D is a divisor in $|\det(\mathcal{F})|$. Then $\mathcal{F}$ is semistable if and only if $\mathcal{F}(-\frac{D}{r})$ is semistable if and only if $\mathcal{F}(-\frac{D}{r})$ is nef.*

Proof. Clearly $\mathcal{F}$ is semistable if and only if $\mathcal{F}(-\frac{D}{r})$ is semistable. If $\mathcal{F}(-\frac{D}{r})$ is nef, by Proposition 62, all the quotient of $\mathcal{F}(-\frac{D}{r})$ has non-negative degree. Therefore $\mathcal{F}$ is semistable.

Conversely, suppose $\mathcal{F}(-\frac{D}{r})$ is not nef. By Theorem 63, we can find $f : C' \to C$ surjective morphism between smooth curves such that $f^*\mathcal{F}(-\frac{D}{r})$ has negative quotient line bundle. Hence $f^*\mathcal{F}(-\frac{D}{r})$ is not semistable and therefore $\mathcal{F}$ is not semistable. $\square$

Remark 39. Using Lemma 37, the proof for Theorem 38 is still valid.

Using above theorem, we get

Corollary 40. *The tensor product of two (strongly) semistable bundles on a smooth curve C is (strongly) semistable. The symmetric power of (strongly) semistable sheaf on C is (strongly) semistable.*

The degree of Frobenius morphism is p, an easy computation gives out $\mu((F^*)^n\mathcal{F}) = p^n\mu(\mathcal{F})$. We will need the approximate behviour when $n \to \infty$. Define

$$L_{max}(\mathcal{F}) = \lim_{n\to\infty} \frac{\mu_{max}(F^{n*}\mathcal{F})}{p^n}, \quad L_{min}(\mathcal{F}) = \lim_{n\to\infty} \frac{\mu_{min}(F^{n*}\mathcal{F})}{p^n}.$$

Pick the maximal destabilizing sheaf $\mathcal{E}$ of $F^{n*}\mathcal{F}$,

$$\mu_{max}(F^{n*}\mathcal{F}) = \mu(\mathcal{E}) = \frac{\mu(F^*(\mathcal{E}))}{p} \leq \mu_{max}((F^*)^{n+1}\mathcal{F}).$$

Therefore $\frac{\mu_{max}(F^{n*}\mathcal{F})}{p^n}$ is increasing in n, and we have similar result for μ_{min}. By definition, we have $L_{max}(\mathcal{F}) \geq \mu_{max}(\mathcal{F})$, $L_{min}(\mathcal{F}) \leq \mu_{min}(\mathcal{F})$, and if $\mathcal{F}$ is strongly semistable, $L_{max}(\mathcal{F}) = L_{min}(\mathcal{F})$.

Let $\alpha(\mathcal{F}) = \max L_{max}(\mathcal{F}) - \mu_{max}(\mathcal{F}), \mu_{min}(\mathcal{F}) - L_{min}(\mathcal{F})$. Our first goal is to give an upper bound for $\alpha(\mathcal{F})$. Fix an set of ample divisors $(H_1, \dots, H_d)$. We can realize $\mu(\mathcal{F})$ under the polarization of $(H_1, \dots, H_{d-1})$ by

$$\mu(\mathcal{F}) = \frac{c_1(\mathcal{F})c_1(\mathcal{O}(H_1))\cdots c_1(\mathcal{O}(H_{d-1}))}{\operatorname{rk}\mathcal{F}} = \frac{c_1(\mathcal{F})H_1\cdots H_{d-1}}{\operatorname{rk}\mathcal{F}}.$$

In the following part we will treat the general polarization.

Theorem 41. *Let $\mathcal{F}$ be a semistable sheaf on X such that $F^*\mathcal{F}$ is not a semistable sheaf on X. Let $0 = HN_0(F^*\mathcal{F}) \subset \cdots HN_l(F^*\mathcal{F}) = F^*\mathcal{F}$ be the Harder-Narasimhan filtration. Then the natural morphism $HN_i(F^*\mathcal{F}) \to F^*\mathcal{F}/HN_i(F^*\mathcal{F}) \otimes \Omega_X$ induced by the canonical connection is nontrivial.*

Proof. We refer to [14]. The proof is far beyond what we should do. $\square$

We can now give the estimation of Harder-Narasimhan filtration of $F^*\mathcal{F}$.

Lemma 42. *Let A be a nef divisor such that Ω_X can be embedded into $\mathcal{O}(A)^{\otimes N}$ for some N. Let $\mathcal{F}$ be a torsion-free semistable sheaf on X, then*

$$\mu_{max}(F^*\mathcal{F}) - \mu_{min}(F^*\mathcal{F}) \leq (\operatorname{rk}(\mathcal{F}) - 1)A \cdot H_1 \cdots H_{n-1}$$

and

$$\frac{\mu_{max}(F^*\mathcal{F})}{p} \leq \mu_{max}(\mathcal{F}) + \frac{\operatorname{rk}\mathcal{F} - 1}{p}AH_1\cdots H_{d-1}.$$

Proof. Let $0 = \mathcal{F}_0 \subset \cdots \mathcal{F}_l = F^*\mathcal{F}$ be the Harder-Narasimhan filtration of $F^*\mathcal{F}$. Assume $F^*\mathcal{F}$ is not semistable. By above theorem, the natural morphism $\mathcal{F}_i \to F^*\mathcal{F}/\mathcal{F}_i \otimes \Omega_X$ is nontrivial. Thus $\mu(\mathcal{F}_i)/\mathcal{F}_{i-1} = \mu_{min}(\mathcal{F}_i) \leq \mu_{max}(F^*\mathcal{F}/\mathcal{F}_i) \otimes \Omega_X$. Note $F^*\mathcal{F}/\mathcal{F}_i \otimes \Omega_X \to (F^*\mathcal{F}/\mathcal{F}_i \otimes \mathcal{O}(A))^N$ is an injection, by Remark 11 we get

$$\begin{aligned}\mu_{max}({}^*\mathcal{F}/\mathcal{F}_i \otimes \Omega_X) &\leq \mu_{max}((F^*\mathcal{F}/\mathcal{F}_i \otimes \mathcal{O}(A))^N) \\ &= \mu_{max}(F^*\mathcal{F}/\mathcal{F}_i \otimes \mathcal{O}(A)) \\ &\leq \mu(\mathcal{F}_{i+1}/\mathcal{F}_i) + AH_1\cdots H_{d-1}.\end{aligned}$$

For each i we can get such an inequality, sum them up, we have

$$\mu_{max}(F^*\mathcal{F}) - \mu_{min}(F^*\mathcal{F}) \leq (\mathrm{rk}(\mathcal{F}) - 1)A \cdot H_1 \cdots H_{n-1}.$$

Substitute $\mathcal{F}$ by $F^*(\mathcal{F}_i/\mathcal{F}_{i-1})$ in above inequality, we have

$$\frac{\mu_{max}(F^*(\mathcal{F}_i/\mathcal{F}_{i-1}))}{p} \leq \mu(\mathcal{F}_i/\mathcal{F}_{i-1}) + \frac{\mathrm{rk}(\mathcal{F}) - 1}{p} AH_1 \cdots H_{d-1}.$$

Again, by Remark 11, $\mu_{max}(F^*\mathcal{F}) \leq \max\{\mu_{max}(F^*(\mathcal{F}_i/\mathcal{F}_{i-1}))\}$ and $\mu(\mathcal{F}_i/\mathcal{F}_{i-1})$ by definition. We immediate get the desired inequality. □

Theorem 43. *Let A be a nef divisor such that Ω_X can be embedded into $\mathcal{O}(A)^{\otimes N}$ for some N. Let $\mathcal{F}$ be a torsion-free semistable sheaf on X, then*

$$\alpha(\mathcal{F}) \leq \frac{\mathrm{rk}\,\mathcal{F} - 1}{p - 1} AH_1 \cdots H_{d-1}.$$

Proof. Apply induction to inequality in Lemma 42, we have

$$\frac{\mu_{max}((F^*)^n\mathcal{F})}{p} \leq \mu_{max}(\mathcal{F}) + (\mathrm{rk}\,\mathcal{F} - 1)AH_1 \cdots H_{d-1}(\frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^n}).$$

Then let $n \to \infty$, we have $L_{max}(\mathcal{F}) - \mu_{max}(\mathcal{F}) \leq \frac{\mathrm{rk}(\mathcal{F})-1}{p-1} AH_1 \cdots H_{d-1}$. The proof for $\mu_{min}(\mathcal{F}) - L_{min}(\mathcal{F})$ is similar. □

The interesting thing is if we pulls back any torsion-free sheaf for sufficiently large times, we will finially reach a "stable" state of Harder-Narasimhan factors. The proof of following theorem involves "Harder-Narasimhan polygen", which is the convex hull with vertex $(\mathrm{rk}(HN_i(\mathcal{F})), \deg(HN_i(\mathcal{F})))$. The proof uses measurement on the area of Harder-Narasimhan polygen.

Theorem 44. *Let $\mathcal{F}$ be a torsion-free sheaf on X. Then there exists a positive integer N such that all the factors in the Harder-Narasimhan filtration of $(F^*)^n\mathcal{F}$ is strongly semistable for all $n \geq N$.*

Proof. See section 2.6 in [9]. □

7 Bogomolov Theorems

In this sesction, we will prove the main theorems in [9], which generalize the Bogomolov inequality to positive characteristic case.

We first give some notations. Assume now X is a smooth projective variety of dimension $n \geq 2$ over some field of characteristic p. Fix an set of ample divisors $(H_1, \ldots, H_n)$ and a nef divisor A such that $\Omega_X \to \mathcal{O}(A)^N$ is injective for some N. For any positive integer r, define the constant $\beta_r(A, H_1, \ldots, H_{d-1}) = (\frac{r(r-1)}{p-1} AH_1 \cdots H_{d-1})^2$.

Definition 45. For any rank r torsion-free sheaf $\mathcal{F}$, the *discriminant* of $\mathcal{F}$ is

$$\Delta(\mathcal{F}) = 2rc_2(\mathcal{F}) - (r-1)c_1(\mathcal{F})^2.$$

In characteristic zero case, we have the well-known Bogomolov inequality for surface X:

$$\Delta(\mathcal{F}) \geq 0.$$

For a nontrivial torsion-free subsheaf $\mathcal{E} \subset \mathcal{F}$, let $\xi_{\mathcal{E},\mathcal{F}} = \frac{c_1(\mathcal{E})}{\operatorname{rk}\mathcal{E}} - \frac{c_1(\mathcal{F})}{\operatorname{rk}\mathcal{F}}$.

Let $\operatorname{Num}(X) = \operatorname{Pic}(X) \otimes \mathbb{R}/\sim$, where the equivalence $\mathcal{L}_1 \sim \mathcal{L}_2$ if $c_1(\mathcal{L}_1)AH_2 \cdots H_{d-1} = c_1(\mathcal{L}_2)AH_2 \cdots H_{d-1}$. Define the open cone

$$K^+ = \{D \in \operatorname{Num}(X) | D^2 H_2 \cdots H_{n-1} > 0 \text{ and } DD'H_2 \cdots H_{n-1} \geq 0 \text{ for all nef } D'\}.$$

The second assumption in the definition of K^+ is used to ensure K^+ is connected. K^+ is an subcone of the cone of effective divisors and contains the ample cone. We can state our main theorem:

Theorem 46. *Let $(H_1, \ldots, H_{n-1})$ be a set of ample divisors on X, the degree $d = H_1^2 H_2 \cdots H_{n-1}$. For any positive integer r, the followings are equivalent:*

1. *Assume $\mathcal{F}$ is a torsion-free strongly semistable sheaf with $\operatorname{rk}\mathcal{F} \leq r$, then we have*

$$\Delta(\mathcal{F}) H_2 \cdots H_{n-1} \geq 0.$$

2. *Let $\mathcal{F}$ be torsion-free strongly semistable sheaf of rank $r' \leq r$. Then we have*

$$d \cdot \Delta(\mathcal{F}) H_2 \cdots H_{n-1} + \beta_{r'} \geq 0.$$

3. *Let $\mathcal{F}$ be a torsion-free sheaf of rank $r' \leq r$. If $d\,\Delta(\mathcal{F})H_2 \cdots H_{n-1} + \beta_{r'} < 0$, then there exists a subsheaf $\mathcal{E} \subset \mathcal{F}$ such that $\mathcal{F}/\mathcal{E}$ is torsion-free and $\xi_{\mathcal{E},\mathcal{F}} \in K^+$.*

Theorem 47. *Let $(H_1, \ldots, H_{n-1})$ be a set of ample line bundles on X, $d = H_1^2 H_2 \cdots H_{n-1}$. Let $\mathcal{F}$ be a torsion-free sheaf with $\operatorname{rk}\mathcal{F} = r$ on X. Assume that H_1 is very ample and the restriction to a very general divisor $H \in |H_1|$ is not semistable (with respect to polarization $(H_2|_H, \ldots, H_{n-1}|_H)$). Let μ_i and r_i be $\mu(\mathcal{F}_i/\mathcal{F}_{i-1})$ and $\operatorname{rk}(\mathcal{F}_i/\mathcal{F}_{i-1})$, where $0 = \mathcal{F}_0 \subset \cdots \mathcal{F}_l = \mathcal{F}|_H$ is the Harder-Narasimhan filtration. Then*

$$\sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 \leq d\,\Delta(\mathcal{F}) H_2 \cdots H_{n-1} + 2r^2 (L_{max}(\mathcal{F}) - \mu(\mathcal{F}))(\mu(\mathcal{F}) - L_{min}(\mathcal{F})).$$

Label each assertion in Theorem 46 by $T_1(r)$, $T_2(r)$, $T_3(r)$ and Theorem 47 by $R(r)$. We shall show $T_1(r) \Rightarrow T_2(r) \Rightarrow T_3(r) \Rightarrow T_1(r)$.

Recall our $\mu(\mathcal{F})$ is defined with respect to ample line sheaves $H_1, \ldots, H_{n-1}$, a natural question is how will Harder-Narasimhan filtration behaves under a slight perturbation. Let H be an ample line sheaf, define $(H_1(t), \ldots, H_{n-1}(t)) = (H_1 + tH, \ldots, H_{n-1} + tH)$.

Theorem 48. *Assume $\mathcal{F}$ is a torsion-free sheaf on X. Then there exists a positive number ϵ and a unique filtration $0 = HN_0(\mathcal{F}) \subset \cdots \subset HN_l(\mathcal{F}) = \mathcal{F}$ such that for any $t \in (0, \epsilon)$, the filtration $HN_i(\mathcal{F})$ is the Harder-Narasimhan filtration with the respect to the polarization $(H_1(t), \ldots, H_{n-1}(t))$.*

Proof. It suffices to show there exists a subsheaf $\mathcal{F}_1 \subset \mathcal{F}$ such that $\mathcal{F}_1$ is the maximal destabilizing sheaf with respect to $(H_1(t), \ldots, H_{n-1}(t))$ for any $t \in (0, \epsilon)$. Note that

$$\mathcal{F} \mapsto \frac{c_1(\mathcal{F})H_1(t)\cdots H_{n-1}(t)}{\operatorname{rk}\mathcal{F}}$$

defines a map from the set of all torsion-free sheaves on X to the set of degree $n-1$ polynomials in t. For any $\mathcal{F}' \subset \mathcal{F}$, expand $\frac{c_1(\mathcal{F}')H_1(t)\cdots H_{n-1}(t)}{\operatorname{rk}\mathcal{F}'}$ as $a_0 + a_1 t + \cdots + a_{n-1}t^{n-1}$, then

$$a_0(\mathcal{F}') = \mu(\mathcal{F}'),$$

$$a_1(\mathcal{F}') = \sum \frac{c_1(\mathcal{F}')HH_1\cdots \hat{H}_i \cdots H_{n-1}}{\operatorname{rk}\mathcal{F}'},$$

$$\vdots$$

$$a_j(\mathcal{F}') = \sum_{1 \le i_1 < \cdots < i_{n-1-j} \le n-1} \frac{c_1(\mathcal{F}')H^j H_{i_1}\cdots H_{i_{n-1-j}}}{\operatorname{rk}\mathcal{F}'}.$$

Each coefficient takes value in $\frac{1}{\operatorname{rk}\mathcal{F}!}\mathbb{Z}$ and bounded by the coefficients of $\frac{c_1(\mathcal{F})H_1(t)\cdots H_{n-1}(t)}{\operatorname{rk}\mathcal{F}}$ since $\mathcal{F}'$ is a subsheaf. Denote the image of such map by S, S is a subset of $\mathbb{Q}[t]$ and has a natural lexicographic order: $\sum_{i=0}^{n-1} a_i t^i \prec \sum_{i=0}^{n-1} b_i t^i$ if $a_0 = b_0, \ldots, a_{i-1} = b_{i-1}$ and $a_i < b_i$ for some i. Since the coefficients of elements in S is bounded above, we can find a maximal polynomial $Q(t)$ in lexicographic order.

Let $\mathcal{F}_1$ be the subsheaf mapsto $Q(t)$. Since the set S is bounded above, we can find $\epsilon > 0$ such that for any $t \in (0, \epsilon)$ and $W(t) \in S$, $Q(t) > R(t)$. Therefore for any $t \in (0, \epsilon)$ and $\mathcal{F}' \subset \mathcal{F}$, $\mu_t(\mathcal{F}_1) \ge \mu_t(\mathcal{F}')$. $\mathcal{F}_1$ is the maximal destabilizing sheaf.

□

First we show that $T_1(r) \Rightarrow T_2(r)$. The key is approximate the polarization $(H_1, \ldots, H_{n-1})$ by $(H_1(t), \ldots, H_{n-1}(t))$. We shall give a bound for $\Delta(\mathcal{F})H_2(t)\cdots H_{n-1}(t)$ by $L_{max,t}$, $L_{min,t}$ and μ_t.

Lemma 49. *Assume for $T_1(r)$. For any $t \ge 0$, let $d(t) = H_1(t)^2 H_2(t)\cdots H_{n-1}(t)$ be the degree with respect to $(H_1(t), \ldots, H_{n-1}(t))$. Then*

$$d(t)\,\Delta(\mathcal{F})H_2(t)\cdots H_{n-1}(t) + r^2(L_{max,t}(\mathcal{F}) - \mu_t(\mathcal{F}))(\mu_t(\mathcal{F}) - L_{max,t}(\mathcal{F})) \ge 0.$$

Proof. Apply Theorem 44 to $\mathcal{F}$, there exists N such that $(F^*)^n\mathcal{F}$ has strongly semistable Harder-Narasimhan factors for all $n \ge N$. Let $0 = \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_l = (F^{N*})\mathcal{F}$ be the Harder-Narasimhan filtration. Denote the rank and slope of $\mathcal{E}_i = \mathcal{F}_i/\mathcal{F}_{i-1}$ by r_i and μ_i. Then

$$\begin{aligned}
&\frac{\Delta((F^{N*})\mathcal{F})H_2(t)\cdots H_{n-1}(t)}{r}\\
=&(c_2(F^{N*}\mathcal{F}) - c_1(F^{N*}\mathcal{F})^2 - \frac{1}{r}c_1(F^{N*}\mathcal{F})^2)H_2(t)\cdots H_{n-1}(t)\\
=&(\sum(c_2(\mathcal{E}_i) - c_1(\mathcal{E}_i)^2) - \sum_{i<j} c_1(\mathcal{E}_i)c_1(\mathcal{E}_j) + \frac{(\sum c_1(\mathcal{E}_i))^2}{\sum r_i})H_2(t)\cdots H_{n-1}(t)\\
=&(\sum(c_2(\mathcal{E}_i) - c_1(\mathcal{E}_i)^2 + \frac{1}{r_i}c_1(\mathcal{E}_i)^2) - \frac{1}{r}\sum_{i<j} r_i r_j(\frac{c_1(\mathcal{F}_i)}{r_i} - \frac{c_1(\mathcal{F}_j)}{r_j})^2)H_2(t)\cdots H_{n-1}(t)\\
=&\sum_i \frac{\Delta(\mathcal{E}_i)H_2(t)\cdots H_{n-1}(t)}{r_i} - \frac{1}{r}\sum_{i<j} r_i r_j(\frac{c_1(\mathcal{F}_i)}{r_i} - \frac{c_1(\mathcal{F}_j)}{r_j})^2 H_2(t)\cdots H_{n-1}(t).
\end{aligned}$$

By Theorem 69,

$$(\frac{c_1(\mathcal{F}_i)}{r_i} - \frac{c_1(\mathcal{F}_j)}{r_j})^2 H_2(t)\cdots H_{n-1}(t) \leq d(\mu_{i,t} - \mu_{j,t}),$$

and by $T_1(r)$,

$$\Delta(\mathcal{E}_i)H_2(t)\cdots H_n(t) \geq 0.$$

So

$$\begin{aligned}
&\Delta(F^{N*}\mathcal{F})H_2(t)\cdots H_{n-1}(t)\\
\geq&\sum_i \frac{r\,\Delta(\mathcal{E}_i)H_2(t)\cdots H_{n-1}(t)}{r_i} - \frac{1}{d}\sum_{i<j} r_i r_j(\mu_{i,t} - \mu_{j,t})^2\\
\geq&-\frac{1}{d}\sum_{i<j} r_i r_j(\mu_{i,t} - \mu_{j,t})^2 \geq -\frac{1}{d}r^2(\mu_{1,t} - \mu)(\mu - \mu_{l,t})\\
\geq&-\frac{1}{d}(L_{max,t} - \mu)(\mu - L_{max,t}).
\end{aligned}$$

$\square$

Proof of Theorem 46 $T_1(r) \Rightarrow T_2(r)$. Apply Theorem 48 to $\mathcal{F}$, we have filtration $0 = \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_l = \mathcal{F}$ which is the Harder-Narasimhan filtration of $\mathcal{F}$ when $t \in (0, \epsilon)$. Since $\mathcal{F}$ is $(H_1, \ldots, H_{n-1})$-semistable, we have

$$\begin{aligned}
\mu(\mathcal{F}) &\geq \mu(\mathcal{F}_1)\\
&= \lim_{t\to 0^+} \mu_t(\mathcal{F}_1)\\
&\geq \lim_{t\to 0^+} \mu_t(\mathcal{F})\\
&= \mu(\mathcal{F}).
\end{aligned}$$

So $\mu(\mathcal{F}) = \lim_{t\to 0^+} \mu_{max,t}(\mathcal{F}) = \lim_{t\to 0^+} \mu_{min,t}(\mathcal{F}) = \lim_{t\to 0^+} \mu_t(\mathcal{F})$. According to Theorem 43, $L_{max,t} - \mu_{max,t} \leq \frac{r-1}{p-1}AH_1\cdots H_{n-1}(t)$ and $\mu_{min,t} - L_{min,t} \leq \frac{r-1}{p-1}AH_1\cdots H_{n-1}(t)$. Therefore

$$\begin{aligned}
&d\,\Delta(\mathcal{F})H_2\cdots H_{n-1} + \beta_r\\
=&\lim_{t\to 0^+}(d\,\Delta(\mathcal{F})H_2\cdots H_{n-1} + (\frac{r(r-1)}{p-1}AH_1(t)\cdots H_{n-1}(t))^2)\\
\geq&\lim_{t\to 0^+}(d(t)\,\Delta(\mathcal{F})H_2(t)\cdots H_{n-1}(t) + r^2(L_{max,t} - \mu_t)(\mu_t - L_{min,t})) \geq 0.
\end{aligned}$$

$\square$

For $T_2(r) \Rightarrow T_3(r)$, we will need the following lemmas:

Lemma 50. *A divisor $D \in K^+$ if and only if $DLH_2 \cdots H_{n-1} > 0$ for all $L \in \bar{K^+} \setminus \{0\}$.*

Proof. By scaling the cone we may assume $H_2, \ldots, H_{n-1}$ very ample. So we can restrict to hypersurfaces and assume X has dimension two. Then $K^+ = \{D | D^2 > 0 \text{ and } DD' \geq 0 \text{ for all nef } D'\}$. By Corollary 70, the index of (D, D') is $(1, h^{1,1}(X) - 1)$. Pick an orthogonal basis $\{K_i\}_{i=1}^r$ of $\mathrm{Num}(X)$ with $K_1^2 = 1$ and $K_i^2 = -1$ for all $i \neq 1$. Let $M = \{D = (x_1, \ldots, x_r) | D^2 > 0, x_1 > 0\}$, we first show $K^+ = M$.

We may assume K_1 is ample. Clearly $K^+ \subset M$. Pick $D \in M$ a nonzero element and any nef divisor D'. Since intersection product restricting to the subspace generated by D and D' has index $(1, -1)$, it is non-degenerate. The continuous map $D \mapsto DD'$ has positive value, M is connected so $D \mapsto DD'$ only takes positive values. Therefore $D \in K^+$ and $M \subset K^+$.

Let $D = (x_1, \ldots, x_r)$ be an element in K^+ and $L = (y_1, \ldots, y_r) \in \bar{K}^+ \backslash \{0\}$. Then we have

$$x_1 > 0 \text{ and } x_1^2 > \sum_{i \geq 2} x_i^2$$

$$y_1 \geq 0 \text{ and } y_1^2 \geq \sum_{i \geq 2} x_i^2$$

$DL \leq 0$ would derive:

$$x_1 y_1 \leq \sum_{i \geq 2} x_i y_i \leq \sqrt{\sum_{i \geq 2} x_i^2} \cdot \sqrt{\sum_{i \geq 2} y_i^2} < x_1 y_1.$$

Therefore $DL > 0$. Conversely, for any $D = (x_1, \ldots, x_r)$ satisfies $DL > 0$ for all $L \in \bar{K^+} \setminus \{0\}$, take $L = (1, 0, \ldots, 0)$, we have $x_1 > 0$. Then take $L = (\sqrt{\sum_{i \geq 2} x_i^2}, x_2, \ldots, x_r)$ we will have $x_1 > \sum_{i \geq 2} x_i^2$. By the definition $D \in M$ so $D \in K^+$. $\square$

Lemma 51. *Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of torsion-free coherent sheaves. Let r' and r'' be the rank of $\mathcal{F}'$ and $\mathcal{F}''$ respectively. Then*

$$\frac{\Delta(\mathcal{F}) H_2 \cdots H_{n-1}}{r} + \frac{rr'}{r''} \xi^2_{\mathcal{F}', \mathcal{F}} H_2 \cdots H_{n-1} = \frac{\Delta(\mathcal{F}') H_2 \cdots H_{n-1}}{r'} + \frac{\Delta(\mathcal{F}'') H_2 \cdots H_{n-1}}{r''}.$$

Proof. Compute the first two Chern classes of $\mathcal{F}$, we have $c_1(\mathcal{F}) = c_1(\mathcal{F}') + c_1(\mathcal{F}'')$ and $c_2(\mathcal{F}) = c_1(\mathcal{F}') c_1(\mathcal{F}'') + c_1(\mathcal{F}')^2 + c_1(\mathcal{F}'')^2$. Thus

$$\begin{aligned}
&\frac{\Delta(\mathcal{F}) H_2 \cdots H_{n-1}}{r} + \frac{rr'}{r''} \xi^2_{\mathcal{F}', \mathcal{F}} H_2 \cdots H_{n-1} \\
=&(2c_2(\mathcal{F}) - c_1(\mathcal{F})^2 + \frac{c_1(\mathcal{F})^2}{r} + \frac{rc_1(\mathcal{F}')^2}{r''r'} + \frac{r'c_1(\mathcal{F})^2}{r''r} - \frac{2c_1(\mathcal{F}')c_1(\mathcal{F})}{r''}) H_2 \cdots H_{n-1} \\
=&(2c_2(\mathcal{F}') - c_1^2(\mathcal{F}') + \frac{c_1(\mathcal{F}')}{r'} + 2c_2(\mathcal{F}'') - c_1^2(\mathcal{F}'') + \frac{c_1(\mathcal{F}')}{r''}) H_2 \cdots H_{n-1} \\
=&\frac{\Delta(\mathcal{F}') H_2 \cdots H_{n-1}}{r'} + \frac{\Delta(\mathcal{F}'') H_2 \cdots H_{n-1}}{r''}.
\end{aligned}$$

$\square$

Lemma 52. *Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of torsion-free coherent sheaves. Let r' and r'' be the rank of $\mathcal{F}'$ and $\mathcal{F}''$ respectively.*

1. *If $\mathcal{E}'$ is a nontrivial torsion-free subsheaf of $\mathcal{F}'$, then $\xi_{\mathcal{E}',\mathcal{F}} = \xi_{\mathcal{E}',\mathcal{F}'} + \xi_{\mathcal{F}',\mathcal{F}}$.*
2. *If $\mathcal{E}''$ is a nontrivial torsion-free subsheaf of $\mathcal{F}''$ of rank s, denote by $\mathcal{E}$ the kernel of the map $\mathcal{F} \to \mathcal{F}''/\mathcal{E}''$, then*
$$\xi_{\mathcal{E},\mathcal{F}} = \frac{r'(r''-s)}{(r'+s)r''}\xi_{\mathcal{F}',\mathcal{F}} + \frac{s}{r'+s}\xi_{\mathcal{E}'',\mathcal{F}''}.$$

Proof. The first equation is obvious. For the second equation,

$$\begin{aligned}
&\frac{r'(r''-s)}{(r'+s)r''}\xi_{\mathcal{F}',\mathcal{F}} + \frac{s}{r'+s}\xi_{\mathcal{E}'',\mathcal{F}''} \\
=&\frac{r''-s}{(r'+s)rr''}(r''c_1(\mathcal{F}) - rc_1(\mathcal{F}'')) + \frac{1}{r''(r'+s)}((r''-s)c_1(\mathcal{F}'') - r''c_1(\mathcal{F}''/\mathcal{E}'')) \\
=&\frac{c_1(\mathcal{E})}{r'+s} - \frac{c_1(\mathcal{F})}{r} = \xi_{\mathcal{E},\mathcal{F}}.
\end{aligned}$$

□

Now we are ready to proof $T_2(r) \Rightarrow T_3(r)$:

Proof of Theorem 46 $T_2(r) \Rightarrow T_3(r)$. The idea is induction on $\operatorname{rk}\mathcal{F} = r$. The base case is clear since for rank one $\mathcal{F}$, $\Delta(\mathcal{F}) = 0$ and $\beta_1 \geq 0$.

Suppose $T_3(r)$ is not true for some $\mathcal{F}$. Using $T_2(r)$, the assumption in $T_3(r)$ tells us that $\mathcal{F}$ is not semistable. Pick the maximal destabilizing sheaf $\mathcal{F}'$ of $\mathcal{F}$ and set $\mathcal{F}'' = \mathcal{F}/\mathcal{F}'$. Assume the rank of $\mathcal{F}$, $\mathcal{F}'$ and $\mathcal{F}''$ are r, r' and r'' respectively. Then we have $\frac{\beta_r}{r} \geq \frac{\beta_{r'}}{r'} + \frac{\beta_{r''}}{r''}$. Apply Lemma 51,

$$\frac{d\,\Delta(\mathcal{F})H_2\cdots H_{n-1}}{r} + \frac{drr'}{r''}\xi^2_{\mathcal{F}',\mathcal{F}}H_2\cdots H_{n-1} = \frac{d\,\Delta(\mathcal{F}')H_2\cdots H_{n-1}}{r'} + \frac{d\,\Delta(\mathcal{F}'')H_2\cdots H_{n-1}}{r''}.$$

So either $\xi^2_{\mathcal{F}',\mathcal{F}}H_2\cdots H_{n-1} > 0$ or one of $d\,\Delta(\mathcal{F}')H_2\cdots H_{n-1} + \beta_{r'}$, $d\,\Delta(\mathcal{F}'')H_2\cdots H_{n-1} + \beta_{r''}$ is negative.

If $\xi^2_{\mathcal{F}',\mathcal{F}}H_2\cdots H_{n-1} > 0$, we show that $\xi_{\mathcal{F}',\mathcal{F}} \in K^+$. For any $L \in \bar{K^+} \setminus \{0\}$, we need to show that $\xi_{\mathcal{F}',\mathcal{F}}LH_2\cdots H_{n-1} > 0$. For $L = H_1$, since $\mu(\mathcal{F}') \geq \mu(\mathcal{F})$, the inequality is true. So it remains to show that the continuous function $L \mapsto \xi_{\mathcal{F}',\mathcal{F}}LH_2\cdots H_{n-1}$ does not change sign on K^+. By Theorem 69

$$(\xi_{\mathcal{F}',\mathcal{F}}LH_2\cdots H_{n-1})^2 \geq \xi^2_{\mathcal{F}',\mathcal{F}}LH_2\cdots H_{n-1} \cdot L^2H_2\cdots H_{n-1} > 0.$$

Since K^+ is connected, the continuity of $L \mapsto \xi_{\mathcal{F}',\mathcal{F}}LH_2\cdots H_{n-1}$ proves the assertion.

If $\xi^2_{\mathcal{F}',\mathcal{F}}H_2\cdots H_{n-1} \leq 0$, without losing of generality we may assume $d\,\Delta(\mathcal{F}')H_2\cdots H_{n-1} + \beta_{r'} < 0$. By $T_3(r')$ and $T_3(r'')$, we can find saturated subsheaves $\mathcal{E}' \subset \mathcal{F}'$ such that $\xi_{\mathcal{E}',\mathcal{F}'} \in K^+$. Define the open subcone $C(\xi) = \{L \in \bar{K^+} \setminus \{0\} | \xi \cdot LH_2\cdots H_{n-1} > 0\}$ of $\bar{K^+} \setminus \{0\}$. By Lemma 52, $\xi_{\mathcal{E}',\mathcal{F}} = \xi_{\mathcal{F}',\mathcal{F}} + \xi_{\mathcal{E}',\mathcal{F}'}$. So for any $L \in C(\xi_{\mathcal{F}',\mathcal{F}})$,

$$\xi_{\mathcal{E}',\mathcal{F}}LH_2\cdots H_{n-1} = \xi_{\mathcal{F}',\mathcal{F}}LH_2\cdots H_{n-1} + \xi_{\mathcal{E}',\mathcal{F}'}LH_2\cdots H_{n-1} > 0.$$

We have $C(\xi_{\mathcal{F}',\mathcal{F}}) \subset C(\xi_{\mathcal{E}',\mathcal{F}})$. By Lemma 50 there exists $L \in \bar{K^+} \setminus \{0\}$, $\xi_{\mathcal{F}',\mathcal{F}} L H_2 \cdots H_{n-1} \leq 0$ and $\xi_{\mathcal{E}',\mathcal{F}'} L H_2 \cdots H_{n-1} > 0$. Note $\xi_{\mathcal{F}',\mathcal{F}} H_1 \cdots H_{n-1} > 0$ and $\xi_{\mathcal{E}',\mathcal{F}'} H_1 \cdots H_{n-1} > 0$, we may slightly change the polarization to $(tH_1 + (1-t)L, H_2, \ldots, H_{n-1})$. There exists $t_0 \in [0,1)$ such that $\xi_{\mathcal{F}',\mathcal{F}}(t_0 H_1 + (1-t_0)L) H_2 \cdots H_{n-1} = 0$. Thus

$$
\begin{aligned}
&\xi_{\mathcal{E}',\mathcal{F}}(t_0 H_1 + (1-t_0)L) H_2 \cdots H_{n-1} \\
=&\xi_{\mathcal{E}',\mathcal{F}'}(t_0 H_1 + (1-t_0)L) H_2 \cdots H_{n-1} \\
>&\xi_{\mathcal{F}',\mathcal{F}}(t_0 H_1 + (1-t_0)L) H_2 \cdots H_{n-1} = 0.
\end{aligned}
$$

Therefore we find an element in $C(\xi_{\mathcal{E}',\mathcal{F}})$ but not in $\xi_{\mathcal{F}',\mathcal{F}}$. After replacing $\mathcal{F}'$ by $\mathcal{E}'$, we get a strict increasing chain of subcones of $\bar{K^+}$.

Finally, we show that we can reach $\xi_{\mathcal{F}',\mathcal{F}} H_2 \cdots H_{n-1} > 0$ in finite steps. Pick a set of ample $\mathbb{R}$-basis $K_1, \ldots, K_m$ of $\mathrm{Num}(X)$ contained in $C(\xi_{\mathcal{F}',\mathcal{F}})$. $\xi_{\mathcal{E}',\mathcal{F}}$ is contained in the lattice $\sum \frac{1}{r!}\mathbb{Z}K_i$. Let $\mu^{K_j}(\mathcal{F})$ be the slope with respect to $(K_j, H_2, \ldots, H_{n-1})$, and we have

$$
0 < \xi_{\mathcal{E}',\mathcal{F}} K_j H_2 \cdots H_{n-1} < \mu^{K_j}_{max}(\mathcal{F}) - \mu^{K_j}(\mathcal{F}).
$$

Thus $\xi_{\mathcal{E}',\mathcal{F}}$ is in the bounded subset of lattice. □

The proof of Theorem 46 $T_2(r) \Rightarrow T_3(r)$ can also be found in [8] Theorem 7.3.3.

Proof of Theorem 46 $T_3(r) \Rightarrow T_1(r)$. Suppose not, we can pick $\mathcal{F}$ to be a torsion-free sheaf such that $\Delta(\mathcal{F}) H_2 \cdots H_{n-1} < 0$. Apply $T_3(r)$ to $F^{l*}\mathcal{F}$ for some l, where F is the absolute Frobenius. We have

$$
d \cdot \Delta((F^*)^l \mathcal{F}) H_2 \cdots H_{n-1} + \beta_r = dp^{2l}\, \Delta(\mathcal{F}) H_2 \cdots H_{n-1} + \beta_r < 0
$$

for l large enough. There exists a saturated subsheaf $\mathcal{E} \subset (F^*)^l \mathcal{F}$ with $\xi_{\mathcal{E}, F^{l*}\mathcal{F}} H_2 \cdots H_{n-1} > 0$. Therefore $F^{l*}\mathcal{F}$ is not semistable and $\mathcal{F}$ is not strongly semistable. □

The remaining part of this section follows [3] and gives the proof of Theorem 46 and Theorem 47. We will do induction on r. The base case $T_3(1)$ and $R(1)$ is clear since any line bundle is semistable. We will show induction steps $R(r) + T_1(r-1) + T_2(r-1) + T_3(r-1) \Rightarrow T_1(r)$ and $T_1(r) + T_2(r) + T_3(r) \Rightarrow R(r)$.

We begin with the proof of $R(r) + T_1(r-1) + T_2(r-1) + T_3(r-1) \Rightarrow T_1(r)$. This is achieved by induction on the dimension of X. We first show for $\dim X = 2$.

Proof for $R(r) + T_1(r-1) + T_2(r-1) + T_3(r-1) \Rightarrow T_1(r)$, $\dim X = 2$ case. Suppose not. Let $\mathcal{F}$ be a torsion-free strongly semistable sheaf such that $\Delta(\mathcal{F}) < 0$. Let $\mathrm{rk}\,\mathcal{F} = r$. X is smooth projective surface so we may replace $\mathcal{F}$ by $(\check{\mathcal{F}})\check{}$ to make $\mathcal{F}$ locally free: This follows from

$$
\Delta(\mathcal{F}) = \Delta((\check{\mathcal{F}})\check{}) + 2r\mathrm{length}((\check{\mathcal{F}})\check{}/\mathcal{F}) \geq \Delta((\check{\mathcal{F}})\check{}).
$$

Since $\mathcal{F}$ is strongly semistable, $\mu = L_{max} = L_{min}$. If $\mathcal{F}|_C$ is not strongly semistable for general $C \in |H_1|$, apply Theorem 47 $R(r)$, one can get

$$
0 \leq \sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 < 0,
$$

which is contradiction. So we conclude $\mathcal{F}|_C$ is also strongly semistable for general $C \in |H_1|$.

Now we consider the symmetric power $S^{kr}\mathcal{F}|_C$. We shall compute $\chi(S^{kr}\mathcal{F}|_C)$ in two different ways and draw contradiction from that. By Corollary 40, $S^{kr}\mathcal{F}|_C$ is strongly semistable. Let $\mathcal{E} = \det(\mathcal{F})$ and pick an divisor D corresponding to $\mathcal{E}$. Consider the sequence

$$0 \to S^{kr}\mathcal{F}(-kD - C) \to S^{kr}\mathcal{F}(-kD) \to S^{kr}\mathcal{F}(-kD)|_C \to 0$$

which is exact for general C by Bertini theorem. Taking cohomologies, we have the inequality

$$h^0(S^{kr}\mathcal{F}(-kD)) \le h^0(S^{kr}\mathcal{F}(-kD)) + h^0(S^{kr}\mathcal{F}|_C).$$

The slope $\mu(S^{kr}\mathcal{F}) = kr\mu(\mathcal{F})$ by the splitting principle. Also, $\mu(\mathcal{O}_X(kD + C)) = kr\mu(\mathcal{F}) + H_1^2 > \mu(S^{kr}\mathcal{F})$. Since they are both semistable, we conclude that there is no morphism $\mathcal{O}_X(kD + C) \to S^{kr}\mathcal{F}$, thus $h^0(S^{kr}\mathcal{F}(-kD - C)) = \dim(\mathrm{Hom}(\mathcal{O}_X(kD + C), S^{kr}\mathcal{F})) = 0$. Therefore we obtain the inequality

$$h^0(S^{kr}\mathcal{F}(-kD)) \le h^0(S^{kr}\mathcal{F}(-kD)|_C).$$

For any semistable vector bundle $\mathcal{G}$ on a curve C, by a similar process we can find the upper bound $h^0(\mathcal{G}) \le [\mathrm{rk}\,\mathcal{G} + c_1(G)H_2|_C]_+$. Therefore the growing speed

$$h^0(S^{kr}\mathcal{F}(-kD)|_C) \sim [\mathrm{rk}(S^{kr}\mathcal{F}|(-kD)_C) + \deg(S^{kr}\mathcal{F}(-kD)|_C)]_+ = O(k^r).$$

Let K be the canonical divisor on X. By Serre duality,

$$h^2(S^{kr}\mathcal{F}(-kD)) = h^0(S^{kr}(\check{\mathcal{F}})(kD) \otimes K).$$

Similarly, take general $C \in |H_1|$, we have

$$h^0(S^{kr}(\check{\mathcal{F}})(kD) \otimes K) \le h^0(S^{kr}(\check{\mathcal{F}})(kD)|_C \otimes K_C) = O(k^r).$$

Consider the fibration $\pi : \mathbb{P}(\mathcal{F}) \to X$, π is affine. Clearly $\mathcal{O}_\pi(kr) = S^{kr}\mathcal{F}$ and by Hirzebruch-Riemann-Roch

$$\begin{aligned}
&\chi(S^{kr}\mathcal{F}(-kD)) = \chi(\mathcal{O}_\pi(kr) \otimes \pi^*\mathcal{O}_X(-kD)) \\
=&\frac{1}{(r+1)!}(r^{r+1}k^{r+1}c_1(\mathcal{O}_\pi(1))^{r+1} - r^{r+1}k^{r+1}\pi^*c_1(\mathcal{F})c_1(\mathcal{O}_\pi(1)) + O(k^r)).
\end{aligned}$$

Since $c_1(\mathcal{O}_\pi(1))^{r+1} - \pi^*c_1(\mathcal{F})c_1(\mathcal{O}_\pi(1))^r + \pi^*c_2(\mathcal{F})c_1(\mathcal{O}_\pi(1))^{r-1} = 0$, we have

$$\begin{aligned}
\chi(S^{kr}\mathcal{F}(-kD)) &= -\frac{r^{r+1}k^{r+1}c_2(\mathcal{F})c_1(\mathcal{O}_\pi(1))^{r+1}}{(r+1)!} + O(k^r) \\
&= -\frac{r^r\,\Delta(\mathcal{F})}{2(r+1)!}k^{r+1} + O(k^r).
\end{aligned}$$

Since $\Delta(\mathcal{F}) < 0$, for sufficiently large k the polynomial is positive and has growing speed $\sim k^{r+1}$, which contradict to the growing speed of h^0 and h^2. □

If we change strongly semistable to semistable, the same proof will give out traditional Bogomolov inequality on a surface over characteristic zero field.

We next do the induction steps in the proof of $R(r)+T_1(r-1)+T_2(r-1)+T_3(r-1) \Rightarrow T_1(r)$.

Proof for $R(r)+T_1(r-1)+T_2(r-1)+T_3(r-1) \Rightarrow T_1(r)$, general case. Using induction on the dimension n, we shall assume $T_i(r-1)$ and $R(r)$ for any dimension and $T_i(r)$ for dimension $< n$, $i = 1, 2, 3$. Suppose not, pick strongly semistable torsion-free sheaf $\mathcal{F}$ such that $\Delta(\mathcal{F})H_2 \cdots H_{n-1} < 0$. For a general divisor $D \in |H_2|$, we have

$$\Delta(\mathcal{F}|_D)H_3|_D \cdots H_{n-1}|_D = \Delta(\mathcal{F})H_2 \cdots H_{n-1}.$$

Consider the shift of polarization. If we are in the case that $\mathcal{F}$ is $(H_2, H_2, H_3, \ldots, H_{n-1})$-semistable, and if $F^{k*}\mathcal{F}|_D$ is not semistable, we can apply $R(r)$ to $F^{k*}\mathcal{F}$. Then

$$\sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 \leq dp^{2k}\, \Delta(\mathcal{F})H_2 \cdots H_{n-1} < 0,$$

which is impossible.

We can first treat the case $\mathcal{F}$ and $\mathcal{F}|_D$ are strongly semistable. According to induction hypothesis $R(r)$, $\Delta(\mathcal{F})H_2 \cdots H_{n-1} = \Delta(\mathcal{F}|_D)H_3 \cdots H_{n-1} \geq 0$.

Then we prove the case that $\mathcal{F}$ is not strongly $(H_2, H_2, H_3, \ldots, H_{n-1})$-semistable. Let $B_t = ((tH_2 + (1-t)H_1))H_2 \cdots H_{n-1}$, then by our assumption $\mathcal{F}$ is strongly B_0-semistable but not B_1-semistable. Strongly semistable is an open condition with respect to t, we can find $t_0 \in [0, 1)$ such that $\mathcal{F}$ is strongly B_{t_0}-semistable but not B_t-semistable for all $t \in (t_0, 1]$. Take k such that $F^{k*}\mathcal{F}$ is not semistable with respect to B_1. By Theorem 48, there exists filtration $0 = \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_l = F^{k*}\mathcal{F}$ which becomes the Harder-Narasimhan filtration for all $t \in (t_0, t_0 + \epsilon)$. Theorem 48 implies that $\mu_t(\mathcal{F}_1) = \mu_t(F^{k*}\mathcal{F})$ for $t \in (t_0, t_0+\epsilon)$. By Remark 6, $\mathcal{F}_1$ and $\mathcal{F}_l/\mathcal{F}_1$ are also semistable with respect to B_{t_0}. Let r, r', r'' denote the rank of $\mathcal{F}_l$, $\mathcal{F}_1$ and $\mathcal{F}_l/\mathcal{F}_1$. According to Lemma 51

$$\frac{\Delta(\mathcal{F}_l)H_2 \cdots H_{n-1}}{r} + \frac{rr'}{r''}\xi^2_{\mathcal{F}_1,\mathcal{F}_l}H_2 \cdots H_{n-1} = \frac{\Delta(\mathcal{F}_1)H_2 \cdots H_{n-1}}{r'} + \frac{\Delta(\mathcal{F}_l/\mathcal{F}_1)H_2 \cdots H_{n-1}}{r''}.$$

Apply Theorem 69, we have

$$\xi^2_{\mathcal{F}_1,\mathcal{F}_l}H_2 \cdots H_{n-1} \cdot (t_0H_1 + (1-t_0)H_2)^2 H_2 \cdots H_{n-1} \leq (\xi_{\mathcal{F}_1,\mathcal{F}_l}B_{t_0})^2 = 0.$$

Since $H_1, \ldots, H_{n-1}$ are all ample, $d(t_0) = (t_k + H_1 + (1-t_0)H_2)^2 H_2 \cdots H_{n-1} > 0$. Define $\beta_r(t) = \beta_r(A, tH_1 + (1-t)H_2, H_2, \ldots, H_{n-1})$, by $T_2(r-1)$, we have

$$\begin{aligned}\frac{\Delta(\mathcal{F}_l)H_2 \cdots H_{n-1}}{r} &\geq \frac{\Delta(\mathcal{F}_1)H_2 \cdots H_{n-1}}{r'} + \frac{\Delta(\mathcal{F}_l/\mathcal{F}_1)H_2 \cdots H_{n-1}}{r''} \\ &\geq -\frac{1}{d(t_0)}(\frac{\beta_{r'}(t_0)}{r'} + \frac{\beta_{r''}(t_0)}{r''}) \\ &\geq -\frac{\beta_r(t_0)}{d(t_0)r}.\end{aligned}$$

Hence

$$\Delta(\mathcal{F})H_2\cdots H_{n-1} \geq -\frac{\beta_r(t_0)}{p^{2k}d(t_0)}.$$

Since $\frac{\beta_r(t)}{d(t)}$ is continuous on $[0,1]$ and thus has bounded image, take $k \to \infty$, we will have $\Delta(\mathcal{F})H_2\cdots H_{n-1} \geq 0$. $\square$

To complete our induction process, we finally prove $T_1(r) + T_2(r) + T_3(r) \Rightarrow R(r+1)$:

Proof of $T_1(r) + T_2(r) + T_3(r) \Rightarrow R(r+1)$. Consider the incident variety Z defined by $\Pi = |H_1|$ and X, $p : Z \to \Pi$ and $q : Z \to X$ are natural projections. Let $0 = \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_l = q^*\mathcal{F}$ be the Harder-Narasimhan filtration with respect to $(p^*\mathcal{O}_\Pi(1), \dots p^*\mathcal{O}_\Pi, q^*H_2, \dots, q^*H_{n-1})$, denote by $\mathcal{E}_i$ the Harder-Narasimhan factor $\mathcal{F}_i/\mathcal{F}_{i-1}$.

For general $s \in \Pi$, by Theorem 10, $\mathcal{F}_i|_{Z_s}$ form the Harder-Narasimhan filtration of $\mathcal{F}|_{Z_s}$. Factors $\mathcal{E}_i$ are $(p^*\mathcal{O}_\Pi(1), \dots p^*\mathcal{O}_\Pi, q^*H_2, \dots, q^*H_{n-1})$-semistable. According to Theorem 44, there exists k such that all the factors $\mathcal{F}'_i$ $(i = 1, \dots, l')$ in the Harder-Narasimhan filtration of $F^{k*}\mathcal{F}$ are strongly semistable. Denote the ranks (resp. slopes) of $\mathcal{F}_i$, $\mathcal{F}'_i$ by r_i and r'_i (resp. μ_i and μ'_i). We show that

$$\sum_{1\leq i<j\leq l} r_i r_j(\mu_i - \mu_j)^2 \leq \sum_{1\leq i<j\leq l'} r'_i r'_j \frac{(\mu'_i - \mu'_j)^2}{p^{2k}}.$$

Consider $\{(\sum_{j\leq i} r_j, \sum_{j\leq i} r_j\mu_j)\}_{i=1}^{l}$ and $\{(\sum_{j\leq i} r'_j, \sum_{j\leq i} \frac{r'_j\mu'_j}{p^k})\}_{i=1}^{l'}$ be the vertex of two convex polygens. Note the convex polygen $\{(\sum_{j\leq i} r_j, \sum_{j\leq i} r_j\mu_j)\}_{i=1}^{l}$ is contained in $\{(\sum_{j\leq i} r'_j, \sum_{j\leq i} \frac{r'_j\mu'_j}{p^k})\}_{i=1}^{l'}$, we will show that for any such convex polygens we have the desired inequality. For simplicity, denote $x_i = r_i$, $y_i = r_i\mu_i$ $(i = 1, \dots, l)$ and $x'_j = r'_j$, $y'_j = \frac{r'_j\mu'_j}{p^k}$ $(j = 1, \dots, l')$.

$$\sum_{i<j} x_i x_j (\frac{y_i}{x_i} - \frac{y_j}{x_j})^2 = (\sum_i x_i)(\sum_j \frac{y_j^2}{x_j}) - (\sum y_i)^2,$$

so it suffices to show

$$\sum_{i=1}^{l} \frac{y_i^2}{x_i} \leq \sum_{i=1}^{l'} \frac{(y'_i)^2}{x'_i}.$$

We shall do it inductively on l. For $l = 1$, the inequality follows from

$$(\sum_{i=1}^{l'} y'_i)^2 \leq (\sum_{i=1}^{l'} x'_i)(\sum_{i=1}^{l'} \frac{(y'_i)^2}{x'_i}).$$

Assume for $l-1$, consider the convex polygen given by $\{(x_i, y_i + t)\}_{i=1}^{l}$, we can take t_0 such that $\{(x_i, y_i + t_0)\}_{i=1}^{l}$ is the maximal one contained in $\{(x'_i, y'_i)\}_{i=1}^{l'}$. There exists at least on vertex $(x_u, y_u + t_0)$ on the edge connecting (x'_v, y'_v) and (x'_{v+1}, y'_{v+1}). Then apply the induction hypothesis to the convex polygens $\{(x'_1, y'_1), \dots, (x'_v, y'_v), (x_u, y_u)\}$, we have

$$\sum_{i=2}^{u} \frac{y_i^2}{x_i} + \frac{(y_1 + t_0)^2}{x_1} \leq \sum_{i=1}^{v} \frac{(y'_i)^2}{x'_i} + \frac{(\sum_{i=1}^{v} y'_i - \sum_{i=1}^{u} y_i)^2}{\sum_{i=1}^{v} x'_i - \sum_{i=1}^{u} x_i}.$$

Similarly,

$$\sum_{i=u+1}^{l-1}\frac{y_i^2}{x_i}+\frac{(y_l-t_0)^2}{x_l}\leq\sum_{i=v+2}^{l'}\frac{(y_i')^2}{x_i'}+\frac{(\sum_{i=u+1}^{l}y_i-\sum_{i=v+2}^{l'}y_i')^2}{\sum_{i=u+1}^{l}x_i-\sum_{i=v+2}^{l'}x_i'}.$$

$$\sum_{i=1}^{l'}\frac{(y_i')^2}{x_i}=\sum_{i=1}^{v}\frac{(y_i')^2}{x_i'}+\frac{(\sum_{i=1}^{v}y_i'-\sum_{i=1}^{u}y_i)^2}{\sum_{i=1}^{v}x_i'-\sum_{i=1}^{u}x_i}+\sum_{i=v+2}^{l'}\frac{(y_i')^2}{x_i'}+\frac{(\sum_{i=u+1}^{l}y_i-\sum_{i=v+2}^{l'}y_i')^2}{\sum_{i=u+1}^{l}x_i-\sum_{i=v+2}^{l'}x_i'}$$
$$\geq\sum_{i=2}^{l-1}\frac{y_i^2}{x_i}+\frac{(y_1+t_0)^2}{x_1}+\frac{(y_l-t_0)^2}{x_l}\geq\sum_{i=1}^{l}\frac{y_i^2}{x_i}$$

This completes the induction process. Therefore we may replace $\mathcal{F}$ by $F^{k*}\mathcal{F}$ to make sure $\mathcal{F}_i$ are all strongly semistable.

Let $\Lambda=\mathbb{P}^1\subset\Pi$ be a general pencil corresponding to a one dimensional linear subsystem of $|H_1|$ and $B\subset X$ be the base point of Λ. Then B is smooth scheme of codimension two in X. Let Y be the incident variety defined by Λ and X which is equal to the closed subscheme $p^{-1}\Lambda$ in Z. The restriction $q|_Y:Y\to X$ is the blow up along B. Let N be its exceptional divisor. If $\dim X\geq 3$, B is smooth variety so the first Chern class of $\mathcal{E}_i|_Y$ can be written as $(q|_Y)^*M_i+b_iN$, where M_i are divisors on X. If $\dim X=2$, B contains distinct points and the blow-up along B has exceptional divisors $N_1,\dots N_m$. Then the first Chern class of $\mathcal{E}_i|_Y$ can be written as $(q|_Y)^*M_i+\sum\limits_j b_{ij}N_j$. We define $b_i=\frac{1}{m}\sum\limits_j b_{ij}$ in this case.

Then we can have the slope of $\mu_i|_Y$ with respect to $p^*\mathcal{O}_\Lambda(1),q^*H_2,\dots,q^*H_{n-1}$:

$$\mu_i=\frac{c_1(\mathcal{E}_i|_Y)p^*\mathcal{O}_\Lambda(1)q^*H_2\cdots q^*H_{n-1}}{r_i}=\frac{M_iH_2\cdots H_{n-1}+db_i}{r_i}.$$

Applying Theorem 46 $T_1(r)$ to each $\mathcal{E}_i$, we have

$$\Delta(\mathcal{E}_i|_Y)(q|_Y)^*H_2\cdots(q|_Y)^*H_{n-1}=\Delta(\mathcal{E}_i)p^*\mathcal{O}_\Pi(1)^{\dim\Pi-1}q^*H_2\cdots q^*H_{n-1}\geq 0.$$

By Theorem 69, one obtain

$$\begin{aligned}
&\frac{d\,\Delta(\mathcal{F})H_2\cdots H_{n-1}}{r}\\
=&\sum_i\frac{d\,\Delta(\mathcal{E}_i|_Y)(q|_Y)^*H_2\cdots H_{n-1}}{r_i}-\frac{d}{r}\sum_{i<j}(\frac{c_1(\mathcal{E}_i|_Y)}{r_i}-\frac{c_1(\mathcal{E}_j|_Y)}{r_j})^2(q|_Y)^*(H_2\cdots H_{n-1})\\
\geq&-\frac{d}{r}\sum_{i<j}(\frac{c_1(\mathcal{E}_i|_Y)}{r_i}-\frac{c_1(\mathcal{E}_j|_Y)}{r_j})^2(q|_Y)^*(H_2\cdots H_{n-1})\\
=&\frac{d}{r}\sum_{i<j}r_ir_j(d(\frac{b_i}{r_i}-\frac{b_j}{r_j})^2-(\frac{M_i}{r_i}-\frac{M_j}{r_j})^2H_2\cdots H_{n-1})\\
\geq&\frac{1}{r}\sum_{i<j}r_ir_j(d^2(\frac{b_i}{r_i}-\frac{b_j}{r_j})^2-(\frac{M_iH_1\cdots H_{n-1}}{r_i}-\frac{M_jH_1\cdots H_{n-1}}{r_j})^2)\\
=&\frac{1}{r}\sum_{i<j}r_ir_j(d^2(\frac{b_i}{r_i}-\frac{b_j}{r_j})^2-(\mu_i-\mu_j+\frac{b_j}{r_j}-\frac{b_i}{r_i})^2)\\
=&\frac{2d}{r}\sum_{i<j}(\mu_i-\mu_j)(b_ir_j-b_jr_i)-\frac{1}{r}\sum_{i<j}r_ir_j(\mu_i-\mu_j)^2.
\end{aligned}$$

Since $\mathcal{E}_i|_Y$ form a filtration of $(q|_Y)^*\mathcal{F}$, we have

$$\sum b_i = c_1((q|_Y)^*\mathcal{F})(q|_Y)^*(H_2 \cdots H_{n-1}) = 0.$$

So

$$\begin{aligned}
&\frac{2d}{r}\sum_{i<j}(\mu_i - \mu_j)(b_i r_j - b_j r_i)\\
=&\frac{2d}{r}((\sum_{i=1}^{l}\mu_i b_i)(\sum_{j=1}^{l} r_j) - (\sum_{i=1}^{l}\mu_i r_i)(\sum_{j=1}^{l} b_j))\\
=&2d\sum b_i\mu_i.
\end{aligned}$$

Since $(q|_Y)_*(\mathcal{F}_i|_Y) \subset \mathcal{F}$, we have the inequalities

$$\frac{\sum_{j\le i} M_j H_1 \cdots H_{n-1}}{\sum_{j\le i} r_i} \le \mu_{max}.$$

So

$$\sum_{j\le i} db_j \ge \sum_{j\le i} r_j(\mu_j - \mu_{max})$$

and

$$\begin{aligned}
d\sum b_i\mu_i &= d\sum_{i=1}^{l-1}(\sum_{j\le i} b_j)(\mu_i - \mu_{i+1}) + d\mu_l\sum_{j=1}^{l} b_j\\
&\ge \sum_{i=1}^{l-1}(\sum_{j\le i} r_j(\mu_j - \mu_{max}))(\mu_i - \mu_{i+1}) + d\mu_l\sum_{j=1}^{l}(\mu_j - \mu_{max})\\
&= \sum_i r_i\mu_i^2 - r\mu\mu_{max}\\
&\ge \sum_i r_i\mu_i^2 - r\mu^2 - r(\mu_{max} - \mu)(\mu - \mu_{min})\\
&= \sum_{i<j}\frac{r_i r_j}{r}(\mu_i - \mu_j)^2 - r(\mu_{max} - \mu)(\mu - \mu_{min}).
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{d\,\Delta(\mathcal{F})}{H_2 \cdots H_{n-1}} &\ge \sum_{i<j}\frac{r_i r_j}{r}(\mu_i - \mu_j)^2 - 2r(\mu_{max} - \mu)(\mu - \mu_{min})\\
&\ge \sum_{i<j}\frac{r_i r_j}{r}(\mu_i - \mu_j)^2 - 2r(L_{max} - \mu)(\mu - L_{min}).
\end{aligned}$$

We complete the proof. □

8 Boundedness in Positive Characteristic

In this section, we will apply Theorem 46 and Theorem 47 to derive the boundedness of semistable sheaves over positive characteristic fields.

Lemma 53. *Let $\mathcal{F}$ be a torsion-free sheaf of rank r on X and H_1 very ample divisor, $H_2, \ldots, H_{n-1}$ ample divisors. Let $d = H_1^2 H_2 \cdots H_{n-1}$ be the degree. Let $H \in |\mathcal{O}(H_1)|$ be a general divisor. Then we have*

$$\frac{r}{2}(L_{max}(\mathcal{F}|_H) - L_{min}(\mathcal{F}|_H))^2 \le d\,\Delta(\mathcal{F}) H_2 \cdots H_{n-1} + 2r^2(L_{max} - \mu)(\mu - L_{min}).$$

Proof. First we consider the case $\mathcal{F}|_H$ is not strongly semistable. Let $0 = HN_0(\mathcal{F}|_H) \subset \cdots \subset HN_l(\mathcal{F}|_H) = \mathcal{F}|_H$ be the Harder-Narasimhan filtration of $\mathcal{F}|_H$. Let $r_1, \ldots, r_l$ and $\mu_1, \ldots, \mu_l$ be the ranks and slopes of $HN_i(\mathcal{F}|_H)/HN_{i-1}(\mathcal{F}|_H)$ with respect to polarization $(H_2|_H, \ldots, H_{n-1}|_H)$. By Theorem 47, we have inequality

$$\sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 \le d\,\Delta(\mathcal{F}) H_2 \cdots H_{n-1} + 2r^2(L_{max}(\mathcal{F}) - \mu(\mathcal{F}))(\mu(\mathcal{F}) - L_{min}(\mathcal{F})).$$

We next show that

$$\sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 \ge \frac{r_1 r_l}{r_1 + r_l}(\mu_1 - \mu_l) \sum_{i=1}^{l} r_i.$$

For $l = 1, 2, 3$, one can directly check this inequality. For $l \ge 3$, let $r_1' = r_1$, $r_3' = r_l$ and $r_2' = \sum_{i=2}^{l-1} r_i$, $\mu_1' = \mu_1$, $\mu_3' = \mu_l$ and $\mu_2' = \frac{1}{r_2'} \sum_{i=2}^{l-1} r_i \mu_i$. Then using $l = 3$ case

$$\begin{aligned}
\sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 &= (\sum r_i)(\sum r_i \mu_i^2) - (\sum r_i \mu_i)^2 \\
&\ge (\sum r_i')(\sum r_i' \mu_i'^2) - (\sum r_i' \mu_i')^2 \\
&= \sum_{i<j} r_i' r_j' (\mu_i' - \mu_j')^2 \\
&\ge \frac{r_1' r_3'}{r_1' + r_3'}(\mu_1' - \mu_3')^2 (\sum r_i') \\
&= \frac{r_1 r_l}{r_1 + r_l}(\mu_1 - \mu_l) \sum_{i=1}^{l} r_i.
\end{aligned}$$

So

$$\frac{r_1 r_l}{r_1 + r_l}(\mu_1 - \mu_l) \sum_{i=1}^{l} r_i \ge \frac{r}{2}(L_{max}(\mathcal{F}|_H) - L_{min}(\mathcal{F}|_H))^2$$

and we have the desired inequality.

For the case that $\mathcal{F}|_H$ is strongly semistable, $L_{max}(\mathcal{F}) = L_{min}(\mathcal{F})$ and the inequality follows from Lemma 49. □

Similar to Theorem 33 in characteristic zero case, we have

Theorem 54. *Let $H_1, \ldots, H_{n-1}$ be a set of very ample divisors and $d = H_1^2 H_2 \cdots H_{n-1}$. Let $X_l = |H_1| \cap \cdots \cap |H_l|$ $(1 \le l \le n-1)$ be a very general complete intersection. Let $\mathcal{F}$ be a torsion-free sheaf of rank r on X. Take a nef divisor A such that $\Omega_{X_l} \to \mathcal{O}_{X_l}(A|_{X_l})^{N_l}$ is injective for all*

$1 \leq l \leq n-1$. *Then*

$$\mu_{max}(\mathcal{F}|_{X_l}) - \mu_{min}(\mathcal{F}|_{X_l}) \leq \frac{r^{l/2}-1}{r-\sqrt{r}}(\sqrt{2[d\,\Delta(\mathcal{F})H_2\cdots H_{n-1}]_+} + 2\sqrt{\beta_r}) + r^{l/2}(\mu_{max}(\mathcal{F}) - \mu_{min}(\mathcal{F})).$$

Proof. For the case $l=1$, apply Lemma 53

$$\begin{aligned}(L_{max,1} - L_{min,1})^2 &\leq \frac{2}{r} d\,\Delta(\mathcal{F})H_2\cdots H_{n-1} + 4r(L_{max}-\mu)(\mu - L_{min}) \\ &\leq \frac{2}{r}[d\,\Delta(\mathcal{F})H_2\cdots H_{n-1}]_+ + 2r^2(\frac{L_{max}-L_{min}}{2})^2.\end{aligned}$$

By Theorem 43, $\mu_{max,1} - \mu_{min,1} \leq L_{max} - L_{min} \leq \mu_{max} - \mu_{min} + \frac{2\sqrt{\beta_r}}{r}$, so

$$\begin{aligned}\mu_{max,1} - \mu_{min,1} &\leq \sqrt{\frac{2}{r}[d\,\Delta(\mathcal{F})H_2\cdots H_{n-1}]_+ + 4r(\frac{\mu_{max}-\mu_{min}}{2} + 2\sqrt{\beta_r})^2} \\ &\leq \frac{1}{\sqrt{r}}(\sqrt{[2d\,\Delta(\mathcal{F})H_2\cdots H_{n-1}]_+} + 2\sqrt{\beta_r}) + \sqrt{r}(\mu_{max} - \mu_{min}).\end{aligned}$$

For $l > 1$, apply induction on l.

$$\begin{aligned}\mu_{max,l} - \mu_{min,l} &\leq \frac{1}{\sqrt{r}}(\sqrt{[2d\,\Delta(\mathcal{F})H_2\cdots H_{n-1}]_+} + 2\sqrt{\beta_r}) + \sqrt{r}(\mu_{max,l-1} - \mu_{min,l-1}) \\ &\leq (\frac{1}{\sqrt{r}} + \frac{r^{(l-1)/2}-1}{\sqrt{r}-1})(\sqrt{[2d\,\Delta(\mathcal{F})H_2\cdots H_{n-1}]_+} + 2\sqrt{\beta_r}) + r^{l/2}(\mu_{max} - \mu_{min}) \\ &= \frac{r^{l/2}-1}{r-\sqrt{r}}(\sqrt{2[d\,\Delta(\mathcal{F})H_2\cdots H_{n-1}]_+} + 2\sqrt{\beta_r}) + r^{l/2}(\mu_{max} - \mu_{min}).\end{aligned}$$

□

We could establish the boundedness theorem for semistable sheaves over positive characteristic field.

Corollary 55. *Let X be a projective variety over algebraically closed field k ($\mathrm{char}\, k = p$). Let $\mathfrak{F}$ be family of semistable sheaves on X with fixed Hilbert polynomial and bounded μ_{max}. Then $\mathfrak{F}$ is bounded.*

Proof. Immediate corollary from Theorem 17 and Theorem 54. □

A Results from Descent Theory

In this section we introduce some basic results from descent of quasicoherent sheaves. We need faithfully flat descent and Galois descent in the Theorem 24 and Lemma 26. The proof of theorems in this section will be omitted, and we refer chapter 14 [4] and [1, Tag 0238] for details.

Let $p : S' \to S$ be a faithfully flat quasicompact morphism of schemes. Let $S'' = S' \times_S S'$ and $S''' = S' \times_S S' \times_S S''$ and the projections be $p_i : S'' \to S'$ and $p_{ij} : S''' \to S''$.

Definition 56. Let $\mathcal{F}$ be a quasicoherent sheaf on S'. A *descent datum* of $\mathcal{F}$ is a $\mathcal{O}_{S'}$-module morphism $\varphi : p_1^*\mathcal{F} \xrightarrow{\simeq} p_2^*\mathcal{F}$ satisfying the cocycle condition

$$p_{23}^*\varphi \circ p_{12}^*\varphi = p_{13}^*\varphi,$$

i.e. φ makes the following diagram commutes:

$$\begin{tikzcd}
 & p_{12}^*p_2^*\mathcal{F} \arrow[r, equal] & p_{23}^*p_1^*\mathcal{F} \arrow[dr, "p_{23}^*\varphi"] & \\
p_{12}^*p_1^*\mathcal{F} \arrow[ur, "p_{12}^*\varphi"] \arrow[dr, equal] & & & p_{23}^*p_2^*\mathcal{F} \\
 & p_{13}^*p_1^*\mathcal{F} \arrow[r, "p_{13}^*\varphi"] & p_{13}^*p_2^*\mathcal{F} \arrow[ur, equal] &
\end{tikzcd}$$

We can define the category $\mathrm{Qcoh}(S'/S)$ of quasicoherent sheaves on S' with the descent datum. More precisely, the object of $\mathrm{Qcoh}(S'/S)$ is pair $(\mathcal{F}, \varphi)$ the quasicoherent sheaves on S' with descent datum φ. The morphism from $(\mathcal{F}, \varphi)$ to $(\mathcal{G}, \psi)$ is a morphism of $\mathcal{O}'_S$-modules $u : \mathcal{F} \to \mathcal{G}$ such that $p_2^*u \circ \varphi = \psi \circ p_1^*u$.

Let $\Phi : \mathrm{Qcoh}(S) \to \mathrm{Qcoh}(S'/S)$ be a functor which maps $\mathcal{F}$ to $p^*\mathcal{F}$ and give rise to a canonical datum $\varphi : p_1^*(p^*\mathcal{F}) \xrightarrow{\simeq} p_2^*(p^*\mathcal{F})$.

Theorem 57. $\Phi : \mathrm{Qcoh}(S) \to \mathrm{Qcoh}(S'/S)$ *defines an equivalence of categories.*

Next we study a specific case of faithfully flat descent, using the action of a Galois group. Let G be a finite group and S a scheme. We can view $\coprod\limits_{g\in G} S$ as the constant group scheme over S: It represents the functor that associates S-scheme T with the locally constant map $T \to G$. Denote the group scheme $\coprod\limits_{g\in G}$ by G_S. The multiplication $\mu : G_S \times_S G_S \to G_S$ is given by transportation of T-points in each component. An action of G_S on a S-scheme S' by S-automorphism is a morphism $\sigma : G_S \times_S S' \to S'$, $\sigma : (g, s') \mapsto gs'$ in T-points.

Definition 58. Let $p_2 : G_S \times_S S' \to S'$ be the projection on the section component. A G_S-*equivariant structure* on quasicoherent $\mathcal{O}_S$-module $\mathcal{F}$ is an isomorphism $\varphi : \sigma^*\mathcal{F} \xrightarrow{\simeq} p_2^*\mathcal{F}$ satisfying the cocycle condition

$$p_{23}^*\varphi \circ (id_{G_S} \times \sigma)^*\varphi = (\mu \times id_{S'})\varphi.$$

Definition 59. A *Galois covering* with Galois group G_S is a faithfully flat morphism $p : S \to S$ with a G_S-action on S' by S-automorphism such that the morphism $G_S \times_S S' \to S' \times_S S'$ which on T-points $(g, s') \mapsto (s', gs')$ is an isomorphism.

Moreover, if $f : Y \to X$ is a finite morphism of normal projective schemes with $K(Y)$ Galois over $K(X)$, then f is a Galois covering.

Let $S' \to S$ be a Galois covering. We can define the G_S-equivariant quasicoherent sheaf category $\mathrm{Qcoh}_G(S'/S)$: The objects in $\mathrm{Qcoh}_G(S'/S)$ are pairs $(\mathcal{F}, \varphi)$ of quasicoherent sheaves on S' with

G_S-equivariant structure. A morphism from $(\mathcal{F},\varphi)$ to $(\mathcal{G},\psi)$ are $\mathcal{O}_{S'}$-morphism $u:\mathcal{F}\to\mathcal{G}$ such that $\psi\circ\sigma^*u = p_2^*u\circ\varphi$.

Similarly, we can define a functor $\Phi:\mathrm{Qcoh}(S)\to\mathrm{Qcoh}_G(S'/S)$ which maps $\mathcal{F}$ to p^* and gives rise to canonical G_S-equivariant structure $p_2^*p^*\mathcal{F}\xrightarrow{\simeq}\sigma^*p^*\mathcal{F}$.

Theorem 60. $\Phi:\mathrm{Qcoh}(S)\to\mathrm{Qcoh}_G(S'/S)$ *defines an equivalence of categories.*

B Ample Vector Bundles

We will summarize the important relations between ample sheaf and positive degree sheaf in this section. We will give the nef version of Hodge Index Theorem and corollaries which will be used in Section 7. The original reference for this section is [6], and we will mainly refer [6], [10] and [11] for the proofs.

Let X be a smooth projective variety over algebraically closed field k, $\mathcal{E}$ is a vector bundle on X and $\pi:\mathbb{P}(\mathcal{E})\to X$ be the projective bundle associated to $\mathcal{E}$. Denote the tautological line sheaf on $\mathbb{P}(\mathcal{E})$ by $\mathcal{O}_\pi(1)$.

Definition 61.
1. $\mathcal{E}$ is *ample* if $\mathcal{O}_\pi(1)$ is ample;
2. $\mathcal{E}$ is *nef* if $\mathcal{O}_\pi(1)$ is nef.

The following proposition is straight from the definition.

Proposition 62. *Let $\mathcal{E}$ be an ample (nef) vector bundle on X, then for any quotient $\mathcal{F}$ of $\mathcal{E}$, $\mathcal{F}$ is ample (nef).*

Theorem 63. *Let $\mathcal{E}$ be a vector bundle on X, f is a morphism from smooth projective curve C to X. Then $\mathcal{E}$ is nef if and only if any quotient line bundle $\mathcal{L}$ of $f^*\mathcal{E}$ has non-negative degree.*

Theorem 64. *Let $\mathcal{E}$ be a vector bundle on X, then the followings are equivalent:*
1. *$\mathcal{E}$ is ample.*
2. *For any coherent sheaf $\mathcal{F}$ on X, there is an integer $m(\mathcal{F})$ such that $H^i(X,S^m\mathcal{E}\otimes\mathcal{F})=0$ for all $i>0$ and $m\geq m(\mathcal{F})$.*
3. *For any coherent sheaf $\mathcal{F}$ on X, there is an integer $n(\mathcal{F})$ such that $S^n\mathcal{E}\otimes\mathcal{F}$ is generated by global sections for all $n\geq n(\mathcal{F})$.*

Using the above theorem, one can show that:

Proposition 65. *Let $0\to\mathcal{E}'\to\mathcal{E}\to\mathcal{E}''\to 0$ be an exact sequence of vector bundles on X. If $\mathcal{E}'$ and $\mathcal{E}''$ are ample, then so is $\mathcal{E}$.*

Proposition 66. *Let $\mathcal{E}$ be a vector bundle on X, then the followings are equivalent:*
1. *$\mathcal{E}$ is ample (resp. nef).*
2. *$S^k\mathcal{E}$ is ample (resp. nef) for some $k\geq 1$.*
3. *$S^k\mathcal{E}$ is ample (resp. nef) for all $k\geq 1$.*

Corollary 67. *Assume $\mathcal{E}$ and $\mathcal{F}$ are ample vector bundles on X. Then $\mathcal{E} \otimes \mathcal{F}$ is ample.*

All propositions above are still true on varieties over k with positive characteristic.

Theorem 68. *Assume k is an algebraically closed field with characteristic zero. Let X be a projective curve over k and $\mathcal{E}$ be a semistable vector bundle of over X. Then*

1. *$\mathcal{E}$ is nef if and only if $\deg(\mathcal{E}) \geq 0$;*
2. *$\mathcal{E}$ is ample if and only if $\deg(\mathcal{E}) > 0$.*

The assumption $\operatorname{char} k = 0$ is required in Theorem 68. In fact, Serre constructed a non-singular curve X of genus 3 over a field of characteristic 3, and a vector bundle $\mathcal{E}$ of rank 2 with $\deg(\mathcal{E}) = 1$, while all the quotient of $\mathcal{E}$ has positive degree but not ample. See [5] for the example.

The following theorems are well-known Hodge index theorem and it's index form.

Theorem 69 (Hodge Index Theorem)**.** *Let X be a smooth projective variety of dimension $n \geq 2$. Assume $D_1, \ldots, D_{n-1}$ is a set of nef divisors such that $d = D_1^2 D_2 \ldots D_{n-1} > 0$. Then for any arbitrary divisor D we have*

$$(D^2 D_2 \ldots D_{n-1}) \cdot (D_1^2 D_2 \ldots D_{n-1}) \leq (D D_1 D_2 \ldots D_{n-1})^2.$$

Corollary 70. *Let X and D_i be as in Theorem 69. Then the intersection product*

$$(D, D') \mapsto D D' D_2 \ldots D_{n-1}$$

has the index $(1, h^{1,1}(X) - 1)$.

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References

[1] The Stacks Project Authors. *Stacks Project*, 2024.

[2] David Gieseker. Stable vector bundles and the frobenius morphism. 6(1):95–101, 1973.

[3] Haoyang Guo, Sanal Shivaprasad, Dylan Spence, and Yueqiao Wu. Boundedness of semistable sheaves, 2021.

[4] Ulrich Görtz and Torsten Wedhorn. *Algebraic Geometry I: Schemes: With Examples and Exercises*. Springer Studium Mathematik - Master. 2020.

[5] Robin Hartshorne. Ample vector bundles on curves. *Nagoya Mathematical Journal*, 43:73–89, 1971.

[6] Robin Hartshorne. *Ample subvarieties of algebraic varieties*, volume 156. Springer, 2006.

[7] Robin Hartshorne. *Algebraic geometry*, volume 52. Springer Science & Business Media, 2013.

[8] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves*. Cambridge University Press, 2010.

[9] Adrian Langer. Semistable sheaves in positive characteristic. *Annals of mathematics*, pages 251–276, 2004.

[10] Robert Lazarsfeld. *Positivity in algebraic geometry I*. Springer.

[11] Robert Lazarsfeld. *Positivity in Algebraic Geometry II*. Springer.

[12] Masaki Maruyama. The theorem of Grauert-Mülich-Spindler. *Mathematische Annalen*, 255(3):317–333, 1981.

[13] David Mumford. *Lectures on Curves on an Algebraic Surface. (AM-59)*. Princeton University Press, 1966.

[14] NI Shepherd-Barron. Semi-stability and reduction mod p. *Topology*, 37(3):659–664, 1998.

[15] Fumio Takemoto. Stable vector bundles on algebraic surfaces. *Nagoya Mathematical Journal*, 47:29–48, 1972.