ABELIAN CATEGORY AND ADDICTIVE FUNCTOR

1. Preadditive and additive categories

Here is the definition of a preadditive category.

Definition 1.1. A category $\mathcal{A}$ is called *preadditive* if each morphism set $\mathrm{Mor}_{\mathcal{A}}(x, y)$ is endowed with the structure of an abelian group such that the compositions

$$\mathrm{Mor}(x, y) \times \mathrm{Mor}(y, z) \longrightarrow \mathrm{Mor}(x, z)$$

are bilinear. A functor $F : \mathcal{A} \to \mathcal{B}$ of preadditive categories is called *additive* if and only if $F : \mathrm{Mor}(x, y) \to \mathrm{Mor}(F(x), F(y))$ is a homomorphism of abelian groups for all $x, y \in \mathrm{Ob}(\mathcal{A})$.

In particular for every x, y there exists at least one morphism $x \to y$, namely the zero map.

Lemma 1.2. *Let $\mathcal{A}$ be a preadditive category. Let x be an object of $\mathcal{A}$. The following are equivalent*

(1) *x is an initial object,*
(2) *x is a final object, and*
(3) *$id_x = 0$ in $\mathrm{Mor}_{\mathcal{A}}(x, x)$.*

Furthermore, if such an object 0 exists, then a morphism $\alpha : x \to y$ factors through 0 if and only if $\alpha = 0$.

Proof. First assume that x is either (1) initial or (2) final. In both cases, it follows that $\mathrm{Mor}(x, x)$ is a trivial abelian group containing id_x, thus $\mathrm{id}_x = 0$ in $\mathrm{Mor}(x, x)$, which shows that each of (1) and (2) implies (3).

Now assume that $\mathrm{id}_x = 0$ in $\mathrm{Mor}(x, x)$. Let y be an arbitrary object of $\mathcal{A}$ and let $f \in \mathrm{Mor}(x, y)$. Denote $C : \mathrm{Mor}(x, x) \times \mathrm{Mor}(x, y) \to \mathrm{Mor}(x, y)$ the composition map. Then $f = C(0, f)$ and since C is bilinear we have $C(0, f) = 0$. Thus $f = 0$. Hence x is initial in $\mathcal{A}$. A similar argument for $f \in \mathrm{Mor}(y, x)$ can be used to show that x is also final. Thus (3) implies both (1) and (2). $\square$

Definition 1.3. In a preadditive category $\mathcal{A}$ we call *zero object*, and we denote it 0 any final and initial object as in Lemma 1.2 above.

Lemma 1.4. *Let $\mathcal{A}$ be a preadditive category. Let $x, y \in \mathrm{Ob}(\mathcal{A})$. If the product $x \times y$ exists, then so does the coproduct $x \amalg y$. If the coproduct $x \amalg y$ exists, then so does the product $x \times y$. In this case also $x \amalg y \cong x \times y$.*

Proof. Suppose that $z = x \times y$ with projections $p : z \to x$ and $q : z \to y$. Denote $i : x \to z$ the morphism corresponding to $(1, 0)$. Denote $j : y \to z$ the morphism corresponding to $(0,1)$. Thus we have the commutative diagram

$$\begin{tikzcd} x \arrow[rr, "1"] \arrow[dr, "i"] & & x \\ & z \arrow[ur, "p"] \arrow[dr, "q"] & \\ y \arrow[rr, "1"] \arrow[ur, "j"] & & y \end{tikzcd}$$

where the diagonal compositions are zero. It follows that $i \circ p + j \circ q : z \to z$ is the identity since it is a morphism which upon composing with p gives p and upon composing with q gives q. Suppose given morphisms $a : x \to w$ and $b : y \to w$. Then we can form the map $a \circ p + b \circ q : z \to w$. In this way we get a bijection $\mathrm{Mor}(z, w) = \mathrm{Mor}(x, w) \times \mathrm{Mor}(y, w)$ which show that $z = x \amalg y$.

We leave it to the reader to construct the morphisms p, q given a coproduct $x \amalg y$ instead of a product. □

Definition 1.5. Given a pair of objects x, y in a preadditive category $\mathcal{A}$, the *direct sum* $x \oplus y$ of x and y is the direct product $x \times y$ endowed with the morphisms i, j, p, q as in Lemma 1.4 above.

Remark 1.6. Note that the proof of Lemma 1.4 shows that given p and q the morphisms i, j are uniquely determined by the rules $p \circ i = \mathrm{id}_x$, $q \circ j = \mathrm{id}_y$, $p \circ j = 0$, $q \circ i = 0$. Moreover, we automatically have $i \circ p + j \circ q = \mathrm{id}_{x \oplus y}$. Similarly, given i, j the morphisms p and q are uniquely determined. Finally, given objects x, y, z and morphisms $i : x \to z$, $j : y \to z$, $p : z \to x$ and $q : z \to y$ such that $p \circ i = \mathrm{id}_x$, $q \circ j = \mathrm{id}_y$, $p \circ j = 0$, $q \circ i = 0$ and $i \circ p + j \circ q = \mathrm{id}_z$, then z is the direct sum of x and y with the four morphisms equal to i, j, p, q.

Lemma 1.7. *Let $\mathcal{A}$, $\mathcal{B}$ be preadditive categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. Then F transforms direct sums to direct sums and zero to zero.*

Proof. Suppose F is additive. A direct sum z of x and y is characterized by having morphisms $i : x \to z$, $j : y \to z$, $p : z \to x$ and $q : z \to y$ such that $p \circ i = \mathrm{id}_x$, $q \circ j = \mathrm{id}_y$, $p \circ j = 0$, $q \circ i = 0$ and $i \circ p + j \circ q = \mathrm{id}_z$, according to Remark 1.6. Clearly $F(x), F(y), F(z)$ and the morphisms $F(i), F(j), F(p), F(q)$ satisfy exactly the same relations (by additivity) and we see that $F(z)$ is a direct sum of $F(x)$ and $F(y)$. Hence, F transforms direct sums to direct sums.

To see that F transforms zero to zero, use the characterization (3) of the zero object in Lemma 1.2. □

Definition 1.8. A category $\mathcal{A}$ is called *additive* if it is preadditive and finite products exist, in other words it has a zero object and direct sums.

Namely the empty product is a finite product and if it exists, then it is a final object.

Definition 1.9. Let $\mathcal{A}$ be a preadditive category. Let $f : x \to y$ be a morphism.

(1) A *kernel* of f is a morphism $i : z \to x$ such that (a) $f \circ i = 0$ and (b) for any $i' : z' \to x$ such that $f \circ i' = 0$ there exists a unique morphism $g : z' \to z$ such that $i' = i \circ g$.

(2) If the kernel of f exists, then we denote this $\mathrm{Ker}(f) \to x$.
(3) A *cokernel* of f is a morphism $p : y \to z$ such that (a) $p \circ f = 0$ and (b) for any $p' : y \to z'$ such that $p' \circ f = 0$ there exists a unique morphism $g : z \to z'$ such that $p' = g \circ p$.
(4) If a cokernel of f exists we denote this $y \to \mathrm{Coker}(f)$.
(5) If a kernel of f exists, then a *coimage of f* is a cokernel for the morphism $\mathrm{Ker}(f) \to x$.
(6) If a kernel and coimage exist then we denote this $x \to \mathrm{Coim}(f)$.
(7) If a cokernel of f exists, then the *image of f* is a kernel of the morphism $y \to \mathrm{Coker}(f)$.
(8) If a cokernel and image of f exist then we denote this $\mathrm{Im}(f) \to y$.

In the above definition, we have spoken of "the kernel" and "the cokernel", tacitly using their uniqueness up to unique isomorphism. This follows from the Yoneda lemma (Categories, Section 3) because the kernel of $f : x \to y$ represents the functor sending an object z to the set $\mathrm{Ker}(\mathrm{Mor}_{\mathcal{A}}(z, x) \to \mathrm{Mor}_{\mathcal{A}}(z, y))$. The case of cokernels is dual.

We first relate the direct sum to kernels as follows.

Lemma 1.10. *Let $\mathcal{C}$ be a preadditive category. Let $x \oplus y$ with morphisms i, j, p, q as in Lemma 1.4 be a direct sum in $\mathcal{C}$. Then $i : x \to x \oplus y$ is a kernel of $q : x \oplus y \to y$. Dually, p is a cokernel for j.*

Proof. Let $f : z' \to x \oplus y$ be a morphism such that $q \circ f = 0$. We have to show that there exists a unique morphism $g : z' \to x$ such that $f = i \circ g$. Since $i \circ p + j \circ q$ is the identity on $x \oplus y$ we see that

$$f = (i \circ p + j \circ q) \circ f = i \circ p \circ f$$

and hence $g = p \circ f$ works. Uniqueness holds because $p \circ i$ is the identity on x. The proof of the second statement is dual. □

Lemma 1.11. *Let $\mathcal{C}$ be a preadditive category. Let $f : x \to y$ be a morphism in $\mathcal{C}$.*

(1) *If a kernel of f exists, then this kernel is a monomorphism.*
(2) *If a cokernel of f exists, then this cokernel is an epimorphism.*
(3) *If a kernel and coimage of f exist, then the coimage is an epimorphism.*
(4) *If a cokernel and image of f exist, then the image is a monomorphism.*

Proof. Part (1) follows easily from the uniqueness required in the definition of a kernel. The proof of (2) is dual. Part (3) follows from (2), since the coimage is a cokernel. Similarly, (4) follows from (1). □

Lemma 1.12. *Let $f : x \to y$ be a morphism in a preadditive category such that the kernel, cokernel, image and coimage all exist. Then f can be factored uniquely as $x \to \mathrm{Coim}(f) \to \mathrm{Im}(f) \to y$.*

Proof. There is a canonical morphism $\mathrm{Coim}(f) \to y$ because $\mathrm{Ker}(f) \to x \to y$ is zero. The composition $\mathrm{Coim}(f) \to y \to \mathrm{Coker}(f)$ is zero, because it is the unique morphism which gives rise to the morphism $x \to y \to \mathrm{Coker}(f)$ which is zero (the uniqueness follows from Lemma 1.11 (3)). Hence $\mathrm{Coim}(f) \to y$ factors uniquely through $\mathrm{Im}(f) \to y$, which gives us the desired map. □

2. Abelian categories

An abelian category is a category satisfying just enough axioms so the snake lemma holds. An axiom (that is sometimes forgotten) is that the canonical map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ of Lemma 1.12 is always an isomorphism. Example 1.13 shows that it is necessary.

Definition 2.1. A category $\mathcal{A}$ is *abelian* if it is additive, if all kernels and cokernels exist, and if the natural map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism for all morphisms f of $\mathcal{A}$.

Lemma 2.2. *Let $\mathcal{A}$ be a preadditive category. The additions on sets of morphisms make $\mathcal{A}^{opp}$ into a preadditive category. Furthermore, $\mathcal{A}$ is additive if and only if $\mathcal{A}^{opp}$ is additive, and $\mathcal{A}$ is abelian if and only if $\mathcal{A}^{opp}$ is abelian.*

Proof. The first statement is straightforward. To see that $\mathcal{A}$ is additive if and only if $\mathcal{A}^{opp}$ is additive, recall that additivity can be characterized by the existence of a zero object and direct sums, which are both preserved when passing to the opposite category. Finally, to see that $\mathcal{A}$ is abelian if and only if $\mathcal{A}^{opp}$ is abelian, observes that kernels, cokernels, images and coimages in $\mathcal{A}^{opp}$ correspond to cokernels, kernels, coimages and images in $\mathcal{A}$, respectively. □

Definition 2.3. Let $f : x \to y$ be a morphism in an abelian category.

(1) We say f is *injective* if $\mathrm{Ker}(f) = 0$.
(2) We say f is *surjective* if $\mathrm{Coker}(f) = 0$.

If $x \to y$ is injective, then we say that x is a *subobject* of y and we use the notation $x \subset y$. If $x \to y$ is surjective, then we say that y is a *quotient* of x.

Lemma 2.4. *Let $f : x \to y$ be a morphism in an abelian category $\mathcal{A}$. Then*

(1) *f is injective if and only if f is a monomorphism, and*
(2) *f is surjective if and only if f is an epimorphism.*

Proof. Proof of (1). Recall that $\mathrm{Ker}(f)$ is an object representing the functor sending z to $\mathrm{Ker}(\mathrm{Mor}_{\mathcal{A}}(z, x) \to \mathrm{Mor}_{\mathcal{A}}(z, y))$, see Definition 1.9. Thus $\mathrm{Ker}(f)$ is 0 if and only if $\mathrm{Mor}_{\mathcal{A}}(z, x) \to \mathrm{Mor}_{\mathcal{A}}(z, y)$ is injective for all z if and only if f is a monomorphism. The proof of (2) is similar. □

In an abelian category, if $x \subset y$ is a subobject, then we denote

$$y/x = \mathrm{Coker}(x \to y).$$

Lemma 2.5. *Let $\mathcal{A}$ be an abelian category. All finite limits and finite colimits exist in $\mathcal{A}$.*

Proof. To show that finite limits exist it suffices to show that finite products and equalizers exist, see Categories, Lemma 18.4. Finite products exist by definition and the equalizer of $a, b : x \to y$ is the kernel of $a - b$. The argument for finite colimits is similar but dual to this. □

Example 2.6. Let $\mathcal{A}$ be an abelian category. Pushouts and fibre products in $\mathcal{A}$ have the following simple descriptions:

(1) If $a : x \to y$, $b : z \to y$ are morphisms in $\mathcal{A}$, then we have the fibre product: $x \times_y z = \mathrm{Ker}((a, -b) : x \oplus z \to y)$.

(2) If $a : y \to x$, $b : y \to z$ are morphisms in $\mathcal{A}$, then we have the pushout: $x \amalg_y z = \mathrm{Coker}((a, -b) : y \to x \oplus z)$.

Definition 2.7. Let $\mathcal{A}$ be an additive category. Consider a sequence of morphisms

$$\ldots \to x \to y \to z \to \ldots \quad \text{or} \quad x_1 \to x_2 \to \ldots \to x_n$$

in $\mathcal{A}$. We say such a sequence is a *complex* if the composition of any two consecutive (drawn) arrows is zero. If $\mathcal{A}$ is abelian then we say a complex of the first type above is *exact at y* if $\mathrm{Im}(x \to y) = \mathrm{Ker}(y \to z)$ and we say a complex of the second kind is *exact at x_i* where $1 < i < n$ if $\mathrm{Im}(x_{i-1} \to x_i) = \mathrm{Ker}(x_i \to x_{i+1})$. We a sequence as above is *exact* or is an *exact sequence* or is an *exact complex* if it is a complex and exact at every object (in the first case) or exact at x_i for all $1 < i < n$ (in the second case). There are variants of these notions for sequences of the form

$$\ldots \to x_{-3} \to x_{-2} \to x_{-1} \quad \text{and} \quad x_1 \to x_2 \to x_3 \to \ldots$$

A *short exact sequence* is an exact complex of the form

$$0 \to A \to B \to C \to 0.$$

In the following lemma we assume the reader knows what it means for a sequence of abelian groups to be exact.

Lemma 2.8. *Let $\mathcal{A}$ be an abelian category. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a complex of $\mathcal{A}$.*

(1) *$M_1 \to M_2 \to M_3 \to 0$ is exact if and only if*

$$0 \to \mathrm{Hom}_{\mathcal{A}}(M_3, N) \to \mathrm{Hom}_{\mathcal{A}}(M_2, N) \to \mathrm{Hom}_{\mathcal{A}}(M_1, N)$$

is an exact sequence of abelian groups for all objects N of $\mathcal{A}$, and

(2) *$0 \to M_1 \to M_2 \to M_3$ is exact if and only if*

$$0 \to \mathrm{Hom}_{\mathcal{A}}(N, M_1) \to \mathrm{Hom}_{\mathcal{A}}(N, M_2) \to \mathrm{Hom}_{\mathcal{A}}(N, M_1)$$

is an exact sequence of abelian groups for all objects N of $\mathcal{A}$.

Proof. Omitted. Hint: See Algebra, Lemma 10.1. □

Definition 2.9. Let $\mathcal{A}$ be an abelian category. Let $i : A \to B$ and $q : B \to C$ be morphisms of $\mathcal{A}$ such that $0 \to A \to B \to C \to 0$ is a short exact sequence. We say the short exact sequence is *split* if there exist morphisms $j : C \to B$ and $p : B \to A$ such that (B, i, j, p, q) is the direct sum of A and C.

Lemma 2.10. *Let $\mathcal{A}$ be an abelian category. Let $0 \to A \to B \to C \to 0$ be a short exact sequence.*

(1) *Given a morphism $s : C \to B$ left inverse to $B \to C$, there exists a unique $\pi : B \to A$ such that (s, π) splits the short exact sequence as in Definition 2.9.*
(2) *Given a morphism $\pi : B \to A$ right inverse to $A \to B$, there exists a unique $s : C \to B$ such that (s, π) splits the short exact sequence as in Definition 2.9.*

Proof. Omitted. □

Lemma 2.11. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{tikzcd} w \arrow[r, "f"] \arrow[d, "g"'] & y \arrow[d, "h"] \\ x \arrow[r, "k"] & z \end{tikzcd}$$

be a commutative diagram.

(1) *The diagram is cartesian if and only if*

$$0 \to w \xrightarrow{(g,f)} x \oplus y \xrightarrow{(k,-h)} z$$

is exact.

(2) *The diagram is cocartesian if and only if*

$$w \xrightarrow{(g,-f)} x \oplus y \xrightarrow{(k,h)} z \to 0$$

is exact.

Proof. Let $u = (g, f) : w \to x \oplus y$ and $v = (k, -h) : x \oplus y \to z$. Let $p : x \oplus y \to x$ and $q : x \oplus y \to y$ be the canonical projections. Let $i : \mathrm{Ker}(v) \to x \oplus y$ be the canonical injection. By Example 2.6, the diagram is cartesian if and only if there exists an isomorphism $r : \mathrm{Ker}(v) \to w$ with $f \circ r = q \circ i$ and $g \circ r = p \circ i$. The sequence $0 \to w \xrightarrow{u} x \oplus y \xrightarrow{v} z$ is exact if and only if there exists an isomorphism $r : \mathrm{Ker}(v) \to w$ with $u \circ r = i$. But given $r : \mathrm{Ker}(v) \to w$, we have $f \circ r = q \circ i$ and $g \circ r = p \circ i$ if and only if $q \circ u \circ r = f \circ r = q \circ i$ and $p \circ u \circ r = g \circ r = p \circ i$, hence if and only if $u \circ r = i$. This proves (1), and then (2) follows by duality. □

Lemma 2.12. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{tikzcd} w \arrow[r, "f"] \arrow[d, "g"'] & y \arrow[d, "h"] \\ x \arrow[r, "k"] & z \end{tikzcd}$$

be a commutative diagram.

(1) *If the diagram is cartesian, then the morphism* $\mathrm{Ker}(f) \to \mathrm{Ker}(k)$ *induced by g is an isomorphism.*

(2) *If the diagram is cocartesian, then the morphism* $\mathrm{Coker}(f) \to \mathrm{Coker}(k)$ *induced by h is an isomorphism.*

Proof. Suppose the diagram is cartesian. Let $e : \mathrm{Ker}(f) \to \mathrm{Ker}(k)$ be induced by g. Let $i : \mathrm{Ker}(f) \to w$ and $j : \mathrm{Ker}(k) \to x$ be the canonical injections. There exists $t : \mathrm{Ker}(k) \to w$ with $f \circ t = 0$ and $g \circ t = j$. Hence, there exists $u : \mathrm{Ker}(k) \to \mathrm{Ker}(f)$ with $i \circ u = t$. It follows $g \circ i \circ u \circ e = g \circ t \circ e = j \circ e = g \circ i$ and $f \circ i \circ u \circ e = 0 = f \circ i$, hence $i \circ u \circ e = i$. Since i is a monomorphism this implies $u \circ e = \mathrm{id}_{\mathrm{Ker}(f)}$. Furthermore, we have $j \circ e \circ u = g \circ i \circ u = g \circ t = j$. Since j is a monomorphism this implies $e \circ u = \mathrm{id}_{\mathrm{Ker}(k)}$. This proves (1). Now, (2) follows by duality. □

Lemma 2.13. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{array}{ccc} w & \xrightarrow{f} & y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ x & \xrightarrow{k} & z \end{array}$$

be a commutative diagram.

(1) *If the diagram is cartesian and k is an epimorphism, then the diagram is cocartesian and f is an epimorphism.*
(2) *If the diagram is cocartesian and g is a monomorphism, then the diagram is cartesian and h is a monomorphism.*

Proof. Suppose the diagram is cartesian and k is an epimorphism. Let $u = (g, f) : w \to x \oplus y$ and let $v = (k, -h) : x \oplus y \to z$. As k is an epimorphism, v is an epimorphism, too. Therefore and by Lemma 2.11, the sequence $0 \to w \xrightarrow{u} x \oplus y \xrightarrow{v} z \to 0$ is exact. Thus, the diagram is cocartesian by Lemma 2.11. Finally, f is an epimorphism by Lemma 2.12 and Lemma 2.4. This proves (1), and (2) follows by duality. □

Lemma 2.14. *Let $\mathcal{A}$ be an abelian category.*

(1) *If $x \to y$ is surjective, then for every $z \to y$ the projection $x \times_y z \to z$ is surjective.*
(2) *If $x \to y$ is injective, then for every $x \to z$ the morphism $z \to z \amalg_x y$ is injective.*

Proof. Immediately from Lemma 2.4 and Lemma 2.13. □

Lemma 2.15. *Let $\mathcal{A}$ be an abelian category. Let $f : x \to y$ and $g : y \to z$ be morphisms with $g \circ f = 0$. Then, the following statements are equivalent:*

(1) *The sequence $x \xrightarrow{f} y \xrightarrow{g} z$ is exact.*
(2) *For every $h : w \to y$ with $g \circ h = 0$ there exist an object v, an epimorphism $k : v \to w$ and a morphism $l : v \to x$ with $h \circ k = f \circ l$.*

Proof. Let $i : \mathrm{Ker}(g) \to y$ be the canonical injection. Let $p : x \to \mathrm{Coim}(f)$ be the canonical projection. Let $j : \mathrm{Im}(f) \to \mathrm{Ker}(g)$ be the canonical injection.

Suppose (1) holds. Let $h : w \to y$ with $g \circ h = 0$. There exists $c : w \to \mathrm{Ker}(g)$ with $i \circ c = h$. Let $v = x \times_{\mathrm{Ker}(g)} w$ with canonical projections $k : v \to w$ and $l : v \to x$, so that $c \circ k = j \circ p \circ l$. Then, $h \circ k = i \circ c \circ k = i \circ j \circ p \circ l = f \circ l$. As $j \circ p$ is an epimorphism by hypothesis, k is an epimorphism by Lemma 2.13. This implies (2).

Suppose (2) holds. Then, $g \circ i = 0$. So, there are an object w, an epimorphism $k : w \to \mathrm{Ker}(g)$ and a morphism $l : w \to x$ with $f \circ l = i \circ k$. It follows $i \circ j \circ p \circ l = f \circ l = i \circ k$. Since i is a monomorphism we see that $j \circ p \circ l = k$ is an epimorphism. So, j is an epimorphisms and thus an isomorphism. This implies (1). □

Lemma 2.16. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{array}{ccccc} x & \xrightarrow{f} & y & \xrightarrow{g} & z \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ u & \xrightarrow{k} & v & \xrightarrow{l} & w \end{array}$$

be a commutative diagram.

(1) *If the first row is exact and k is a monomorphism, then the induced sequence* $\mathrm{Ker}(\alpha) \to \mathrm{Ker}(\beta) \to \mathrm{Ker}(\gamma)$ *is exact.*
(2) *If the second row is exact and g is an epimorphism, then the induced sequence* $\mathrm{Coker}(\alpha) \to \mathrm{Coker}(\beta) \to \mathrm{Coker}(\gamma)$ *is exact.*

Proof. Suppose the first row is exact and k is a monomorphism. Let $a : \mathrm{Ker}(\alpha) \to \mathrm{Ker}(\beta)$ and $b : \mathrm{Ker}(\beta) \to \mathrm{Ker}(\gamma)$ be the induced morphisms. Let $h : \mathrm{Ker}(\alpha) \to x$, $i : \mathrm{Ker}(\beta) \to y$ and $j : \mathrm{Ker}(\gamma) \to z$ be the canonical injections. As j is a monomorphism we have $b \circ a = 0$. Let $c : s \to \mathrm{Ker}(\beta)$ with $b \circ c = 0$. Then, $g \circ i \circ c = j \circ b \circ c = 0$. By Lemma 2.15 there are an object t, an epimorphism $d : t \to s$ and a morphism $e : t \to x$ with $i \circ c \circ d = f \circ e$. Then, $k \circ \alpha \circ e = \beta \circ f \circ e = \beta \circ i \circ c \circ d = 0$. As k is a monomorphism we get $\alpha \circ e = 0$. So, there exists $m : t \to \mathrm{Ker}(\alpha)$ with $h \circ m = e$. It follows $i \circ a \circ m = f \circ h \circ m = f \circ e = i \circ c \circ d$. As i is a monomorphism we get $a \circ m = c \circ d$. Thus, Lemma 2.15 implies (1), and then (2) follows by duality. □

Lemma 2.17. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{tikzcd} & x \arrow[r, "f"] \arrow[d, "\alpha"] & y \arrow[r, "g"] \arrow[d, "\beta"] & z \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & u \arrow[r, "k"] & v \arrow[r, "l"] & w & \end{tikzcd}$$

be a commutative diagram with exact rows.

(1) *There exists a unique morphism $\delta : \mathrm{Ker}(\gamma) \to \mathrm{Coker}(\alpha)$ such that the diagram*

$$\begin{tikzcd} y \arrow[d, "\beta"'] & y \times_z \mathrm{Ker}(\gamma) \arrow[l, "\pi'"'] \arrow[r, "\pi"] & \mathrm{Ker}(\gamma) \arrow[d, "\delta"] \\ v \arrow[r, "\iota'"] & \mathrm{Coker}(\alpha) \amalg_u v & \mathrm{Coker}(\alpha) \arrow[l, "\iota"'] \end{tikzcd}$$

commutes, where π and π' are the canonical projections and ι and ι' are the canonical coprojections.
(2) *The induced sequence*

$$\mathrm{Ker}(\alpha) \xrightarrow{f'} \mathrm{Ker}(\beta) \xrightarrow{g'} \mathrm{Ker}(\gamma) \xrightarrow{\delta} \mathrm{Coker}(\alpha) \xrightarrow{k'} \mathrm{Coker}(\beta) \xrightarrow{l'} \mathrm{Coker}(\gamma)$$

is exact. If f is injective then so is f', and if l is surjective then so is l'.

Proof. As π is an epimorphism and ι is a monomorphism by Lemma 2.13, uniqueness of δ is clear. Let $p = y \times_z \mathrm{Ker}(\gamma)$ and $q = \mathrm{Coker}(\alpha) \amalg_u v$. Let $h : \mathrm{Ker}(\beta) \to y$, $i : \mathrm{Ker}(\gamma) \to z$ and $j : \mathrm{Ker}(\pi) \to p$ be the canonical injections. Let $\pi'' : u \to \mathrm{Coker}(\alpha)$ be the canonical projection. Keeping in mind Lemma 2.13 we get a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(\pi) \arrow[r, "j"] & p \arrow[r, "\pi"] \arrow[d, "\pi'"'] & \mathrm{Ker}(\gamma) \arrow[r] \arrow[d, "i"'] & 0 \\
 & x \arrow[r, "f"] \arrow[d, "\alpha"'] & y \arrow[r, "g"] \arrow[d, "\beta"'] & z \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & u \arrow[r, "k"] \arrow[d, "\pi''"'] & v \arrow[r, "l"] \arrow[d, "\iota'"'] & w & \\
0 \arrow[r] & \mathrm{Coker}(\alpha) \arrow[r, "\iota"] & q & &
\end{tikzcd}$$

As $l \circ \beta \circ \pi' = \gamma \circ i \circ \pi = 0$ and as the third row of the diagram above is exact, there is an $a : p \to u$ with $k \circ a = \beta \circ \pi'$. As the upper right quadrangle of the diagram above is cartesian, Lemma 2.12 yields an epimorphism $b : x \to \mathrm{Ker}(\pi)$ with $\pi' \circ j \circ b = f$. It follows $k \circ a \circ j \circ b = \beta \circ \pi' \circ j \circ b = \beta \circ f = k \circ \alpha$. As k is a monomorphism this implies $a \circ j \circ b = \alpha$. It follows $\pi'' \circ a \circ j \circ b = \pi'' \circ \alpha = 0$. As b is an epimorphism this implies $\pi'' \circ a \circ j = 0$. Therefore, as the top row of the diagram above is exact, there exists $\delta : \mathrm{Ker}(\gamma) \to \mathrm{Coker}(\alpha)$ with $\delta \circ \pi = \pi'' \circ a$. It follows $\iota \circ \delta \circ \pi = \iota \circ \pi'' \circ a = \iota' \circ k \circ a = \iota' \circ \beta \circ \pi'$ as desired.

As the upper right quadrangle in the diagram above is cartesian there is a $c : \mathrm{Ker}(\beta) \to p$ with $\pi' \circ c = h$ and $\pi \circ c = g'$. It follows $\iota \circ \delta \circ g' = \iota \circ \delta \circ \pi \circ c = \iota' \circ \beta \circ \pi' \circ c = \iota' \circ \beta \circ h = 0$. As ι is a monomorphism this implies $\delta \circ g' = 0$.

Next, let $d : r \to \mathrm{Ker}(\gamma)$ with $\delta \circ d = 0$. Applying Lemma 2.15 to the exact sequence $p \xrightarrow{\pi} \mathrm{Ker}(\gamma) \to 0$ and d yields an object s, an epimorphism $m : s \to r$ and a morphism $n : s \to p$ with $\pi \circ n = d \circ m$. As $\pi'' \circ a \circ n = \delta \circ d \circ m = 0$, applying Lemma 2.15 to the exact sequence $x \xrightarrow{\alpha} u \xrightarrow{p} \mathrm{Coker}(\alpha)$ and $a \circ n$ yields an object t, an epimorphism $\varepsilon : t \to s$ and a morphism $\zeta : t \to x$ with $a \circ n \circ \varepsilon = \alpha \circ \zeta$. It holds $\beta \circ \pi' \circ n \circ \varepsilon = k \circ \alpha \circ \zeta = \beta \circ f \circ \zeta$. Let $\eta = \pi' \circ n \circ \varepsilon - f \circ \zeta : t \to y$. Then, $\beta \circ \eta = 0$. It follows that there is a $\vartheta : t \to \mathrm{Ker}(\beta)$ with $\eta = h \circ \vartheta$. It holds $i \circ g' \circ \vartheta = g \circ h \circ \vartheta = g \circ \pi' \circ n \circ \varepsilon - g \circ f \circ \zeta = i \circ \pi \circ n \circ \varepsilon = i \circ d \circ m \circ \varepsilon$. As i is a monomorphism we get $g' \circ \vartheta = d \circ m \circ \varepsilon$. Thus, as $m \circ \varepsilon$ is an epimorphism, Lemma 2.15 implies that $\mathrm{Ker}(\beta) \xrightarrow{g'} \mathrm{Ker}(\gamma) \xrightarrow{\delta} \mathrm{Coker}(\alpha)$ is exact. Then, the claim follows by Lemma 2.16 and duality. □

Lemma 2.18. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{tikzcd}
 & & x \arrow[rr] \arrow[dl] \arrow[dd, "\alpha"] & & y \arrow[rr] \arrow[dl] \arrow[dd, "\beta"] & & z \arrow[rr] \arrow[dl] \arrow[dd, "\gamma"] & & 0 \\
 & x' \arrow[rr] \arrow[dd, "\alpha'"] & & y' \arrow[rr] \arrow[dd, "\beta'"] & & z' \arrow[rr] \arrow[dd, "\gamma'"] & & 0 & \\
 & 0 \arrow[r] & u \arrow[rr] \arrow[dl] & & v \arrow[rr] \arrow[dl] & & w \arrow[dl] & & \\
0 \arrow[r] & u' \arrow[rr] & & v' \arrow[rr] & & w' & & &
\end{tikzcd}$$

be a commutative diagram with exact rows. Then, the induced diagram

$$\begin{tikzcd}
\mathrm{Ker}(\alpha) \arrow[r] \arrow[d] & \mathrm{Ker}(\beta) \arrow[r] \arrow[d] & \mathrm{Ker}(\gamma) \arrow[r, "\delta"] \arrow[d] & \mathrm{Coker}(\alpha) \arrow[r] \arrow[d] & \mathrm{Coker}(\beta) \arrow[r] \arrow[d] & \mathrm{Coker}(\gamma) \arrow[d] \\
\mathrm{Ker}(\alpha') \arrow[r] & \mathrm{Ker}(\beta') \arrow[r] & \mathrm{Ker}(\gamma') \arrow[r, "\delta'"] & \mathrm{Coker}(\alpha') \arrow[r] & \mathrm{Coker}(\beta') \arrow[r] & \mathrm{Coker}(\gamma')
\end{tikzcd}$$

commutes.

Proof. Omitted. □

Lemma 2.19. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{tikzcd}
w \arrow[r] \arrow[d, "\alpha"] & x \arrow[r] \arrow[d, "\beta"] & y \arrow[r] \arrow[d, "\gamma"] & z \arrow[d, "\delta"] \\
w' \arrow[r] & x' \arrow[r] & y' \arrow[r] & z'
\end{tikzcd}$$

be a commutative diagram with exact rows.

(1) *If α, γ are surjective and δ is injective, then β is surjective.*
(2) *If β, δ are injective and α is surjective, then γ is injective.*

Proof. Assume α, γ are surjective and δ is injective. We may replace w' by $\mathrm{Im}(w' \to x')$, i.e., we may assume that $w' \to x'$ is injective. We may replace z by $\mathrm{Im}(y \to z)$, i.e., we may assume that $y \to z$ is surjective. Then we may apply Lemma 2.17 to

$$\begin{tikzcd}
 & \mathrm{Ker}(y \to z) \arrow[r] \arrow[d] & y \arrow[r] \arrow[d] & z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ker}(y' \to z') \arrow[r] & y' \arrow[r] & z' &
\end{tikzcd}$$

to conclude that $\mathrm{Ker}(y \to z) \to \mathrm{Ker}(y' \to z')$ is surjective. Finally, we apply Lemma 2.17 to

$$\begin{tikzcd}
 & w \arrow[r] \arrow[d] & x \arrow[r] \arrow[d] & \mathrm{Ker}(y \to z) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & w' \arrow[r] & x' \arrow[r] & \mathrm{Ker}(y' \to z') &
\end{tikzcd}$$

to conclude that $x \to x'$ is surjective. This proves (1). The proof of (2) is dual to this. □

Lemma 2.20. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{tikzcd}
v \arrow[r] \arrow[d, "\alpha"] & w \arrow[r] \arrow[d, "\beta"] & x \arrow[r] \arrow[d, "\gamma"] & y \arrow[r] \arrow[d, "\delta"] & z \arrow[d, "\epsilon"] \\
v' \arrow[r] & w' \arrow[r] & x' \arrow[r] & y' \arrow[r] & z'
\end{tikzcd}$$

be a commutative diagram with exact rows. If β, δ are isomorphisms, ϵ is injective, and α is surjective then γ is an isomorphism.

Proof. Immediate consequence of Lemma 2.19. □

3. Extensions

Definition 3.1. Let $\mathcal{A}$ be an abelian category. Let $A, B \in \mathrm{Ob}(\mathcal{A})$. An *extension* E *of* B *by* A is a short exact sequence

$$0 \to A \to E \to B \to 0.$$

An *morphism of extensions* between two extensions $0 \to A \to E \to B \to 0$ and $0 \to A \to F \to B \to 0$ means a morphism $f : E \to F$ in $\mathcal{A}$ making the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{id}"] & E \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & A \arrow[r] & F \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

commutative. Thus, the extensions of B by A form a category.

By abuse of language we often omit mention of the morphisms $A \to E$ and $E \to B$, although they are definitively part of the structure of an extension.

Definition 3.2. Let $\mathcal{A}$ be an abelian category. Let $A, B \in \mathrm{Ob}(\mathcal{A})$. The set of isomorphism classes of extensions of B by A is denoted

$$\mathrm{Ext}_{\mathcal{A}}(B, A).$$

This is called the Ext-*group*.

This definition works, because by our conventions $\mathcal{A}$ is a set, and hence $\mathrm{Ext}_{\mathcal{A}}(B, A)$ is a set. In any of the cases of "big" abelian categories listed in Categories, Remark 2.2 one can check by hand that $\mathrm{Ext}_{\mathcal{A}}(B, A)$ is a set as well. Also, we will see later that this is always the case when $\mathcal{A}$ has either enough projectives or enough injectives. Insert future reference here.

Actually we can turn $\mathrm{Ext}_{\mathcal{A}}(-, -)$ into a functor

$$\mathcal{A} \times \mathcal{A}^{opp} \longrightarrow \textit{Sets}, \quad (A, B) \longmapsto \mathrm{Ext}_{\mathcal{A}}(B, A)$$

as follows:

(1) Given a morphism $B' \to B$ and an extension E of B by A we define $E' = E \times_B B'$ so that we have the following commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & E' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & E \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

The extension E' is called the *pullback of* E *via* $B' \to B$.

(2) Given a morphism $A \to A'$ and an extension E of B by A we define $E' = A' \amalg_A E$ so that we have the following commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & E' \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

The extension E' is called the *pushout of* E *via* $A \to A'$.

To see that this defines a functor as indicated above there are several things to verify. First of all functoriality in the variable B requires that $(E\times_B B')\times_{B'} B'' = E\times_B B''$ which is a general property of fibre products. Dually one deals with functoriality in the variable A. Finally, given $A \to A'$ and $B' \to B$ we have to show that

$$A' \amalg_A (E \times_B B') \cong (A' \amalg_A E) \times_B B'$$

as extensions of B' by A'. Recall that $A' \amalg_A E$ is a quotient of $A' \oplus E$. Thus the right hand side is a quotient of $A' \oplus E \times_B B'$, and it is straightforward to see that the kernel is exactly what you need in order to get the left hand side.

Note that if E_1 and E_2 are extensions of B by A, then $E_1 \oplus E_2$ is an extension of $B \oplus B$ by $A \oplus A$. We push out by the sum map $A \oplus A \to A$ and we pull back by the diagonal map $B \to B \oplus B$ to get an extension $E_1 + E_2$ of B by A.

$$\begin{tikzcd}
0 \arrow[r] & A \oplus A \arrow[r] \arrow[d, "\Sigma"'] & E_1 \oplus E_2 \arrow[r] \arrow[d] & B \oplus B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & E' \arrow[r] & B \oplus B \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u] & E_1 + E_2 \arrow[r] \arrow[u] & B \arrow[r] \arrow[u, "\Delta"] & 0
\end{tikzcd}$$

The extension $E_1 + E_2$ is called the *Baer sum* of the given extensions.

Lemma 3.3. *The construction $(E_1, E_2) \mapsto E_1 + E_2$ above defines a commutative group law on $\mathrm{Ext}_{\mathcal{A}}(B, A)$ which is functorial in both variables.*

Proof. Omitted. ☐

Lemma 3.4. *Let $\mathcal{A}$ be an abelian category. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence in $\mathcal{A}$.*

(1) *There is a canonical six term exact sequence of abelian groups*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(M_3, N) \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(M_2, N) \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(M_1, N) \arrow[dll] \\
& \mathrm{Ext}_{\mathcal{A}}(M_3, N) \arrow[r] & \mathrm{Ext}_{\mathcal{A}}(M_2, N) \arrow[r] & \mathrm{Ext}_{\mathcal{A}}(M_1, N)
\end{tikzcd}$$

for all objects N of $\mathcal{A}$, and

(2) *there is a canonical six term exact sequence of abelian groups*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(N, M_1) \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(N, M_2) \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(N, M_3) \arrow[dll] \\
& \mathrm{Ext}_{\mathcal{A}}(N, M_1) \arrow[r] & \mathrm{Ext}_{\mathcal{A}}(N, M_2) \arrow[r] & \mathrm{Ext}_{\mathcal{A}}(N, M_3)
\end{tikzcd}$$

for all objects N of $\mathcal{A}$.

Proof. Omitted. Hint: The boundary maps are defined using either the pushout or pullback of the given short exact sequence. ☐

4. Additive functors

First a completely silly lemma characterizing additive functors between additive categories.

Lemma 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. The following are equivalent*

1. *F is additive,*
2. *$F(A) \oplus F(B) \to F(A \oplus B)$ is an isomorphism for all $A, B \in \mathcal{A}$, and*
3. *$F(A \oplus B) \to F(A) \oplus F(B)$ is an isomorphism for all $A, B \in \mathcal{A}$.*

Proof. Additive functors commute with direct sums by Lemma 1.7 hence (1) implies (2) and (3). On the other hand (2) and (3) are equivalent because the composition $F(A) \oplus F(B) \to F(A \oplus B) \to F(A) \oplus F(B)$ is the identity map. Assume (2) and (3) hold. Let $f, g : A \to B$ be maps. Then $f + g$ is equal to the composition

$$A \to A \oplus A \xrightarrow{\mathrm{diag}(f,g)} B \oplus B \to B$$

Apply the functor F and consider the following diagram

$$\begin{tikzcd}
F(A) \arrow[r] \arrow[dr] & F(A \oplus A) \arrow[rr, "F(\mathrm{diag}(f,g))"'] & & F(B \oplus B) \arrow[r] \arrow[d] & F(B) \\
& F(A) \oplus F(A) \arrow[u] \arrow[rr, "\mathrm{diag}(F(f),F(g))"] & & F(B) \oplus F(B) \arrow[ur] &
\end{tikzcd}$$

We claim this is commutative. For the middle square we can verify it separately for each of the four induced maps $F(A) \to F(B)$ where it follows from the fact that F is a functor (in other words this square commutes even if F does not satisfy any properties beyond being a functor). For the triangle on the left, we use that $F(A \oplus A) \to F(A) \oplus F(A)$ is an isomorphism to see that it suffice to check after composition with this map and this check is trivial. Dually for the other triangle. Thus going around the bottom is equal to $F(f + g)$ and we conclude. $\square$

Recall that we defined, in Categories, Definition 23.1 the notion of a “right exact”, “left exact” and “exact” functor in the setting of a functor between categories that have finite (co)limits. Thus this applies in particular to functors between abelian categories.

Lemma 4.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor.*

1. *If F is either left or right exact, then it is additive.*
2. *F is left exact if and only if for every short exact sequence $0 \to A \to B \to C \to 0$ the sequence $0 \to F(A) \to F(B) \to F(C)$ is exact.*
3. *F is right exact if and only if for every short exact sequence $0 \to A \to B \to C \to 0$ the sequence $F(A) \to F(B) \to F(C) \to 0$ is exact.*
4. *F is exact if and only if for every short exact sequence $0 \to A \to B \to C \to 0$ the sequence $0 \to F(A) \to F(B) \to F(C) \to 0$ is exact.*

Proof. If F is left exact, i.e., F commutes with finite limits, then F sends products to products, hence F preserved direct sums, hence F is additive by Lemma 4.1. On the other hand, suppose that for every short exact sequence $0 \to A \to B \to C \to 0$ the sequence $0 \to F(A) \to F(B) \to F(C)$ is exact. Let A, B be two objects. Then we have a short exact sequence

$$0 \to A \to A \oplus B \to B \to 0$$

see for example Lemma 1.10. By assumption, the lower row in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F(A) \arrow[r] \arrow[d] & F(A) \oplus F(B) \arrow[r] \arrow[d] & F(B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F(A) \arrow[r] & F(A \oplus B) \arrow[r] & F(B) &
\end{tikzcd}$$

is exact. Hence by the snake lemma (Lemma 2.17) we conclude that $F(A) \oplus F(B) \to F(A \oplus B)$ is an isomorphism. Hence F is additive in this case as well. Thus for the rest of the proof we may assume F is additive.

Denote $f : B \to C$ a map from B to C. Exactness of $0 \to A \to B \to C$ just means that $A = \mathrm{Ker}(f)$. Clearly the kernel of f is the equalizer of the two maps f and 0 from B to C. Hence if F commutes with limits, then $F(\mathrm{Ker}(f)) = \mathrm{Ker}(F(f))$ which exactly means that $0 \to F(A) \to F(B) \to F(C)$ is exact.

Conversely, suppose that F is additive and transforms any short exact sequence $0 \to A \to B \to C \to 0$ into an exact sequence $0 \to F(A) \to F(B) \to F(C)$. Because it is additive it commutes with direct sums and hence finite products in $\mathcal{A}$. To show it commutes with finite limits it therefore suffices to show that it commutes with equalizers. But equalizers in an abelian category are the same as the kernel of the difference map, hence it suffices to show that F commutes with taking kernels. Let $f : A \to B$ be a morphism. Factor f as $A \to I \to B$ with $f' : A \to I$ surjective and $i : I \to B$ injective. (This is possible by the definition of an abelian category.) Then it is clear that $\mathrm{Ker}(f) = \mathrm{Ker}(f')$. Also $0 \to \mathrm{Ker}(f') \to A \to I \to 0$ and $0 \to I \to B \to B/I \to 0$ are short exact. By the condition imposed on F we see that $0 \to F(\mathrm{Ker}(f')) \to F(A) \to F(I)$ and $0 \to F(I) \to F(B) \to F(B/I)$ are exact. Hence it is also the case that $F(\mathrm{Ker}(f'))$ is the kernel of the map $F(A) \to F(B)$, and we win.

The proof of (3) is similar to the proof of (2). Statement (4) is a combination of (2) and (3). □

Lemma 4.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. For every pair of objects A, B of $\mathcal{A}$ the functor F induces an abelian group homomorphism*

$$\mathrm{Ext}_{\mathcal{A}}(B, A) \longrightarrow \mathrm{Ext}_{\mathcal{B}}(F(B), F(A))$$

which maps the extension E to $F(E)$.

Proof. Omitted. □

The following lemma is used in the proof that the category of abelian sheaves on a site is abelian, where the functor b is sheafification.

Lemma 4.4. *Let $a : \mathcal{A} \to \mathcal{B}$ and $b : \mathcal{B} \to \mathcal{A}$ be functors. Assume that*

1. *$\mathcal{A}$, $\mathcal{B}$ are additive categories, a, b are additive functors, and a is right adjoint to b,*
2. *$\mathcal{B}$ is abelian and b is left exact, and*
3. *$ba \cong id_{\mathcal{A}}$.*

Then $\mathcal{A}$ is abelian.

Proof. As $\mathcal{B}$ is abelian we see that all finite limits and colimits exist in $\mathcal{B}$ by Lemma 2.5. Since b is a left adjoint we see that b is also right exact and hence exact, see Categories, Lemma 24.6. Let $\varphi : B_1 \to B_2$ be a morphism of $\mathcal{B}$. In particular, if $K = \mathrm{Ker}(B_1 \to B_2)$, then K is the equalizer of 0 and φ and hence bK is the equalizer of 0 and $b\varphi$, hence bK is the kernel of $b\varphi$. Similarly, if $Q = \mathrm{Coker}(B_1 \to B_2)$, then Q is the coequalizer of 0 and φ and hence bQ is the coequalizer of 0 and $b\varphi$, hence bQ is the cokernel of $b\varphi$. Thus we see that every morphism of the form $b\varphi$ in $\mathcal{A}$ has a kernel and a cokernel. However, since $ba \cong \mathrm{id}$ we see that every morphism of $\mathcal{A}$ is of this form, and we conclude that kernels and cokernels exist in $\mathcal{A}$. In fact, the argument shows that if $\psi : A_1 \to A_2$ is a morphism then

$$\mathrm{Ker}(\psi) = b\,\mathrm{Ker}(a\psi), \quad \text{and} \quad \mathrm{Coker}(\psi) = b\,\mathrm{Coker}(a\psi).$$

Now we still have to show that $\mathrm{Coim}(\psi) = \mathrm{Im}(\psi)$. We do this as follows. First note that since $\mathcal{A}$ has kernels and cokernels it has all finite limits and colimits (see proof of Lemma 2.5). Hence we see by Categories, Lemma 24.6 that a is left exact and hence transforms kernels (=equalizers) into kernels.

$$\begin{aligned}
\mathrm{Coim}(\psi) &= \mathrm{Coker}(\mathrm{Ker}(\psi) \to A_1) && \text{by definition} \\
&= b\,\mathrm{Coker}(a(\mathrm{Ker}(\psi) \to A_1)) && \text{by formula above} \\
&= b\,\mathrm{Coker}(\mathrm{Ker}(a\psi) \to aA_1)) && a \text{ preserves kernels} \\
&= b\,\mathrm{Coim}(a\psi) && \text{by definition} \\
&= b\,\mathrm{Im}(a\psi) && \mathcal{B} \text{ is abelian} \\
&= b\,\mathrm{Ker}(aA_2 \to \mathrm{Coker}(a\psi)) && \text{by definition} \\
&= \mathrm{Ker}(baA_2 \to b\,\mathrm{Coker}(a\psi)) && b \text{ preserves kernels} \\
&= \mathrm{Ker}(A_2 \to b\,\mathrm{Coker}(a\psi)) && ba = \mathrm{id}_{\mathcal{A}} \\
&= \mathrm{Ker}(A_2 \to \mathrm{Coker}(\psi)) && \text{by formula above} \\
&= \mathrm{Im}(\psi) && \text{by definition}
\end{aligned}$$

Thus the lemma holds. $\square$