Computer-assisted enumeration and classification of multi-qubit doilies

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Introduction

- Contextual geometries
- The Mermin-Peres magic square
- Multi-qubit doilies

Contributions

- Some properties of (numbers of) multi-qubit doilies
- Numbers of multi-qubit doilies
- Doily generation program

Contextuality

Kochen-Specker theorem

No non-contextual hidden-variable theory can reproduce the outcomes predicted by quantum physics

Without loss of generality, a non-contextual hidden-variable (NCHV) theory admits the existence of a function $v : \mathcal{P}_N \to \{-1, 1\}$ that determines (as $v(M)$) the result of any measurement with the multi-qubit Pauli observable M (among its two eigenvalues -1 and 1) independently of former measurements, even when they are compatible (commuting)

Mermin-Peres square proves Kochen-Specker theorem by describing experiments with nine two-qubit Pauli observables which contradict the NCHV hypothesis

Kochen and Specker. “The Problem of Hidden Variables in Quantum Mechanics”. *Indiana Univ. Math. J.*. 1968.

The Mermin-Peres magic square

Finite geometry with 9 points and 6 lines

- Each point $\equiv$ an observable
- Each line $\equiv$ a measurement context

$$
\begin{array}{ccccccc}
-1 & & -1 & & 1 & & \\
X \otimes I & - & I \otimes X & - & X \otimes X & & I \otimes I \\
1 \mid & & 1 \mid & & 1 \parallel & & \\
I \otimes Y & - & Y \otimes I & - & Y \otimes Y & & I \otimes I \\
-1 \mid & & -1 \mid & & ? \parallel & & \\
X \otimes Y & - & Y \otimes X & - & Z \otimes Z & & I \otimes I \\
 & & & & & & \\
I \otimes I & & I \otimes I & & -(I \otimes I) & &
\end{array}
$$

Quantum geometries

Definition of a quantum geometry(O, C):

- O is a finite set of observables (points): hermitian operators ($M = M^\dagger$) of finite dimension.
- C is a finite set of sub-sets of O called contexts (lines) such that:
  - each observable $M \in O$ satisfies $M^2 = Id$ (eigenvalues in $\{-1, 1\}$)
  - every observable M and N of a context commute ($MN = NM$)
  - The product of all observables of a context is the identity matrix Id or $-Id$

Holweck. "Testing Quantum Contextuality of Binary Symplectic Polar Spaces on a Noisy Intermediate Scale Quantum Computer". *Quantum Information Processing*. 2021.

Contextual finite quantum geometries

$$
\begin{array}{ccccccc}
o_1 & & o_2 & & o_3 & & \\
X \otimes I & - & I \otimes X & - & X \otimes X & & l_1 \\
| \; o_3 & & | \; o_4 & & \| \; o_5 & & \\
I \otimes Y & - & Y \otimes I & - & Y \otimes Y & & l_2 \\
| \; o_7 & & | \; o_8 & & \| \; o_9 & & \\
X \otimes Y & - & Y \otimes X & - & Z \otimes Z & & l_3 \\
 & & & & & & \\
l_4 & & l_5 & & l_6 & &
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{ccccccccc} o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 \end{array} \\
A = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{pmatrix}
\begin{array}{c} l_1 \\ l_2 \\ l_3 \\ l_4 \\ l_5 \\ l_6 \end{array}
\qquad
E = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
\begin{array}{c} I \otimes I \\ I \otimes I \\ I \otimes I \\ I \otimes I \\ I \otimes I \\ -(I \otimes I) \end{array}
\end{array}
$$

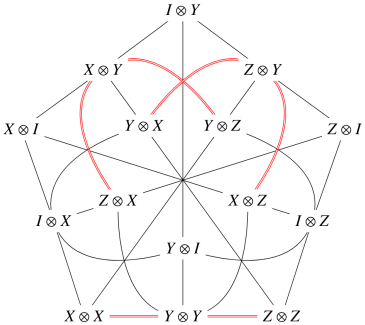
the product of observables on l_i is $(-1)^{E_i} I$

The geometry is *contextual* if $\nexists\, x . Ax = E$

Abramsky and Brandenburger. “The Sheaf-Theoretic Structure of Non-Locality and Contextuality”. *New Journal of Physics*. 2011.

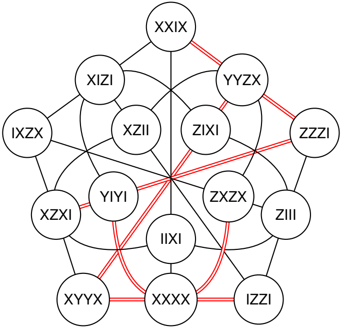
The W_2 doily

The doily is the contextual geometry of all the 2-qubit observables using Pauli observables except $I \otimes I$



N-qubit doilies

N-qubit doily: Contextual geometry on N qubits with the same point/line structure as the W_2 doily

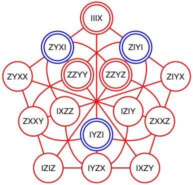


Example of 4-qubit doily

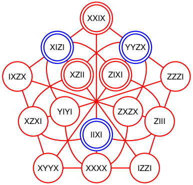
Doily Classification

- **Signature**: number of Is per observable (A: $N-1$ I per observable, B: $N-2$, C: $N-3\ldots$)
- **Nature** ν of the doily

For any unicentric triad (3 observables collinear with only one common observable)

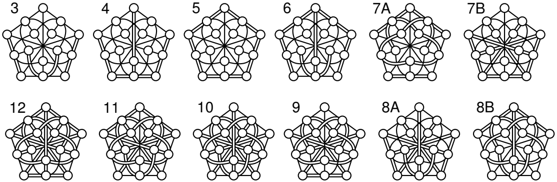


$ZYXI.IYZI.ZIYI = I^4 \Leftrightarrow$ Linear



$XIZI.IIXI.YYZX \neq I^4 \Leftrightarrow$ Quadratic

- **Configuration** of the negative lines



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Some properties of (numbers of) multi-qubit doilies

Numbers of multi-qubit doilies

Doily generation program

Contextuality degree

For every contextual geometry with an incidence matrix A, and for the valuation vector E related to the value of each line, we have

$$\nexists\, x; Ax = E$$

We are looking for the minimal difference between E and a vector Ax called the Hamming distance:

$$d_H\left(\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right) = 2$$

Property: every N-qubit doily has a contextuality degree of 3

de Boutray, Holweck, Giorgetti, Masson, and Saniga. "Contextuality degree of quadrics in multi-qubit symplectic polar spaces". 2022.

Numbers of multi-qubit doilies

Numbers $D(N)$ (resp. $D_l(N)$, $D_q(N)$) of (resp. linear, quadratic) N-qubit doilies

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=1}^{k} \frac{q^{n-k+i}-1}{q^i-1}$$

$$D(N) = D_l(N) + D_q(N)$$

$$D_l(N) = \begin{bmatrix} 2N \\ 4 \end{bmatrix}_2 - \begin{bmatrix} N \\ 4 \end{bmatrix}_2 \prod_{i=1}^{4}\left(2^{N+1-i}+1\right) - 7\begin{bmatrix} N \\ 3 \end{bmatrix}_2 2^{2N-6}\prod_{i=1}^{3}\left(2^{N+1-i}+1\right)/3$$

$$D_q(N) = 16\left(\begin{bmatrix} 2N \\ 5 \end{bmatrix}_2 - \begin{bmatrix} N \\ 5 \end{bmatrix}_2 \prod_{i=1}^{5}\left(2^{N+1-i}+1\right) - 15\begin{bmatrix} N \\ 4 \end{bmatrix}_2 2^{2N-8}\prod_{i=1}^{4}\left(2^{N+1-i}+1\right)/3\right)$$

N	$D_l(N)$	$D_q(N)$	$D(N)$
2	1	_	1
3	336	1 008	1 344
4	91 392	1 370 880	1 462 272
5	23 744 512	1 495 904 256	1 519 648 768
6	6 100 942 848	1 555 740 426 240	1 561 841 369 088
7	1 563 272 675 328	1 599 227 946 860 544	1 600 791 219 535 872
8	400 289 425 260 544	1 639 185 196 441 927 680	1 639 585 485 867 188 224
9	102 479 956 839 235 584	1 678 929 132 897 196 572 672	1 679 031 612 854 035 808 256

Doily generation program

Goal: Generate all N-qubit doilies for a given N in order to classify and check various properties about them

The *C* language is used because it allows for

- quick execution
- the use of test and proof tools

Execution time (Intel® Core™ i7-8665U CPU @ 1.90GHz, 8 cores):

- **4 qubits**: 1 462 272 doilies in 0.5s and 1.4 Mo
- **5 qubits**: 1 519 648 768 doilies in 12min and 1.8 Mo

Muller, Saniga, Giorgetti, de Boutray, and Holweck. “Multi-Qubit Doilies: Enumeration for All Ranks and Classification for Ranks Four and Five”. *Journal of Computational Science*. 2022.

Observables

The N-qubit observable $G_1 G_2 \cdots G_N$, with

$$G_j \leftrightarrow (g_j, g_{j+N}),\ j \in \{1, 2, \ldots, N\},$$

knowing that

$.$	I	X	Y	Z
I	I	X	Y	Z
X	X	I	iZ	$-iY$
Y	Y	$-iZ$	I	iX
Z	Z	iY	$-iX$	I

Multiplication table of the Pauli operators

$$I \leftrightarrow (0, 0),\ X \leftrightarrow (0, 1),\ Y \leftrightarrow (1, 1),\ \text{and } Z \leftrightarrow (1, 0).$$

no need to know the phase

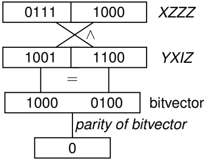
Operations

Product of observables

$$ZZZZ.XYZI = 11110000_2 \oplus 01101100_2 = 10011100_2 = p.YXIZ$$

Symplectic product

$$\langle a, b\rangle = a_1 b_{N+1} + a_{N+1} b_1 + a_2 b_{N+2} + a_{N+2} b_2 + \ldots + a_N b_{2N} + a_{2N} b_N$$

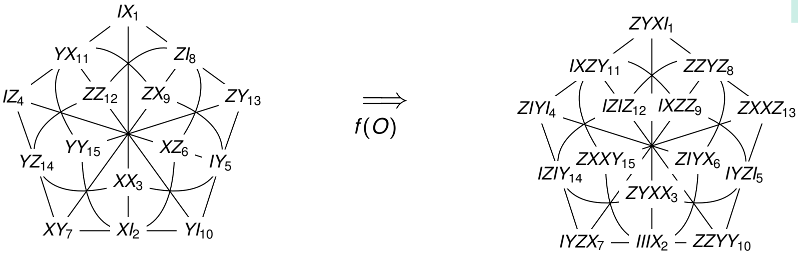


Computing process of the symplectic product

Representation of the doily

Every N-qubit doily is an injective labeling of the W_2 doily

We use the binary representation of the observables as array indices



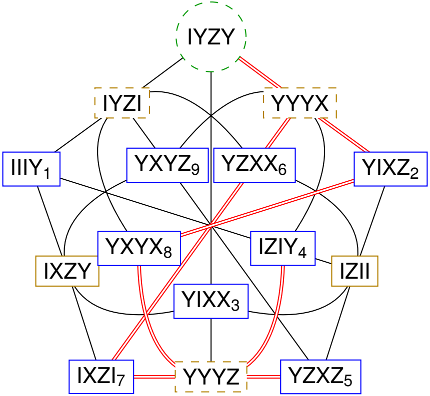
Representation of the 2-qubit doily with the bitvector used in the program

O	II	IX	XI	XX	IZ	IY	XZ	XY	ZI	ZX	YI	YX	ZZ	ZY	YZ	YY
bv	0000	0001	0010	0011	0100	0101	0110	0111	1000	1001	1010	1011	1100	1101	1110	1111
index	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$f(O)$	$\emptyset$	ZYXI	IIIX	ZYXX	ZIYI	IYZI	ZIYX	IYZX	ZZYZ	IXZZ	ZZYY	IXZY	IZIZ	ZXXZ	IZIY	ZXXY

Doily generation process steps

main → find_ovoids → generate_center → generate_doily → classify_doily

main	find_ovoids	generate_center	generate_doily	classify_doily
Main function initialization of the variables	Generation of every ovoid	For a pre-defined triad, compute every possible center	Identify all the other observables of the doily	Calculation of the properties and insertion in the results table



Doily generation algorithm

Algorithm 1 Description of the doily generation algorithm

for each ovoid $O = \{o_1, o_2, o_3, o_4, o_5\}$ in W_N, with $o_1 < o_2 < o_3 < o_4 < o_5$ **do**

  $f(IX) \leftarrow o_1 \;||\; f(IZ) \leftarrow o_2 \;||\; f(XY) \leftarrow o_3 \;||\; f(ZY) \leftarrow o_4 \;||\; f(YY) \leftarrow o_5$

  for each center c of $\{o_1, o_2, o_3\}$ in W_N that anticommutes with o_4 and o_5 **do**

    $f(XI) \leftarrow c$

    for each line (p, q, r) in the order of the sequence S **do** $f(r) \leftarrow |f(p).f(q)|$ **end for**

    if O is not the smallest ovoid of f **then** discard f **end if**

    ... ▷ Classification of f

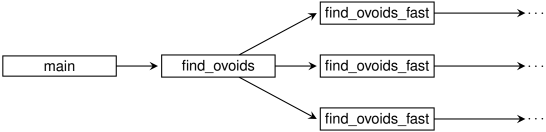
  end for

end for

$$S \equiv (XI, IX, XX), (XI, IZ, XZ), (XI, XY, IY), (ZY, XX, YZ), (ZY, XZ, YX), (ZY, IY, ZI), (YY, XX, ZZ), (YY, XZ, ZX), (YY, IY, YI)$$

Parallelization

Use of the parallelization library *OpenMP*



Doily generation process

```
#pragma omp parallel for schedule(dynamic,1) //dynamic distribution
for (bv i = 1; i < BV_LIMIT; i++) {//For all observables of Wn
  bv bv1[OVOID_COUNT];//We initialize an array representing the doily
  bv1[0] = i;//we assign its first value
  find_ovoid_fast(bv1);//We keep on generating the doily in this thread
}
```

The parallelization occurs at the generation of the first observable o_1 of the ovoid: 4^N iterations

Classification results

	Observables					Configuration of negative lines											
Type	A	B	C	D	ν	3	4	5	6	7A	7B	8A	8B	9	10	11	12
1	0	3	0	12	q	216				648				648			
2	0	4	0	11	q				3888			3888					
3	0	5	0	10	q	972		1944		4860	1944			1944			
4	1	0	5	9	q	648								648			
5	3	0	3	9	l	144											
6	0	6	0	9	q		1296		5184								
7	0	1	6	8	q	972				3888						972	
8	1	1	5	8	q				7776								
9	2	1	4	8	q	1944		1944									
10	2	1	4	8	l	972					972						
11	0	7	0	8	q			1944		972							
12	0	2	6	7	q				15552			11664	19440				
13	1	2	5	7	q	7776		13608			15552			1944			
14	1	2	5	7	l	3888					7776						
15	2	2	4	7	q		11664						3888				
⋮																	
95	6	9	0	0	l	6											

Partial results for the number of 4-qubit doilies

https://quantcert.github.io/doilies/

Conclusion

Review

- Work leading to a recent publication
- Results available at
  `https://quantcert.github.io/doilies`

Perspectives

- Extend the scope of the program to other contextual geometries
- Formal proof of the properties found

Muller, Saniga, Giorgetti, de Boutray, and Holweck. "Multi-Qubit Doilies: Enumeration for All Ranks and Classification for Ranks Four and Five". *Journal of Computational Science*. 2022.

Questions?

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