The common logical structure of classical and quantum mechanics

(and all scientific theories)

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Assumptions of Physics

The paper

The common logical structure of classical and quantum mechanics

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Project website: https://assumptionsofphysics.org/
YouTube channel: https://www.youtube.com/user/gcarcassi
Facebook page: https://www.facebook.com/AssumptionsOfPhysics

Main goal: to identify a handful of physical principles from which the basic laws can be rigorously derived

Sub-goal: a sturdier mathematical foundation for physical theories (all mathematical definitions justified from physical requirements)

Thesis

When the proper comparison is set up, classical and quantum mechanics follow the same logical structure

- Both theories have a non-distributive lattice embedded in a distributive one

More in general: all physical theories must follow the same logical structure

- Topology (Open sets) $\Longleftrightarrow$ Heyting algebra of experimentally verifiable statements
- σ-algebra (Borel sets) $\Longleftrightarrow$ Boolean algebra of statements associated with a test
- Statistical mixing $\Rightarrow$ Non-distributive lattice of closed subspaces

Plan

- Supposed failures of classical logic
  - Distributive law and conjunction (logical OR) in QM; the role of temporal evaluation and statistical considerations in showing how quantum mechanics obeys classical logic
- Common logical structure of CM and QM
  - The lattice of quantum logic (closed subspaces) does not contain all physically relevant statements; QM already has a classical lattice of statements (σ-algebra) which does; if we compare to a classical statistical theory, the situation is the same (lattice of closed sub-spaces embedded in σ-algebra)
- General logical structure for all scientific theories
  - Requirements from experimental verification: statements verifiable in finite time, countable at maximum; the lattice of verifiable statements identifies a topology; the lattice of theoretical statements identifies a σ-algebra; any theory that allows statistical mixing will also provide lattice of closed subspaces

Disclaimer

- Andrea Oldofredi is the one fluent in the philosophical and quantum logic literature
- My interest and expertise lies in understanding how the details of the mathematical structure map to physical concepts

Supposed failures of classical logic

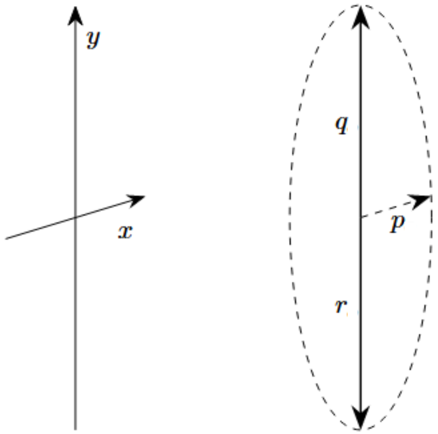
Distributivity law $p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$

Consider the following statements

p – "the electron has x-spin up"

q – "the electron has y-spin up"

r – "the electron has y-spin down"



Claim: spin in the y direction is either up or down – $q \vee r = \top$

Claim: x and y directions are incompatible – $p \wedge q = p \wedge r = \bot$

Suppose $p = \top$

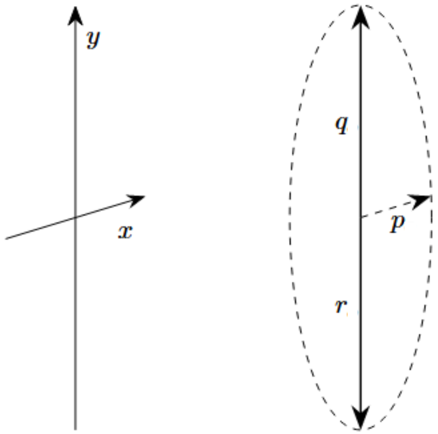
Distributivity violated!

$$p \wedge (q \vee r) = \top \leftrightarrow (p \wedge q) \vee (p \wedge r) = \bot$$

Disjunction $q \vee r = \top$ if and only if $q = \top$ or $r = \top$

Claim: spin in the y direction is either up or down – $q \vee r = \top$

Claim: this is true even if we prepare x-spin up

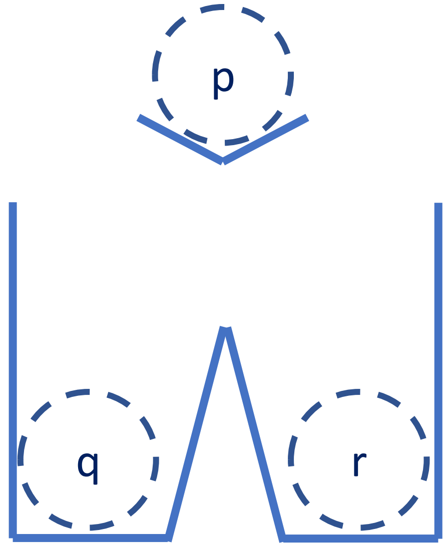


But if $p = \top$, then $q = r = \perp$ even though $q \vee r = \top$

Quantum disjunction works differently!

Distributivity law $p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$

Analog scenarios can be constructed with classical systems



A ball can be placed in position p, above a hatch; when the hatch is open, the ball will land in either position q or r with equal probability (due to the symmetry)

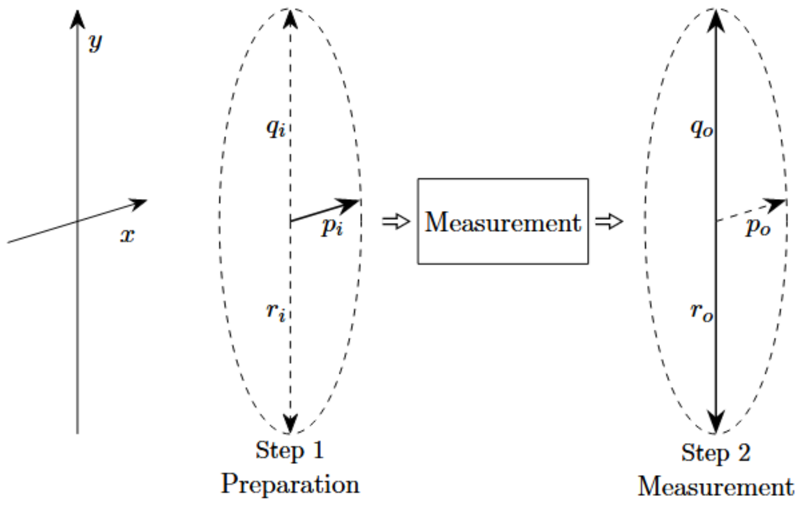
We can make similar misleading claims
- the ball must land either on q or r: $q \vee r = \top$
- the ball cannot be in two places at once: $p \wedge q = p \wedge r = \bot$

Classical mechanics does not follow classical logic!

Distributivity law $p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$

Need to take into account time:
use i for preparation (input)
and o for measurement (output)

The spin in the y direction is either up or down after a y measurement: $q_o \vee r_o = \top$



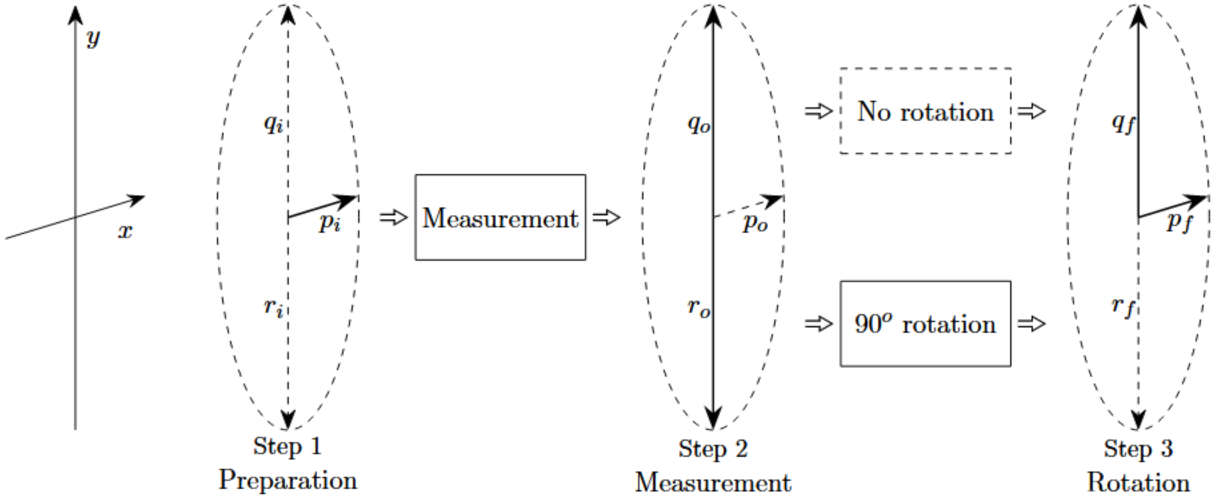
There is **no incompatibility** between x direction *before* the measurement and y direction *after* the measurement: $p_i \wedge q_o = \top$ if we prepared x-up and measured y-up

Distributivity not violated!

Disjunction $q \vee r = \mathrm{T}$ if and only if $q = \mathrm{T}$ or $r = \mathrm{T}$

Yes, if we measure y, we either have y-spin up or y-spin down

Suppose you rotate y-spin down after the measurement: $p_f \vee q_f = \mathrm{T}$



It does not follow that we must always have x-spin up or y-spin up:
$p_f \vee q_f = \mathrm{T} \nrightarrow p_i \vee q_i = \mathrm{T}$ and $p_f \vee q_f = \mathrm{T} \nrightarrow p_o \vee q_o = \mathrm{T}$

So, how does the quantum disjunction work?

Disjunction $q \vee r = \mathrm{T}$ if and only if $q = \mathrm{T}$ or $r = \mathrm{T}$

Since we have no idea how the disjunction in QM should work,
let's find equivalent statements for which we know how the disjunction works!

q – "the electron has y-spin up"

r – "the electron has y-spin down"

q' – "the expectation of y-spin is $\hbar/2$"

r' – "the expectation of y-spin is $-\hbar/2$"

Because those are extremal values, the statements are equivalent: $q \leftrightarrow q'$ and $r \leftrightarrow r'$

$q' \vee r'$ – "the expectation of y-spin is $\pm\hbar/2$"

This is not always true! True and only true if y-spin is up or down!

Quantum disjunction works the same!

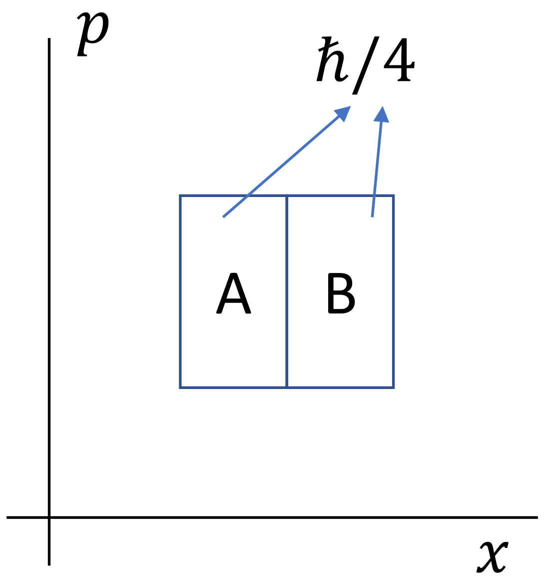
Disjunction

$q \vee r = \mathsf{T}$ if and only if $q = \mathsf{T}$ or $r = \mathsf{T}$

q – "the state of the particle is in A"

r – "the state of the particle is in B"

Claim: the statement $q \vee r$ can be true but, because of the uncertainty principle, $q = r = \perp$



Quantum disjunction works differently!

Disjunction

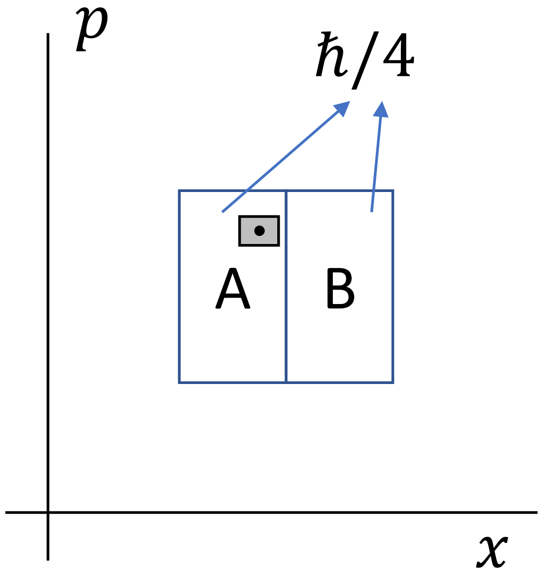
$q \vee r = \mathrm{T}$ if and only if $q = \mathrm{T}$ or $r = \mathrm{T}$

Consider a classical distribution $\rho(x,p)$

Either statistical or probabilistic

q – "the state of the particle is in A"

Ambiguous



q_c – "the center of mass is in A"

$[\bar{x},\bar{p}] \in A$

q_d – "one stdev of the distr. is in A"

$$(\bar{x}-\sigma_x/2, \bar{x}+\sigma_x/2) \times \left(\bar{p}-\sigma_p/2, \bar{p}+\sigma_p/2\right) \subseteq A$$

$q_c \vee r_c =$ "the center of mass is in A∪B"

$q_d \vee r_d \neq$ "one stdev of the distr. is in A∪B"

Half-sigma in A and half-sigma in B

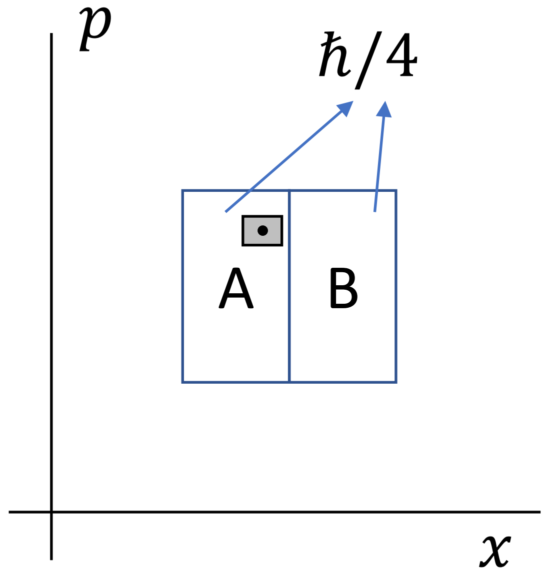
The disjunction does not always map to the union of the region

(nothing to do with quantum mechanics)

Disjunction $\quad q \lor r = \mathrm{T}$ if and only if $q = \mathrm{T}$ or $r = \mathrm{T}$

For a quantum state ψ

q – "the state of the particle is in A" Ambiguous



q_c – "the center of mass is in A"

$$[\bar{x}, \bar{p}] \in A$$

q_d – "one stdev of the distr. is in A"

$$(\bar{x} - \sigma_x/2, \bar{x} + \sigma_x/2) \times \left(\bar{p} - \sigma_p/2, \bar{p} + \sigma_p/2\right) \subseteq A$$

$q_c \lor r_c =$ "the center of mass is in AUB"

$q_d \lor r_d \neq$ "one stdev of the distr. is in AUB"

Half-sigma in A and half-sigma in B

Quantum disjunction works the same!

(just map to quantum expectations)

How to not get confused

- Be explicit on the process and the temporal ordering
  - What processes (e.g. preparations and measurements) is the system subjected to?
  - When are the statements evaluated?
- Be explicit about statistical/probabilistic attributes
  - Are we discussing center of mass (always well defined), support, standard deviation, … ?
- When comparing classical and quantum mechanics, compare apples to apples (i.e. expectation values with expectation values) and oranges to oranges (i.e. support/stdDev with support/stdDev)

Common logical structure of CM and QM

Link between logic and set theory

Consider various scientific statements such as:

"the mass of the electron is within 500 ± 50 keV"
"the earth-moon distance is $400{,}000$ km $\pm 50{,}000$ km"
"there are 6 flavors of quarks"
…

Statement $\Leftrightarrow$ Subset

The object we are describing

They can be modelled mathematically with the following pattern:

$$x \in U \subseteq X$$

The set all possible ways the object can be

A subset of possible ways the object can be

Link between logic and probability theory

A probability space is a triple $(\Omega, \Sigma_\Omega, P)$

Sample space Ω – the set of all possible outcomes ($\{1,2,3,4,5,6\}$ for a die)
A σ-algebra Σ_Ω – collection of all events we consider (even, odd, less than 3,…)
A probability measure P – assigns a probability to each event

Statement $\Leftrightarrow$ Subset

Mathematically, a σ-algebra is a collection of subsets closed under complement, countable union and countable intersection

Σ_Ω is countably complete Boolean algebra of statements

Probability theory requires classical logic

Do Hilbert spaces have a σ-algebra? Yes

Hilbert space ⇒ normed vector space ⇒ metric space ⇒ topological space ⇒ equipped with a Borel σ-algebra

Is the Borel σ-algebra something intelligible? Yes

A metric space defines a distance. A distance allows us to write statements of the type:

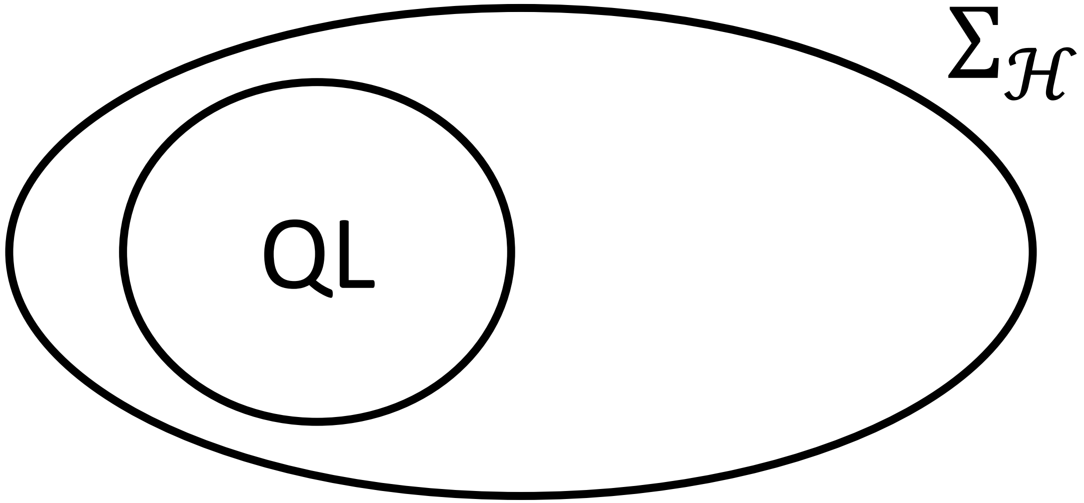
"The object x is within ϵ of reference y" $\quad \boldsymbol{s}(\epsilon, y) = \text{"}d(x,y)^2 < \epsilon^2\text{"}$

Any element in the σ-algebra is constructed from statements of that type using countable disjunction/conjunction and negation

What is the relationship with quantum logic?

The statements of quantum logic correspond to the closed subspaces of the Hilbert space. The σ-algebra contains all closed subsets.

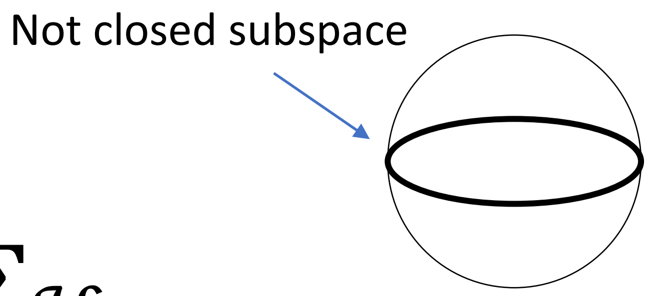
The lattice of quantum logic is a proper subset of the σ-algebra



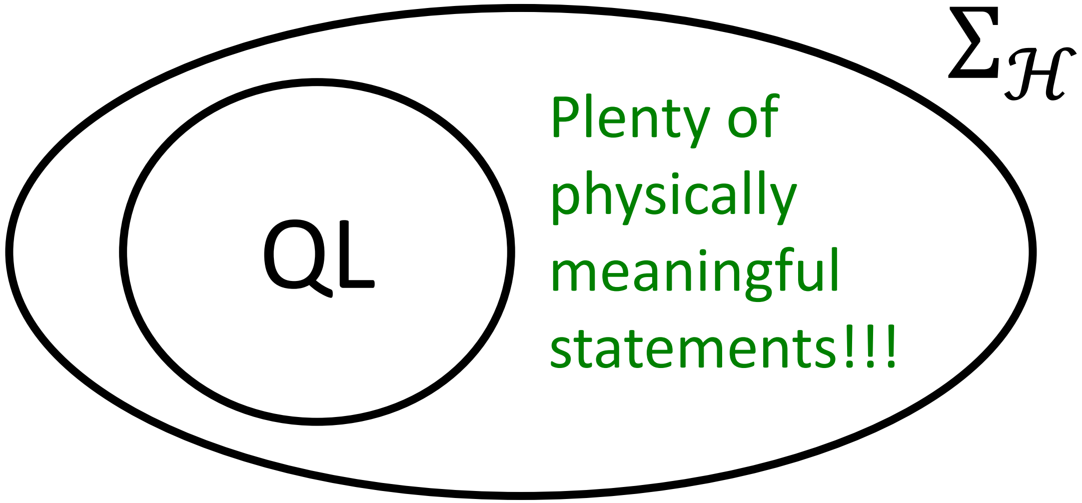
Are the extra statements physically useful? Yes

Statements of the type "the expectation of observable A is within U" are part of $\Sigma_{\mathcal{H}}$ but not QL

E.g.: the average z-spin is zero



The expectation of A given ψ is given by $E[A|\psi] = \langle\psi|A\psi\rangle$, which is continuous if A is bounded. This means the pre-image of Borel sets are Borel sets.



Quantum mechanics	Classical mechanics
$\mathcal{H}_{\mathbb{C}}$ - $\psi(x)$	$\mathcal{H}_{\mathbb{R}}$ - $\sqrt{\rho}(x,p)$
Metric induced topology $\mathsf{T}_{\mathcal{H}_{\mathbb{C}}}$	Metric induced topology $\mathsf{T}_{\mathcal{H}_{\mathbb{R}}}$
Countably complete Boolean algebra of statements $\Sigma_{\mathcal{H}_{\mathbb{C}}}$	Countably complete Boolean algebra of statements $\Sigma_{\mathcal{H}_{\mathbb{R}}}$
Non-distributive lattice of closed subspaces $L(\mathcal{H}_{\mathbb{C}})$	Non-distributive lattice of closed subspaces $L(\mathcal{H}_{\mathbb{R}})$

To summarize

- Probability theory requires a countably complete Boolean (classical) lattice of statements
- Quantum mechanics comes already equipped with such a lattice of statements (the Borel algebra)
- The Borel algebra includes all QL statements (everything QL does can be done in the Borel algebra)
- The Borel algebra includes statistical statements, which are of physical interest, that are not part of the QL statements (QL does not allow us to capture all physical statements)
- The same structure exists for classical distributions over phase-space
- Quantum mechanics and classical mechanics, when properly compared, share a common logical structure
    - The measure theoretic structure, however, will be different

Common logical structure of all scientific theories

We want to capture the scientific requirement of experimental verification. The basic notion will be **verifiable statements**: assertions that can be experimentally verified in a finite time

Examples:

- The mass of the photon is less than 10^{-13} eV
- If the height of the mercury column is between 24 and 25 millimeters then its temperature is between 24 and 25 Celsius
- If I take 2 ± 0.01 Kg of Sodium-24 and wait 15 ± 0.01 hours there will be only 1 ± 0.01 Kg left

Counterexamples:

- The mass of the photon is exactly 0 eV (not verifiable due to infinite precision)
- There is no extra-terrestrial life (absence of evidence is not evidence of absence)

We have to keep in mind that the meaning of the statements, their relationships and what truth values are allowed **depends on context** (e.g. premise, theory, etc…)

The mass of the electron is 511 $\pm$ 0.5 KeV

When measuring the mass, it is a verifiable hypothesis

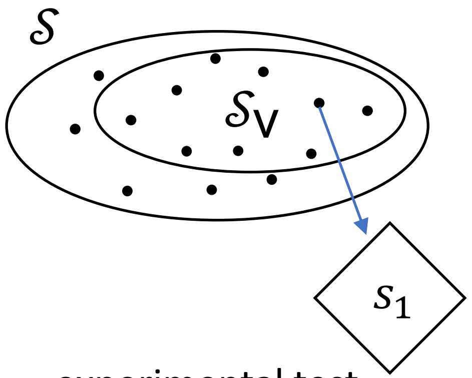
When performing particle identification, it is assumed to be true

Properties of verifiable statements

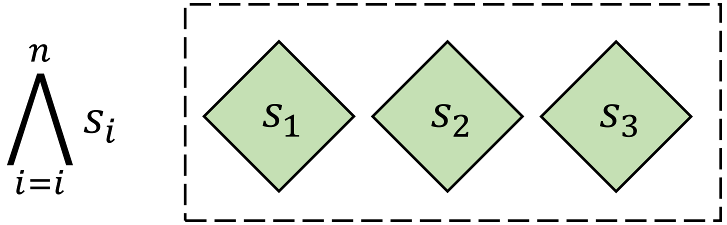
Axiom 1.27 (Axiom of verifiability). *A **verifiable statement** is a statement that, if true, can be shown to be so experimentally. Formally, each logical context $\mathcal{S}$ contains a set of statements $\mathcal{S}_{\mathsf{v}} \subseteq \mathcal{S}$ whose elements are said to be verifiable. Moreover, we have the following properties:*

- *every certainty $\top \in \mathcal{S}$ is verifiable*
- *every impossibility $\bot \in \mathcal{S}$ is verifiable*
- *a statement equivalent to a verifiable statement is verifiable*

Remark. The **negation or logical NOT** of a verifiable statement is not necessarily a verifiable statement.



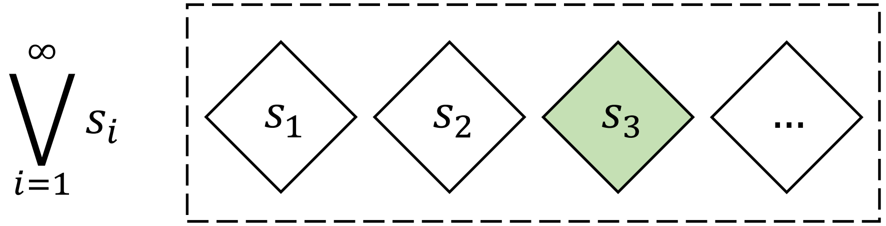
s_1	Test Result
T	SUCCESS (in finite time)
F	FAILURE (in finite time)
	UNDEFINED



All tests must succeed

Axiom 1.31 (Axiom of finite conjunction verifiability). *The conjunction of a finite collection of verifiable statements is a verifiable statement. Formally, let $\{\mathsf{s}_i\}_{i=1}^{n} \subseteq \mathcal{S}_{\mathsf{v}}$ be a finite collection of verifiable statements. Then the conjunction $\bigwedge_{i=1}^{n} \mathsf{s}_i \in \mathcal{S}_{\mathsf{v}}$ is a verifiable statement.*

Axiom 1.32 (Axiom of countable disjunction verifiability). *The disjunction of a countable collection of verifiable statements is a verifiable statement. Formally, let $\{\mathsf{s}_i\}_{i=1}^{\infty} \subseteq \mathcal{S}_{\mathsf{v}}$ be a countable collection of verifiable statements. Then the disjunction $\bigvee_{i=1}^{\infty} \mathsf{s}_i \in \mathcal{S}_{\mathsf{v}}$ is a verifiable statement.*



One successful test is sufficient

Requirements of experimental verifiability:

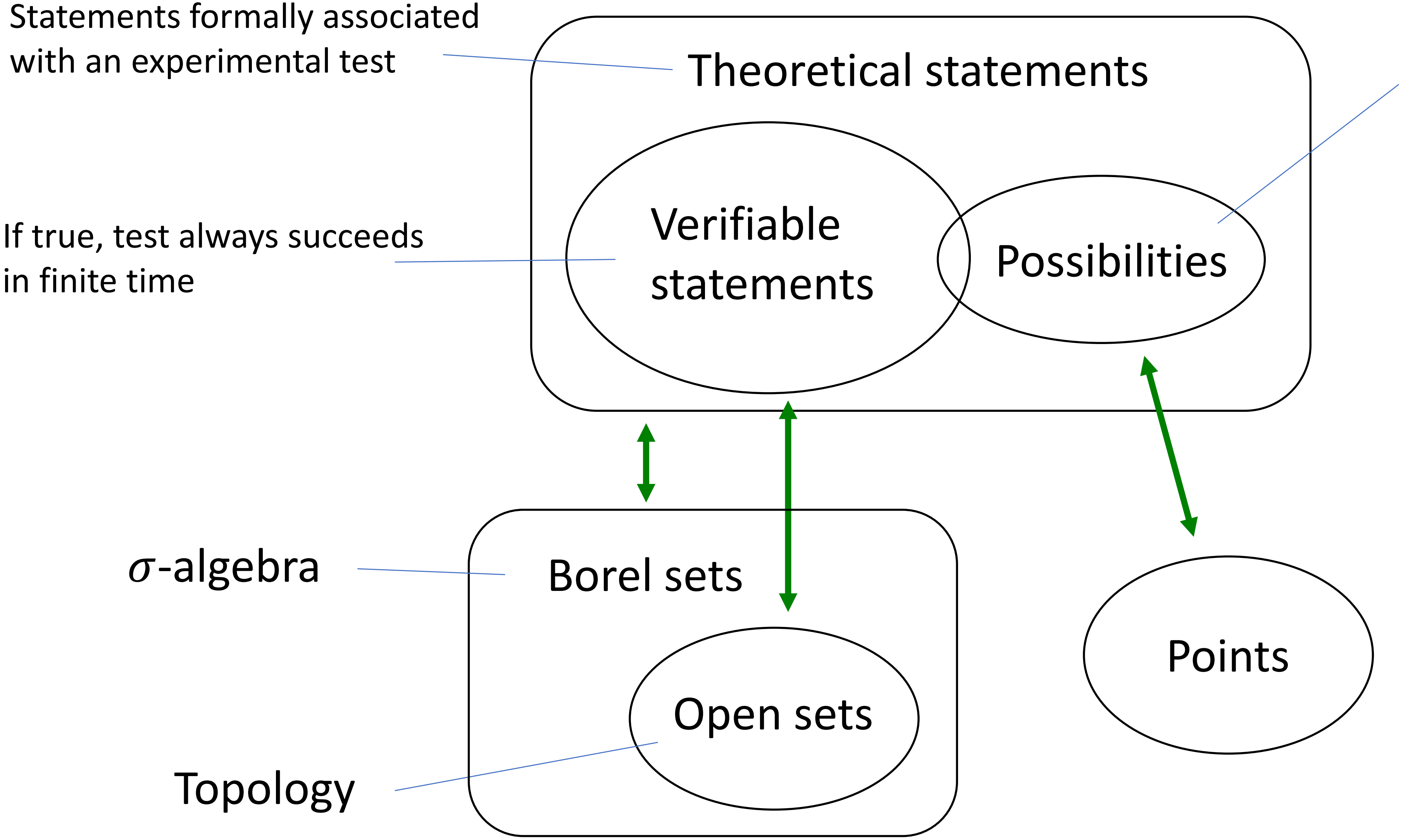
1. Verifiable statements are closed under finite conjunction and countable disjunction.
2. We can at most verify countably many statements (in the limit of arbitrarily long time).

Experimental domain $\mathcal{D}_{\boldsymbol{X}}$: a set of verifiable statements

1. closed under finite conjunction and countable disjunction
2. generated by countably many verifiable statements

Theoretical domain $\bar{\mathcal{D}}_{\boldsymbol{X}} \supseteq \mathcal{D}_{\boldsymbol{X}}$: all statements with a test, regardless of termination (closure of an experimental domain under negation and countable conjunction/disjunction)

Possibilities $\boldsymbol{X} \subset \bar{\mathcal{D}}_{\boldsymbol{X}}$: the statements that give the complete picture (i.e. every theoretical statement s, they either imply s or $\neg s$ – mathematically, the atoms of the lattice $\bar{\mathcal{D}}_{\boldsymbol{X}}$)



Precise map between physical concepts and their mathematical representation

All proofs can be "translated" into physically meaningful language

Some interesting results

Every set of physically distinguishable cases is a T_0 second-countable topological space

Every set of physically distinguishable cases can have up to the cardinality of the continuum

Every relationship between two sets of physically distinguishable cases must be topologically continuous, as inference can be used for experimental verification

Every relationship between two real quantities must be analytically continuous with up to countably many discontinuities, and the region of those discontinuities must be verifiable (e.g. we can verify water is in the triple point or undergoing phase transition)

We identified a set of necessary and sufficient physical conditions that lead to ordered quantities (these cannot be assumed valid at Planck scale)

These are **general results** that **must be valid for all theories**

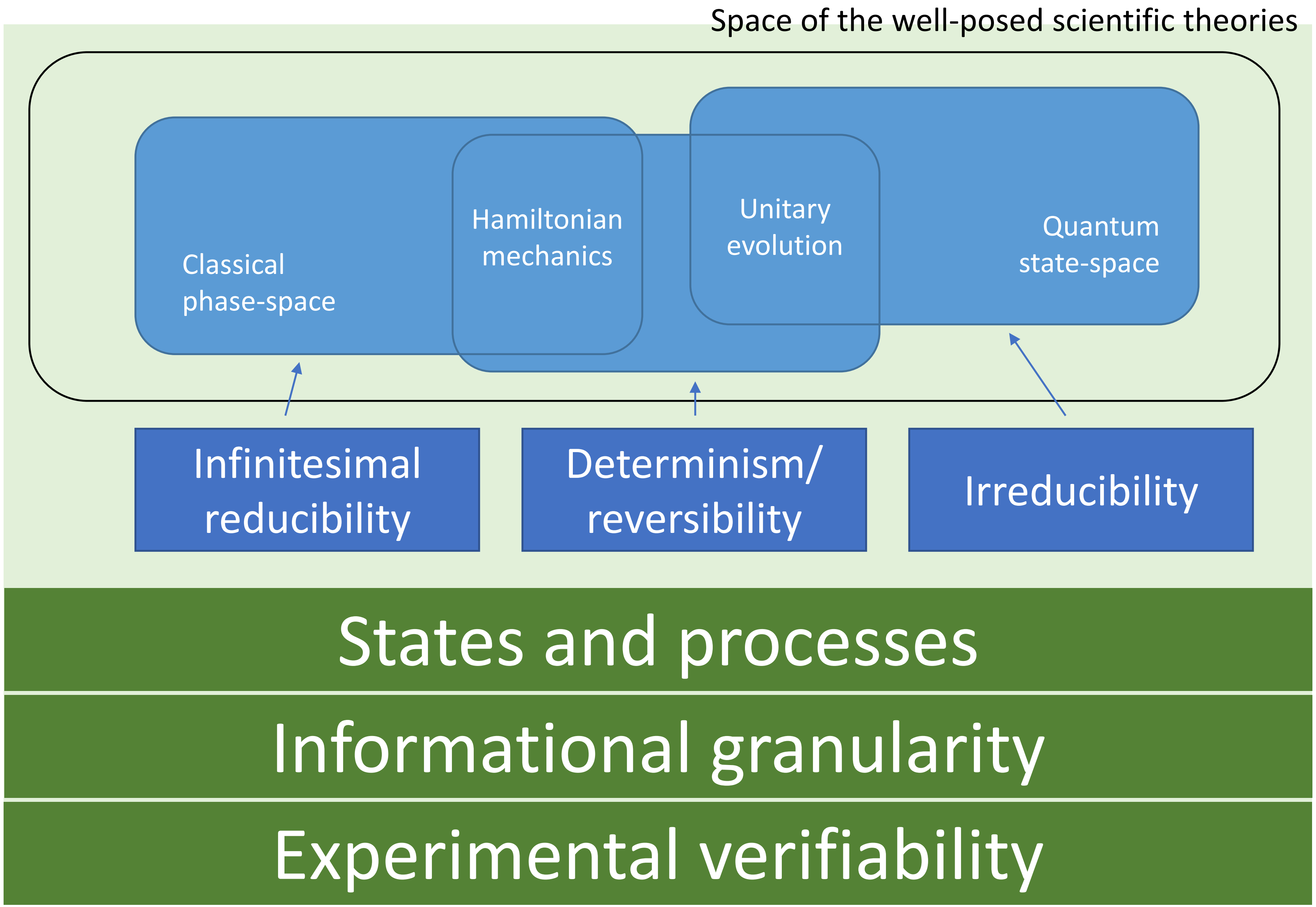
Physical theories

Specializations of the general theory under the different assumptions

Assumptions

General theory

Basic requirements and definitions valid in all theories



We want to describe the space of states for a system

$\Rightarrow$ Set of physically distinguishable elements

$\Rightarrow$ Topology $\left(\substack{\text{verifiable}\\\text{statements}}\right)$ and σ-algebra $\left(\substack{\text{theoretical}\\\text{statements}}\right)$

Must allow for ensembles

$p_a s_a + p_b s_b$

$\Rightarrow$ Statistical mixture imposes a linear operation

$\Rightarrow$ Lattice of subspaces

These are necessary structures for theories that describe objects that can be experimentally defined

Conclusions

- Once the full meaning of quantum propositions is properly taken into account, quantum mechanics follows classical logic
- Quantum logic propositions only apply to “single-shot” measurements, and therefore are of very limited use
- Hilbert spaces are already equipped with a classical logic lattice (the σ-algebra) that includes all quantum logic statements plus all statistical statements
- The space of classical statistical distributions is also a Hilbert space, which is equipped with a similar “quantum logic” lattice
- This pattern stems from a requirement of experimental verification, and therefore is common to all physical theories

  Topology (Open sets) $\Longleftrightarrow$ Heyting algebra of experimentally verifiable statements

  σ-algebra (Borel sets) $\Longleftrightarrow$ Boolean algebra of statements associated with a test

  Statistical mixing $\Rightarrow$ Non-distributive lattice of closed subspaces

Resources

Project website: https://assumptionsofphysics.org/

Papers, presentation slides, list of open problems, …

YouTube channel: https://www.youtube.com/user/gcarcassi

Popularize results of our research, recorded presentations, …

"Reverse physics: from laws to physical assumptions"

https://arxiv.org/abs/2111.09107 (Foundations of Physics 2022)

"Geometrical and physical interpretation of the action principle"

https://arxiv.org/abs/2208.06428 (pre-print)

"The four postulates of quantum mechanics are three"

https://arxiv.org/abs/2003.11007 (Physical Review Letters 2021)

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