Generalized ensemble spaces

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Abstract

We present an approach we are developing to define a theory of physical states that applies to all physical systems. Classical and quantum systems are seen as further specializations of this structure, while some properties/theorems are proven in the more abstract setting. The fundamental axioms of this framework can be justified on physical grounds, so that all mathematical objects have a clear physical correspondence. The work connects ideas from convex spaces, information geometry, and other fields.

Assumptions of Physics

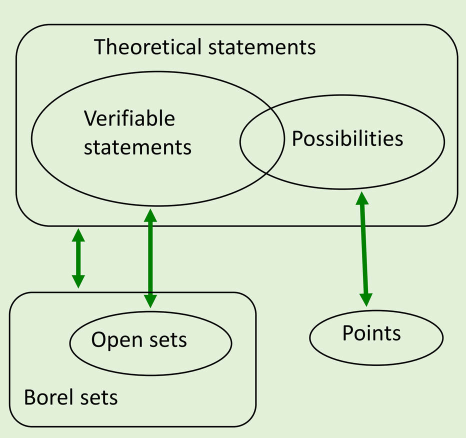
Axiom of ensemble

Since a physical theory needs to provide repeatedly testable results, it must be able to describe statistical ensembles that are distinguishable experimentally.

⇒ Topological structure

Every ensemble space must be a T_0 second countable topological space. The open sets represent statements that are experimentally verifiable: there is a test and the test succeeds in finite time if and only if the statement is true.

The Borel sets represent statements that are associated to a test, regardless of termination.



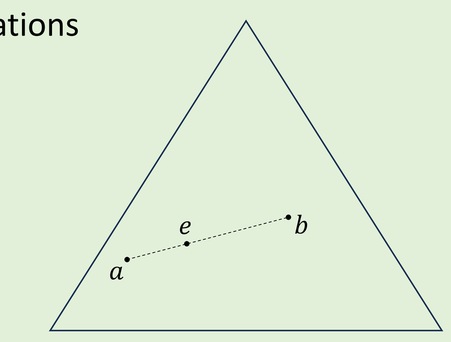
Axioms of mixture

Given two ensembles, we can always obtain new ones using statistical mixtures (e.g. selecting one 40% of the times and the other 60%).

⇒ Convex structure

Ensemble spaces allow convex combinations

$$e = pa + \bar{p}b$$
$$\bar{p} = 1 - p$$



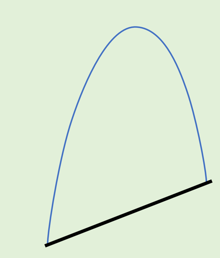
Axioms of entropy

Every ensemble must have a well defined entropy that represents the variability of the elements within the ensemble.

⇒ Entropic structure

Entropy is strictly concave

$$S(p_1 e_1 + p_2 e_2) \geq \sum p_i S(e_i)$$



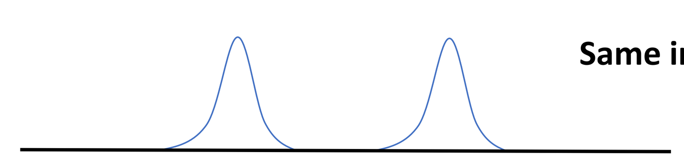
Upper bound on entropy increase

$$S(p_1 e_1 + p_2 e_2) \leq \sum p_i S(e_i) - p_i \log p_i$$

No common component vs orthogonal

No ensemble is part of both

$$\nexists e \; s.t. \; e_1 = p_1 e + \bar{p}_1 e_3$$
$$e_2 = p_2 e + \bar{p}_2 e_4$$



Same in classical mechanics

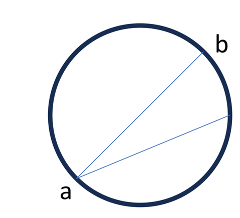
⇒ disjoint support

Entropy maximized during mixture

$$S(p_1 e_1 + p_2 e_2) = \sum p_i S(e_i) - p_i \log p_i$$

Different in quantum mechanics

a and b orthogonal
a and c no common component but not orthogonal



Complemented space

Space is complemented if we can “invert” mixing

$$e, e_1, p \Rightarrow e_2 \; s.t. \; e = pe_1 + \bar{p}e_2$$

⇒ vector space

Statistical quantities

A statistical quantity is a continuous linear functional

$$F : \mathcal{E} \to \mathbb{R}$$

Quantifiable ensemble space: ensembles identified by quantities

⇒ locally convex topological vector space

Orth decomposability

If an ensemble is a mixture, it is a mixture of orthogonal ensembles.

$$e = pe_1 + \bar{p}e_2 \Rightarrow e = \lambda a + \bar{\lambda} a_\perp$$

⇒ inner product space (?)

⇑ ⇑ ⇑

Above properties satisfied by classical and quantum spaces, not obviously justifiable a priori: possible new physical theories?

Fraction capacity

Generalized non-additive probability

Given an ensemble e and a set of ensembles A, what is the biggest component of e that can be achieved with a mixture of A?

$$\mathrm{fcap}_e(a) = \sup(\{p \in [0,1] \mid e = pa + \bar{p}e_1\})$$

$$\mathrm{fcap}_e(A) = \sup\left(\mathrm{fcap}_e\big(\mathrm{hull}(A)\big)\right)$$

⇒ non-negative, unit bounded, monotonic, sub-additive set function ⇒ fuzzy measure

Recovers probability (additive) in classical mechanics and quantum measurements

State capacity

Generalized non-additive state count

Given a set of ensembles A, quantifies how many distinguishable cases can be found in that subspace

$$\mathrm{scap}(A) = \sup\left(2^{S(\mathrm{hull}(A))}\right)$$

since $2^{S(p_1 e_1 + p_2 e_2)} \leq 2^{S(e_1)} + 2^{S(e_2)}$

⇒ non-negative, monotonic, sub-additive set function ⇒ fuzzy measure

Additive across orthogonal subspaces, recovers the classical Liouville measure

Ensemble subspaces

Define subspaces from the entropy upper bound (orthogonality)

$$A^\perp = \{e \in \mathcal{E} \mid \forall a \in A \;\; e \perp a\}$$
$$X = (X^\perp)^\perp$$

Recovers subspaces in both classical and quantum mechanics

Entropic geometry

Pseudo-distance (recovers Jensen-Shannon Divergence)

$$0 \leq S\left(\frac{1}{2}e_1 + \frac{1}{2}e_2\right) - \frac{1}{2}\left(S(e_1) + S(e_2)\right) \leq 1$$

Strict concavity of entropy ⇒ Hessian negative definite (recovers Fisher-Rao metric and Bures metric)

$$g(\delta e_1, \delta e_2) = -\frac{\partial^2 S}{\partial e^2}(\delta e_1, \delta e_2)$$

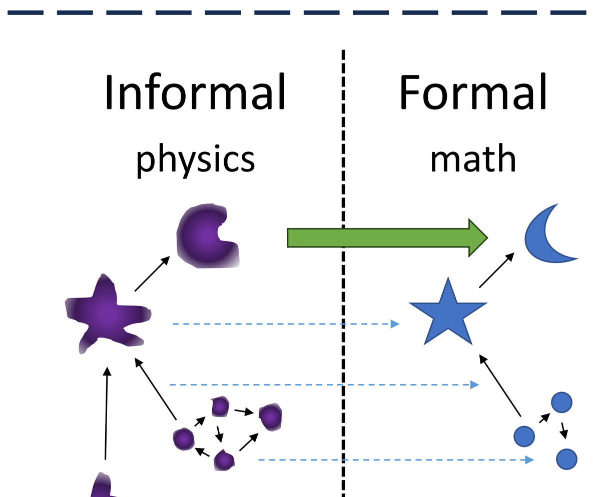
Physical mathematics

Derive mathematical structures from physical requirements

Axioms Necessary – map object from the informal (physical) realm into the formal (mathematical) system

Definitions Optional – further categorize objects that are already within the formal (mathematical) system

Physical justifications must be provided for both axioms and definitions: must show that the mathematical characterization is suitable for the given physical objects in given conditions



All math derived from physical requirements

Actively looking for collaborators

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Contexts and spectra

Context: lattice of subspaces where disjoint subspaces are orthogonal

$$X \cap Y = \emptyset \Rightarrow X \perp Y$$

Points: limits of subspaces as they become small (ultrafilters)

Recovers classical state (lattice of subspace is a context) and quantum contexts (?)