Towards a consistent mathematical foundation for all physical theories

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https://assumptionsofphysics.org

Main goal of the project

Identify a handful of physical starting points from which the basic laws can be rigorously derived

Assumptions of Physics

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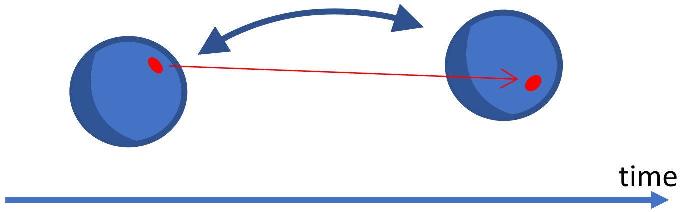
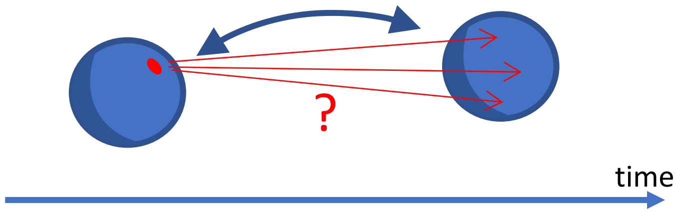
For example:

Infinitesimal reducibility ⇒ Classical state



Irreducibility ⇒ Quantum state



This also requires rederiving all mathematical structures from physical requirements

For example:

Science is evidence based ⇒ scientific theory must be characterized by experimentally verifiable statements ⇒ topology and σ-algebras

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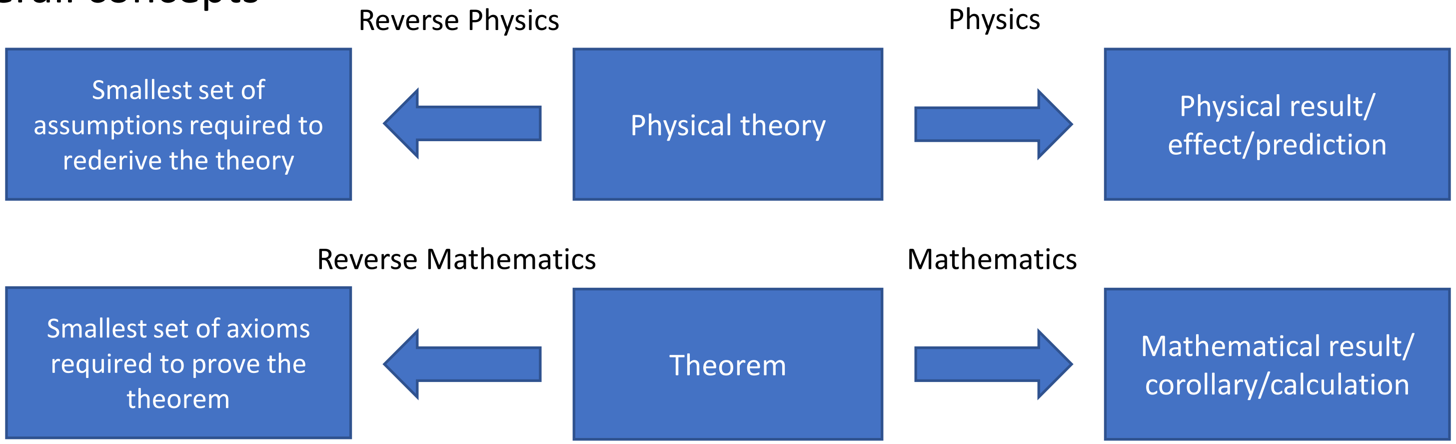
Assumptions of Physics

Find the right overall concepts

Reverse physics:

Start with the equations, reverse engineer physical assumptions/principles

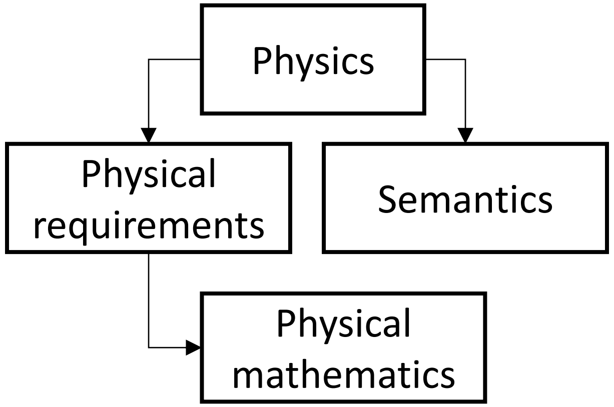
Found Phys **52**, 40 (2022)



Goal: find the right overall physical concepts, “elevate” the discussion from mathematical constructs to physical principles

Physical mathematics:

Start from scratch and rederive all mathematical structures from physical requirements



Goal: get the details right, perfect one-to-one map between mathematical and physical objects

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Assumptions of Physics

Hamiltonian mechanics $\iff$ Determinism/Reversibility + DOF independence

Assumption DR (Determinism and Reversibility). *The system undergoes deterministic and reversible evolution. That is, specifying the state of the system at a particular time is equivalent to specifying the state at a future (determinism) or past (reversibility) time.*

The displacement field is divergenceless: $\partial_a S^a = 0$	(DR-DIV)
The Jacobian of time evolution is unitary: $\left|\partial_b \hat{\xi}^a\right| = 1$	(DR-JAC)
Densities are conserved through the evolution: $\hat{\rho}(\hat{\xi}^a) = \rho(\xi^b)$	(DR-DEN)
Volumes are conserved through the evolution: $d\hat{\xi}^1 \cdots d\hat{\xi}^n = d\xi^1 \cdots d\xi^n$	(DR-VOL)
The evolution is deterministic and reversible.	(DR-EV)
The evolution is deterministic and thermodynamically reversible	(DR-THER)
The evolution conserves information entropy	(DR-INFO)
The evolution conserves the uncertainty of peaked distributions	(DR-UNC)

Assumption IND (Independent DOFs). *The system is decomposable into independent degrees of freedom. That is, the variables that describe the state can be divided into groups that have independent definition, units and count of states.*

The system is decomposable into independent DOFs	(IND-DOF)
The system allows statistically independent distributions over each DOF	(IND-STAT)
The system allows informationally independent distributions over each DOF	(IND-INFO)
The system allows peaked distributions where the uncertainty is the product of the uncertainty on each DOF	(IND-UNC)

Assumptions of Physics,
Michigan Publishing (v2 2023)

$\Downarrow$ $\Downarrow$

The evolution leaves ω_{ab} invariant: $\hat{\omega}_{ab} = \omega_{ab}$	(DI-SYMP)
The evolution leaves the Poisson brackets invariant	(DI-POI)
The rotated displacement field is curl free: $\partial_a S_b - \partial_b S_a = 0$	(DI-CURL)

$\Updownarrow$

$$d_t q^i = \partial_{p_i} H$$
$$d_t p_i = -\partial_{q^i} H$$

$$S_a = S^b \omega_{ba} = \partial_a H$$

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Assumptions of Physics

Variation of the action measures the flow of states
Variation = 0 ⇒ flow of states tangent to the path

Geometric and physical interpretation of the action principle, *Sci Rep* **13**, 12138 (2023)

DR

$$\nabla \cdot \vec{S} = 0$$

No state is "lost" or "created" as time evolves

$[p, 0, -H]$

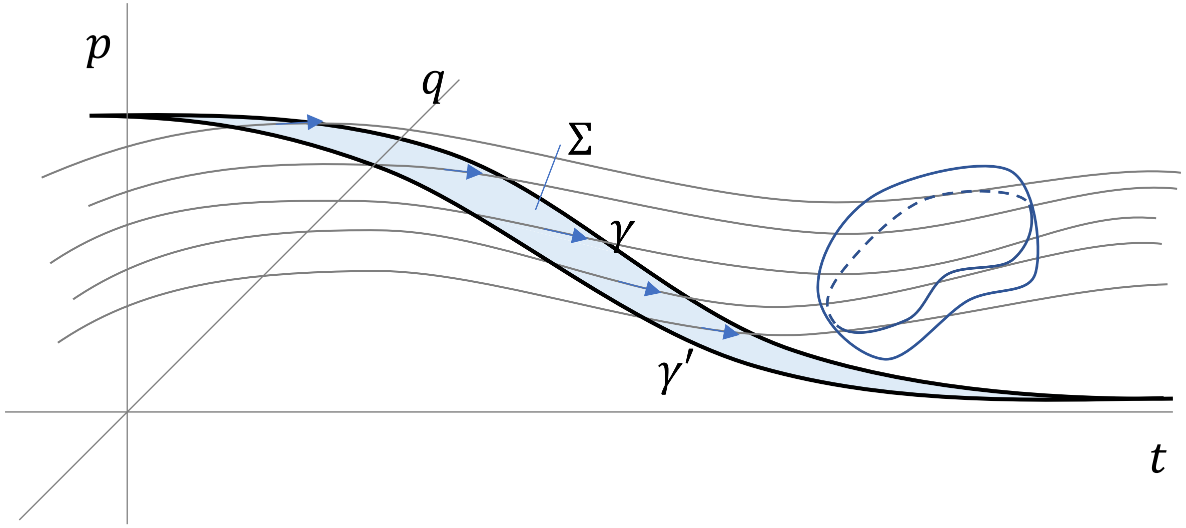
$$\vec{S} = -\nabla \times \vec{\theta}$$

(Minus sign to match convention)

$$\mathcal{S}[\gamma] = \int_\gamma L dt = \int_\gamma \vec{\theta} \cdot d\vec{\gamma}$$

KE

The action is the line integral of the vector potential (unphysical)



Variation of the action

$$\delta\mathcal{S}[\gamma] = \oint_{\partial\Sigma} \vec{\theta} \cdot d\vec{\gamma}$$

$$= - \iint_\Sigma \vec{S} \cdot d\vec{\Sigma}$$

Gauge independent, physical!

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Assumptions of Physics

Other results

- https://assumptionsofphysics.org/papers.html
- Classical mechanics is the high-entropy limit of quantum mechanics (2024)
  - The limit $\hbar \to 0$ is equivalent to $S(\rho) \to \infty$
- The unphysicality of Hilbert spaces (2023)
  - Closure under Cauchy sequences is physically untenable
- On the Common Logical Structure of Classical and Quantum Mechanics (2022)
  - Quantum mechanics does NOT require a departure from classical logic
- Hamiltonian mechanics is conservation of information entropy (2020)
  - Conservation of the entropy (and marginals) is equivalent to Hamiltonian evolution
- The four postulates of quantum mechanics are three (2020)
  - The tensor product postulate is redundant
- Variability as a better characterization of Shannon entropy (2019)
  - It is conceptually better and recovers the formula

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Possible contribution: generalize results to (classical) field theories

- Find/develop suitable Hamiltonian/symplectic formulation for classical field theories (e.g. classical EM, general relativity)
- Find/develop suitable definition of entropy in the context of field theory
- Extend results from classical mechanics to field theories

Physical mathematics

- Goal: find the most general mathematical structures that are still physically meaningful
- ⇒ Derive mathematical structure from physically justifiable premises

Axiom 1.7 (Axiom of mixture). *The statistical mixture of two ensembles is an ensemble. Formally, an ensemble space $\mathcal{E}$ is equipped with an operation $+ : [0,1] \times \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ called* ***mixing****, noted with the infix notation $p\mathsf{a} + \bar{p}\mathsf{b}$, with the following properties:*

- ***Continuity****: the map $+(p, \mathsf{a}, \mathsf{b}) \to p\mathsf{a} + \bar{p}\mathsf{b}$ is continuous (with respect to the product topology of $[0,1] \times \mathcal{E} \times \mathcal{E}$)*
- ***Identity****: $1\mathsf{a} + 0\mathsf{b} = \mathsf{a}$*
- ***Idempotence****: $p\mathsf{a} + \bar{p}\mathsf{a} = \mathsf{a}$ for all $p \in [0,1]$*
- ***Commutativity****: $p\mathsf{a} + \bar{p}\mathsf{b} = \bar{p}\mathsf{b} + p\mathsf{a}$ for all $p \in [0,1]$*
- ***Associativity****: $p_1\mathsf{e}_1 + \bar{p}_1\left(\left(\overline{\frac{p_3}{\bar{p}_1}}\right)\mathsf{e}_2 + \frac{p_3}{\bar{p}_1}\mathsf{e}_3\right) = \bar{p}_3\left(\frac{p_1}{\bar{p}_3}\mathsf{e}_1 + \left(\overline{\frac{p_1}{\bar{p}_3}}\right)\mathsf{e}_2\right) + p_3\mathsf{e}_3$ where $p_1 + p_3 \leq 1$*

Justification. This axiom captures the ability to create a mixture merely by selecting between the output of different processes. Let e_1 and e_2 be two ensembles that represent the output of two different processes P_1 and P_2. Let a selector S_p be a process that outputs two symbols, the first with probability p and the second with probability $\bar{p}$. Then we can create another process P that, depending on the selector, outputs either the output of P_1 or P_2. All possible preparations of such a procedure will form an ensemble. Therefore we are justified in equipping an ensemble space with a mixing operation that takes a real number from zero to one, and two ensembles.

Given that mixing represents an experimental relationship, and all experimental relationships must be continuous in the natural topology, mixing must be a continuous function. Note that p is a continuously ordered quantity, where no value is perfectly experimentally verifiable, and therefore the natural topology is the one of the reals. This justifies continuity.

If $p = 1$, the output of P will always be the output of P_1. This justifies the identity property. If P_1 and P_2 are the same process, then the output of P will always be the output of P_1. This justifies the idempotence property. The order in which the processes are given does not matter as long as the same probability is matched to the same process. The process P is identical under permutation of P_1 and P_2. This justifies commutativity. If we are mixing three processes P_1, P_2 and P_3, as long as the final probabilities are the same, it does not matter if we mix P_1 and P_2 first or P_2 and P_3. This justifies associativity. □

Informal intuitive statement

(something that makes sense to a physicist or an engineer)

Formal requirement

(something a mathematician will find precise)

Show that the formal requirement follows from the intuitive statement

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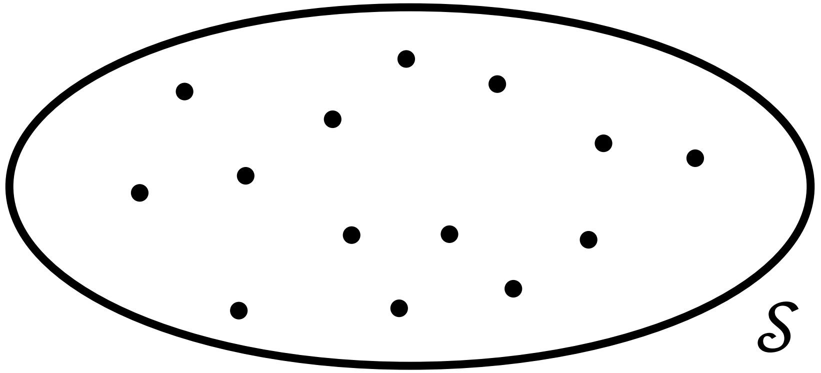
Assumptions of Physics

Science is about statements that can be connected to experimental evidence ⇒

Logic of experimental verifiability

Axiom 1.2 (Axiom of context). *A* ***statement*** s *is an assertion that is either true or false. A* ***logical context*** $\mathcal{S}$ *is a collection of statements with well defined logical relationships. Formally, a logical context* $\mathcal{S}$ *is a collection of elements called statements upon which is defined a function* $\mathrm{truth} : \mathcal{S} \to \mathbb{B}$.



	s_1	s_2	s_3	…
a →	T	T	F	…
	T	F	T	…
	T	F	F	…

$\mathcal{A}_{\mathcal{S}}$

Axiom 1.4 (Axiom of possibility). *A* ***possible assignment*** *for a logical context* $\mathcal{S}$ *is a map* $a : \mathcal{S} \to \mathbb{B}$ *that assigns a truth value to each statement in a way consistent with the content of the statements. Formally, each logical context comes equipped with a set* $\mathcal{A}_{\mathcal{S}} \subseteq \mathbb{B}^{\mathcal{S}}$ *such that* $\mathrm{truth} \in \mathcal{A}_{\mathcal{S}}$. *A map* $a : \mathcal{S} \to \mathbb{B}$ *is a possible assignment for* $\mathcal{S}$ *if* $a \in \mathcal{A}_{\mathcal{S}}$.

⇒ Statements form a complete Boolean algebra

Axiom 1.9 (Axiom of closure). *We can always find a statement whose truth value arbitrarily depends on an arbitrary set of statements. Formally, let* $S \subseteq \mathcal{S}$ *be a set of statements and* $f_{\mathbb{B}} : \mathbb{B}^S \to \mathbb{B}$ *an arbitrary function from an assignment of* S *to a truth value. Then we can always find a statement* $\bar{\mathsf{s}} \in \mathcal{S}$ *that depends on* S *through* $f_{\mathbb{B}}$.

s_1	s_2	s_3	…
T	T	F	…
T	F	T	…
T	F	F	…

→ $f_{\mathbb{B}}$ →

$f(s_1, s_2, s_3)$
T
T
F

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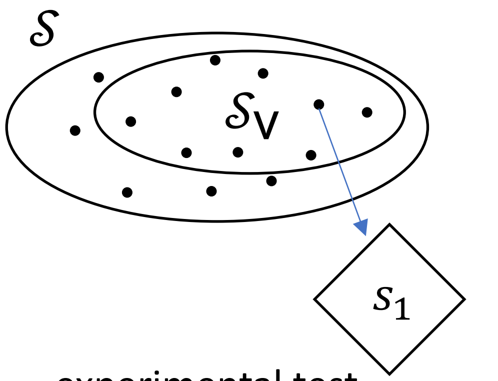
Axiom 1.27 (Axiom of verifiability). *A **verifiable statement** is a statement that, if true, can be shown to be so experimentally. Formally, each logical context $\mathcal{S}$ contains a set of statements $\mathcal{S}_{\mathsf{v}} \subseteq \mathcal{S}$ whose elements are said to be verifiable. Moreover, we have the following properties:*

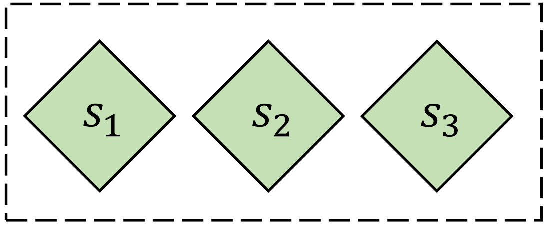
- *every certainty $\top \in \mathcal{S}$ is verifiable*
- *every impossibility $\bot \in \mathcal{S}$ is verifiable*
- *a statement equivalent to a verifiable statement is verifiable*

Remark. The **negation or logical NOT** of a verifiable statement is not necessarily a verifiable statement.



s_1	Test Result
T	SUCCESS (in finite time)
F	FAILURE (in finite time)
	UNDEFINED

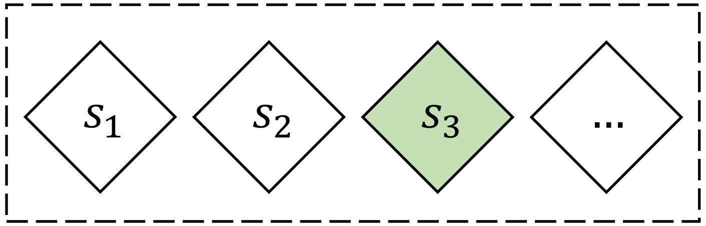
$$\bigwedge_{i=i}^{n} s_i$$



All tests must succeed

Axiom 1.31 (Axiom of finite conjunction verifiability). *The conjunction of a finite collection of verifiable statements is a verifiable statement. Formally, let $\{\mathsf{s}_i\}_{i=1}^{n} \subseteq \mathcal{S}_{\mathsf{v}}$ be a finite collection of verifiable statements. Then the conjunction $\bigwedge\limits_{i=1}^{n} \mathsf{s}_i \in \mathcal{S}_{\mathsf{v}}$ is a verifiable statement.*

$$\bigvee_{i=1}^{\infty} s_i$$



One successful test is sufficient

Axiom 1.32 (Axiom of countable disjunction verifiability). *The disjunction of a countable collection of verifiable statements is a verifiable statement. Formally, let $\{\mathsf{s}_i\}_{i=1}^{\infty} \subseteq \mathcal{S}_{\mathsf{v}}$ be a countable collection of verifiable statements. Then the disjunction $\bigvee\limits_{i=1}^{\infty} \mathsf{s}_i \in \mathcal{S}_{\mathsf{v}}$ is a verifiable statement.*

$\Rightarrow$ Verifiable statements form a Heyting algebra

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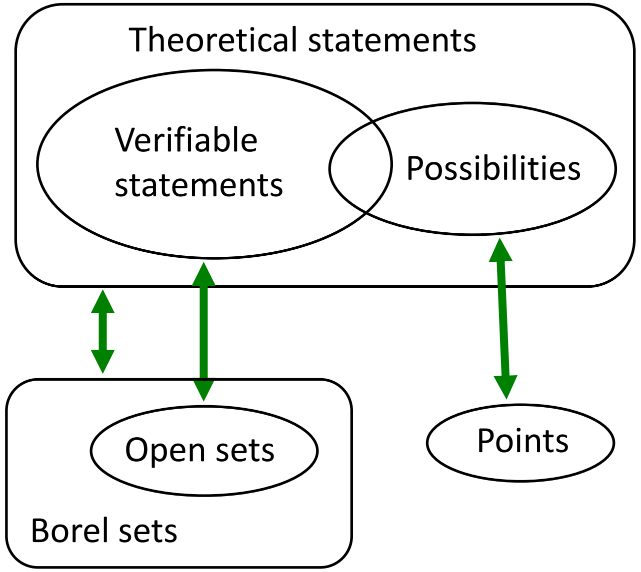
Assumptions of Physics

Topology and σ-algebra



s_1	Test Result
T	SUCCESS (in finite time)
	UNDEFINED
F	UNDEFINED
	FAILURE (in finite time)

$int(A)$ corresponds to the verifiable part of a statement

∂A corresponds to the undecidable part of a statement

$ext(A)$ corresponds to the falsifiable part of a statement

Experimental verifiability $\Rightarrow$ topology and σ-algebras (foundation of geometry, probability, …)

Perfect map between math and physics

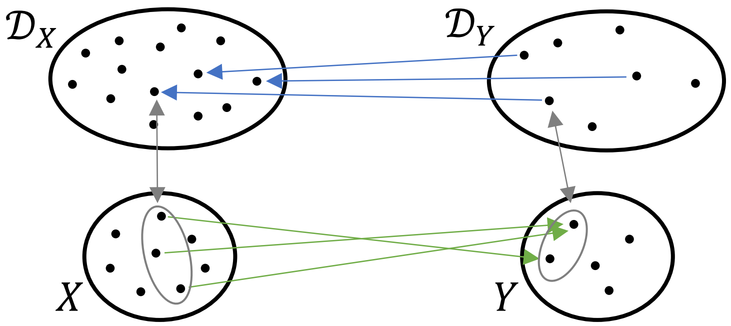
NB: in physics, topology and σ-algebra are parts of the **same** logic structure

Open set $(509.5, 510.5) \Leftrightarrow$ Verifiable "the mass of the electron is 510 ± 0.5 KeV"

Closed set $[510] \Leftrightarrow$ Falsifiable "the mass of the electron is exactly 510 KeV"

Borel set $\mathbb{Q}$ $(int(\mathbb{Q}) \cup ext(\mathbb{Q}) = \emptyset) \Leftrightarrow$ Theoretical "the mass of the electron in KeV is a rational number" (undecidable)

Inference relationship $r: \mathcal{D}_Y \to \mathcal{D}_X$ such that $r(s) \equiv s$

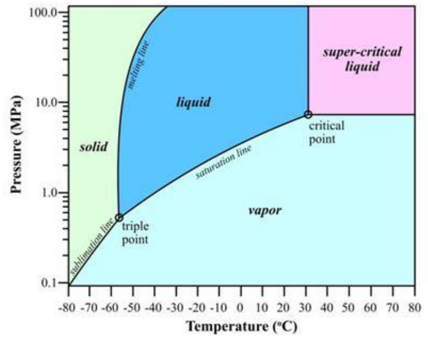


Causal relationship $f: X \to Y$ such that $x \preccurlyeq f(x)$

Inference relationship
$\Leftrightarrow$
Causal relationship

Relationships must be topologically continuous

Topologically continuous consistent with analytic discontinuity on isolated points



Phase transition $\Leftrightarrow$ Topologically isolated regions

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Assumptions of Physics

Ensemble spaces
(generalized state spaces)

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Assumptions of Physics

Physical laws are about reproducible relationships

statement of the type "whenever we prepare this we have that"

we must be able to "test one more time"

Experimentally, we always prepare ensembles with finite uncertainty

Empirical theories are about ensembles

A physical theory must AT LEAST describe which ensembles are possible within the theory

Minimum requirements for ensembles

Must be experimentally well-defined ⇒ Axiom of ensemble
Responsible for all logical/topological structures

Must allow statistical mixtures ⇒ Axiom of mixture
Responsible for all linear structures

Must have a well-defined variability ⇒ Axiom of entropy
Responsible for all geometric structures

Goal: recover as much as possible from just these three axioms

	State space	Ensembles
Classical discrete	$X = \{x_1, x_2, \ldots\}$	$\mathcal{E} = \{\, p_i \mid \sum_i p_i = 1\}$
Classical continuum	Phase space Symplectic manifold $X = \{\mathbb{R}^{2n}, \omega\}$	$\mathcal{E} = \{\, \rho \in L^1(\mathbb{R}^{2n}) \mid \int \rho\, dq^n dp_n = 1\}$
Quantum mechanics	Projective complex Hilbert space $X = P(\mathcal{H})$	$\mathcal{E} = \{$ positive semi-definite Hermitian with $\mathrm{tr}(\rho) = 1\,\}$

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Assumptions of Physics

Axiom 1.4 (Axiom of ensemble). *The state of a system is represented by an* ***ensemble****, which represents all possible preparations of equivalent systems prepared according to the same procedure. The set of all possible ensembles for a particular system is an* ***ensemble space****. Formally, an ensemble space is a T_0 second countable topological space where each element is called an ensemble.*

Topology is responsible for handling limits and infinite operations

All other axioms are on finite elements

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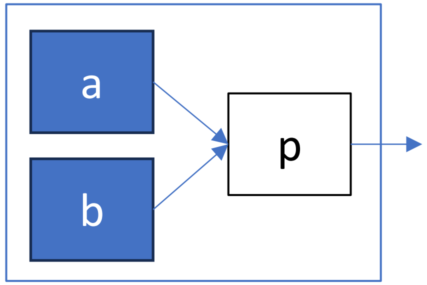
Assumptions of Physics

Axiom 1.7 (Axiom of mixture). *The statistical mixture of two ensembles is an ensemble. Formally, an ensemble space $\mathcal{E}$ is equipped with an operation $+ : [0,1] \times \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ called* ***mixing****, noted with the infix notation $\mathsf{pa} + \bar{\mathsf{p}}\mathsf{b}$, with the following properties:*

- ***Continuity****: the map $+(p, \mathsf{a}, \mathsf{b}) \to \mathsf{pa} + \bar{\mathsf{p}}\mathsf{b}$ is continuous (with respect to the product topology of $[0,1] \times \mathcal{E} \times \mathcal{E}$)*
- ***Identity****:* $1\mathsf{a} + 0\mathsf{b} = \mathsf{a}$
- ***Idempotence****:* $\mathsf{pa} + \bar{\mathsf{p}}\mathsf{a} = \mathsf{a}$ *for all* $p \in [0,1]$
- ***Commutativity****:* $\mathsf{pa} + \bar{\mathsf{p}}\mathsf{b} = \bar{\mathsf{p}}\mathsf{b} + \mathsf{pa}$ *for all* $p \in [0,1]$
- ***Associativity****:* $p_1 \mathsf{e}_1 + \bar{p}_1 \left(\overline{\left(\frac{p_3}{\bar{p}_1}\right)} \mathsf{e}_2 + \frac{p_3}{\bar{p}_1} \mathsf{e}_3 \right) = \bar{p}_3 \left(\frac{p_1}{\bar{p}_3} \mathsf{e}_1 + \overline{\left(\frac{p_1}{\bar{p}_3}\right)} \mathsf{e}_2 \right) + p_3 \mathsf{e}_3$ *where* $p_1 + p_3 \leq 1$

Ensembles can be mixed

⇒ Convex structure



Axiom 1.21 (Axiom of entropy). *Every element of the ensemble is associated with an* ***entropy*** *which quantifies the variability of the preparations of the ensemble. Formally, an ensemble space $\mathcal{E}$ is equipped with a function $S : \mathcal{E} \to \mathbb{R}$, defined up to a positive multiplicative constant representing the unit numerical value. The entropy has the following properties:*

- ***Continuity***[a]
- ***Strict concavity****: $S(p\mathsf{a} + \bar{p}\mathsf{b}) \geq pS(\mathsf{a}) + \bar{p}S(\mathsf{b})$ with the equality holding if and only if $\mathsf{a} = \mathsf{b}$*
- ***Upper variability bound****: there exists a universal function $I(p_1, p_2)$ (i.e. the same for all ensemble spaces) such that $S(p\mathsf{a} + \bar{p}\mathsf{b}) \leq I(p, \bar{p}) + pS(\mathsf{a}) + \bar{p}S(\mathsf{b})$; if the equality holds, a and b are* ***non-overlapping*** *or* ***orthogonal****, noted $\mathsf{a} \perp \mathsf{b}$*
- ***Mixtures preserve orthogonality****:*[b] *$\mathsf{a} \perp \mathsf{b}$ and $\mathsf{a} \perp \mathsf{c}$ if and only if $\mathsf{a} \perp p\mathsf{b} + \bar{p}\mathsf{c}$ for any $p \in (0, 1)$*

Ensembles must have a well-defined entropy

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Assumptions of Physics

Axiom 1.21 (Axiom of entropy). *Every element of the ensemble is associated with an* ***entropy*** *which quantifies the variability of the preparations of the ensemble. Formally, an ensemble space $\mathcal{E}$ is equipped with a function $S : \mathcal{E} \to \mathbb{R}$, defined up to a positive multiplicative constant representing the unit numerical value. The entropy has the following properties:*

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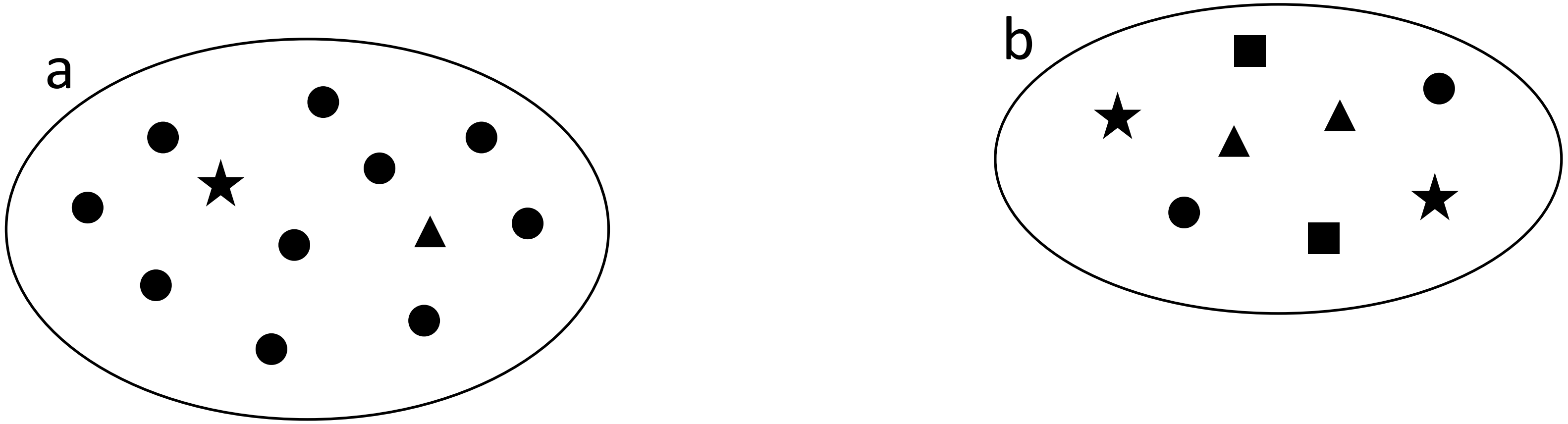

Ensembles represent a collection of preparations which are, in general, not identical

Variability: how similar are the preparations to each other?

Entropy is a measure of variability

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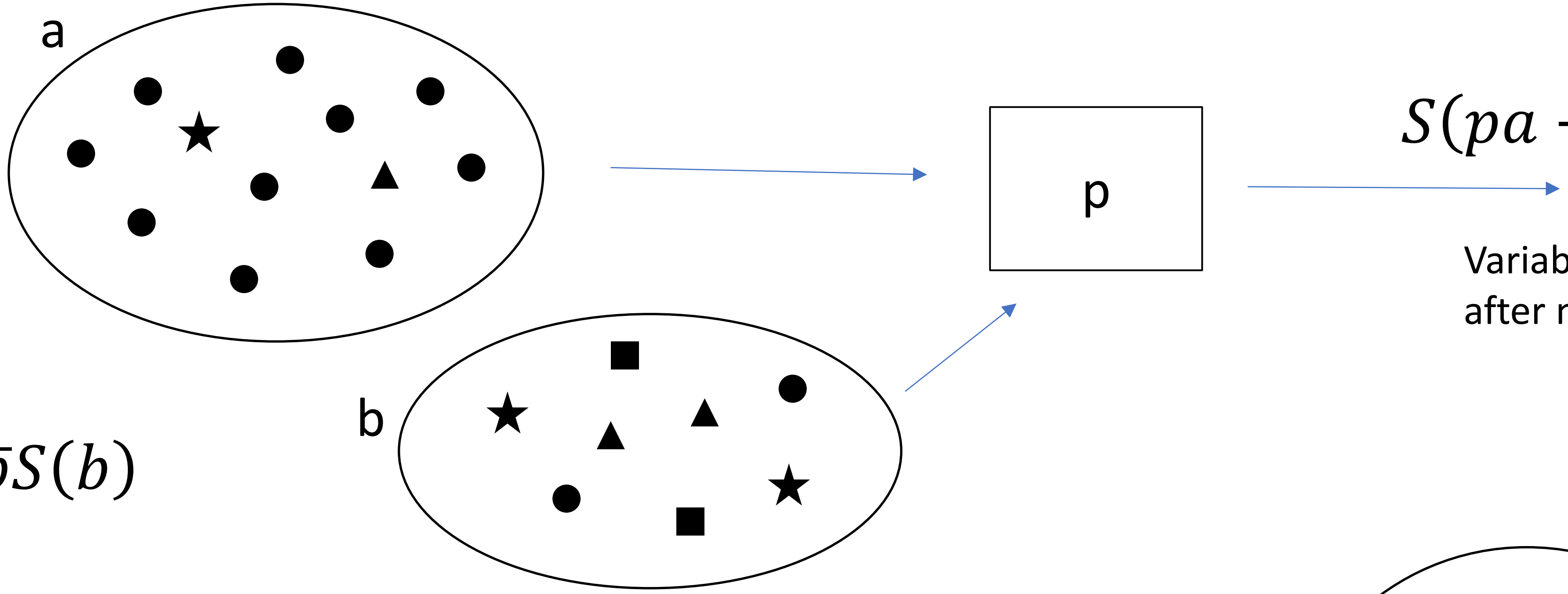
Assumptions of Physics

- ***Strict concavity:*** $S(p\mathsf{a} + \bar{p}\mathsf{b}) \geq pS(\mathsf{a}) + \bar{p}S(\mathsf{b})$ *with the equality holding if and only if* $\mathsf{a} = \mathsf{b}$



$$S(pa + \bar{p}b)$$

Variability
after mixture

Average variability
before mixture

$$pS(a) + \bar{p}S(b)$$

Final variability will be greater than average variability before mixture

If we mix an ensemble with itself, the variability is unchanged

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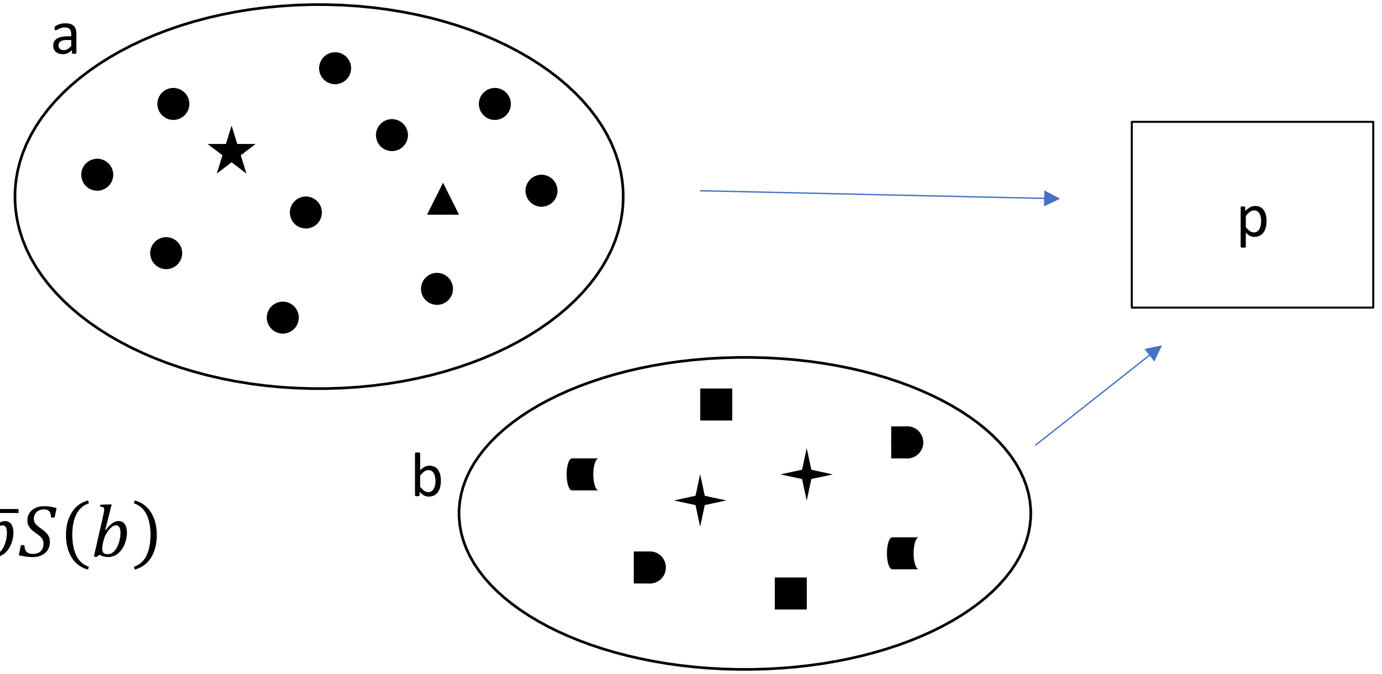
Assumptions of Physics

- ***Upper variability bound***: *there exists a universal function* $I(p_1, p_2)$ *(i.e. the same for all ensemble spaces) such that* $S(p\mathsf{a} + \bar{p}\mathsf{b}) \leq I(p, \bar{p}) + pS(\mathsf{a}) + \bar{p}S(\mathsf{b})$; *if the equality holds,* a *and* b *are* ***non-overlapping*** *or* ***orthogonal***, *noted* $\mathsf{a} \perp \mathsf{b}$



$$S(pa + \bar{p}b)$$

Variability after mixture

Average variability before mixture

$$pS(a) + \bar{p}S(b)$$

Maximum increase when the ensembles are "completely different"

Increase is only a function of the mixing coefficient

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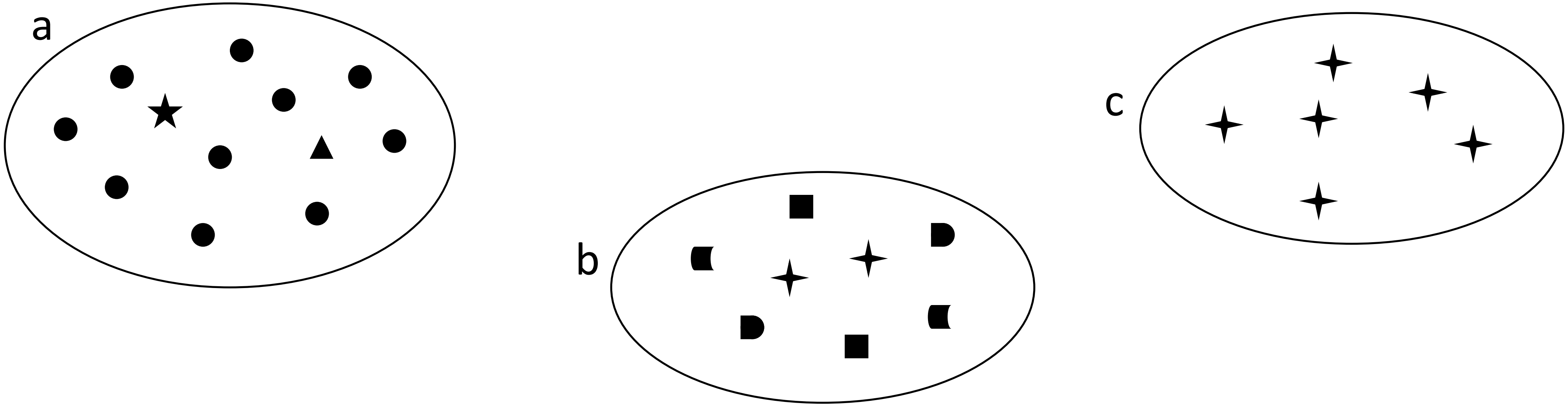
Assumptions of Physics

- ***Mixtures preserve orthogonality:***[b] *$\mathsf{a} \perp \mathsf{b}$ and $\mathsf{a} \perp \mathsf{c}$ if and only if $\mathsf{a} \perp p\mathsf{b} + \bar{p}\mathsf{c}$ for any $p \in (0,1)$*



If a has no elements in common with b or c,
it has no elements in common with any mixture

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Assumptions of Physics

The entropy of the coefficients $I(p, \bar{p})$ is uniquely determined

Theorem 1.25 (Uniqueness of entropy). *The entropy of the coefficients $I(p, \bar{p})$ is the Shannon entropy. That is, $I(p, \bar{p}) = -\kappa\,(p \log p + \bar{p} \log \bar{p})$ where $\kappa > 0$ is the arbitrary multiplicative constant for the entropy. For a mixture of arbitrarily many elements, $I(\{p_i\}) = -\kappa \sum_i p_i \log p_i$.*

Shannon entropy

Proof "does not know" whether we are dealing with classical ensembles, quantum ensembles, or ensembles for a theory yet to be discovered

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Assumptions of Physics

Separate ensembles

$$\mathsf{a} ⫪ \mathsf{b}$$

No “common component” $\mathsf{c} \in \mathcal{E}$

such that
$$\mathsf{a} = p_1 \mathsf{c} + \bar{p}_1 \mathsf{e}_1$$
$$\mathsf{b} = p_2 \mathsf{c} + \bar{p}_2 \mathsf{e}_2$$

$\Leftarrow$

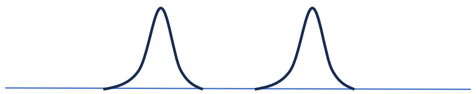
Orthogonal ensembles

$$\mathsf{a} \perp \mathsf{b}$$

Saturate upper bound

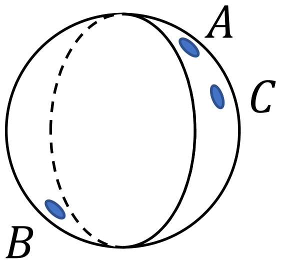
$$S(p\mathsf{a} + \bar{p}\mathsf{b}) = I(p, \bar{p}) + pS(\mathsf{a}) + \bar{p}S(\mathsf{b})$$

Coincide in
classical ensemble spaces



disjoint support

Different in
quantum ensemble spaces



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Assumptions of Physics

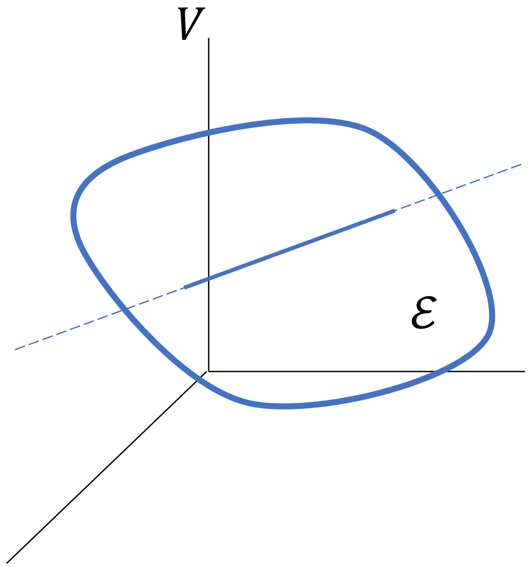
Ensemble spaces embed in a compact subset of a vector space

Proposition 1.35. *A convex space X embeds into a vector space if and only if it is* ***cancellative****, that is $\mathsf{pa} + \bar{\mathsf{p}}\mathsf{e} = \mathsf{pb} + \bar{\mathsf{p}}\mathsf{e}$ for some $p \in (0,1)$ implies $\mathsf{a} = \mathsf{b}$.*

Theorem 1.36 (Vector space embedding)**.** *Let $\mathsf{a}, \mathsf{b}, \mathsf{e} \in \mathcal{E}$ such that $p\mathsf{a} + \bar{p}\mathsf{e} = p\mathsf{b} + \bar{p}\mathsf{e}$ for some $p \in (0,1)$. Then $\mathsf{a} = \mathsf{b}$. Therefore any ensemble space embeds into a real vector space.*

Definition 1.50. *A* ***line*** *$A \subseteq \mathcal{E}$ is a convex subset such that for any three elements one can be expressed as a mixture of the other two. That is, for all $\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3 \in A$ there exists a permutation $\sigma : \{1,2,3\} \to \{1,2,3\}$ and $p \in [0,1]$ such that $\mathsf{e}_{\sigma(1)} = p\mathsf{e}_{\sigma(2)} + \bar{p}\mathsf{e}_{\sigma(3)}$.*

Theorem 1.52 (Lines are bounded)**.** *Let $A \subseteq \mathcal{E}$ be a line. Then we can find a bounded interval $V \subseteq \mathbb{R}$ and an invertible function $f : A \to V$ such that $f(p\mathsf{a} + \bar{p}\mathsf{b}) = pf(\mathsf{a}) + \bar{p}f(\mathsf{b})$ for all $\mathsf{a}, \mathsf{b} \in A$.*



The entropy bounds are responsible for these constraints

Is it a topological vector space? Is it locally convex?

How much does the entropy increase during mixture?

$$MS(\mathsf{a},\mathsf{b}) = S\left(\frac{1}{2}\mathsf{a} + \frac{1}{2}\mathsf{b}\right) - \left(\frac{1}{2}S(\mathsf{a}) + \frac{1}{2}S(\mathsf{b})\right)$$

Recovers the Jensen-Shannon divergence
(both classical and quantum)

1. *non-negativity:* $MS(\mathsf{a},\mathsf{b}) \geq 0$
2. *identity of indiscernibles:* $MS(\mathsf{a},\mathsf{b}) = 0 \iff \mathsf{a} = \mathsf{b}$
3. *unit boundedness:* $MS(\mathsf{a},\mathsf{b}) \leq 1$
4. *maximality of orthogonals:* $MS(\mathsf{a},\mathsf{b}) = 1 \iff \mathsf{a} \perp \mathsf{b}$
5. *symmetry:* $MS(\mathsf{a},\mathsf{b}) = MS(\mathsf{b},\mathsf{a})$

Pseudo-distance from the entropy

Entropy imposes a metric on the affine structure

$$\|\delta \mathrm{e}\|_{\mathrm{e}} = \sqrt{8MS(\mathrm{e}, \mathrm{e} + \delta \mathrm{e})}$$

$$g_{\mathrm{e}}(\delta \mathrm{e}_1, \delta \mathrm{e}_2) = \frac{1}{2}\left(\|\delta \mathrm{e}_1 + \delta \mathrm{e}_2\|_{\mathrm{e}}^2 - \|\delta \mathrm{e}_1\|_{\mathrm{e}}^2 - \|\delta \mathrm{e}_2\|_{\mathrm{e}}^2\right)$$

$$\Rightarrow \quad g_{\mathrm{e}}(\delta \mathrm{e}_1, \delta \mathrm{e}_2) = -\frac{\partial^2 S}{\partial \mathrm{e}^2}(\delta \mathrm{e}_1, \delta \mathrm{e}_2).$$

Entropy strict concavity means the Hessian is negative definite

Recovers Fisher-Rao information metric
(both classical and quantum)

How much of *e* is a mixture of other ensembles?

Definition 1.83. *Let* $\mathsf{e}, \mathsf{a} \in \mathcal{E}$ *be two ensembles. The* ***fraction*** *of* a *in* e *is the greatest mixing coefficient for which* e *can be expressed as a mixture of* a*. That is,* $\mathrm{frac}_{\mathsf{e}}(\mathsf{a}) = \sup(\{p \in [0,1] \mid \exists\, \mathsf{b} \in \mathcal{E} \text{ s.t. } \mathsf{e} = p\mathsf{a} + \bar{p}\mathsf{b}\})$.

Definition 1.85. *Let* $\mathsf{e} \in \mathcal{E}$ *be an ensemble and* $A \subseteq \mathcal{E}$ *a Borel set. The* ***fraction capacity*** *of* A *for* e *is the biggest fraction achievable with convex combinations of* A*. That is,* $\mathrm{fcap}_{\mathsf{e}}(A) = \sup(\mathrm{frac}_{\mathsf{e}}(\mathrm{hull}(A)) \cup \{0\})$.

Proposition 1.87. *The fraction capacity for an ensemble is a set function that is*

1. *non-negative and unit bounded:* $\mathrm{fcap}_{\mathsf{e}}(A) \in [0,1]$
2. *monotone:* $A \subseteq B \implies \mathrm{fcap}_{\mathsf{e}}(A) \leq \mathrm{fcap}_{\mathsf{e}}(B)$
3. *subadditive:* $\mathrm{fcap}_{\mathsf{e}}(A \cup B) \leq \mathrm{fcap}_{\mathsf{e}}(A) + \mathrm{fcap}_{\mathsf{e}}(B)$
4. *continuous from below:* $\mathrm{fcap}_{\mathsf{e}}(\lim_{i\to\infty} A_i) = \lim_{i\to\infty} \mathrm{fcap}_{\mathsf{e}}(A_i)$ *for any increasing sequence* $\{A_i\}$
5. *continuous from above:* $\mathrm{fcap}_{\mathsf{e}}(\lim_{i\to\infty} A_i) = \lim_{i\to\infty} \mathrm{fcap}_{\mathsf{e}}(A_i)$ *for any decreasing sequence* $\{A_i\}$

fuzzy measure

Fraction capacity is a non-additive probability measure

additive over orthogonal sets

What is the spread of an ensemble in terms of distinguishable states?

Proposition 1.153 (Exponential entropy subadditivity). *Let* $\mathsf{e}_1, \mathsf{e}_2 \in \mathcal{E}$*. Let* $S_1 = S(\mathsf{e}_1)$ *and* $S_2 = S(\mathsf{e}_2)$*. Let* $\mathsf{e} = p\mathsf{e}_1 + \bar{p}\mathsf{e}_2$ *for some* $p \in [0,1]$ *and* $S = S(\mathsf{e})$*. Then* $2^S \leq 2^{S_1} + 2^{S_2}$*, with the equality if and only if* e_1 *and* e_2 *are orthogonal and* $p = \frac{2^{S_1}}{2^{S_1}+2^{S_2}}$.

Definition 1.156. *Let* $U \subseteq \mathcal{E}$ *be the subset of an ensemble space. The* ***state capacity*** *of* U *is defined as* $\mathrm{scap}(U) = \sup(2^{S(\mathrm{hull}(U))})$ *if* $U \neq \varnothing$ *and* $\mathrm{scap}(U) = 0$ *otherwise.*

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Inspiring Awe & Wonder

Proposition 1.157. *The state capacity is a set function that is*

1. *non-negative:* $\mathrm{scap}(U) \in [0, +\infty]$
2. *monotone:* $U \subseteq V \implies \mathrm{scap}(U) \leq \mathrm{scap}(V)$
3. *subadditive:* $\mathrm{scap}(U \cup V) \leq \mathrm{scap}(U) + \mathrm{scap}(V)$
4. *additive over orthogonal sets:* $U \perp V \implies \mathrm{scap}(U \cup V) = \mathrm{scap}(U) + \mathrm{scap}(V)$

Recall statistical mechanics:

$$S(\rho_U) = \log \mu(U)$$

additive over orthogonal sets

State capacity is a non-additive measure

Recovers Liouville measure in classical mechanics and dimensionality of Hilbert subspaces in quantum mechanics

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Assumptions of Physics

Statistical quantities (i.e. expectation values on ensembles)

Definition 1.126. *A* ***statistical property****, or simply property, is an attribute that allows statistical handling. Formally, it is a continuous map* $F : \mathcal{E} \to \mathcal{Q}$ *where* $\mathcal{Q}$ *is a convex topological space such that* $F(p\mathsf{e}_1 + \bar{p}\mathsf{e}_2) = pF(\mathsf{e}_1) + \bar{p}F(\mathsf{e}_2)$.

A ***statistical quantity****, or statistical variable, or simply variable, is a numerical statistical property. That is, it is a continuous linear real valued operator* $F : \mathcal{E} \to \mathbb{R}$.

If ensembles are identified by a set of statistical quantities, the ambient vector space is a locally convex topological vector space (the quantities induce semi-norms)

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Assumptions of Physics

Classical probability contexts

Ensembles closed under affine combinations

Definition 1.103. *Let $\mathcal{E}$ be an ensemble space. A* ***classical probability context*** *is a flat $C \subseteq \mathcal{E}$ where each decomposable element in C is separately decomposable in C but not separately multidecomposable in C.*

Single decomposition in terms of separate ensembles

Theorem 1.113. *Let $C \subseteq \mathcal{E}$ be a probability context and $\mathsf{e} \in C$ be an ensemble in the context. The set function $p^*_{\mathsf{e}} : T_{\sigma(C)} \to [0,1]$, defined such that $p^*_{\mathsf{e}}(U) = \inf(\{\mathrm{fcap}(A) \mid \sigma(A) \supseteq U\})$, is a topological measure. Therefore ensemble in a classical probability context can be represented by a unique measure over its spectra that is continuous with respect to its topology.*

Classical probability is recovered over classical probability contexts

Classical ensemble spaces are probability contexts

The mixed states that commute with a maximal set of observables form a classical probability context

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Assumptions of Physics

New math needed!

We need to make sure that probability is assigned only to tests that are actually terminating

Definition 1.89. *Let X be a topological space. We say a measure $\mu : \Sigma_X \to \mathbb{R}$ is compatible with the topology if $\mu(U) = \mu(\bar{U})$ for every open set U.*

Definition 1.94. *Let X be a topological space. A* ***topological measure*** *is a set function μ over the topology that satisfies the following:*

1. *non-negative: $\mu(U) \geq 0$*
2. *topologically additive: $\mu\left(\mathrm{int}\left(\overline{\bigcup_{i \in I} U_i}\right)\right) = \sum_{i \in I} \mu(U_i)$ whenever $U_i \cap U_j = \varnothing$ for all $i \neq j$*
3. *locally finite: for any U, we can find $V \in \mathsf{T}_X$ such that $\varnothing \subset V \subseteq U$ and $\mu(V) < +\infty$.*

The measure is finite if $\mu(X) < \infty$.

Theorem 1.96 (Topological measure extension theorem)**.** *Let X be a topological space and μ a topological measure on that space. Then there exists a measure $\overline{\mu}$ defined on the Borel algebra such that $\overline{\mu}|_{\mathsf{T}_X} = \mu$. If X is second countable or if μ is finite the extension is unique.*

Proof. The strategy is first to extend the topological measure μ to a pre-measure μ' over the algebra of nicely bounded sets generated by the open sets and then use Carathéodory's extension theorem to extend to a measure $\bar{\mu}$ on the whole Borel algebra.

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Assumptions of Physics

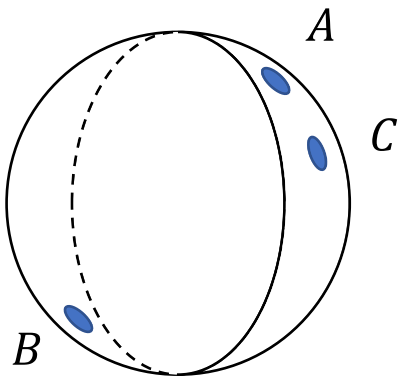
Still to understand: Poisson structure

Both classical and quantum mechanics have a symplectic/Poisson structure

Poisson brackets for classical observables

Commutators for quantum observables

Need to define one on ensemble spaces



One requirement is that the only quantities that “kill” the Poisson structure are the entropy and all functions of the entropy

This means that entropy provides a foliation of symplectic “manifolds”

Wrapping it up

- Assumptions of Physics aims to find a minimal set of physical assumptions from which we can rederive the laws
- Physical mathematics aims to derive mathematical structures from well-defined physical requirements: not just mathematics, but physical mathematics
- Looking for area experts that want to contribute to this unified goal
- Different levels of commitment
  - Evangelize
  - Review proofs and have in depth discussions
  - Prove a conjecture
  - Literature research for papers (to publish low hanging fruits)
  - …

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Assumptions of Physics

Some open problems

- https://assumptionsofphysics.org/problems/index.html
- Extend current findings (e.g. action principle interpretation) to classical field theory
  - Find the right Hamiltonian/symplectic formulation of field theories with a suitable entropy
- Reformulate results from classical and quantum information theory within ensemble spaces
- Reframe spectral theory in the context of ensemble spaces
- Turn results for topological measures into a full theory
- Help develop a non-additive measure theory,
  with a generalization of the Radon-Nikodym derivative
- …

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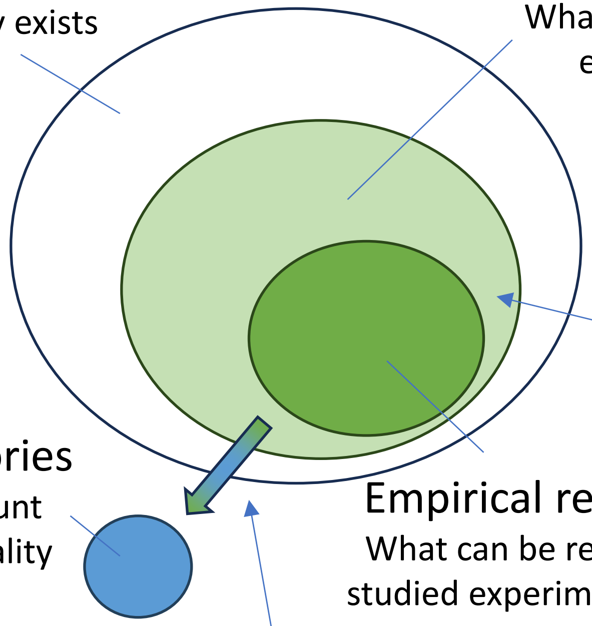
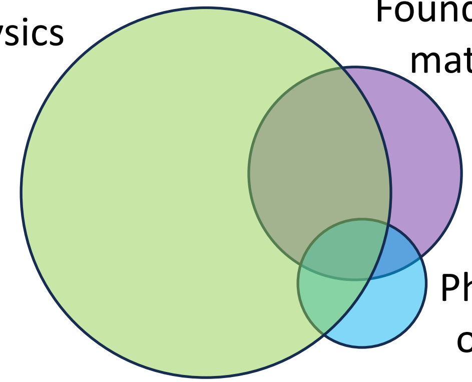
Underlying perspective



What is the boundary?
What are the requirements?

How exactly does the abstraction/idealization process work?



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Assumptions of Physics

If physics is about creating models of empirical reality, the foundations of physics should be a theory of models of empirical reality

Requirements of experimental verification, assumptions of each theory, realm of validity of assumptions, …

We need people that care about details, eager to acquire a wide range of technical skill, working toward the common goal of a physically meaningful, mathematically precise and philosophically consistent foundations for physics

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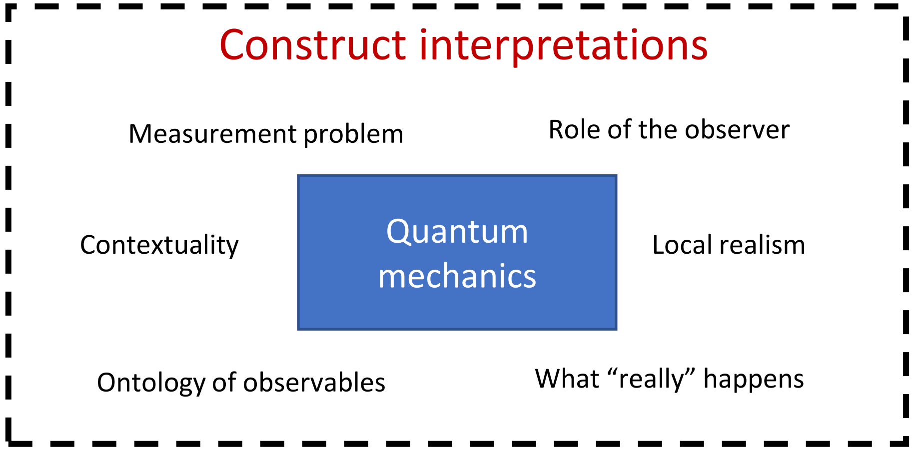
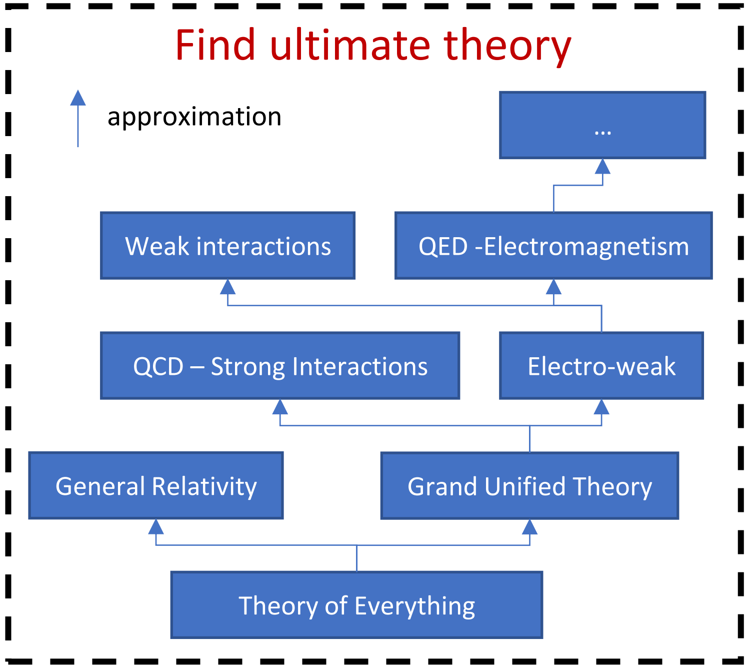
Assumptions of Physics

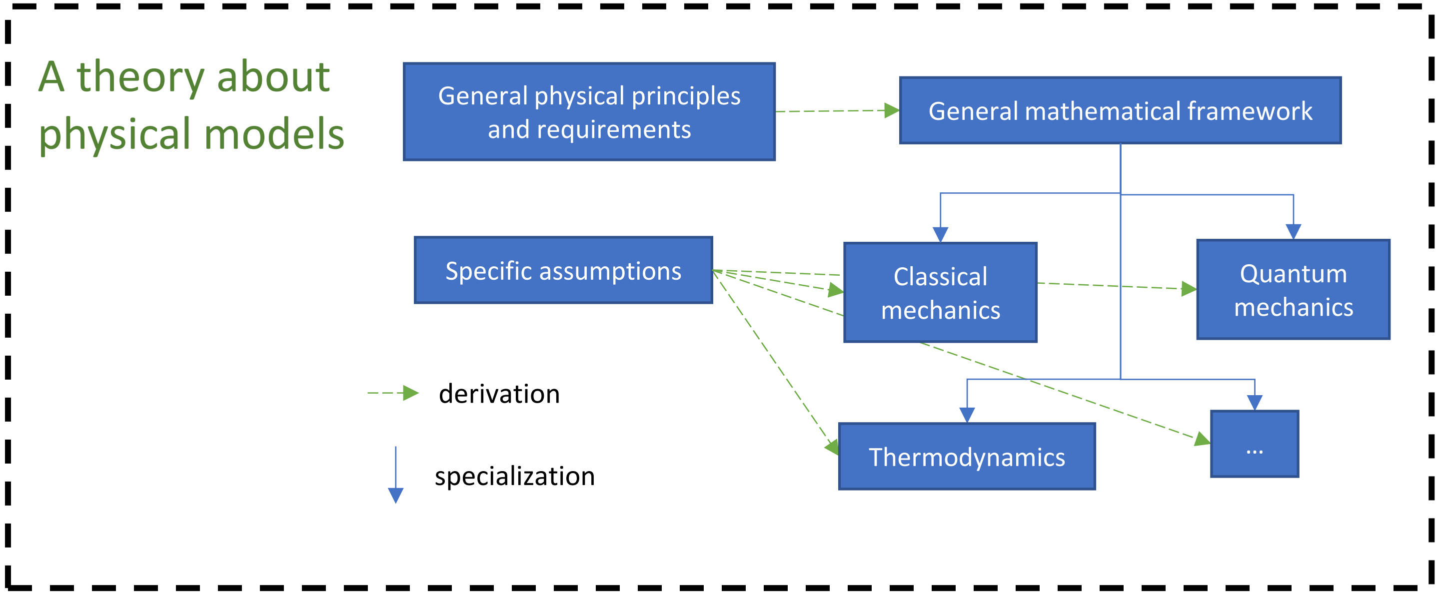
Different approach to the foundations of physics

Typical approaches





Our approach



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