Overview of the Assumptions of Physics program

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Reverse Physics and Physical Mathematics

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The current connection between
math and physics is “suboptimal”

A physically meaningful and
mathematically precise approach exists

New mathematics needs
to be developed

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Assumptions
of
Physics

To gain insight into the foundations of mathematics, it is useful to understand exactly what axioms lead to which result

Reverse mathematics

When the theorem is derived from the right axioms, the axioms can be derived from the theorem

Friedman

Axiom of Choice $\Longleftrightarrow$ Zorn's lemma

Base theory

Zermelo-Fraenkel set theory

To have a physically meaningful and mathematically precise foundation for physics, we need to understand exactly what mathematical property corresponds to which physical requirement

When the math is derived from the right physical assumptions,
the physical assumptions can be derived from the math

Reverse physics

Mathematical relationships $\Longleftrightarrow$ Physical assumptions

Base theory

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Assumptions of Physics

$d_t\mu(U) = 0$ $\Longleftrightarrow$ $d_t \log \mu(U) = 0$ $\Longleftrightarrow$ $d_t \int -\rho \log \rho \, d\mu = 0$

Conservation of state count

Conservation of thermodynamic entropy

Conservation of information entropy

$d_t\omega_{ab} = 0$
symplectomorphism

Determinism and reversibility

$d_t\left|cov(\xi^a, \xi^b)\right| \approx 0$
(for peaked distributions)

Conservation of uncertainty

All equivalent conditions!!!

$d_t q = \partial_p H$
$d_t p = -\partial_q H$
Hamilton's equations $\Longleftrightarrow$ $d_t\xi^a\omega_{ab} = \partial_b H$ $\Longleftrightarrow$ $\partial_a \wedge d_t\xi^b\omega_{bc} = 0$

Base theory: classical states for 1 degree of freedom

State: $\xi^a = [\, q \; p \,] \in \mathbb{R}^2$

Count of states: $\mu(U) = \int_U dqdp = \int_U \omega$

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Assumptions of Physics

$\psi(0) = \phi$ iff $\psi(t) = e^{\frac{H\Delta t}{\imath\hbar}}\phi$ $\Longleftrightarrow$ $d_t \mathrm{tr}(-\rho \log \rho) = 0$ $\Longleftrightarrow$ ...

Bijective map of state and probability

Conservation of information entropy

...

Determinism and reversibility

$U(\Delta t)|\psi\rangle = e^{\frac{H\Delta t}{\imath\hbar}}|\psi\rangle$

unitary evolution

$\imath\hbar d_t|\psi\rangle = H|\psi\rangle$

Schrödinger equation

In both cases, the law of evolution is the same physical assumption!!!

Base theory: quantum mechanics

State: $\psi \in P(\mathcal{H})$

Probability of ψ given ϕ: $p(\psi|\phi) = |\langle\psi|\phi\rangle|^2$

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Assumptions of Physics

Is there a more general base theory in which we can define and prove the correspondence?

$\Rightarrow$ What are the basic axioms for physical theories?

$$d_t e = H?\, e$$

Generalized Hamilton's equations

$\Longleftrightarrow$

Determinism and reversibility

Base theory: ???

State: $e\,???$

Entropy: $S(e)\,???$

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Assumptions of Physics

Physical world

(informal system)

Mathematical representation

(formal system)

How does this mapping work?

What standards of rigor are desirable/attainable?

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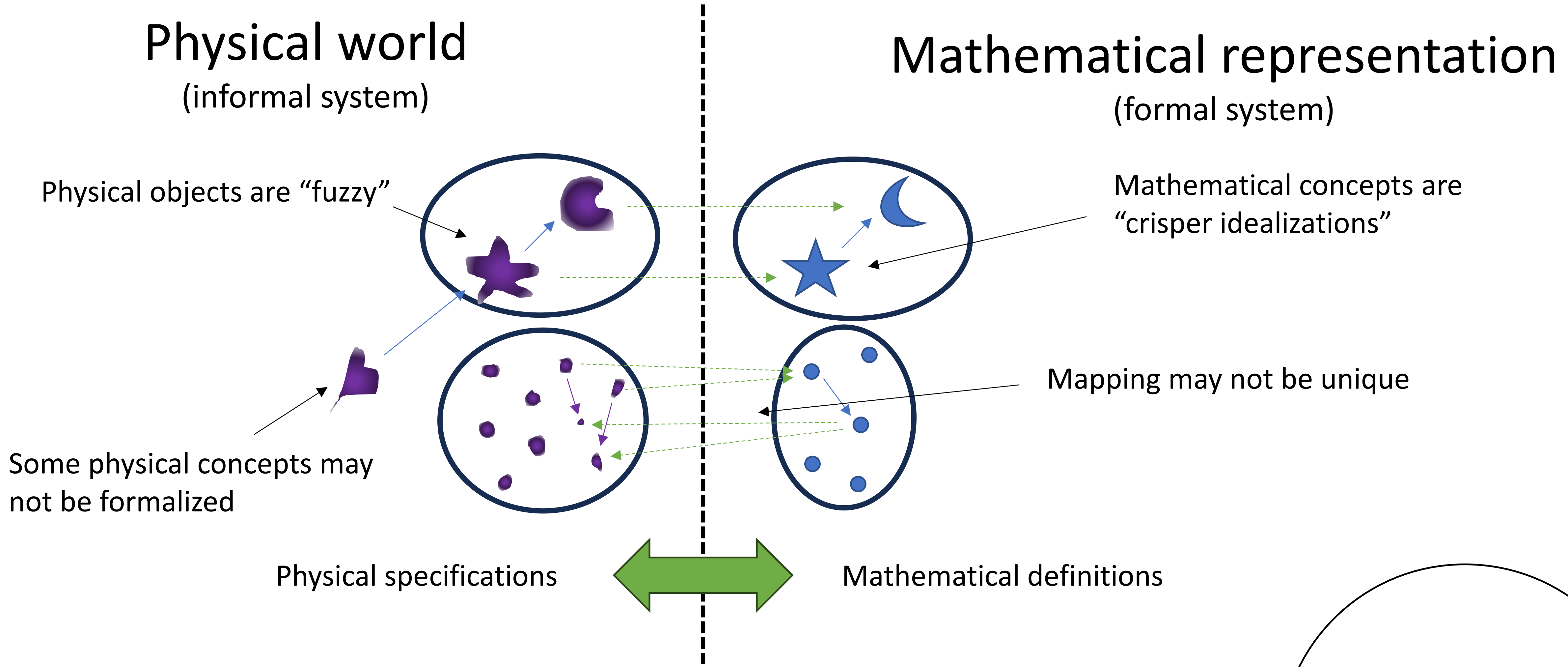
Assumptions of Physics



When physical objects are mapped to the right mathematical objects, the physical specification maps to the mathematical definitions

Physical mathematics

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Assumptions of Physics

Axiom 1.7 (Axiom of mixture). *The statistical mixture of two ensembles is an ensemble.*

Informal intuitive requirement

(something that makes sense to a physicist or an engineer)

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Assumptions of Physics

Axiom 1.7 (Axiom of mixture). *The statistical mixture of two ensembles is an ensemble. Formally, an ensemble space $\mathcal{E}$ is equipped with an operation $+ : [0,1] \times \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ called* ***mixing****, noted with the infix notation $p\mathsf{a} + \bar{p}\mathsf{b}$, with the following properties:*

- ***Continuity****: the map $+(p, \mathsf{a}, \mathsf{b}) \to p\mathsf{a} + \bar{p}\mathsf{b}$ is continuous (with respect to the product topology of $[0,1] \times \mathcal{E} \times \mathcal{E}$)*
- ***Identity****:* $1\mathsf{a} + 0\mathsf{b} = \mathsf{a}$
- ***Idempotence****: $p\mathsf{a} + \bar{p}\mathsf{a} = \mathsf{a}$ for all $p \in [0,1]$*
- ***Commutativity****: $p\mathsf{a} + \bar{p}\mathsf{b} = \bar{p}\mathsf{b} + p\mathsf{a}$ for all $p \in [0,1]$*
- ***Associativity****: $p_1\mathsf{e}_1 + \bar{p}_1 \left(\left(\overline{\frac{p_3}{\bar{p}_1}}\right)\mathsf{e}_2 + \frac{p_3}{\bar{p}_1}\mathsf{e}_3 \right) = \bar{p}_3 \left(\frac{p_1}{\bar{p}_3}\mathsf{e}_1 + \left(\overline{\frac{p_1}{\bar{p}_3}}\right)\mathsf{e}_2 \right) + p_3\mathsf{e}_3$ where $p_1 + p_3 \le 1$*

Informal intuitive requirement

(something that makes sense to a physicist or an engineer)

Formal definition

(something a mathematician will find precise)

Axiom 1.7 (Axiom of mixture). *The statistical mixture of two ensembles is an ensemble. Formally, an ensemble space $\mathcal{E}$ is equipped with an operation $+ : [0,1] \times \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ called **mixing**, noted with the infix notation $p\mathsf{a} + \bar{p}\mathsf{b}$, with the following properties:*

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Justification. This axiom captures the ability to create a mixture merely by selecting between the output of different processes. Let e_1 and e_2 be two ensembles that represent the output of two different processes P_1 and P_2. Let a selector S_p be a process that outputs two symbols, the first with probability p and the second with probability $\bar{p}$. Then we can create another process P that, depending on the selector, outputs either the output of P_1 or P_2. All possible preparations of such a procedure will form an ensemble. Therefore we are justified in equipping an ensemble space with a mixing operation that takes a real number from zero to one, and two ensembles.

Given that mixing represents an experimental relationship, and all experimental rela-

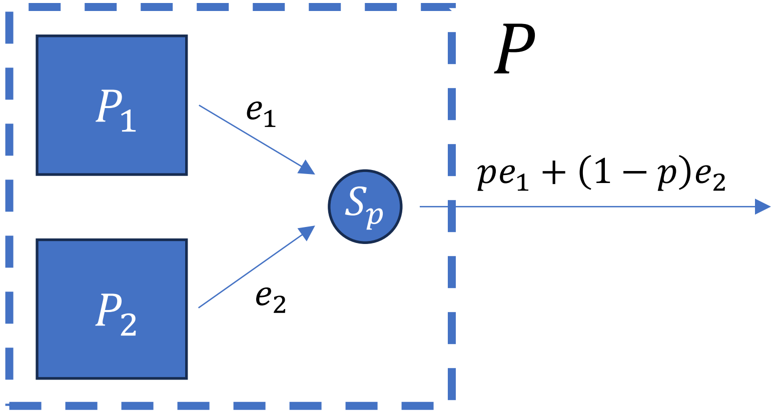
Informal intuitive requirement

(something that makes sense to a physicist or an engineer)

Formal definition

(something a mathematician will find precise)

Show that the formal definition follows from the intuitive requirement



Clear idea of what is being modelled

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Assumptions of Physics

Axiom 1.7 (Axiom of mixture). *The statistical mixture of two ensembles is an ensemble. Formally, an ensemble space $\mathcal{E}$ is equipped with an operation $+ : [0,1] \times \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ called* ***mixing****, noted with the infix notation $p\mathsf{a} + \bar{p}\mathsf{b}$, with the following properties:*

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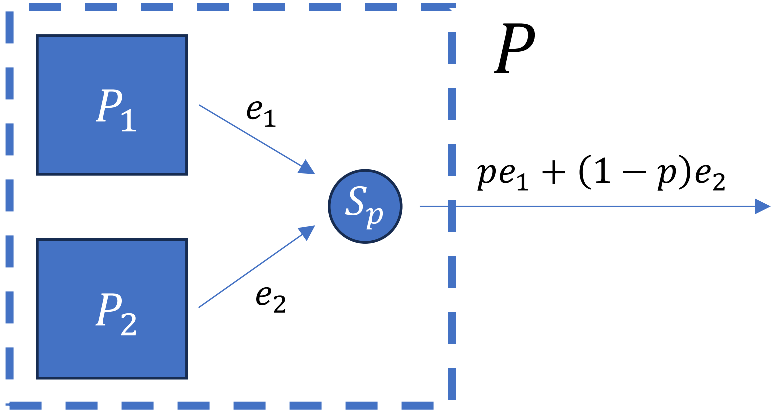
If $p = 1$, the output of P will always be the output of P_1. This justifies the identity property. If P_1 and P_2 are the same process, then the output of P will always be the output of P_1. This justifies the idempotence property. The order in which the processes are given does not matter as long as the same probability is matched to the same process. The process P is identical under permutation of P_1 and P_2. This justifies commutativity. If we are mixing three processes P_1, P_2 and P_3, as long as the final probabilities are the same, it does not matter if we mix P_1 and P_2 first or P_2 and P_3. This justifies associativity. □

Informal intuitive requirement
(something that makes sense to a physicist or an engineer)

Formal definition
(something a mathematician will find precise)

Show that the formal definition follows from the intuitive requirement

$$(1-p_1)\left(1-\frac{p_3}{1-p_1}\right) = 1 - p_1 - p_3 = (1-p_3)\left(1-\frac{p_1}{1-p_3}\right)$$



Properties justified by understanding the model

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Assumptions of Physics

Axiom 1.7 (Axiom of mixture). *The statistical mixture of two ensembles is an ensemble.* $\boxed{\textit{Formally,}}$ *an ensemble space $\mathcal{E}$ is equipped with an operation $+ : [0,1] \times \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ called* ***mixing****, noted with the infix notation $p\mathsf{a} + \bar{p}\mathsf{b}$, with the following properties:*

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Informal intuitive requirement
(something that makes sense to a physicist or an engineer)

Formal definition
(something a mathematician will find precise)

Show that the formal definition follows from the intuitive requirement

There is no question as to what the math describes

The properties are justified by, are a consequence of, what the model describes

Every math proof can be understood physically

⇒ The math describes and only describes physically meaningful concepts

It’s physical mathematics

We have a methodology!

What are the most basic objects for our base theory?

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Assumptions of Physics

Principle of scientific objectivity. Science is universal, non-contradictory and evidence based.

⇒ Science is about statements that are associated to experimental tests

Statements must be either true or false for everybody

Tests may or may not terminate (i.e. be conclusive)

Statement	Test Result
T	SUCCESS (in finite time)
	UNDEFINED
F	UNDEFINED
	FAILURE (in finite time)

Verifiable statement	Test Result
T	SUCCESS (in finite time)
F	UNDEFINED
	FAILURE (in finite time)

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Assumptions of Physics

Axioms of logic

Axiom 1.2 (Axiom of context). *A **statement** s is an assertion that is either true or false. A **logical context** $\mathcal{S}$ is a collection of statements with well defined logical relationships. Formally, a logical context $\mathcal{S}$ is a collection of elements called statements upon which is defined a function* $\text{truth} : \mathcal{S} \to \mathbb{B}$.

Axiom 1.4 (Axiom of possibility). *A **possible assignment** for a logical context $\mathcal{S}$ is a map $a : \mathcal{S} \to \mathbb{B}$ that assigns a truth value to each statement in a way consistent with the content of the statements. Formally, each logical context comes equipped with a set $\mathcal{A}_{\mathcal{S}} \subseteq \mathbb{B}^{\mathcal{S}}$ such that* $\text{truth} \in \mathcal{A}_{\mathcal{S}}$. *A map $a : \mathcal{S} \to \mathbb{B}$ is a possible assignment for $\mathcal{S}$ if $a \in \mathcal{A}_{\mathcal{S}}$.*

Axiom 1.9 (Axiom of closure). *We can always find a statement whose truth value arbitrarily depends on an arbitrary set of statements. Formally, let $S \subseteq \mathcal{S}$ be a set of statements and $f_{\mathbb{B}} : \mathbb{B}^{S} \to \mathbb{B}$ an arbitrary function from an assignment of S to a truth value. Then we can always find a statement $\bar{\mathsf{s}} \in \mathcal{S}$ that depends on S through $f_{\mathbb{B}}$.*

Lead to standard logic
(i.e. Boolean algebra)

two-valued logic

Axioms of verifiability

Axiom 1.27 (Axiom of verifiability). *A **verifiable statement** is a statement that, if true, can be shown to be so experimentally. Formally, each logical context $\mathcal{S}$ contains a set of statements $\mathcal{S}_{\mathsf{v}} \subseteq \mathcal{S}$ whose elements are said to be verifiable. Moreover, we have the following properties:*

- *every certainty $\top \in \mathcal{S}$ is verifiable*
- *every impossibility $\perp \in \mathcal{S}$ is verifiable*
- *a statement equivalent to a verifiable statement is verifiable*

Axiom 1.31 (Axiom of finite conjunction verifiability). *The conjunction of a finite collection of verifiable statements is a verifiable statement. Formally, let $\{\mathsf{s}_i\}_{i=1}^{n} \subseteq \mathcal{S}_{\mathsf{v}}$ be a finite collection of verifiable statements. Then the conjunction $\bigwedge\limits_{i=1}^{n} \mathsf{s}_i \in \mathcal{S}_{\mathsf{v}}$ is a verifiable statement.*

Axiom 1.32 (Axiom of countable disjunction verifiability). *The disjunction of a countable collection of verifiable statements is a verifiable statement. Formally, let $\{\mathsf{s}_i\}_{i=1}^{\infty} \subseteq \mathcal{S}_{\mathsf{v}}$ be a countable collection of verifiable statements. Then the disjunction $\bigvee\limits_{i=1}^{\infty} \mathsf{s}_i \in \mathcal{S}_{\mathsf{v}}$ is a verifiable statement.*

Lead to intuitionist logic
(i.e. Heyting algebra)

three-valued logic

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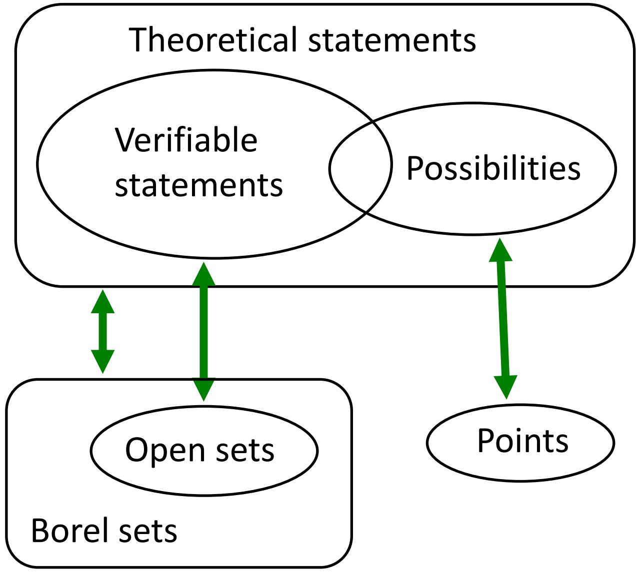
Assumptions of Physics

Topology and σ-algebra

Experimental verifiability ⇒ topology and σ-algebras (foundation of geometry, probability, …)

Perfect map between math and physics



s_1	Test Result
T	SUCCESS (in finite time)
T	UNDEFINED
F	UNDEFINED
F	FAILURE (in finite time)

$int(A)$ corresponds to the verifiable part of a statement

∂A corresponds to the undecidable part of a statement

$ext(A)$ corresponds to the falsifiable part of a statement

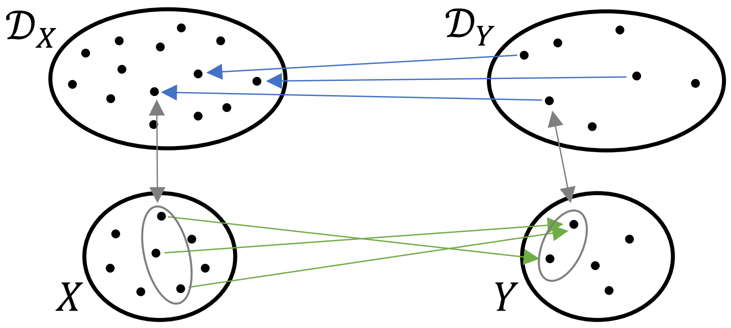
NB: in physics, topology and σ-algebra are parts of the **same** logic structure

Open set $(509.5, 510.5)$ ⟺ Verifiable "the mass of the electron is 510 ± 0.5 KeV"

Closed set $[510]$ ⟺ Falsifiable "the mass of the electron is exactly 510 KeV"

Borel set $\mathbb{Q}$ $(int(\mathbb{Q}) \cup ext(\mathbb{Q}) = \emptyset)$ ⟺ Theoretical "the mass of the electron in KeV is a rational number" (undecidable)

Inference relationship $r: \mathcal{D}_Y \to \mathcal{D}_X$ such that $r(s) \equiv s$

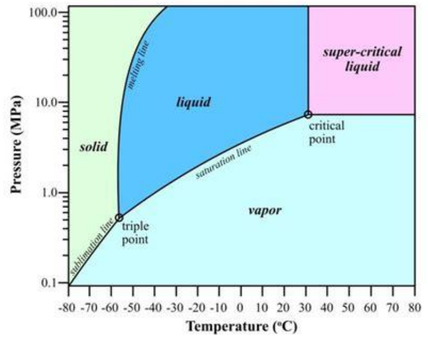


Causal relationship $f: X \to Y$ such that $x \preccurlyeq f(x)$

Inference relationship
⟺
Causal relationship

Relationships must be topologically continuous

Topologically continuous consistent with analytic discontinuity on isolated points



Phase transition ⟺ Topologically isolated regions

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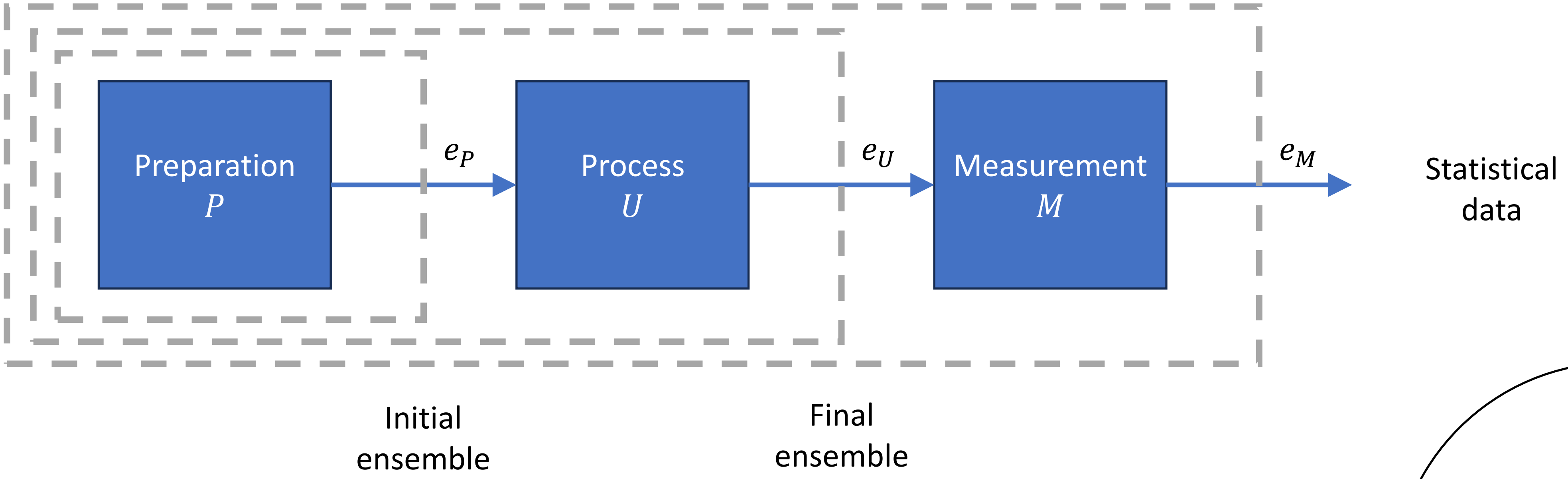
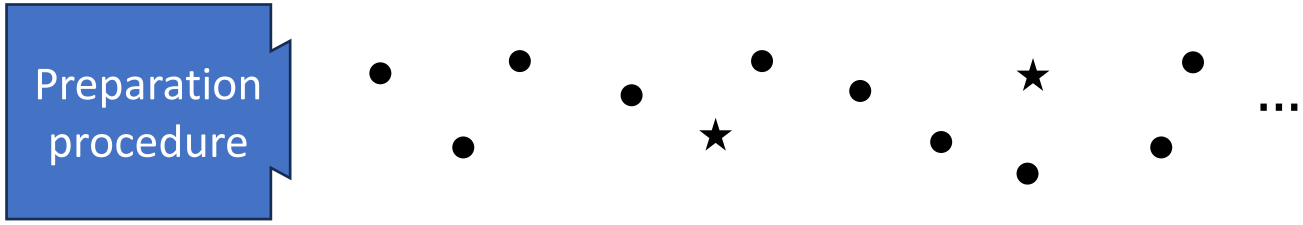
Assumptions of Physics

Principle of scientific reproducibility. Scientific laws describe relationships that can always be experimentally reproduced.

⇒ Scientific laws are relationships between ensembles





Ensemble of outputs

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Assumptions of Physics

Minimal requirements for an ensemble space

Axiom 1.4 (Axiom of ensemble). *The state of a system is represented by an* ***ensemble****, which represents all possible preparations of equivalent systems prepared according to the same procedure. The set of all possible ensembles for a particular system is an* ***ensemble***

Experimental verifiability ⇒ Topology

Responsible for all topological structures

Axiom 1.7 (Axiom of mixture). *The statistical mixture of two ensembles is an ensemble.*

Ensembles can be mixed ⇒ Convex structure

Responsible for all linear structures

Axiom 1.21 (Axiom of entropy). *Every element of the ensemble is associated with an* ***entropy*** *which quantifies the variability of the preparations of the ensemble. Formally, an*

Ensemble variability ⇒ Entropy

Responsible for all geometric structures

Still trying to find the right starting point for Poisson/commutator structure

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Assumptions of Physics

Some general results/constructions

Theorem 1.25 (Uniqueness of entropy). *The entropy of the coefficients $I(p, \bar{p})$ is the Shannon entropy. That is, $I(p, \bar{p}) = -\kappa (p \log p + \bar{p} \log \bar{p})$ where $\kappa > 0$ is the arbitrary multiplicative constant for the entropy. For a mixture of arbitrarily many elements, $I(\{p_i\}) = -\kappa \sum_i p_i \log p_i$.*

The maximal entropy increase $I(p, \bar{p})$ is uniquely determined, independently of physical theory

Ensembles embed into a vector space
(if continuously, we have a foundation for calculus)

Theorem 1.65 (Differences from a vector space). *Let $\mathsf{a} \in \mathcal{E}$ be an interior point and let $V = \{[r(\mathsf{b} - \mathsf{a})]\}$ be the set of equivalence classes of ensemble differences from a. Then V is a vector space under the scalar multiplication and addition.*

$$MS(\mathsf{a}, \mathsf{b}) = S\left(\frac{1}{2}\mathsf{a} + \frac{1}{2}\mathsf{b}\right) - \left(\frac{1}{2}S(\mathsf{a}) + \frac{1}{2}S(\mathsf{b})\right)$$

Entropy increase during mixing generalizes Jensen-Shannon Divergence (pseudo-distance)

Hessian of the entropy generalizes Fisher-Rao metric from information geometry

$$g_{\mathsf{e}}(\delta\mathsf{e}_1, \delta\mathsf{e}_2) = -\frac{\partial^2 S}{\partial \mathsf{e}^2}(\delta\mathsf{e}_1, \delta\mathsf{e}_2).$$

Definition 1.85. *Let $\mathsf{e} \in \mathcal{E}$ be an ensemble and $A \subseteq \mathcal{E}$ a Borel set. The **fraction capacity** of A for e is the biggest fraction achievable with convex combinations of A. That is, $\mathrm{fcap}_{\mathsf{e}}(A) = \sup(\mathrm{frac}_{\mathsf{e}}(\mathrm{hull}(A)) \cup \{0\})$.*

Non-additive generalization of probability

Generalization of $S(\rho_U) = \log \mu(U)$

Definition 1.156. *Let $U \subseteq \mathcal{E}$ be the subset of an ensemble space. The **state capacity** of U is defined as $\mathrm{scap}(U) = \sup(2^{S(\mathrm{hull}(U))})$ if $U \neq \varnothing$ and $\mathrm{scap}(U) = 0$ otherwise.*

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Assumptions of Physics

Open problems

What theorems that hold in topological vector spaces also hold in topological convex spaces?

E.g. Is every T_1 second countable TCS metrizable? True for TVS.

Can we show the entropy induces a metric, which induces the topology?

Can we prove continuous embedding in a TVS?

Can we find a non-additive integral/derivative to generalize probability theory?

Standard approaches (e.g. Sugeno and Choquet integrals) do not work

How do we make field theories fit the framework?

E.g. What is the correct function space, on top of which we can define
(cylinder?) measures, a notion of entropy and a “symplectic” structure?

...

Conclusion

- The current connection between math and physics is “suboptimal”
  - Physicists typically ignore all the hard problems; mathematicians cannot know which choices are physically significant and which aren’t
- A physically meaningful and mathematically precise approach exists
  - Reverse physics allows us to understand what role each mathematical structure is playing in a physical theory; physical mathematics allows us to design mathematical structures that are physically justified and therefore map perfectly to the physics
- New mathematics needs to be developed
  - We have a number of mathematical questions that can be extracted from the framework and studied independently; we are actively looking for interested mathematicians

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Assumptions of Physics

To learn more

- Project website
  - https://assumptionsofphysics.org for papers, presentations, …
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- YouTube channels
  - https://www.youtube.com/@gcarcassi
    Videos with results and insights from the research
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