Non-additive measures for quantum probability?

Gabriele Carcassi and Christine Aidala

University of Michigan

UNIVERSITY OF MICHIGAN

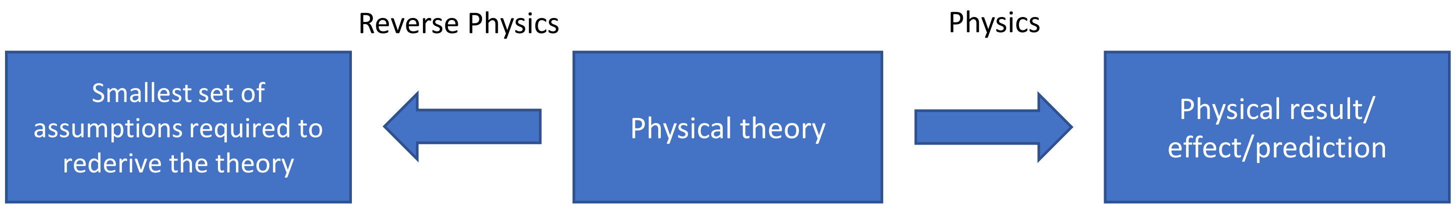
JOHN TEMPLETON FOUNDATION

Inspiring Awe & Wonder

Assumptions of Physics

Goal of the overall project

Identify a handful of physical starting points from which the basic laws can be rigorously derived



Assumptions of Physics

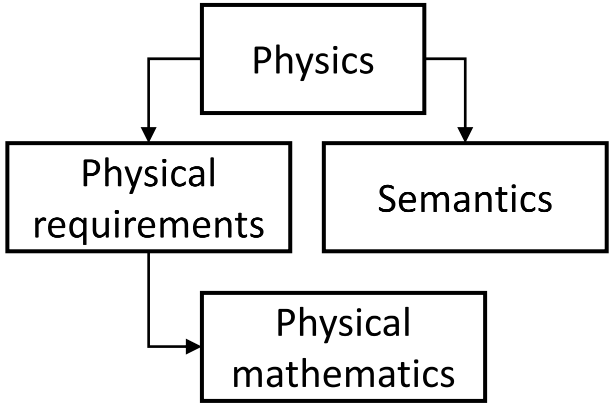
https://assumptionsofphysics.org

Reverse physics:

Start from the equations and identify principles and assumptions

Physical mathematics:

Start from scratch and rederive all mathematical structures from physical requirements



https://assumptionsofphysics.org/

Assumptions of Physics

1. Quantum mechanics violates classical probability

2. Classical probability is additive measure theory
⇒ quantum mechanics violates additive measure theory

3. Can we use non-additive measure theory
for a physically meaningful generalization?

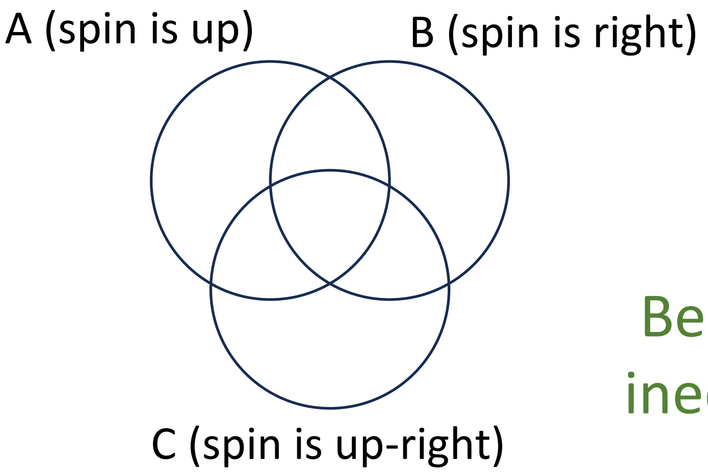
4. What tools would (or would not) work?

https://assumptionsofphysics.org/

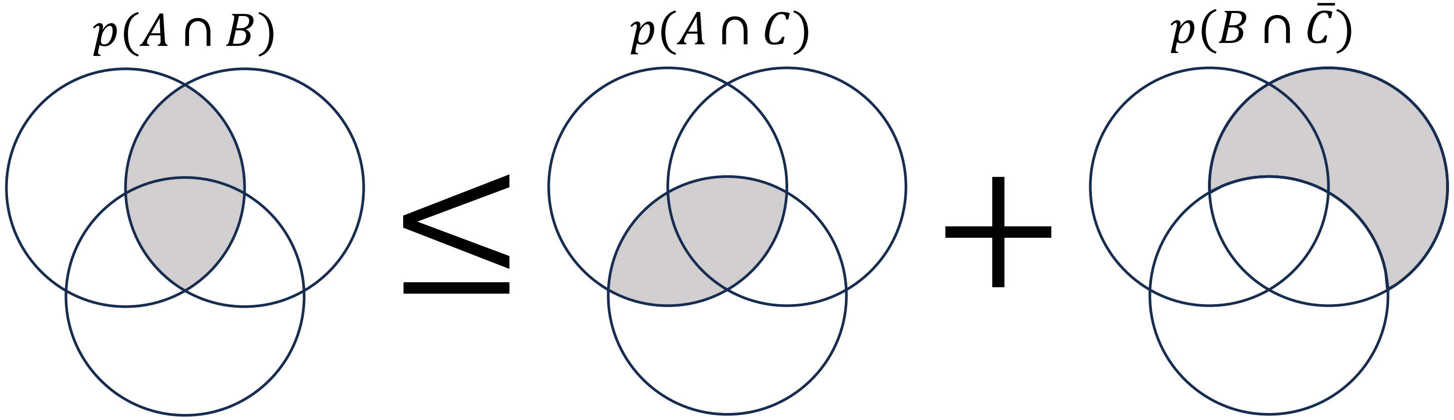
Assumptions of Physics

Classical probability and quantum mechanics

https://assumptionsofphysics.org/



Bell type inequality



Violated by quantum mechanics

CHSH inequality

$A, a, B, b \in \{-1, +1\}$ $\quad$ $(A - a, A + a) \in \{(0, \pm 2), (\pm 2, 0)\}$

$$|\langle AB \rangle - \langle Ab \rangle - \langle aB \rangle - \langle ab \rangle| \leq 2$$

In quantum mechanics $2\sqrt{2}$

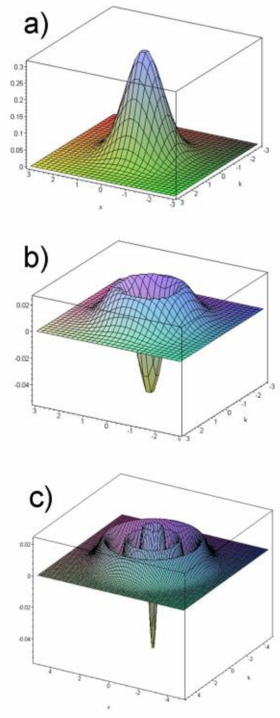
$\Rightarrow$ Failure of classical probability (of standard measure theory)

https://assumptionsofphysics.org/

Assumptions of Physics

Quasi-probability distribution

Wigner function for first two solutions of the harmonic oscillator



Wigner function

Sample space is not the state space

$$W(x,p) = \frac{1}{\pi\hbar}\int_X \psi^*(x+y)\psi(x-p)e^{2\iota py/\hbar}dy$$

Can be negative: no clear meaning

Recovers marginal distributions

$$|\psi(x)|^2 = \int W(x,p)dp \quad |\psi(p)|^2 = \int W(x,p)dx$$

$$\langle G\rangle = \int g(x,p)W(x,p)dxdp$$

Acts as a probability density to recover probabilities and expectation values
Additive under statistical mixing

Density first (not measure first)
Not clear what densities are allowed or not

https://assumptionsofphysics.org/

Assumptions of Physics

Can we find a generalization of probability theory that works in the same way in both classical and quantum mechanics?

Sample space is the respective state space
Probability defined as a set function on the sample space
Expectations recovered as integral of some function of the sample space
…

What set function for probability?

What integral for expectation?

What derivative for probability density?

https://assumptionsofphysics.org/

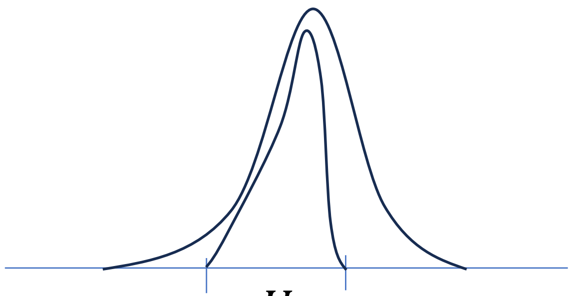
Assumptions of Physics

Ensemble spaces and mixing probability

https://assumptionsofphysics.org/

Assumptions of Physics

Probability as biggest coefficient when mixing measures



Measure over whole space

$$p(A) = p(A|U)p(U) + p(A|U^C)p(U^C)$$

Measure over U

Coefficient of convex combination

$\mathcal{E}$ convex space of probability measures

$\mathcal{E}_U$ convex subspace of $\mathcal{E}$ with support U

Given $\mu \in \mathcal{E}$, $\mu(U) = \sup(\{p \in [0,1] \mid \exists x \in \mathcal{E}_U, y \in \mathcal{E}$
$\mu = px + (1-p)y\})$

This we can generalize to quantum mechanics (to any convex set)

https://assumptionsofphysics.org/

Assumptions of Physics

General structure for space of ensembles

Convex structure $px + (1-p)y \in \mathcal{E}$ $p \in [0,1]$ $x, y \in \mathcal{E}$

Concave entropy $S : \mathcal{E} \to \mathbb{R}$

Increase of entropy due to mixing is bound

$$0 \leq S(px + (1-p)y) - \left(pS(x) + (1-p)S(y)\right) \leq I(p, 1-p)$$

No increase

Semi-metric

Max increase = Shannon entropy

Equality

Orthogonality ≡ No overlap

https://assumptionsofphysics.org/

Assumptions of Physics

Mixing probability

All convex combinations

Hull $\quad \text{hull}(U) = \{\sum p_i e_i, e_i \in U\}$

$\approx$ All probability distributions over U

Probability $\quad p_e(U) = \sup(\{p \in [0,1] \mid \exists x \in \text{hull}(U), y \in \mathcal{E}$
$e = px + (1-p)y\})$

Biggest coefficient reachable by a mixture of elements of U

Set function

1. non-negative and unit bounded: $p_e(U) \in [0,1]$
2. monotone: $U \subseteq V \Rightarrow p_e(U) \le p_e(V)$
3. subadditive: $p_e(U \cup V) \le p_e(U) + p_e(V)$

Additive on subspaces that can be orthogonally decomposed

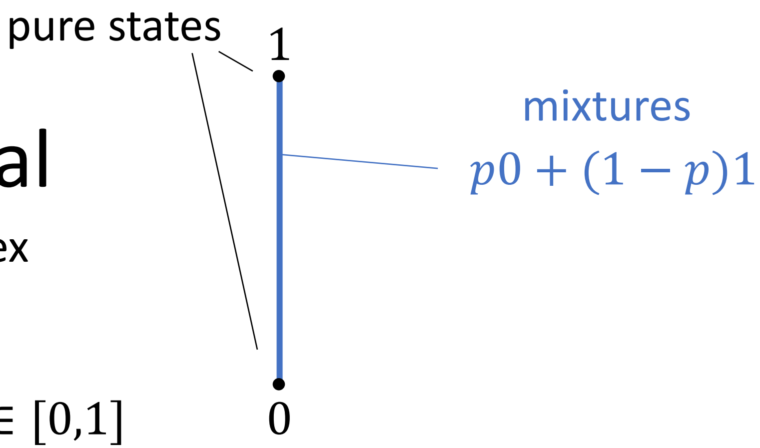
https://assumptionsofphysics.org/

Assumptions of Physics

Two-state system (bit and qubit)

Classical

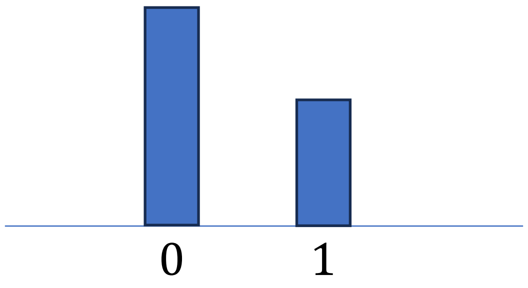
2D simplex



$e \sim (p) \in [0,1]$

$p_e(\emptyset) = 0 \qquad p_e(\{0,1\}) = 1$

$p_e(0) = p \qquad p_e(1) = 1 - p$



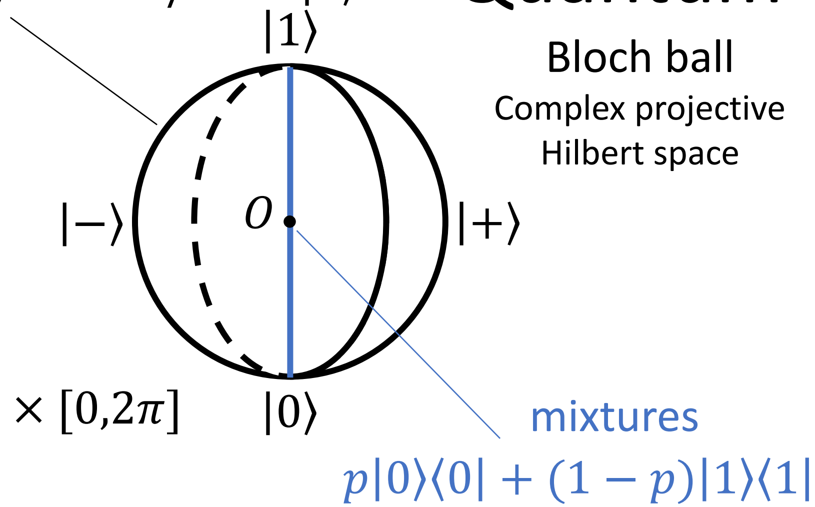
Quantum

Bloch ball

Complex projective Hilbert space

pure states (superpositions)

$|\psi\rangle = \cos\hat{\theta}/2\,|0\rangle + \sin\hat{\theta}/2\, e^{\imath\hat{\varphi}}\,|1\rangle$



$e \sim (r, \theta, \varphi) \in [0,1] \times [0,\pi] \times [0,2\pi]$

$p_e(\emptyset) = 0 \qquad p_e(X) = 1$

$$p_e(\{|\psi\rangle\}) = 1 - \frac{r^2 \sin^2 \Delta\theta + (1 - r\cos\Delta\theta)^2}{2(1 - r\cos\Delta\theta)}$$

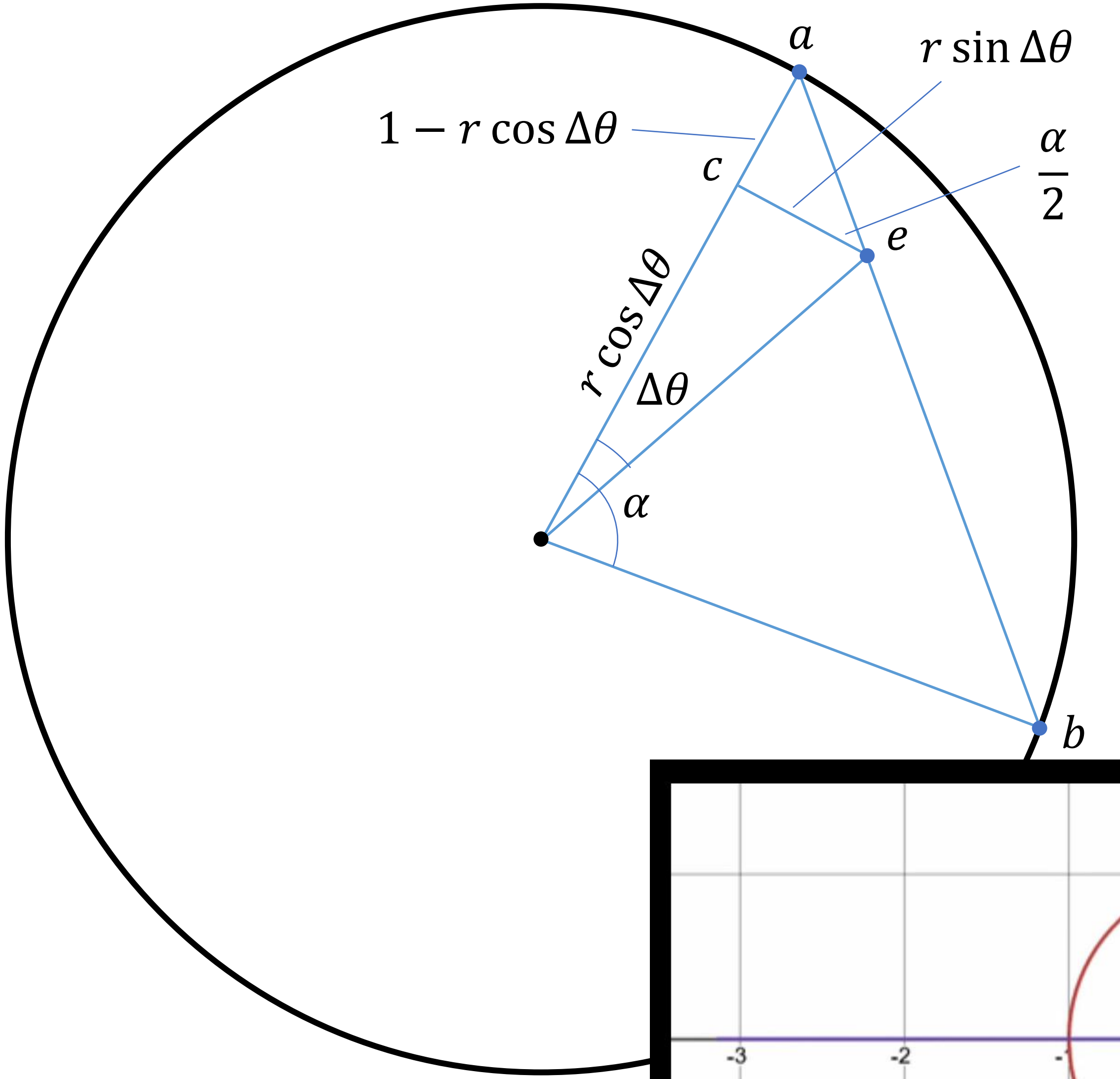
$$p_{|\psi\rangle}(|\phi\rangle) = \begin{cases} 1 & \psi = \phi \\ 0 & \psi \neq \phi \end{cases} \qquad p_O(|\phi\rangle) = \frac{1}{2}$$

lowest entropy

highest entropy

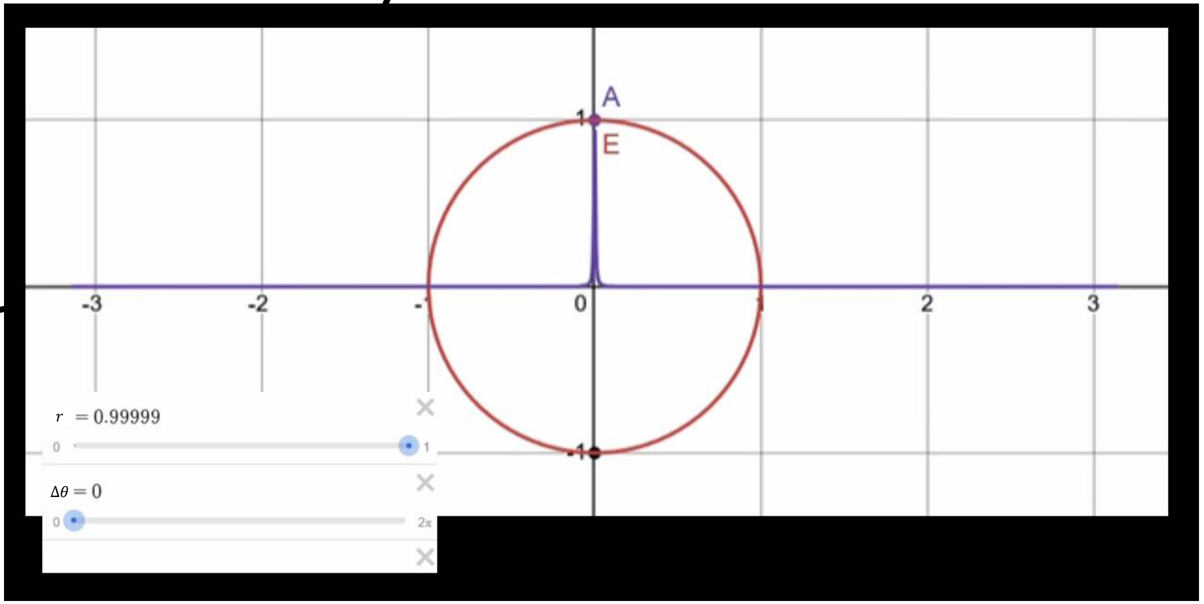
https://assumptionsofphysics.org/

Assumptions of Physics



$$p = 1 - \frac{\overline{ae}}{\overline{ab}} = 1 - \frac{\overline{ae}}{2\sin\frac{\alpha}{2}} = 1 - \frac{\overline{ae}}{2\frac{\overline{ac}}{\overline{ae}}} = 1 - \frac{\overline{ae}^2}{2\overline{ac}}$$

$$= 1 - \frac{(r\sin\Delta\theta)^2 + (1 - r\cos\Delta\theta)^2}{2(1 - r\cos\Delta\theta)}$$

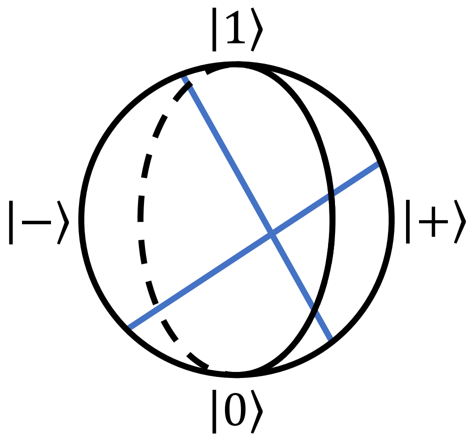


https://assumptionsofphysics.org/

Assumptions of Physics

The non-additivity is caused by the possible multiple decompositions

Classical probability:
single decomposition
in pure states



Quantum probability:
multiple decompositions
in pure states

Finite probability on each point of a continuum

High degree of non-additivity

Uncountably many possible decompositions

$$p_O(X) = 1 \leq \sum_X p_O(\{x\}) = \frac{1}{2}|\mathbb{R}|$$

E.g. for maximally mixed state

https://assumptionsofphysics.org/

Assumptions of Physics

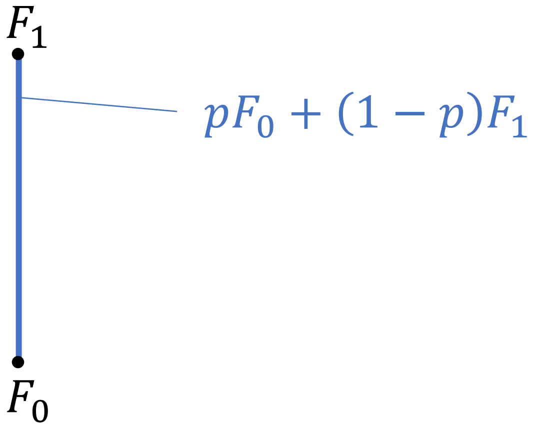
Random variables

$F: \mathcal{E} \to \mathbb{R}$

$$F(px + (1-p)y) = pF(x) + (1-p)F(y)$$

Classical

2D simplex



$f(x) = F(\{x\})$

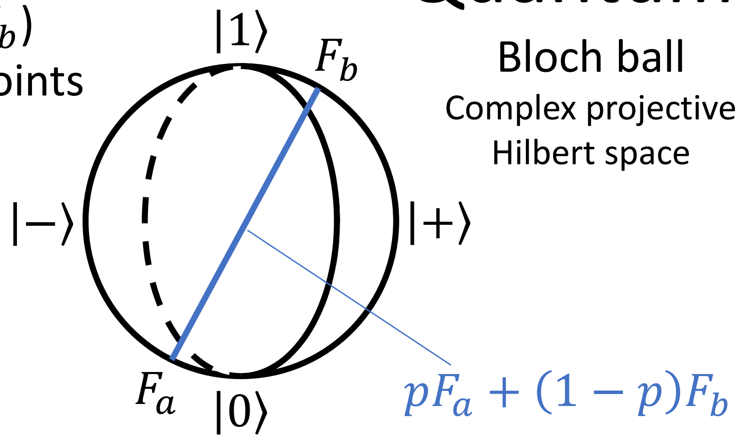
$E[F] = \sum f(x)p(x) = F(e)$

Quantum

Bloch ball

Complex projective Hilbert space

Max and min (F_a and F_b) must be on opposite points



$f(x) = F(\{x\})$

$E[F] = \widehat{\sum} f(x)p(x) = F(e)$

???

This is what needs to be generalized to a continuum for integral/derivative

https://assumptionsofphysics.org/

Assumptions of Physics

Entropy and count of states

https://assumptionsofphysics.org/

Assumptions of Physics

Classical statistical mechanics links count of states and entropy

$$S(\rho_U) = \log \mu(U)$$

Count of states

Shannon/Gibbs entropy

Uniform distribution over U

Fundamental postulate of statistical mechanics

Quantum statistical mechanics has a somewhat related expression

$$S(\rho_U) \leq \log(\dim_{\mathbb{C}}(\mathrm{span}(U)))$$

Equal for a uniform distribution over $\mathrm{span}(U)$

Von Neumann entropy

Uniform distribution over U

Dimensionality of subspace

Generalized expression?

https://assumptionsofphysics.org/

Assumptions of Physics

Proposition 1.24 (Exponential entropy subadditivity). *Let* $\mathsf{e}_1, \mathsf{e}_2 \in \mathcal{E}$. *Let* $S_1 = S(\mathsf{e}_1)$ *and* $S_2 = S(\mathsf{e}_2)$. *Let* $\mathsf{e} = p\mathsf{e}_1 + \bar{p}\mathsf{e}_2$ *for some* $p \in [0,1]$ *and* $S = S(\mathsf{e})$. *Then* $2^S \leq 2^{S_1} + 2^{S_2}$, *with the equality if and only if* e_1 *and* e_2 *are disjunct and* $p = \frac{2^{S_1}}{2^{S_1}+2^{S_2}}$.

Exponential of the entropy has key property

Proof. If p is fixed, the upper variability bound of entropy is saturated only if e_1 and e_2 are disjunct by definition. The entropy maximum for the mixed ensemble can only be achieved when the elements are disjunct, for some value of p.

Proof is mere calculation

$$\begin{aligned}
0 = \frac{dS}{dp} &= \frac{d}{dp}S(\mathsf{e}) = \frac{d}{dp}\left(-p\log p - \bar{p}\log\bar{p} + pS_1 + \bar{p}S_2\right) \\
&= -\log p - 1 + \log\bar{p} + 1 + S_1 - S_2 \\
\log\frac{p}{\bar{p}} &= \log 2^{S_1} - \log 2^{S_2} \\
\log\frac{p}{1-p} &= \log\frac{2^{S_1}}{2^{S_2}} \\
p2^{S_2} &= (1-p)2^{S_1} \\
p(2^{S_1} + 2^{S_2}) &= 2^{S_1} \\
p &= \frac{2^{S_1}}{2^{S_1} + 2^{S_2}}
\end{aligned}$$

$$\begin{aligned}
\bar{p} &= 1 - \frac{2^{S_1}}{2^{S_1} + 2^{S_2}} = \frac{2^{S_2}}{2^{S_1} + 2^{S_2}} \\
S = S(\mathsf{e}) &= -p\log p - \bar{p}\log\bar{p} + pS_1 + \bar{p}S_2 \\
&= -\frac{2^{S_1}}{2^{S_1} + 2^{S_2}}\log\frac{2^{S_1}}{2^{S_1} + 2^{S_2}} - \frac{2^{S_2}}{2^{S_1} + 2^{S_2}}\log\frac{2^{S_2}}{2^{S_1} + 2^{S_2}} \\
&+ \frac{2^{S_1}}{2^{S_1} + 2^{S_2}}\log 2^{S_1} + \frac{2^{S_2}}{2^{S_1} + 2^{S_2}}\log 2^{S_2} \\
&= \frac{2^{S_1}}{2^{S_1} + 2^{S_2}}\log\left(2^{S_1} + 2^{S_2}\right) + \frac{2^{S_2}}{2^{S_1} + 2^{S_2}}\log\left(2^{S_1} + 2^{S_2}\right) \\
&= \frac{2^{S_1} + 2^{S_2}}{2^{S_1} + 2^{S_2}}\log\left(2^{S_1} + 2^{S_2}\right) \\
\log 2^S &= \log\left(2^{S_1} + 2^{S_2}\right) \\
2^S &= 2^{S_1} + 2^{S_2}
\end{aligned}$$

https://assumptionsofphysics.org/

Assumptions of Physics

Define $\quad \text{stateCount}\,(U) = \sup\left(2^{S(\text{hull}(U))}\right)$

Exponential of the highest entropy reachable through convex combinations

Set function

1. non-negative: $\text{stateCount}(U) \in [0, +\infty]$
2. monotone: $\mathrm{U} \subseteq V \Rightarrow \text{stateCount}(U) \leq \text{stateCount}(V)$
3. subadditive: $\text{stateCount}(U \cup V) \leq \text{stateCount}(U) + \text{stateCount}(V)$

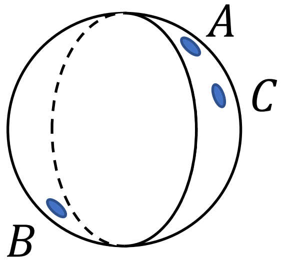
Additive on subspaces that can be orthogonally decomposed

Recovers number of states for discrete classical spaces,
Liouville measure for continuous classical spaces,
dimension of the subspaces in quantum mechanics

https://assumptionsofphysics.org/

Assumptions of Physics

Need for non-additive measure



$\mu(\{A\}) = 2^0 = 1$

$\mu(\{A, B\}) = 2^1 = 2$

not additive

$\mu(\{A, C\}) < 2 = \mu(\{A\}) + \mu(\{C\})$

In quantum mechanics, literally $1 + 1 \leq 2$

	Single point		Finite continuous range	
	$\mu(U)$	$\log \mu(U)$	$\mu(U)$	$\log \mu(U)$
Counting measure $\mu(U) = \#U$ Number of points	1	0	$+\infty$	$+\infty$
Lebesgue measure $\mu([a, b]) = b - a$ Interval size	0	$-\infty$	$< \infty$	$< \infty$
"Quantized" measure $\mu(U) = \sup(2^{S(\mathrm{hull}(U))})$ Entropy over uniform distribution	1	0	$< \infty$	$< \infty$

Pick two!

1. Single point is a single case (i.e. $\mu(\{\psi\}) = 1$)
2. Finite range carries finite information (i.e. $\mu(U) < \infty$)
3. Measure is additive for disjoint sets (i.e. $\mu(\cup U_i) = \sum \mu(U_i)$)

Physically, we count states all else equal

Contextuality $\Longleftrightarrow$ non-additive measure

https://assumptionsofphysics.org/

Assumptions of Physics

Summary

- In Assumptions of Physics (https://assumptionsofphysics.org) we are looking for a generalized setting for both classical and quantum probability
  - The setting is the convex space of ensembles at equilibrium (i.e. statistical distributions): can be mixed and have an entropy defined
- This generalization requires non-additive measures to deal with multiple decompositions (i.e. quantum contextuality)
  - Non-additive measures are not typically used in quantum foundations
  - Looking for discussion/advice/collaboration/…
- Same generalization connects other areas of math and physics
  - Information theory and information geometry
  - Functional analysis and spectral theory
  - Order theory and quantum logic

https://assumptionsofphysics.org/

Assumptions of Physics

For more information

Gabriele Carcassi

carcassi@umich.edu

Instructive videos for undergrads and above

https://www.youtube.com/@gcarcassi

https://www.youtube.com/@AssumptionsofPhysicsResearch

Videos about active research

Assumptions of Physics

https://assumptionsofphysics.org

Thanks to all participants that have shared insights and ideas

Special thanks to Vicenç Torra and Zuzana Ontkovičová

https://assumptionsofphysics.org/

Assumptions of Physics

https://assumptionsofphysics.org/

Assumptions of Physics