The geometry of quantum mechanics

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Assumptions of Physics

Vector — Hilbert space

$$|\psi\rangle \in \mathcal{H}$$

Same state (state is not a vector)

$$|\psi\rangle \sim \lambda|\psi\rangle$$

States are represented by one-dimensional subspaces

i.e. “ray” in Hilbert space $\{\lambda|\psi\rangle\} \subset \mathcal{H}$

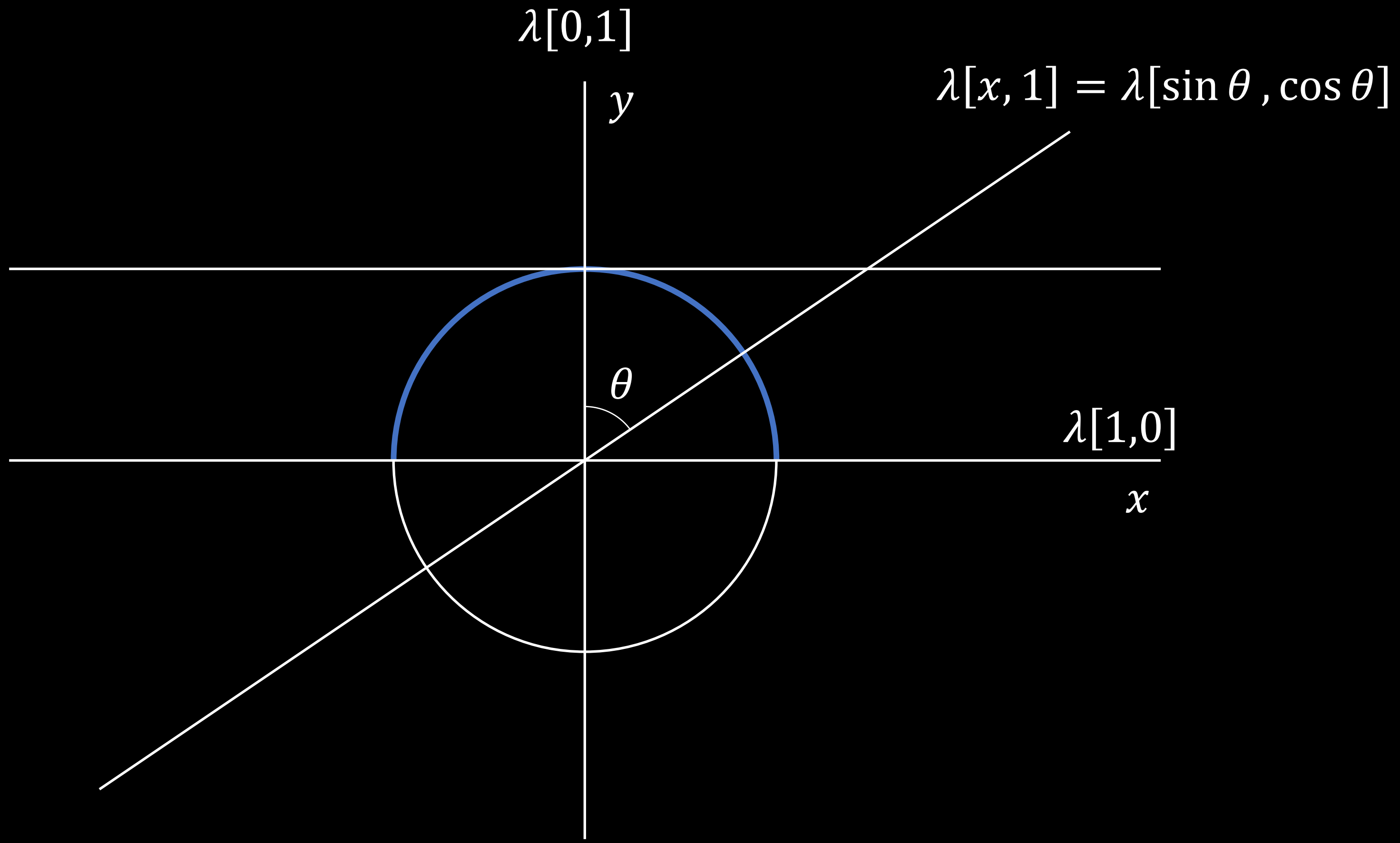
Understand real projective spaces

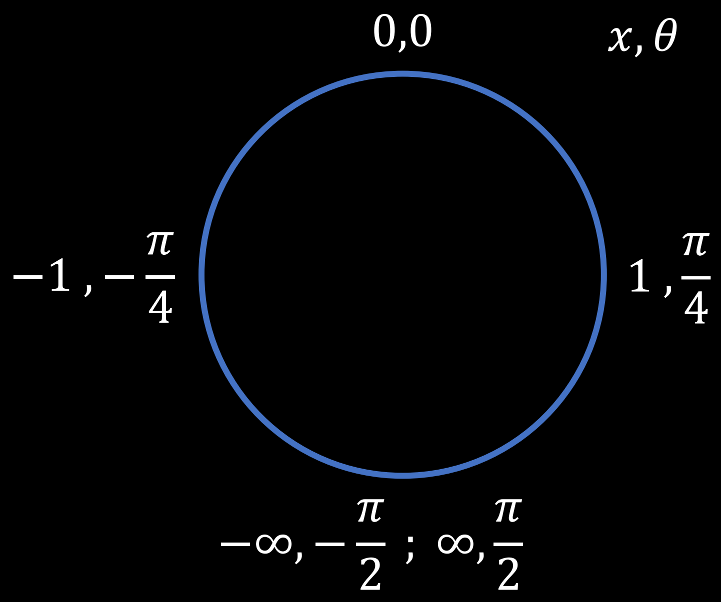
Understand complex projective spaces

What the inner product means geometrically, how a complex vector space is different from a real one with double dimension, …

Real projective line

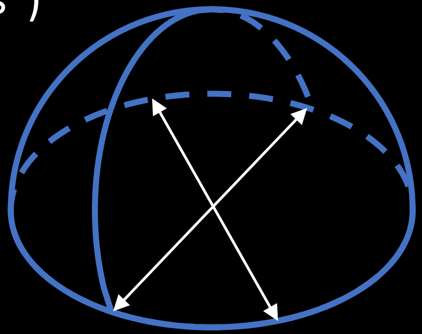
Set of all lines that pass through the origin
(one dimensional-subspace, “rays”)

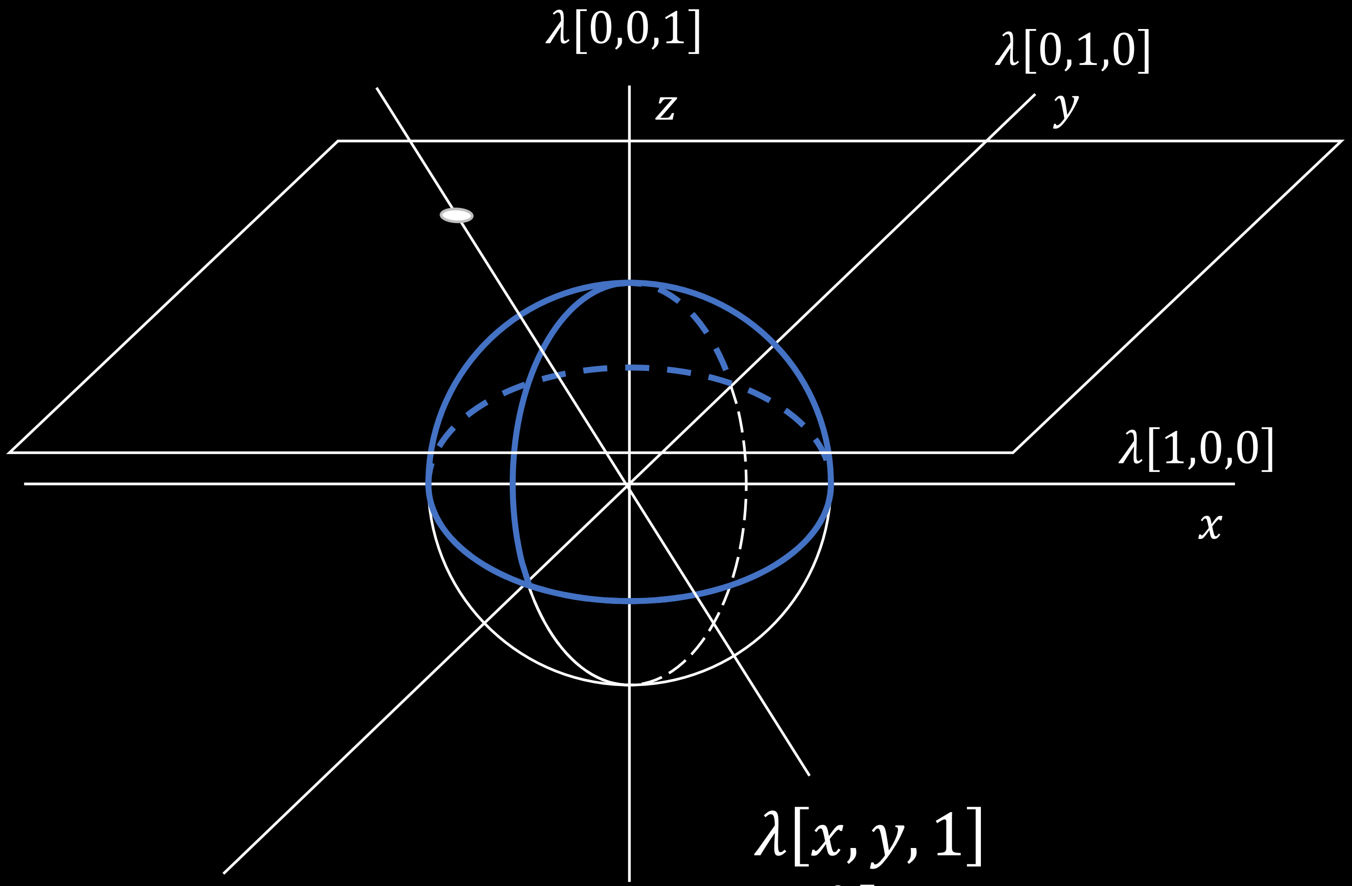




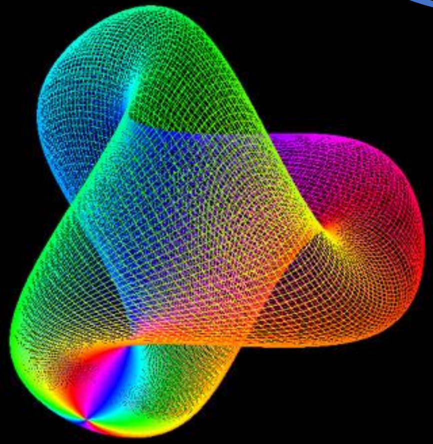
Real projective plane

Set of all lines that pass through the origin (one dimensional-subspace, “rays”)

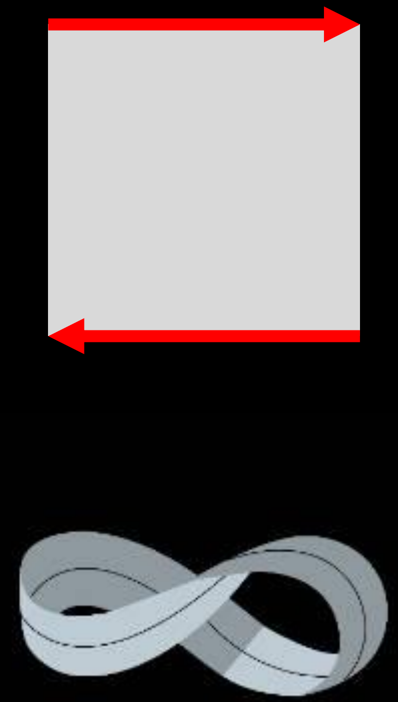




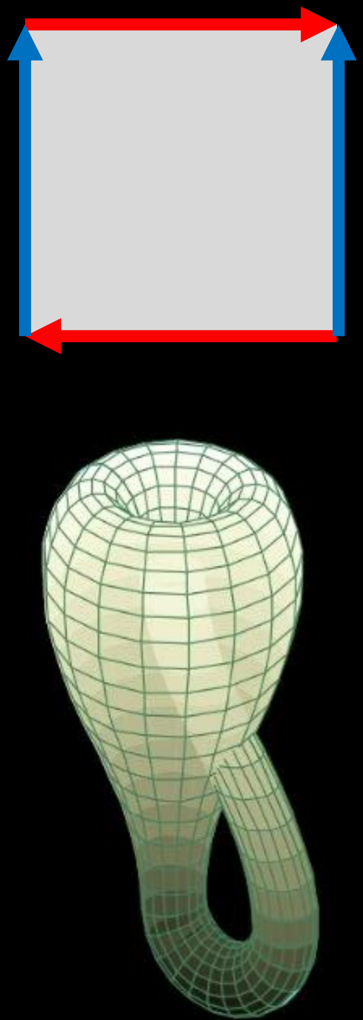
$$\lambda[x, y, 1]$$
$$= \lambda[\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta]$$



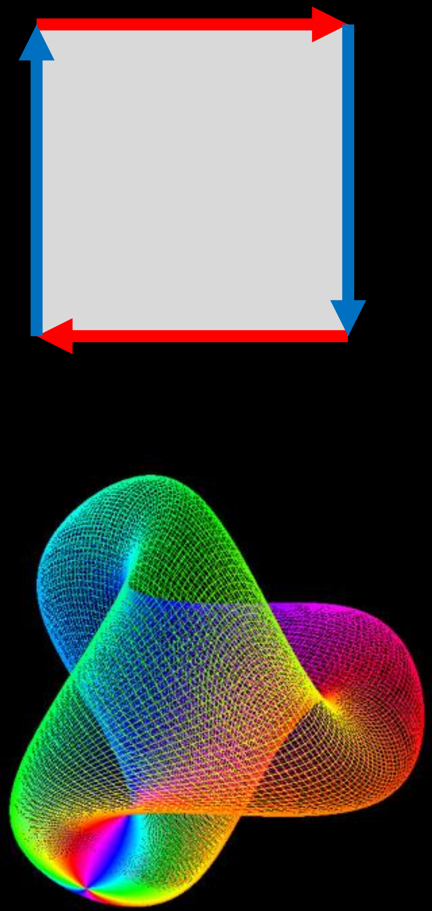
Möbius strip



Klein bottle



Real projective plane



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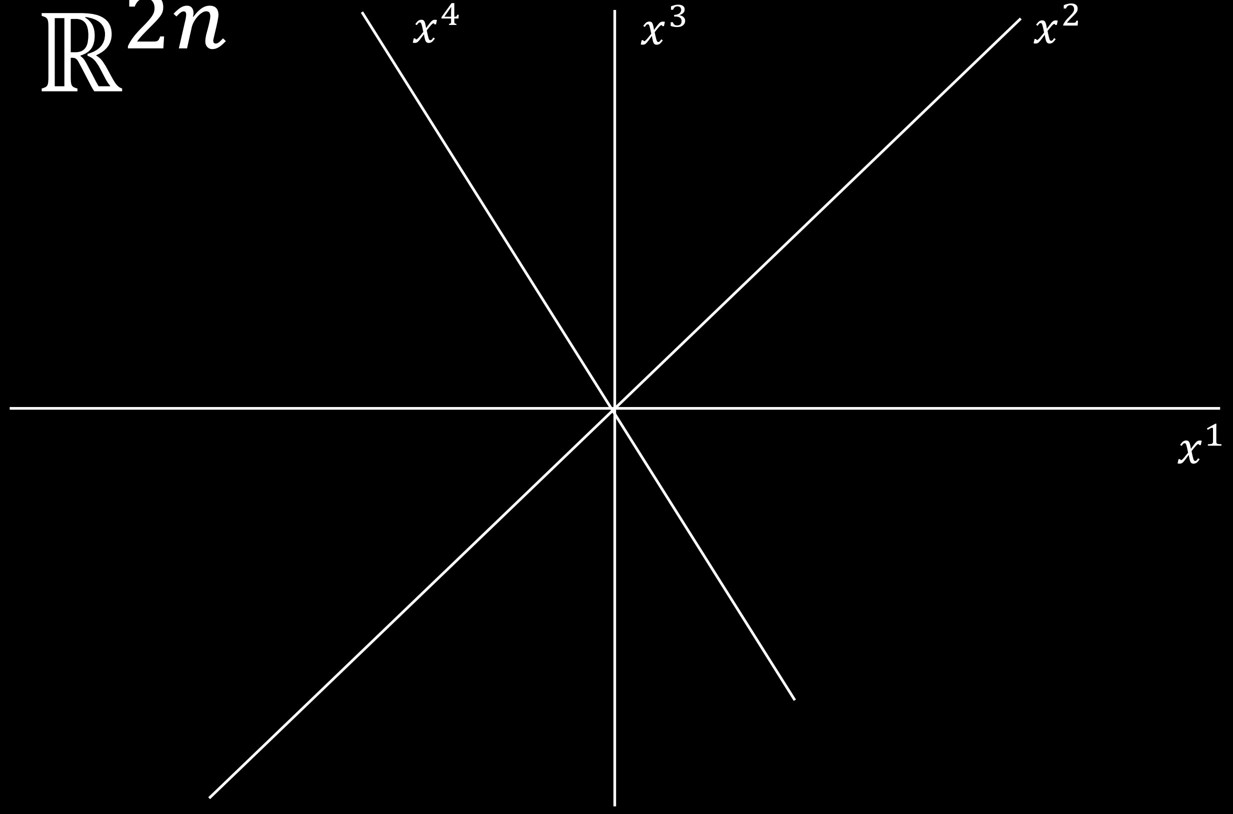
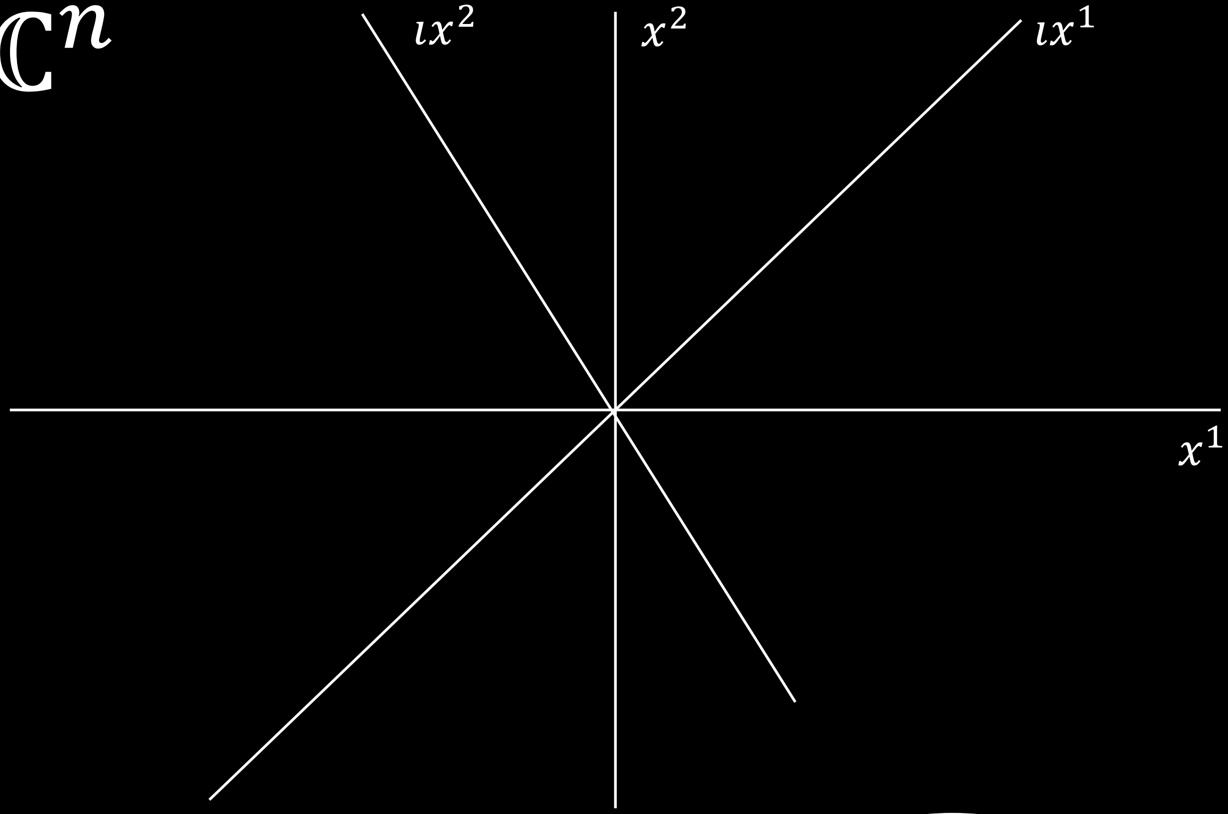
Assumptions of Physics

$\mathbb{R}^{2n}$



$\mathbb{C}^n$



$$v \cdot w = |v||w| \cos\theta$$

Each direction is independent. One angle defined between two vectors. Can rotate any direction onto any direction.

$$\langle \psi, \phi \rangle = |\psi||\phi| \cos\theta \, e^{i\varphi}$$

Directions are bundled into planes. One angle within the plane (phase) and one angle across planes. Can only rotate planes onto planes.

Real space

"Ray" = real line that passes through the origin

$$v \cdot w = |v||w| \cos\theta$$

Complex space

"Ray" = complex plane that passes through the origin

$$\langle \psi, \phi \rangle = |\psi||\phi| \cos\theta \, e^{i\varphi}$$

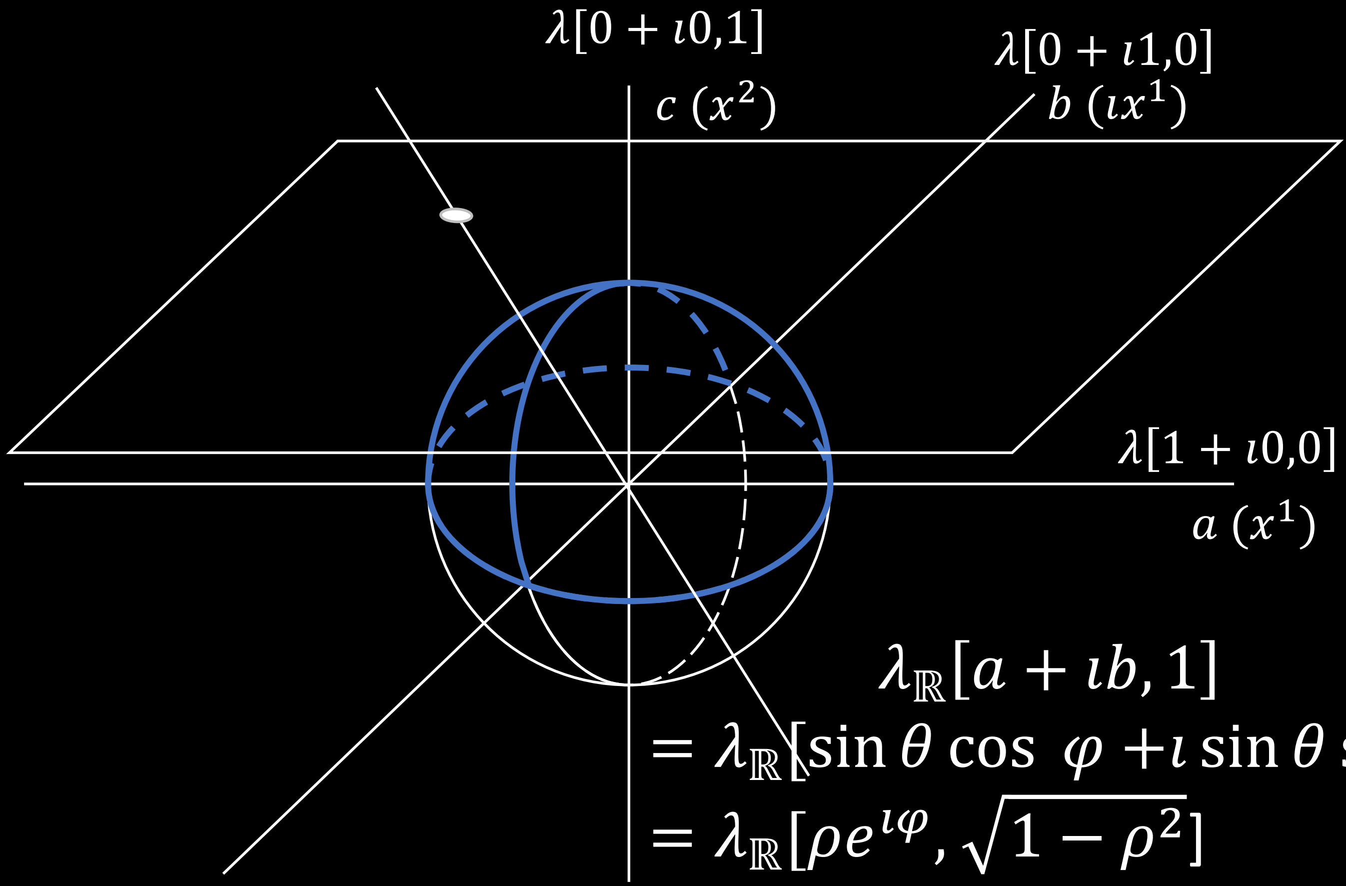
retained in the projective space

NOT retained in the projective space

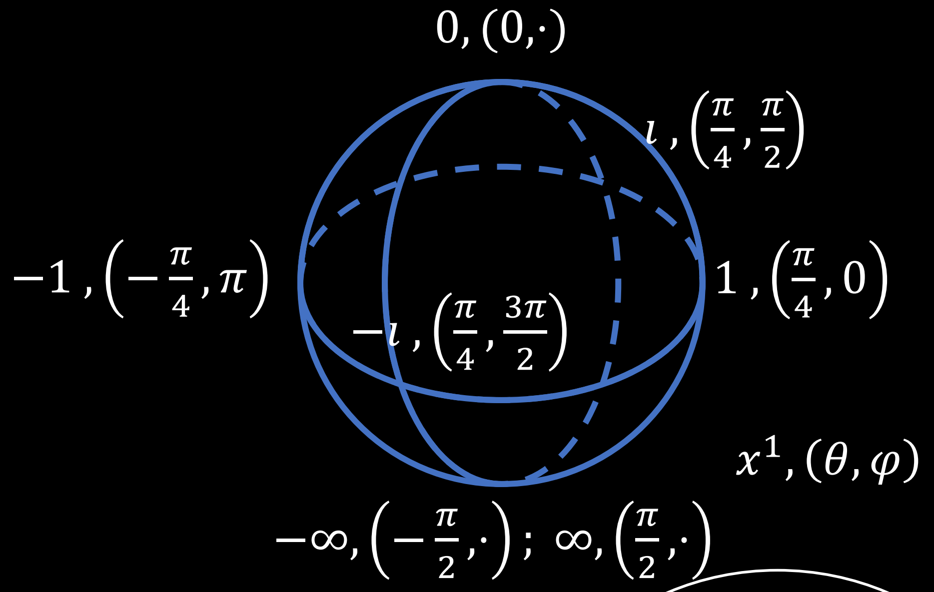
Complex projective line

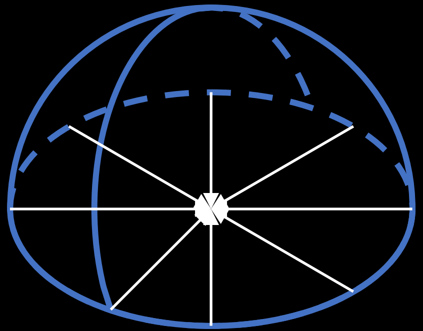
$$\lambda_{\mathbb{C}}[\overset{x^1}{a + \imath b}, \overset{x^2}{c + \imath d}] \to \lambda_{\mathbb{R}}[a + \imath b, c + \imath 0]$$

if $c \neq 0$ and $d \neq 0$



$$\lambda_{\mathbb{R}}[a + \imath b, 1]$$
$$= \lambda_{\mathbb{R}}[\sin\theta \cos\varphi + \imath \sin\theta \sin\varphi, \cos\theta]$$
$$= \lambda_{\mathbb{R}}[\rho e^{\imath\varphi}, \sqrt{1-\rho^2}]$$

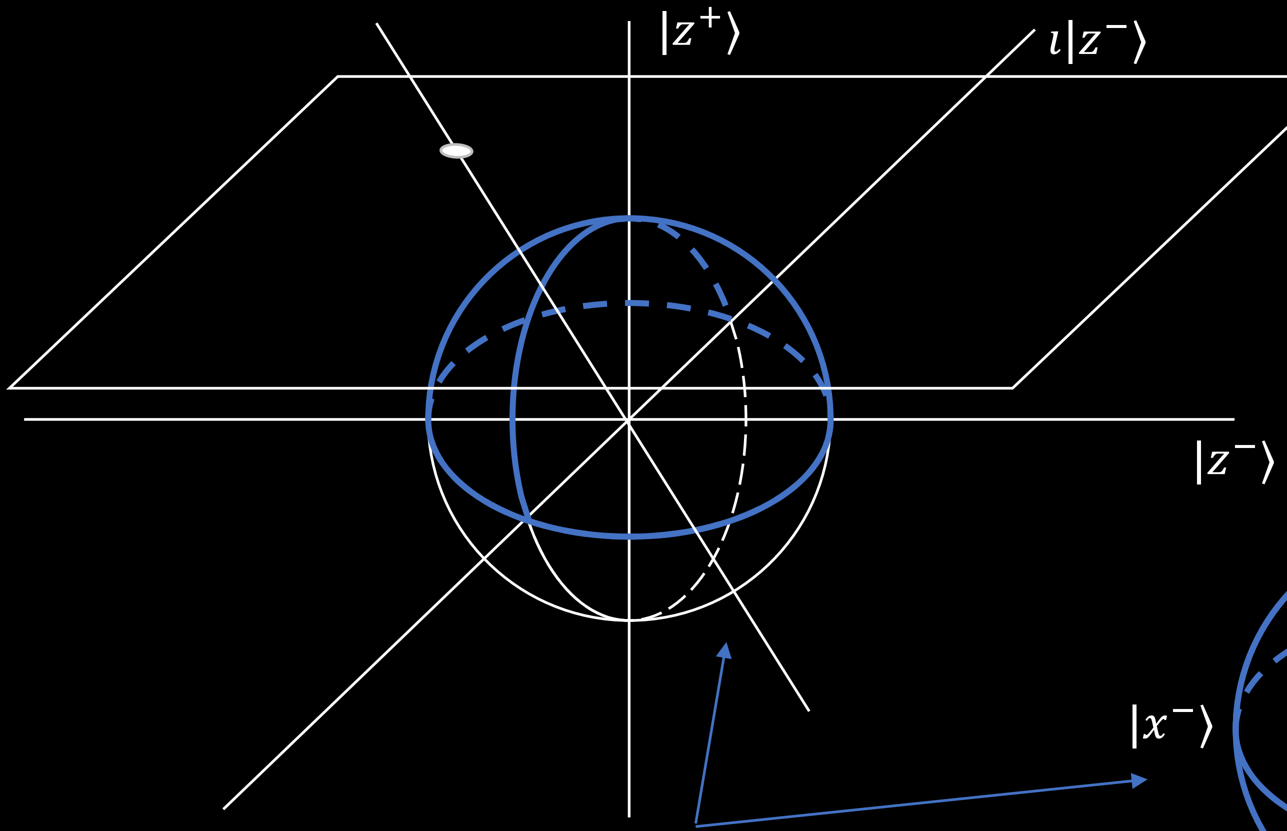




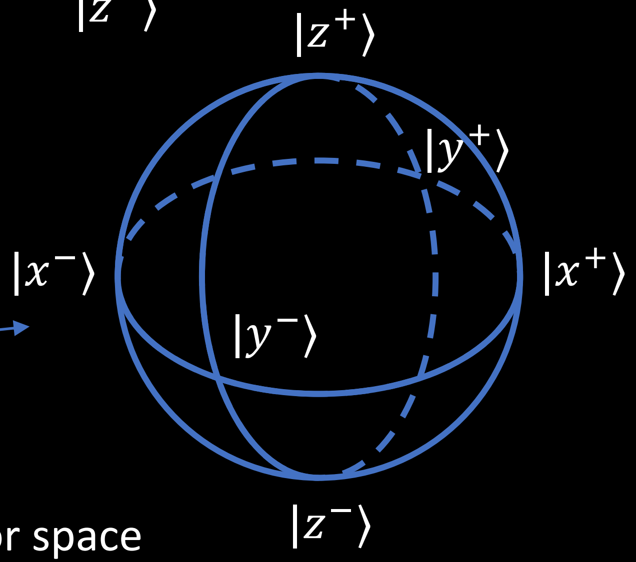
Spin 1/2 – qubit

$$|\psi\rangle = \cos\theta/2\,|z^+\rangle + \sin\theta/2\,e^{\iota\varphi}|z^-\rangle$$
$$= \cos\theta/2\,e^{-\iota\varphi/2}|z^+\rangle + \sin\theta/2\,e^{\iota\varphi/2}|z^-\rangle$$

$$|x^+\rangle = \sqrt{2}/2|z^+\rangle + \sqrt{2}/2|z^-\rangle$$
$$|y^+\rangle = \sqrt{2}/2|z^+\rangle + \iota\sqrt{2}/2|z^-\rangle$$
$$|x^-\rangle = \sqrt{2}/2|z^+\rangle - \sqrt{2}/2|z^-\rangle$$
$$|y^-\rangle = \sqrt{2}/2|z^+\rangle - \iota\sqrt{2}/2|z^-\rangle$$



spin is double cover



angle in vector space
is half the angle in physical space

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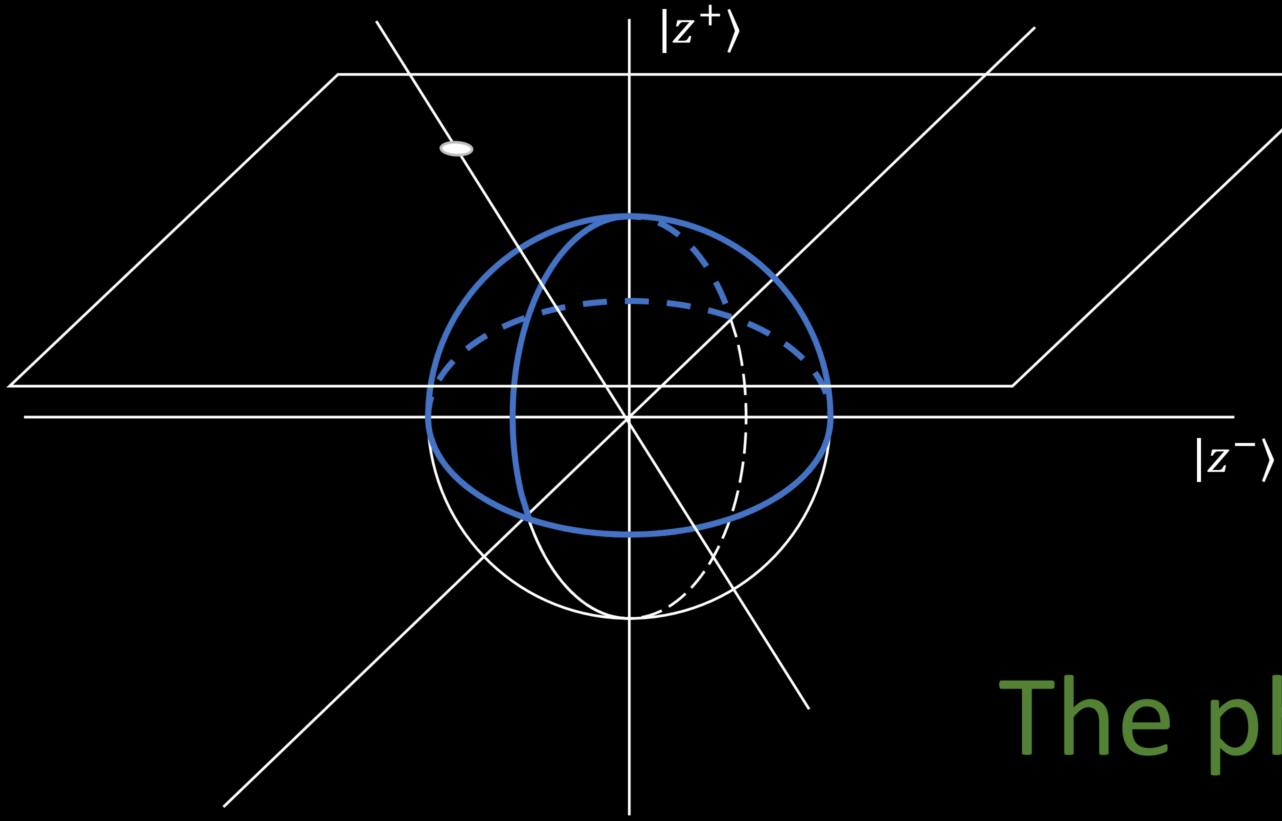
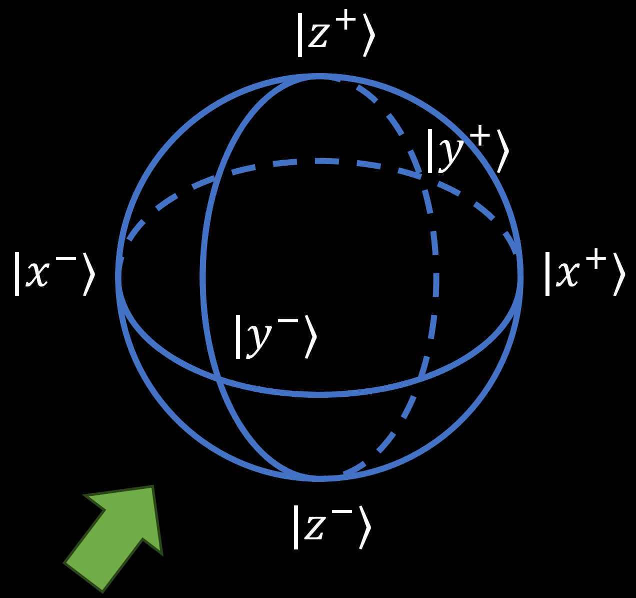
Assumptions of Physics

Vector space

Projective space





The physics is here!

$$\langle \psi, \phi \rangle = |\psi||\phi| \cos \theta_V \, e^{i\varphi}$$

$$p(\psi|\phi) = \frac{\langle \psi, \phi \rangle \langle \phi, \psi \rangle}{\langle \psi, \psi \rangle \langle \phi, \phi \rangle} = \cos^2 \theta_V = \frac{1 + \cos \theta_P}{2}$$

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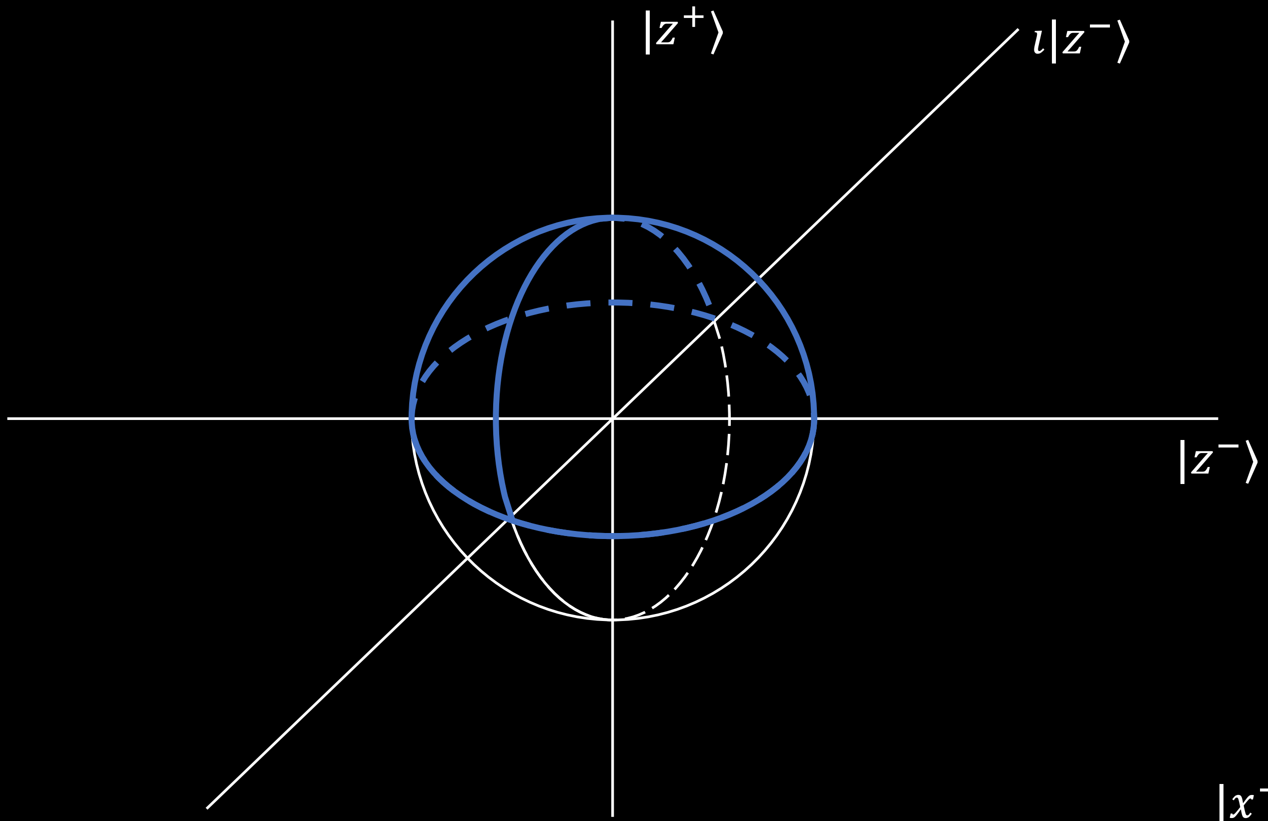
Assumptions of Physics

Superposition of states ≠ probability distribution

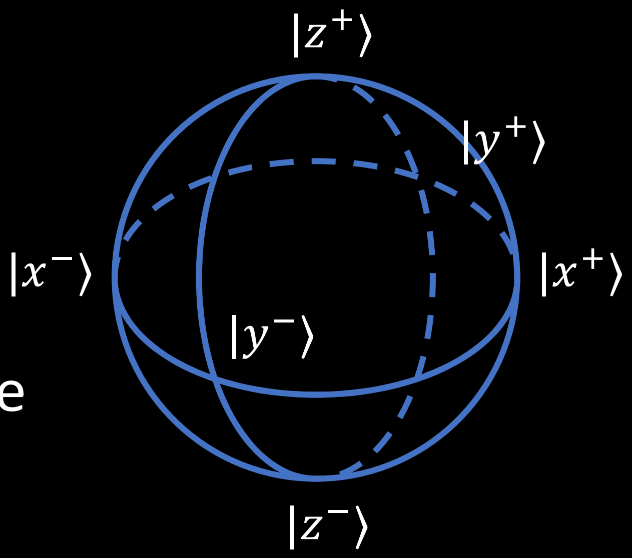


Superpositions are linear decompositions

$$|\psi\rangle = c_+|z^+\rangle + c_-|z^-\rangle$$

$$F = f_x e^x + f_y e^y$$

diagonal force is a superposition of vertical and horizontal force



Everything is a superposition of everything else

$$|x^+\rangle = \sqrt{2}/2|z^+\rangle + \sqrt{2}/2|z^-\rangle$$

$$|z^+\rangle = \sqrt{2}/2|x^+\rangle + \sqrt{2}/2|x^-\rangle$$

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Assumptions of Physics

Schrödinger equation – (unitary) time evolution

$$H|\psi\rangle = \imath\hbar\partial_t|\psi\rangle$$

Hamiltonian

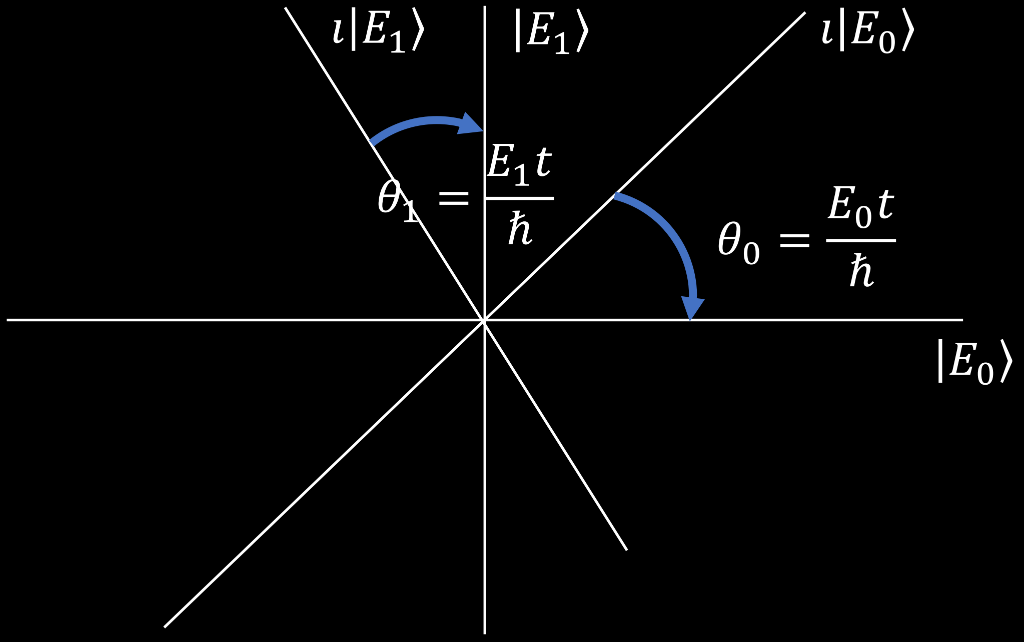
$$H = \begin{bmatrix} E_1 & 0 \\ 0 & E_0 \end{bmatrix}$$

diagonalized

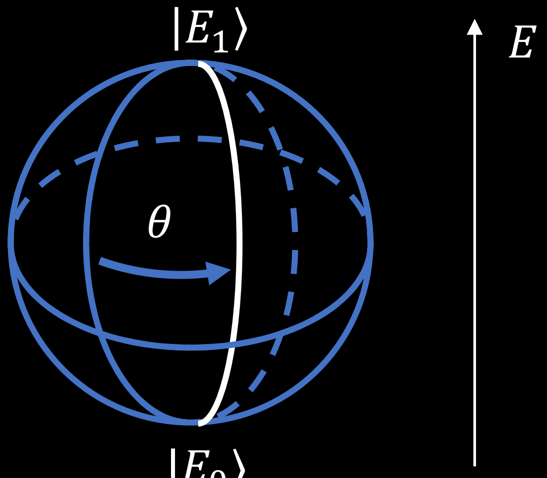
$$|\psi(t)\rangle = U(t)|\psi_0\rangle = e^{\frac{Ht}{\imath\hbar}}|\psi_0\rangle$$

Time evolution operator

$$U(t) = e^{\frac{Ht}{\imath\hbar}} = \begin{bmatrix} e^{\frac{E_1 t}{\imath\hbar}} & 0 \\ 0 & e^{\frac{E_0 t}{\imath\hbar}} \end{bmatrix}$$



$$\theta = \theta_1 - \theta_0 = \frac{(E_1 - E_0)t}{\hbar}$$



$E_1 \geq E_0$

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Assumptions of Physics

Superposition is a property of ANY linear system

https://en.wikipedia.org/wiki/Superposition_principle

The **superposition principle**,[1] also known as **superposition property**, states that, for all linear systems, the net response caused by two or more stimuli is the sum of the responses that would have been caused by each stimulus individually. So that if input *A* produces response *X* and input *B* produces response *Y* then input (*A* + *B*) produces response (*X* + *Y*).

Note: linearity is a property of the VECTOR space, not of the projective space

Quantum superposition is NOT a physical property!

It is a property of the vector space representation

$$
\begin{aligned}
|x^+\rangle &= \sqrt{2}/2|z^+\rangle + \sqrt{2}/2|z^-\rangle \\
|y^+\rangle &= \sqrt{2}/2|z^+\rangle + \imath\sqrt{2}/2|z^-\rangle \\
|x^-\rangle &= \sqrt{2}/2|z^+\rangle - \sqrt{2}/2|z^-\rangle \\
|y^-\rangle &= \sqrt{2}/2|z^+\rangle - \imath\sqrt{2}/2|z^-\rangle
\end{aligned}
$$

Coefficients are representation dependent

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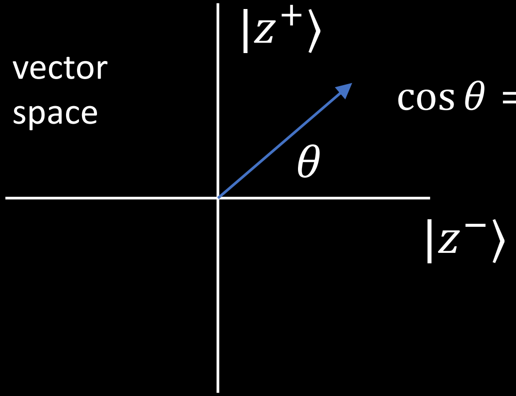
Assumptions of Physics

$$m(c_1|z^+\rangle + c_2|z^-\rangle) = c_1|z^+\rangle + c_2 e^{\iota\frac{|c_2|}{\sqrt{|c_1|^2+|c_2|^2}}}|z^-\rangle$$

phase shift that depends
on both components

vector
space



$$\cos\theta = \frac{|c_2|}{\sqrt{|c_1|^2 + |c_2|^2}}$$

Non linear map

$$m(|z^+\rangle) = |z^+\rangle \quad m(|z^-\rangle) = e^{\iota}|z^-\rangle$$

$$m(|z^+\rangle + |z^-\rangle) = |z^+\rangle + e^{\iota/\sqrt{2}}|z^-\rangle$$

Preserves the rays: colinear map

$$m(\lambda|\psi\rangle) = m\big(\lambda(c_1|z^+\rangle + c_2|z^-\rangle)\big)$$

$$= m\big((\lambda c_1)|z^+\rangle + (\lambda c_2)|z^-\rangle\big)$$

$$= \lambda c_1|z^+\rangle + \lambda c_2 e^{\iota\frac{|\lambda c_2|}{\sqrt{|\lambda c_1|^2+|\lambda c_2|^2}}}|z^-\rangle$$

$$= \lambda c_1|z^+\rangle + \lambda c_2 e^{\iota\frac{|\lambda||c_2|}{|\lambda|\sqrt{|c_1|^2+|c_2|^2}}}|z^-\rangle = \lambda m(|\psi\rangle)$$

$$\langle m(\psi), m(\phi)\rangle$$

$$= |\psi||\phi|\cos\theta\, e^{i\hat{\varphi}}$$

phase of the inner product
will change

The main difference in quantum mechanics is not the use of complex vector spaces, but the use of projective spaces

A quantum state is not a vector in the Hilbert space, but a one-dimensional subspace, a complex plane (i.e. a “ray”)

For a spin 1/2 system, angles in Hilbert space are half the physical angles (half-sphere is “stretched” to a full sphere)

Superposition (linearity) is a property of the vector space, not of the projective space, and therefore not “fully” physical