Open Problems in Physical Mathematics

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Outline

- Goal: find people that are interested/willing to help
- Very brief introduction to Assumptions of Physics, Reverse Physics and Physical Mathematics
- Brief introduction to Ensemble Spaces as core structure required by all physical theories
- Open problems

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Assumptions of Physics

Introduction to the Assumptions of Physics research program

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Assumptions of Physics

Main goal of the project

Identify a handful of physical starting points from which the basic laws can be rigorously derived

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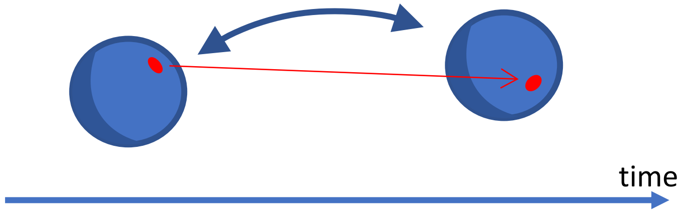
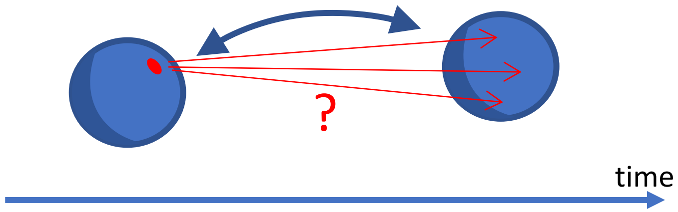
For example:

Infinitesimal reducibility ⇒ Classical state



Irreducibility ⇒ Quantum state



This also requires rederiving all mathematical structures from physical requirements

For example:

Science is evidence based ⇒ scientific theory must be characterized by experimentally verifiable statements ⇒ topology and σ-algebras

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Types of things one does in the foundations of mathematics

Axiom of Choice $\Longleftrightarrow$ Zorn's lemma

Base theory

Zermelo-Fraenkel set theory

Reverse mathematics

When the theorem is derived from the right axioms,
the axioms can be derived from the theorem

Friedman

Types of things we should do in the foundations of physics

$$d_t q = \partial_p H$$
$$d_t p = -\partial_q H$$

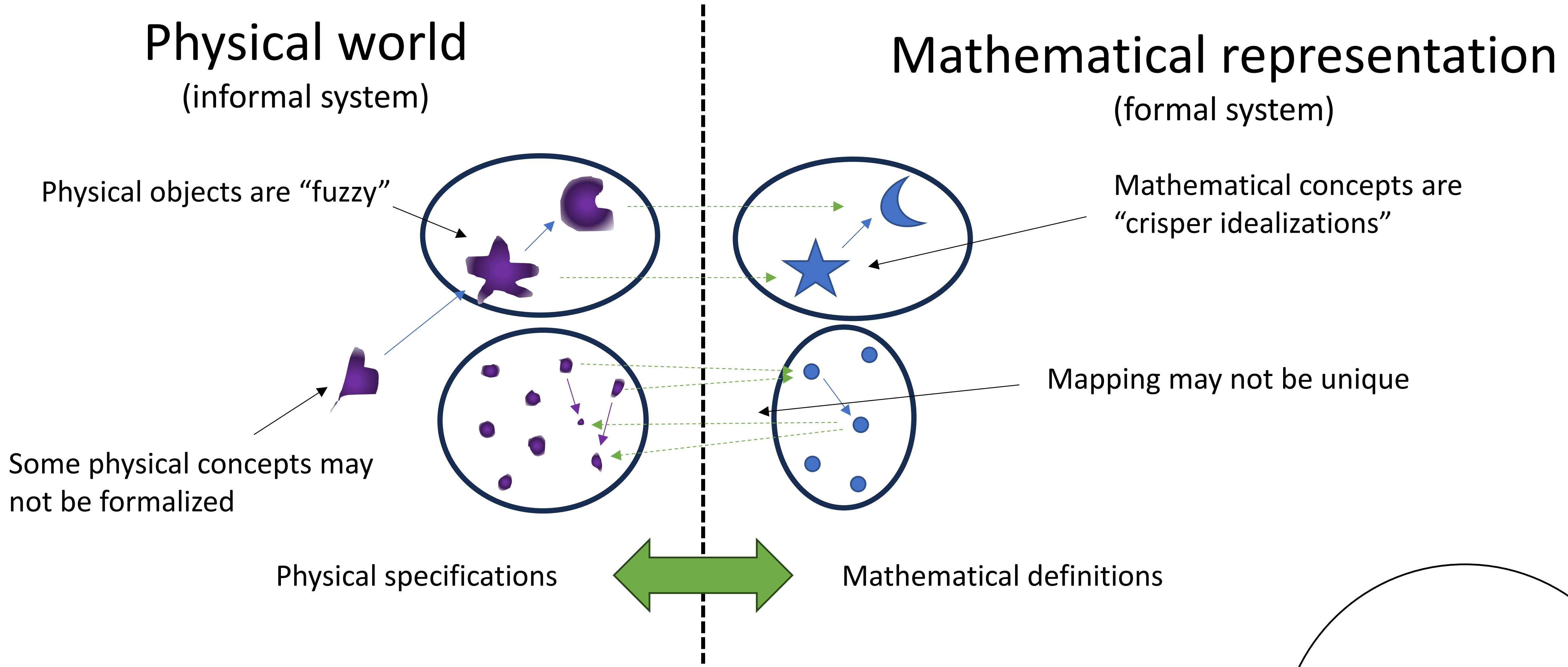
Hamilton's equations $\Longleftrightarrow$ Determinism and reversibility

Base theory

Classical states for one degree of freedom

Reverse physics

When the math is derived from the right physical assumptions,
the physical assumptions can be derived from the math



When physical objects are mapped to the right mathematical objects, the physical specification maps to the mathematical definitions

Physical mathematics

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Assumptions of Physics

Axiom 1.7 (Axiom of mixture). *The statistical mixture of two ensembles is an ensemble.*

Informal intuitive requirement

(something that makes sense to a physicist or an engineer)

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Assumptions of Physics

Axiom 1.7 (Axiom of mixture). *The statistical mixture of two ensembles is an ensemble. Formally, an ensemble space $\mathcal{E}$ is equipped with an operation $+ : [0,1] \times \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ called **mixing**, noted with the infix notation $p\mathsf{a} + \bar{p}\mathsf{b}$, with the following properties:*

- ***Continuity**: the map $+(p, \mathsf{a}, \mathsf{b}) \to p\mathsf{a} + \bar{p}\mathsf{b}$ is continuous (with respect to the product topology of $[0,1] \times \mathcal{E} \times \mathcal{E}$)*
- ***Identity**:* $1\mathsf{a} + 0\mathsf{b} = \mathsf{a}$
- ***Idempotence**:* $p\mathsf{a} + \bar{p}\mathsf{a} = \mathsf{a}$ *for all* $p \in [0,1]$
- ***Commutativity**:* $p\mathsf{a} + \bar{p}\mathsf{b} = \bar{p}\mathsf{b} + p\mathsf{a}$ *for all* $p \in [0,1]$
- ***Associativity**:* $p_1\mathsf{e}_1 + \bar{p}_1\left(\left(\overline{\frac{p_3}{\bar{p}_1}}\right)\mathsf{e}_2 + \frac{p_3}{\bar{p}_1}\mathsf{e}_3\right) = \bar{p}_3\left(\frac{p_1}{\bar{p}_3}\mathsf{e}_1 + \left(\overline{\frac{p_1}{\bar{p}_3}}\right)\mathsf{e}_2\right) + p_3\mathsf{e}_3$ *where* $p_1 + p_3 \leq 1$

Informal intuitive requirement

(something that makes sense to a physicist or an engineer)

Formal definition

(something a mathematician will find precise)

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Assumptions of Physics

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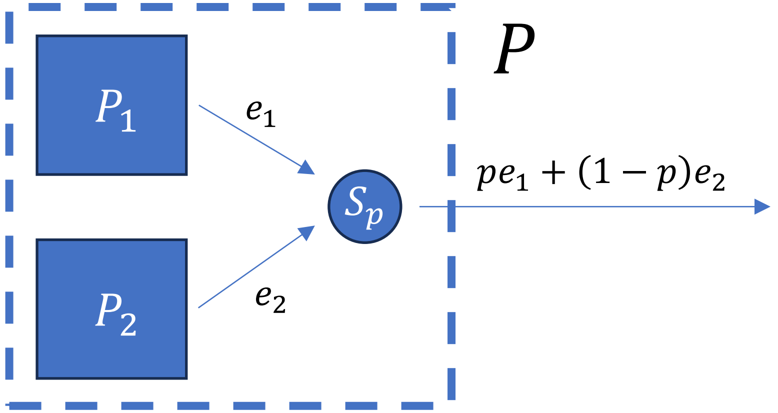
Justification. This axiom captures the ability to create a mixture merely by selecting between the output of different processes. Let e_1 and e_2 be two ensembles that represent the output of two different processes P_1 and P_2. Let a selector S_p be a process that outputs two symbols, the first with probability p and the second with probability $\bar{p}$. Then we can create another process P that, depending on the selector, outputs either the output of P_1 or P_2. All possible preparations of such a procedure will form an ensemble. Therefore we are justified in equipping an ensemble space with a mixing operation that takes a real number from zero to one, and two ensembles.

Given that mixing represents an experimental relationship, and all experimental rela-

Informal intuitive requirement
(something that makes sense to a physicist or an engineer)

Formal definition
(something a mathematician will find precise)

Show that the formal definition follows from the intuitive requirement



Clear idea of what is being modelled

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Ensemble Spaces

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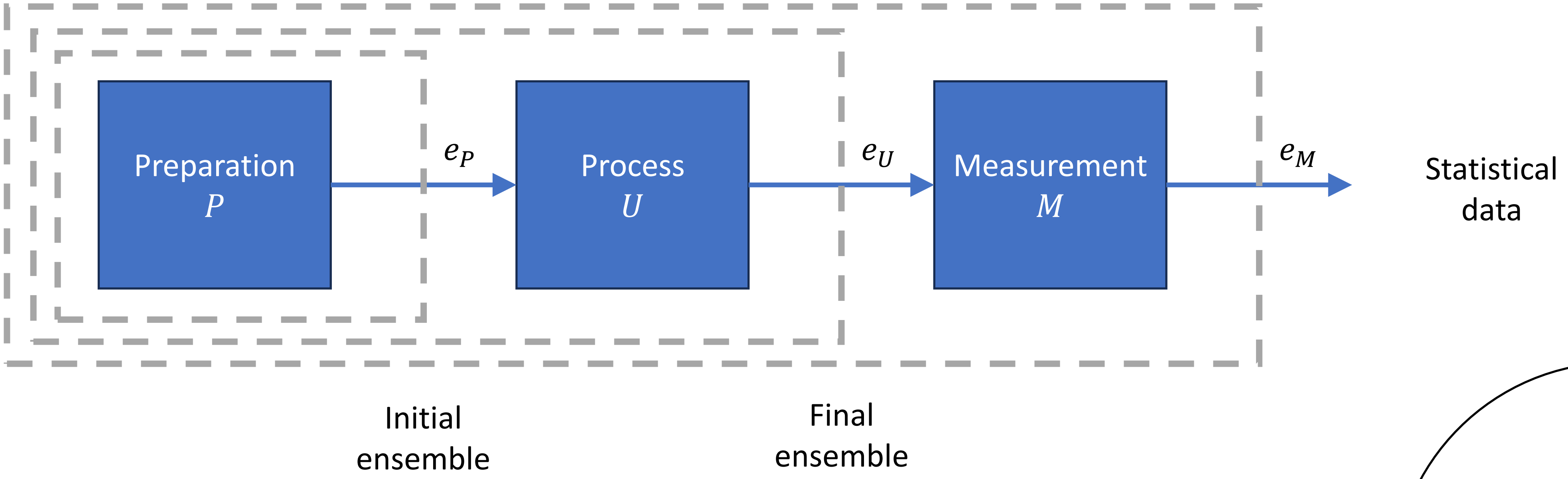
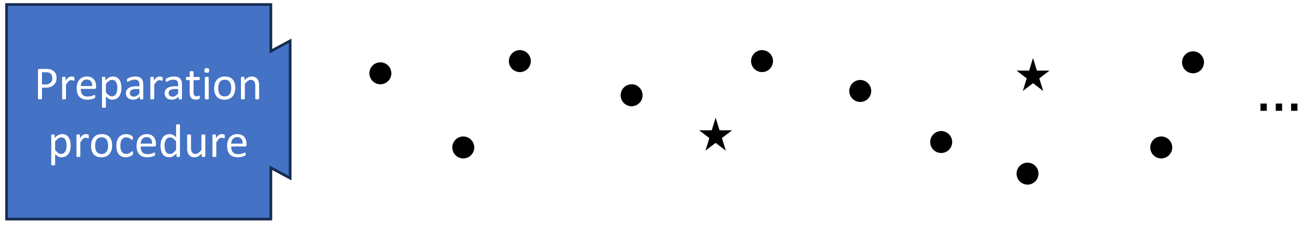
Principle of scientific reproducibility. Scientific laws describe relationships that can always be experimentally reproduced.

⇒ Scientific laws are relationships between ensembles





Ensemble of outputs

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Minimal requirements for an ensemble space

Axiom 1.4 (Axiom of ensemble). *The state of a system is represented by an* ***ensemble****, which represents all possible preparations of equivalent systems prepared according to the same procedure. The set of all possible ensembles for a particular system is an* ***ensemble***

Experimental verifiability ⇒ Topology

Responsible for all topological structures

Axiom 1.7 (Axiom of mixture). *The statistical mixture of two ensembles is an ensemble.*

Ensembles can be mixed ⇒ Convex structure

Responsible for all linear structures

Axiom 1.21 (Axiom of entropy). *Every element of the ensemble is associated with an* ***entropy*** *which quantifies the variability of the preparations of the ensemble. Formally, an*

Ensemble variability ⇒ Entropy

Responsible for all geometric structures

Still trying to find the right starting point for Poisson/commutator structure

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Some general results/constructions

Theorem 1.25 (Uniqueness of entropy). *The entropy of the coefficients $I(p,\bar{p})$ is the Shannon entropy. That is, $I(p,\bar{p}) = -\kappa\,(p\log p + \bar{p}\log\bar{p})$ where $\kappa > 0$ is the arbitrary multiplicative constant for the entropy. For a mixture of arbitrarily many elements, $I(\{p_i\}) = -\kappa\sum_i p_i \log p_i$.*

The maximal entropy increase $I(p,\bar{p})$ is uniquely determined, independently of physical theory

Ensembles embed into a vector space

Theorem 1.65 (Differences from a vector space). *Let $\mathsf{a} \in \mathcal{E}$ be an interior point and let $V = \{[r(\mathsf{b}-\mathsf{a})]\}$ be the set of equivalence classes of ensemble differences from a. Then V is a vector space under the scalar multiplication and addition.*

$$MS(\mathsf{a},\mathsf{b}) = S\left(\frac{1}{2}\mathsf{a} + \frac{1}{2}\mathsf{b}\right) - \left(\frac{1}{2}S(\mathsf{a}) + \frac{1}{2}S(\mathsf{b})\right)$$

Entropy increase during mixing generalizes Jensen-Shannon Divergence (pseudo-distance)

Hessian of the entropy generalizes Fisher-Rao metric from information geometry

$$g_{\mathsf{e}}(\delta\mathsf{e}_1, \delta\mathsf{e}_2) = -\frac{\partial^2 S}{\partial \mathsf{e}^2}(\delta\mathsf{e}_1, \delta\mathsf{e}_2).$$

Definition 1.85. *Let $\mathsf{e} \in \mathcal{E}$ be an ensemble and $A \subseteq \mathcal{E}$ a Borel set. The **fraction capacity** of A for e is the biggest fraction achievable with convex combinations of A. That is,* $\mathrm{fcap}_{\mathsf{e}}(A) = \sup(\mathrm{frac}_{\mathsf{e}}(\mathrm{hull}(A)) \cup \{0\})$.

Non-additive generalization of probability

Generalization of $S(\rho_U) = \log\mu(U)$

Definition 1.156. *Let $U \subseteq \mathcal{E}$ be the subset of an ensemble space. The **state capacity** of U is defined as* $\mathrm{scap}(U) = \sup(2^{S(\mathrm{hull}(U))})$ *if $U \neq \varnothing$ and* $\mathrm{scap}(U) = 0$ *otherwise.*

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Many open problems and conjectures listed in the book

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Assumptions of Physics

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ABOUT THE BOOK

The aim of the book is to show that the basic laws of physics can be derived from a handful of physical principles and assumptions. The project is run in the open, with the full history publicly available under version control, to allow for early feedback and discussion. Comments/questions/requests can be sent to my personal address

PUBLISHED VERSION (V3.0) LATEST VERSION (V3.0) CURRENT DRAFT SOURCE REPOSITORY

Bibtex information - All versions: - 3.0 - 2.0 - 1.0 - 0.3 - 0.2 - 0.1

Working chapter drafts: Ensemble spaces - Quantum mechanics

Book is open access/open source

Open problems and conjectures are included in the book

of $a_2 + b_2$. Therefore e is not multidecomposable. Since this applies to all decomposable ensembles e, $\mathcal{E}$ is monodecomposable. □

These definitions may be enough to prove that every finite-dimensional separately monodecomposable convex space is a simplex. For the infinite case, it would be nice to compare this characterization to a Choquet simplex.

Conjecture 4.33. *A finite-dimensional (i.e. there is a set of finitely many elements whose hull has non-empty interior) separately monodecomposable convex space is a simplex.*

Convex subsets and convex hull

In many cases, we will need to discuss the sets that contain all their possible mixtures. One typically distinguishes two cases. A set is convex if it allows all possible finite mixtures. This

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Open problems

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From classical mechanics to field theories

- Formally, the state changes from (q^i, p_i) to $(\phi(x), \Pi(x))$
  - The position takes the role of the index of the degree of freedom
  - The fields take the place of the coordinates

- To properly generalize results to classical field theory we need to understand
  - How do we define probability distribution on function spaces?
  - How do we define a symplectic form on function spaces?
  - How do we define a Shannon-like entropy on function spaces?
- These three need to be done in a consistent way which may help constrain the problem

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Topological convex spaces

- The axiom of ensemble guarantee a T_0 second countable topology
- The axiom of mixture tells us an ensemble space must be a convex space (i.e. allow convex combinations $pa + (1-p)b$ with $p \in [0,1]$)
- Mixture must be continuous ⇒ topological convex space
- The axiom of ensemble forces convex space to be embeddable in a vector space

- Can we prove a continuous embedding in a topological vector space?
- Can we prove the ambient topological vector space must be second countable?
  - Since it would make it metrizable, any ensemble space would admit a metric, regardless of the specific physical theory
- We can show that the ensemble space must be bounded along each direction: does it make it a bounded set?

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Entropy

- The axiom of entropy is able to recover the entropy univocally ONLY under the assumption that the space is orthogonally decomposable
  - I.e. every ensemble is the convex combination of two ensemble that, when mixed, maximize the entropy
- We lack a “three-point” relationship that would characterize the entropy more in general

- The pseudo-distance $MS(a, b)$ generalizes the Jensen-Shannon divergence, which is the square of a distance function: what requirements must the entropy satisfy such that $\sqrt{MS}$ is a distance?
- Given that $MS(a, b)$ is continuous and it is zero if and only if $a = b$, can we show that $MS(a, a_i) \to 0$ if and only if $a_i \to 0$?
- The open balls of the JSD are convex in both classical and quantum: what requirements must the entropy satisfy such that the open balls of MS are convex?
- If these conjectures are provable from physically justifiable axioms, it would mean every ensemble space is locally convex and metrizable, and therefore the entropy gives us enough structure to define calculus

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Lie algebraic structure

- In both classical mechanics and quantum mechanics, relationships between generators and observables are given by a Lie algebra
  - The Poisson brackets in classical mechanics
  - The commutators in quantum mechanics
- The symplectic nature of both spaces is tied to this structure, and with the identification of generators and observables
- We need a generalization that work on a generic ensemble space

- How can we define this structure on the ensembles, instead of the pure states?
- What exactly is the relationship to the entropy? In both classical and quantum mechanics, Hamiltonian evolution is connected to conservation of entropy
- How much of this structure is already contained in the previous axioms? In classical mechanics, we saw that the existence of the symplectic structure is equivalent to be able to define an entropy on the states themselves, as opposed to separately to each coordinate

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Recover the points from the ensembles

- In general, states do not necessarily correspond to ensembles. For example, given that position and momentum cannot be prepared with infinite precision, Dirac measures over the continuum are not ensembles
  - They also would have minus infinite entropy
- This means that, in general, the ensemble space does not contain its own extreme points
- What recipe can we use to construct the extreme points?
  - One attempt was to construct a lattice based on the notion of orthogonality as maximization of entropy during mixing, and use the Stone representation theorem for Boolean algebras, but this returns points that are too fine
  - Are the points obtained from the statistical variable (i.e. from linear functionals?)

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Non-additive measure theory

- Based on the current axioms, it is straight-forward to define non-additive measures for probability and count of states that reduce to the usual classical mechanics ones

Definition 4.173. *Let* $\mathsf{e} \in \mathcal{E}$ *be an ensemble and* $A \subseteq \mathcal{E}$ *a subset. The* ***fraction capacity*** *of* A *with respect to* e *is the biggest fraction achievable with convex combinations of* A. *That is,* $\mathrm{fcap}_{\mathsf{e}}(A) = \sup(\mathrm{frac}_{\mathsf{e}}(\mathrm{hull}(A)) \cup \{0\})$.

Definition 4.162. *Let* $A \subseteq \mathcal{E}$ *be a subset of an ensemble space. The* ***state capacity*** *of* A *is defined as* $\mathrm{scap}(A) = \sup(2^{S(\mathrm{hull}(A))} \cup \{0\})$.

- Can we find a non-additive integral that recovers the expectation of statistical variables? That is, if X are the extreme points, for every continuous linear functional F

$$E[F \mid \mathsf{e}] = F(\mathsf{e}) = (E)\int_X F(x) d\mathrm{fcap}_{\mathsf{e}}|_X$$

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