Quantum probability and quantum information theory require a novel approach to measure theory

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Assumptions of Physics

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Lead a project called Assumptions of Physics

https://assumptionsofphysics.org/

Find a set of minimal physical assumptions from which the laws can be rederived

Reverse Physics: Start with the equations,
reverse engineer physical assumptions/principles

What are the basic concepts/idealizations
behind the different physical theories?

Physical Mathematics: Start from scratch and
rederive everything from physical requirements

Which mathematical structures (or which parts) are physical?

Goal and Outline

- What exactly is the difference between classical and quantum mechanics?
  - Can we find a key feature that is physically meaningful, is significant in an obvious way, and is sufficient to imply other features?

- Outline
  - Logical structure is the same between classical and quantum mechanics
  - Entropic/geometric/probabilistic structures form a joint structure, different between classical and quantum mechanics
  - The difference is in how states are counted
    - Quantum states do not describe different cases at-all-else-being equal
    - Need new type of measure theory to bring that to light

Is the difference between classical and quantum mechanics due to different rules of logic?

NO: they implement the same logical structure.

On the Common Logical Structure of Classical and Quantum Mechanics

Andrea Oldofredi, Gabriele Carcassi, Christine A. Aidala
Erkenntnis (2022)

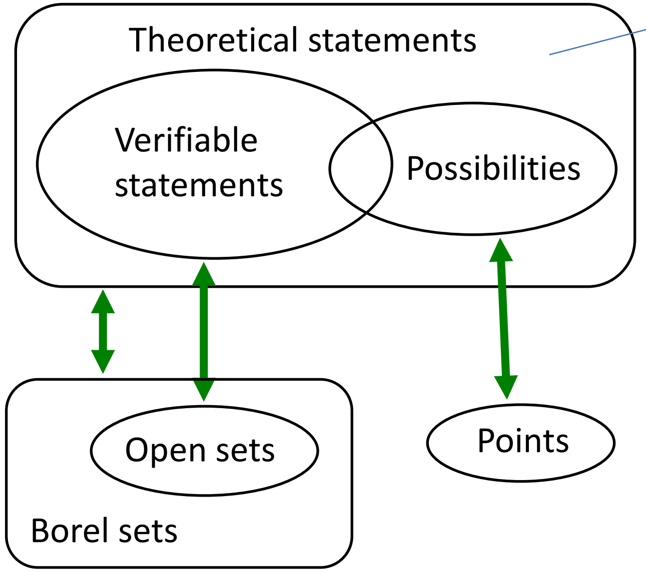
In-depth comparison

Assumptions of Physics (Open Access Book)

Gabriele Carcassi, Christine A. Aidala
https://doi.org/10.3998/mpub.12204707

Develop a common logic framework for all scientific theories

Topology and the logic of experimental verifiability



Statements formally associated with an experimental test

	s_1	Test Result
T		SUCCESS (in finite time)
T		UNDEFINED
F		UNDEFINED
F		FAILURE (in finite time)

$int(A)$ corresponds to the verifiable part of a statement

∂A corresponds to the undecidable part of a statement

$ext(A)$ corresponds to the falsifiable part of a statement

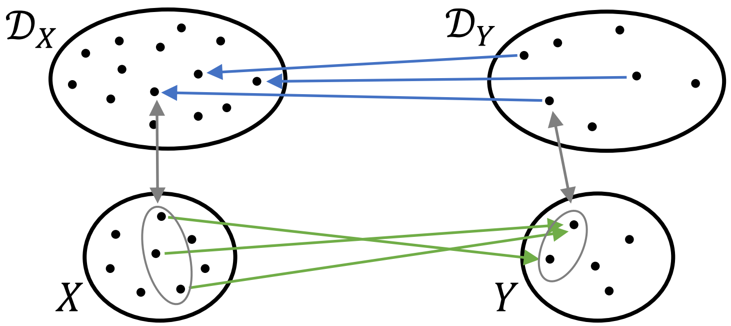
Experimental verifiability ⇒ topology and σ-algebras (foundation of geometry, probability, …)

Open set $(509.5, 510.5)$ ⟺ Verifiable "the mass of the electron is 510 ± 0.5 KeV"

Closed set $[510]$ ⟺ Falsifiable "the mass of the electron is exactly 510 KeV"

Borel set $\mathbb{Q}$ ($int(\mathbb{Q}) \cup ext(\mathbb{Q}) = \emptyset$) ⟺ Theoretical "the mass of the electron in KeV is a rational number" (undecidable)

Inference relationship $r: \mathcal{D}_Y \to \mathcal{D}_X$ such that $r(s) \equiv s$

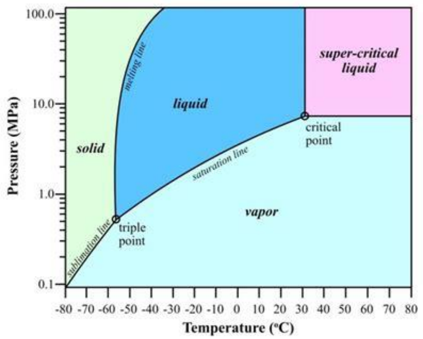


Inference relationship
⟺
Causal relationship

Relationships must be topologically continuous

Causal relationship $f: X \to Y$ such that $x \preccurlyeq f(x)$

Topologically continuous consistent with analytic discontinuity on isolated points



Phase transition ⟺ Topologically isolated regions

Perfect map between math and physics

⇒ Logic/topology/σ-algebra are different aspects of the same structure, and they follow the same rules in CM and QM

Where is the difference?

Rules of probability?

Rules of information theory?

Rules of geometry?

$\mu(U)$

Symplectic manifold

ρ_U uniform over U

Classical probability

Information theory

$$\rho_U(x) = \frac{1}{\mu(U)}$$

Hamiltonian mechanics is conservation of information entropy

Gabriele Carcassi, Christine A. Aidala
https://doi.org/10.1016/j.shpsb.2020.04.004

$$H(\rho_U) = \log \mu(U)$$

Geometry/probability/information theory are different aspects of the same structure

No other structure!
Difference must be here.

$\langle\psi|\phi\rangle$

Projective Hilbert space

$$\rho = \frac{1}{2}\rho_\psi + \frac{1}{2}\rho_\phi$$

Quantum probability

Quantum information theory

$$p(\psi|\phi) = |\langle\psi|\phi\rangle|^2$$

$$H(\rho) = H\left(\frac{1+\sqrt{p}}{2}, \frac{1-\sqrt{p}}{2}\right)$$

Easiest way to show the difference?

Quantum information theory:
entropy cannot be negative but conditional entropy can

Classical information theory (discrete):
neither entropy nor conditional entropy can be negative

Quantum mechanics “mixes” discrete and continuous features

Classical information theory (continuous):
both entropy and conditional entropy can be negative

Neither classical/quantum can fully contain (and only contain) the other

⇒ rules of geometry/probability/information theory are different in classical and quantum mechanics

But how EXACTLY are they different?

Note that classical geometry/probability/information theory are based on the notion of measure – no measure underneath quantum mechanics

OK!

No use of measure theory

$$p(k|P,M) = \mathrm{tr}(\rho E_k)$$

Maybe measure theory is the problem?

Uses measure theory

KO!

"Classical" hidden variables

$$p(k|P,M) = \int_\Lambda p(k|\lambda)\rho_\lambda(\lambda)d\mu(\lambda)$$

However, we do use measures on preparations and on measurements

$$\rho = \int_X dx\rho_x(x)|\psi_x\rangle\langle\psi_x|$$

$$p(U_X) = \mathrm{tr}(\rho I_{U_X})$$

WHEN is measure theory a problem?

What IS measure theory anyway?

Measure theory defines "how big a set is", "how many elements there are in a set", "how we count"

$$\mu(U) \to [0, +\infty]$$

Countable additivity is a fundamental axiom of measure theory

Disjoint sets

$$\mu(\cup_i U_i) = \sum_i \mu(U_i)$$

A density is a ratio between two measures

$$\rho = \frac{d\mu}{dx}$$

Units of μ over units of x

Radon–Nikodym derivative

What do we use measures for?

Probability $p(U_\Sigma)$: how likely "$x \in U_\Sigma$" is true

Physical quantities $x(U_X)$: size of the set in units

Used both in CM and QM

therefore

System configurations $\mu(U_S)$: state count

Present in CM, absent in QM

$$\rho = \int_X dx\rho(x)|\psi_x\rangle\langle\psi_x|$$

$$p(k|P,M) = \int_\Lambda p(k|\lambda)\rho_\lambda(\lambda)d\mu(\lambda)$$

Invalid in QM

$$p(U_X) = \mathrm{tr}\left(\rho I_{U_X}\right)$$

Valid in QM

Quantum mechanics does not provide a measure to count states!

How can we count states in QM?

In classical mechanics, count of states and entropy are linked by

$$H(\rho_U) = \log \mu(U)$$

Suppose we keep this relationship in quantum mechanics.

What type of “measure” do we get?

	Single point		Finite continuous range	
	$\mu(U)$	$\log \mu(U)$	$\mu(U)$	$\log \mu(U)$
Counting measure $\mu(U) = \#U$ Number of points	1	0	$+\infty$	$+\infty$
Lebesgue measure $\mu([a,b]) = b - a$ Interval size	0	$-\infty$	$< \infty$	$< \infty$
"Quantized" measure $\mu(U) = 2^{H(\rho_U)}$ Entropy over uniform distribution	1	0	$< \infty$	$< \infty$

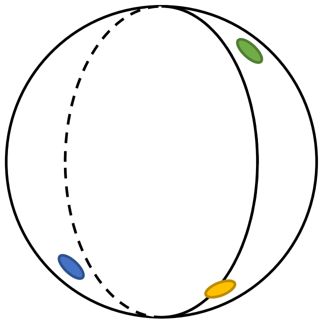
1. Single point is a single case (i.e. $\mu(\{\psi\}) = 1$)
2. Finite range carries finite information (i.e. $\mu(U) < \infty$)
3. Measure is additive for disjoint sets (i.e. $\mu(\cup_i U_i) = \sum_i \mu(U_i)$)

Pick two!

Discrete classical mechanics (counting measure) picks 1 and 3
Continuum classical mechanics (Lebesgue measure) picks 2 and 3
Quantum mechanics picks 1 and 2

What is quantized? The count of states!

In quantum mechanics



Additive on orthogonal states

$$1 + 1 = 2$$

Measure not additive

$$1 + 1 < 2$$

Measure not monotonic

$$\overset{1+1}{2} + 1 < 2$$

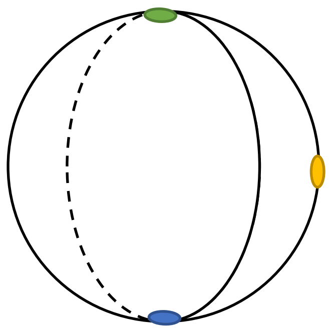
Quantum mechanics requires a new approach to measure theory (which is the basis of probability theory, information theory, theory of integration, differential geometry, …)

ONLY

This is the source of “quantum weirdness”

What is the physical meaning of this?

Why do $|z^+\rangle$ and $|z^-\rangle$ correspond to two cases,
but $|z^+\rangle$ and $|x^+\rangle$ correspond to fewer cases?



The state of a system is never completely independent from the environment

There is a cat inside a box $\Rightarrow$ the box is not on the surface of the sun

Preparing (or measuring) $|z^+\rangle$ and $|x^+\rangle$ implies different boundary conditions (i.e. different preparation or measurement setup)
$\Rightarrow$ they are not two cases all else being equal

Contextuality $\Leftrightarrow$ non-additive measure

Why should we be interested in these “quantized measures”? They are equivalent to what we already have in QM!

Discrete measure and Lebesgue measure are (in a sense) unique

Are quantized measures (in some sense) unique as well?

In a field theory, one (independent) DOF at each point (i.e. local commutation relationship)

⇒ Measure of volume “counts” independent DOFs in a region

Lebesgue measure gives us regions with measure less than one ⇒ less than one DOF

To quantize space-time, we may need a quantized measure

Conclusion

- Both classical and quantum mechanics are composed of two macro parts
  - Logic/Topology/σ-algebra keeps track of the logic of experimentally verifiable statements
  - Geometry/Probability/Information Theory keeps track of the count of states
- The first part has the same structure (no difference in logic)
- The second part (as formulated right now) has a different structure
  - Discrete, continuous and quantum spaces each have a different structure precisely because the count of states is defined differently
- We should study these "quantized measures" on their own merit
  - See if they are unique
  - Likely needed to quantize space-time