The Assumptions of Physics project

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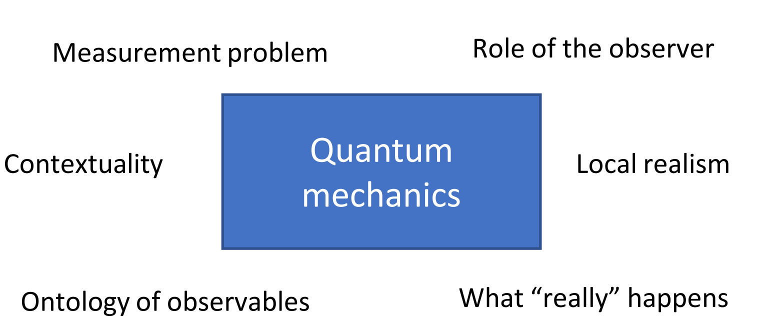
Led by Gabriele Carcassi + Christine A. Aidala
https://assumptionsofphysics.org/

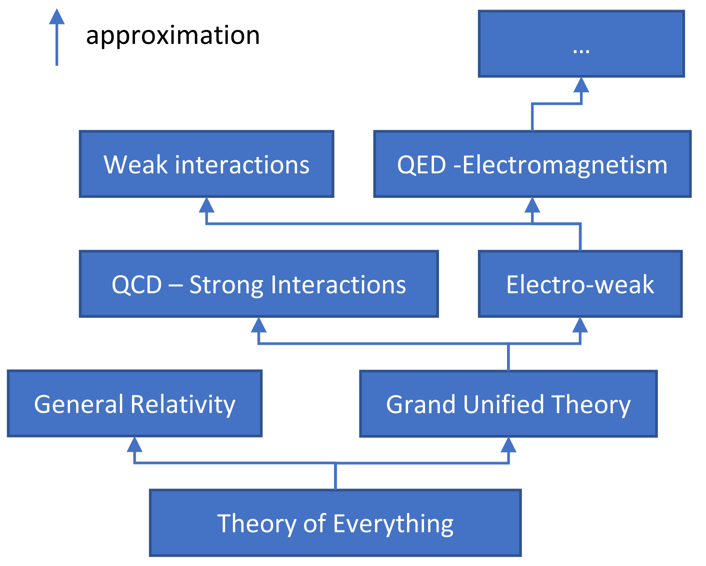
Different approach to the foundations of physics

Typical approaches

Construct interpretations

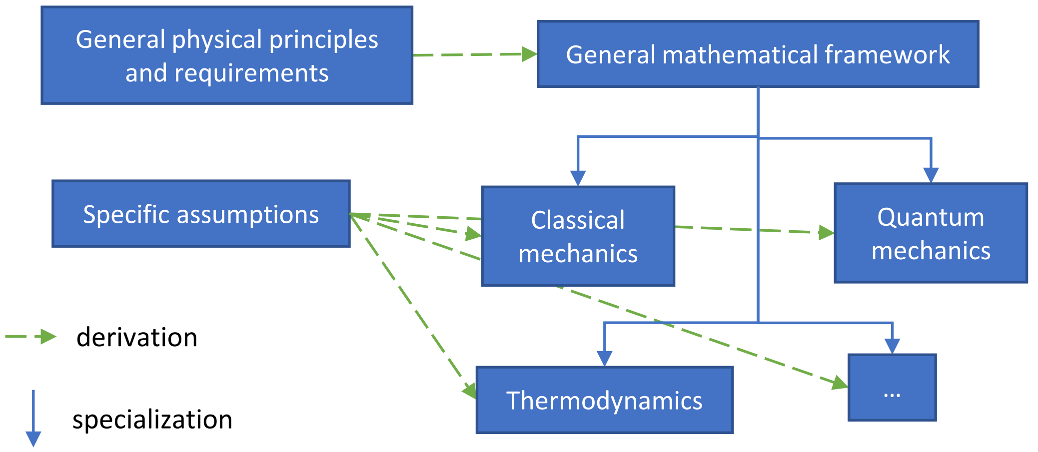


Find ultimate theory



Our approach

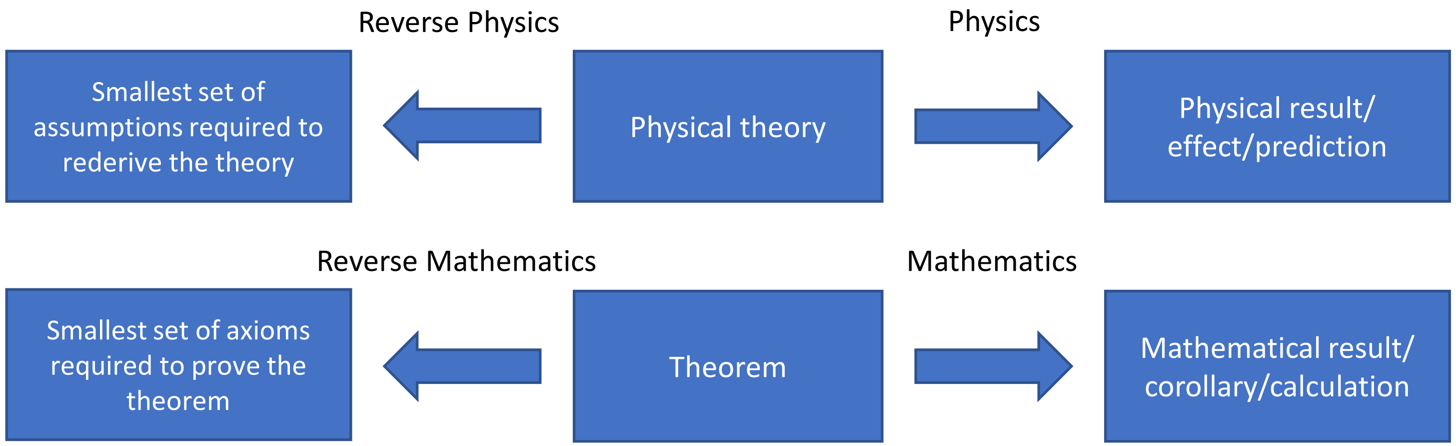
Find a minimal set of physical assumptions from which to rigorously rederive the laws



- Clarify our assumptions
- Put physics back at the center of the discussion
- Give science sturdier mathematical grounds
- Foster connections between different fields of knowledge
- Provide a way to pose deep questions

Reverse physics:

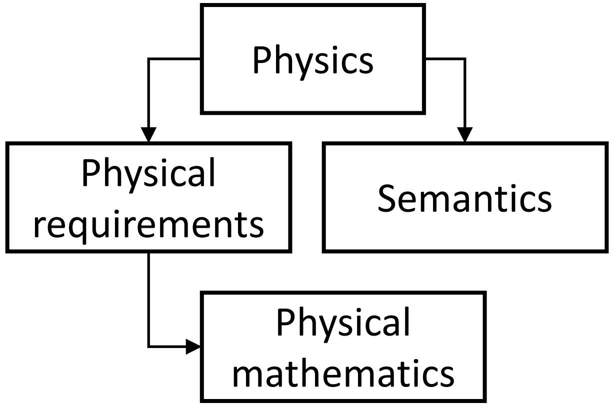
Start with the equations, reverse engineer physical assumptions/principles



Goal: “Elevate” the discussion from mathematical constructs to physical principles, assumptions and requirements

Physical mathematics:

Start from scratch and rederive all mathematical structures from physical requirements



Goal: Construct a perfect one-to-one map between mathematical and physical objects

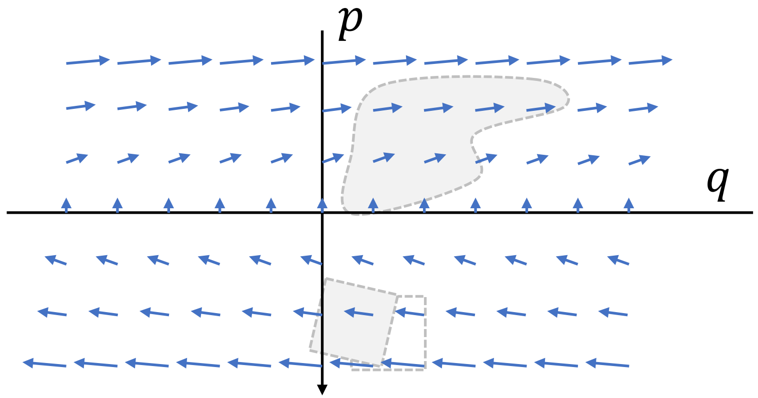
Reverse physics

Reverse Physics: From Laws to Physical Assumptions

Gabriele Carcassi, Christine A. Aidala
Foundations of Physics (2022) 52:40
https://arxiv.org/abs/2111.09107

7 equivalent characterizations of Hamiltonian mechanics

one DOF



(1) Hamilton's equations

$$S^q = \frac{dq}{dt} = \frac{\partial H}{\partial p}$$

$$S^p = \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

(2) Divergenceless displacement

$$div(S^a) = \frac{\partial S^q}{\partial q} + \frac{\partial S^p}{\partial p} = 0$$

(3) Area conservation ($|J| = 1$)

$$dQdP = |J|dqdp$$

(4) Deterministic and reversible evolution

Area conservation ⟺ state count conservation

⟺ deterministic and reversible evolution

(5) Deterministic and thermodynamically reversible evolution

$$S = k_B \log W$$

Area conservation ⟺ entropy conservation

⟺ thermodynamically reversible evolution

(6) Information conservation

$$I[\rho(t + dt)] = I[\rho(t)] - \int \rho \log |J| \, dqdp$$

(7) Uncertainty conservation

$$|\Sigma(t + dt)| = |J||\Sigma(t)||J|$$

for peaked distributions

A full understanding of classical mechanics means understanding these connections

Reversing the principle of least action

$$\nabla \cdot \vec{S} = 0$$

No state is "lost" or "created" as time evolves

$$\vec{S} = -\nabla \times \vec{\theta}$$

(Minus sign to match convention)

$$\mathcal{S}[\gamma] = \int_\gamma L dt = \int_\gamma \vec{\theta} \cdot d\vec{\gamma}$$

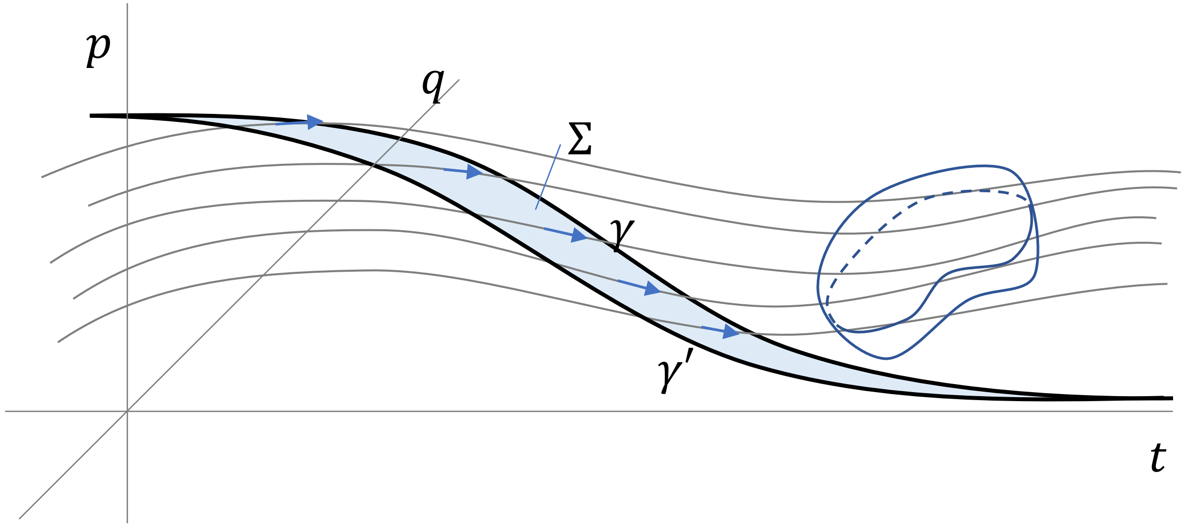
The action is the line integral of the vector potential (unphysical)

Variation of the action

$$\delta \mathcal{S}[\gamma] = \oint_{\partial\Sigma} \vec{\theta} \cdot d\vec{\gamma}$$

$$= -\iint_\Sigma \vec{S} \cdot d\vec{\Sigma}$$

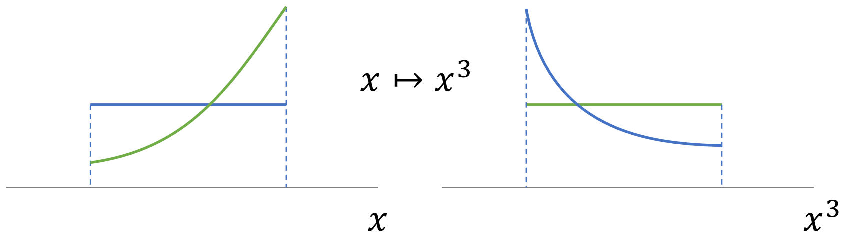
Gauge independent, physical!



https://arxiv.org/abs/2208.06428

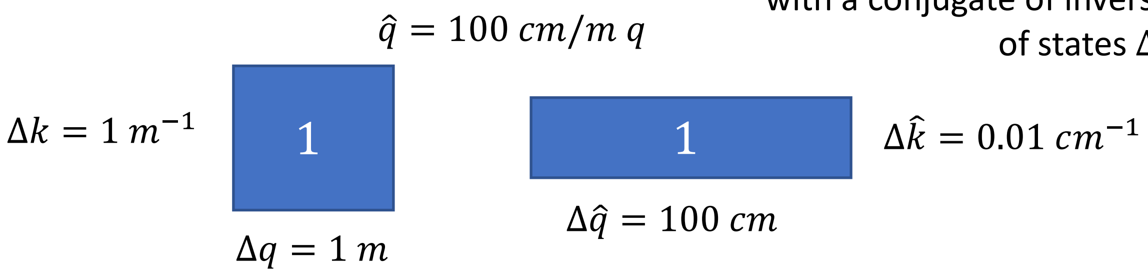
Variation of the action measures the flow of states (physical).
Variation = 0 ⇒ flow of states tangent to the path.

Reversing phase-space



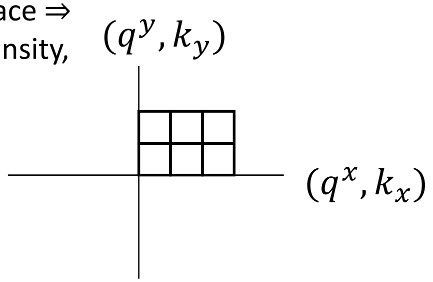
Density, entropy, uniform distributions NOT in general coordinate invariant

Each unit variable (i.e. coordinate) paired with a conjugate of inverse units: number of states $\Delta q \Delta k$ is invariant



Phase space (symplectic) structure is the only one that supports coordinate invariant density, entropy, state count

Independence of DOFs ⇒ independence of units ⇒ orthogonality in phase-space ⇒ invariant marginals (for density, entropy, state count)



Total number of states = product of number of cases in each independent DOF

Hamiltonian mechanics preserves count of states and DOF independence over time

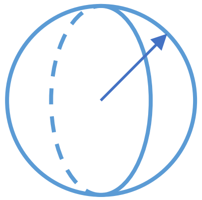
$$\omega_{ab} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \delta^i_j$$

Symplectic form (geometric structure of phase space)

Areas/possibilities in each DOF

Orthogonality/independence across DOFs

Directional DOF



2-sphere the only symplectic manifold

Only 3 spatial dimensions are possible

Invariance at equal time (relativity) gives us the structure of phase space

Massive particles under potential forces

Kinematic equivalence assumption: the state can be recovered from space-time trajectories

$$\frac{\partial p_i}{\partial \dot{q}^j} \equiv m g_{ij}$$

Must be a linear transformation in terms of coordinates

Fixes the units

Integration of the previous expression

$$p_i = m g_{ij} \dot{q}^j + q A_i(q^k)$$

$$\dot{q} = \frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} = \frac{1}{m} g^{ij} (p_j - q A_j)$$

$$H = \frac{1}{2m} (p_i - q A_i) g^{ij} (p_j - q A_j) + q V(q^k)$$

Hamiltonian for massive particles under potential forces

Mass quantifies number of states per unit of velocity

Higher mass ⇒ more states to go through ⇒ harder to accelerate

BUT

Zero mass ⇒ zero states within finite range of velocity ⇒ velocity is fixed

The laws themselves are highly constrained by simple assumptions

Relativistic mechanics

Relativistic aspects without space-time (i.e. without Kinematic Equivalence)

potential of the displacement

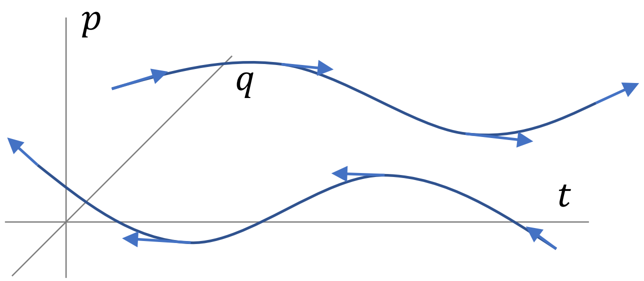
$$\theta = [p^i, -H, 0, 0]$$

energy-momentum co-vector

$$F = \frac{d\hat{t}}{dt}\hat{F} = \frac{d}{dt}\left(\hat{m}\frac{dt}{d\hat{t}}\frac{dx}{dt}\right) = \frac{d}{dt}\left(m\frac{dx}{dt}\right)$$

rest mass scaled by time dilation

Classical antiparticles



$$\frac{dt}{ds} = \frac{\partial \mathcal{H}}{\partial E}$$

Affine parameter anti-aligned with time:
parameterization "goes back" in time

Geometric connections

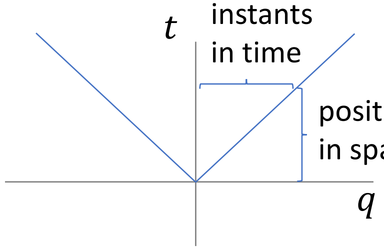
Force quantifies states charted by position across DOF

$$\theta = [m g_{\alpha\beta} u^\beta + q A_\alpha, 0]$$

Metric tensor quantifies states charted by position and velocity

$$\omega_{ab} = \begin{bmatrix} -m G_{\alpha\beta\gamma} u^\gamma + q F_{\alpha\beta} & g_{\alpha\beta} \\ -g_{\alpha\beta} & 0 \end{bmatrix}$$

No clear idea what $G_{\alpha\beta\gamma}$ is…
Inertial forces?



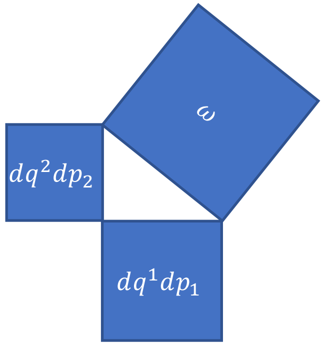
Constant c converts state count between space and time

Lorentzian relativity is the only "correct" one

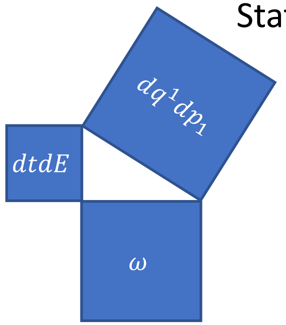
Minkowski signature appears on the extended phase space

$$\omega = dq^1 dp_1 + dq^2 dp_2 \qquad \omega = dq^1 dp_1 - dt dE$$

$$dq^1 dp_1 = \omega + dt dE$$

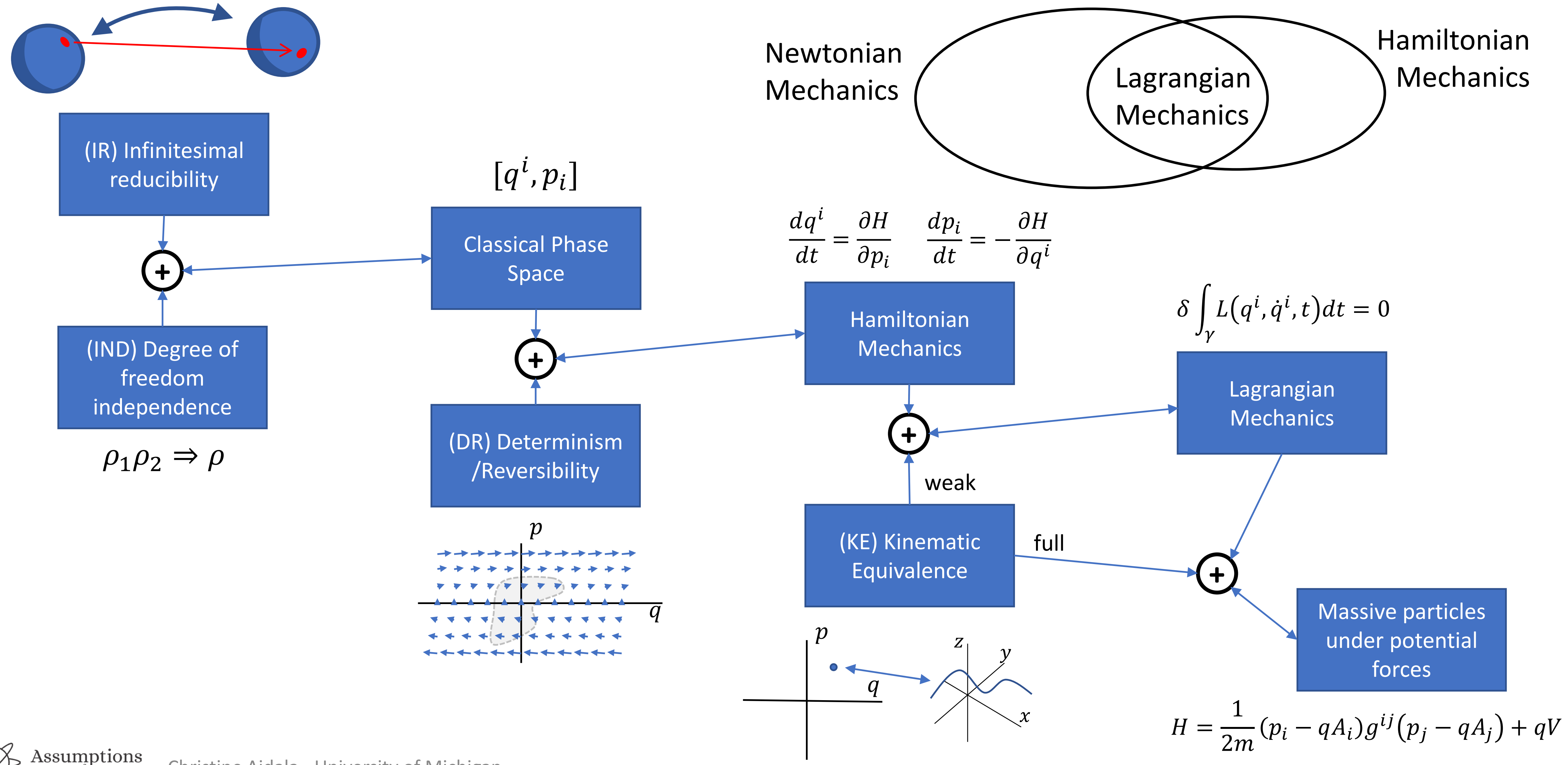


Indep DOF are orthogonal



States are counted at equal time: temporal DOF orthogonal to ω

Assumptions of classical mechanics



Reverse physics: Understanding links between theories

Deterministic and reversible evolution

$\Rightarrow$ existence and conservation of energy (Hamiltonian)

Why?

Stronger version of the first law of thermodynamics

Deterministic and reversible evolution

$\Rightarrow$ past and future depend only on the state of the system

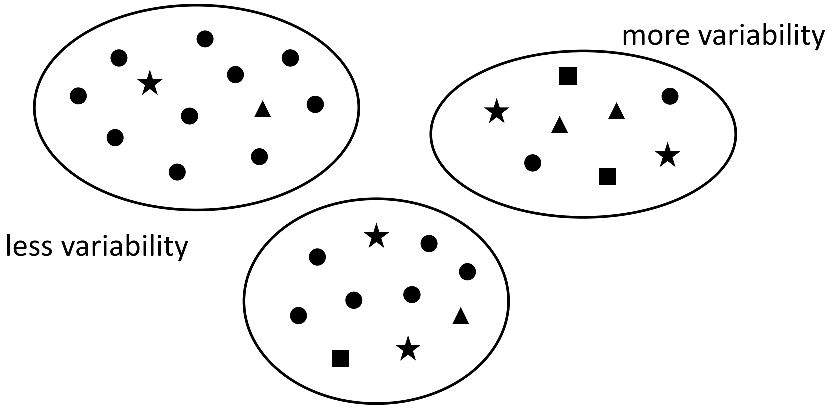
$\Rightarrow$ the evolution does not depend on anything else

$\Rightarrow$ the system is isolated

First law of thermodynamics!

$\Rightarrow$ the system conserves energy

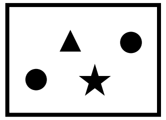
Shannon entropy as variability

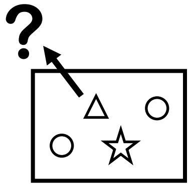


Shannon entropy quantifies the variability of the elements within a distribution

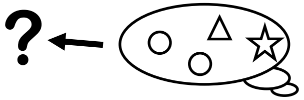
Meaning depends on the type of distribution

Statistical distribution:
variability of what is there





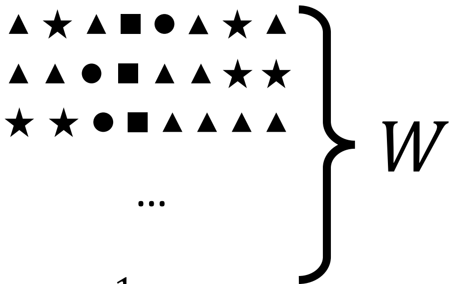
Probability distribution:
variability of what could be there



Credence distribution:
variability of what one believes to be there

$-\sum p_i \log p\_i$ only indicator of variability that satisfies simple requirements

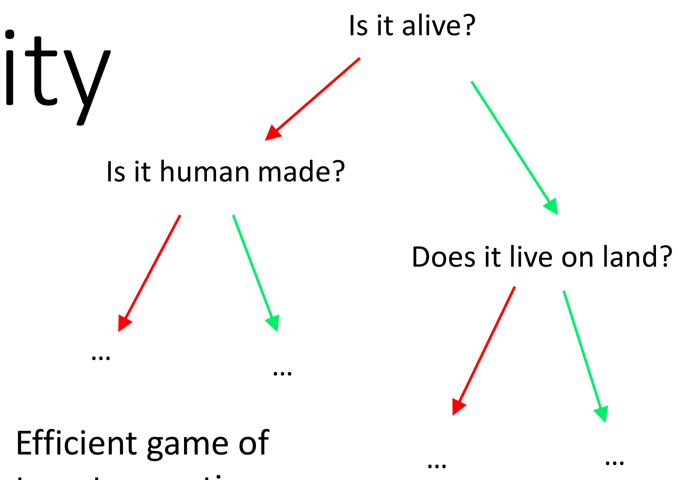
1) Continuous function of p_i only
2) Increase when number cases increase
3) Linear in p_i



$$\frac{1}{N}\log W \approx -\sum p_i \log p_i$$

More variability, more permutations

Variability is also quantified by the logarithm of the number of possible permutations per element



Efficient game of twenty questions

More variability, more questions

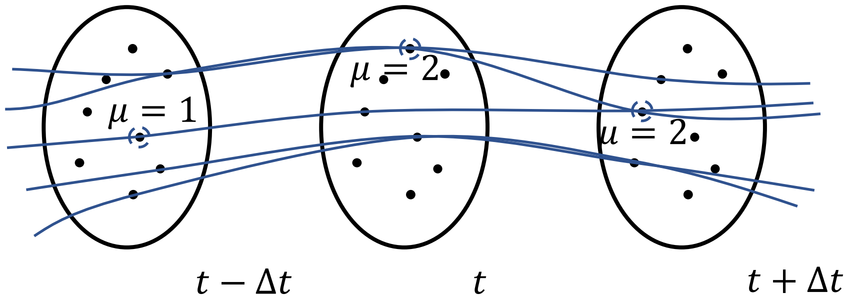
Variability is quantified by the expected minimum number of questions required to identify an element

More variability for a distribution at equilibrium, more fluctuations, more physical entropy

This characterization works across disciplines

Eur. J. Phys. 42, 045102 (2021)

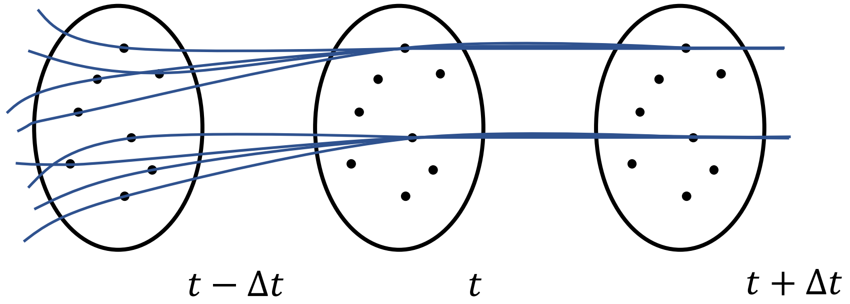
Entropy as logarithm of evolution count



$$P(s_{t+\Delta t}|s_t) = \frac{\mu(s_t \cap s_{t+\Delta t})}{\mu(s_t)}$$

Determinism: evolutions cannot split $\mu\big(s(t+\Delta t)\big) \geq \mu\big(s(t)\big)$

Reversibility: evolutions cannot merge $\mu\big(s(t+\Delta t)\big) \leq \mu\big(s(t)\big)$



For a deterministic process

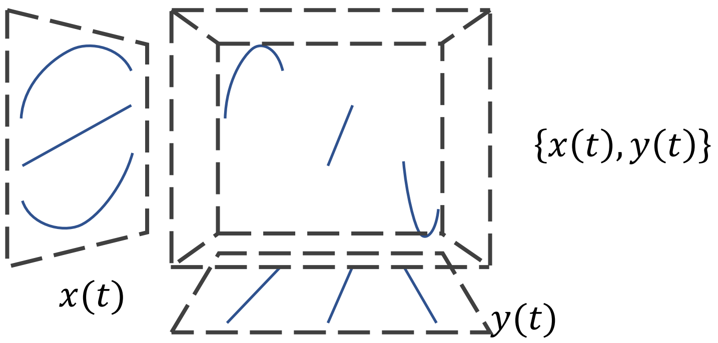
$$\mu\big(s(t+\Delta t)\big) \geq \mu\big(s(t)\big)$$

(equal if reversible)
(maximum at equilibrium)

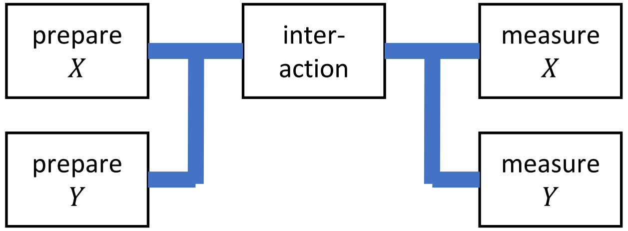
Process entropy defined as $S = \log \mu$
The log of the count of evolutions per state

It is additive for independent systems
$S = S_1 + S_2$

For a deterministic process
$S\big(s(t+\Delta t)\big) \geq S\big(s(t)\big)$
(equal if reversible)
(maximum at equilibrium)



System independence:
evolutions of the composite are the product of individual systems



The flow of the displacement field and Shannon entropy also agree with this definition
(respectively, for det/rev evolution and at equilibrium)

“Reversing” thermodynamics

Assume states are equilibria of faster scale processes

Assume states identified by extensive properties

Assume one of these quantities is energy U

$$S(U, x^i)$$

Existence of equation of state

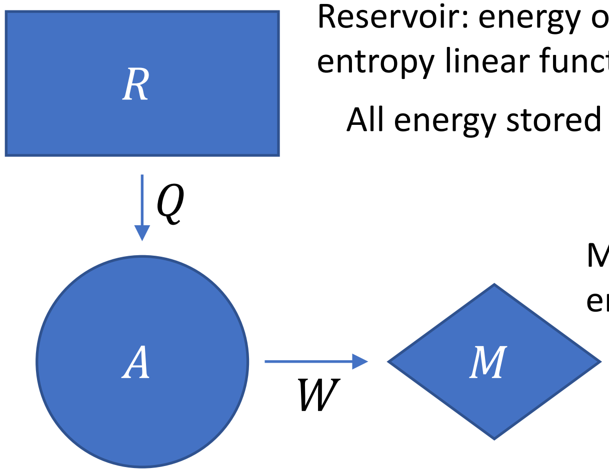
$$\beta = \frac{1}{k_B T} = \frac{\partial S}{\partial U} \quad \text{and} \quad -\beta X_i = \frac{\partial S}{\partial x^i}$$

Define intensive quantities

$$dS = \frac{\partial S}{\partial U} dU + \frac{\partial S}{\partial x^i} dx^i = \beta dU - \beta X_i dx^i$$
$$k_B T dS = dU - X_i dx^i$$
$$dU = T(k_B dS) + X_i dx^i$$

Recover usual relationships

Study interplay of changes of energy and entropy



Reservoir: energy only state variable, entropy linear function of energy

All energy stored in entropy

Mechanical system: same entropy for all states

No energy stored in entropy

$$\beta = \frac{1}{k_B T} = 0$$

$$\Delta U = 0 = \Delta U_A + \Delta U_R + \Delta U_M$$
$$= \Delta U_A - Q + W$$

Recover first law

$$0 \leq \Delta S = \Delta S_A + \Delta S_\mathrm{R} + \Delta S_\mathrm{M}$$
$$= \Delta S_A + \beta_R \Delta U_R + 0 = \Delta S_A + \frac{-Q}{k_B T_R}$$

Recover second law

First law recovered from existence and conservation of Hamiltonian

Second law recovered from definition of entropy as count of evolutions

3rd law and principle of maximal description

Can be formulated as:

Every substance has a finite positive entropy, but at the absolute zero of temperature the entropy may become zero, and does so become in the case of perfect crystalline substances.

G. N. Lewis and M. Randall

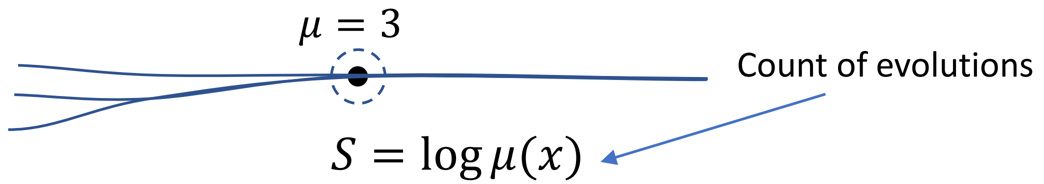
Better “special case” than “crystalline substance”

Null state $\emptyset$: system is absent (e.g. gas with zero particles)

$$A = A \cup \emptyset$$

$$S_A = S_{A\cup\emptyset} = S_A + S_\emptyset \Rightarrow S_\emptyset = 0$$

Entropy for the null state of any system must be 0



Count of evolution can’t be < 1 therefore S can’t be < 0

3rd law can be restated as:

No state can describe a system more accurately than stating the system is not there in the first place.

Principle of maximal description

We can reformulate the 3rd law of thermodynamics as a logical necessity

Classical uncertainty principle

NB: Quantum mechanics has a lower bound on entropy: zero for a pure state.

Classical mechanics has no lower bound on entropy ⇒ violates third law!

What happens if we impose one?

Take the space of all possible distributions $\rho(q, p)$ and order them by Shannon/Gibbs entropy

Consider all distributions with the same entropy S_0. They satisfy

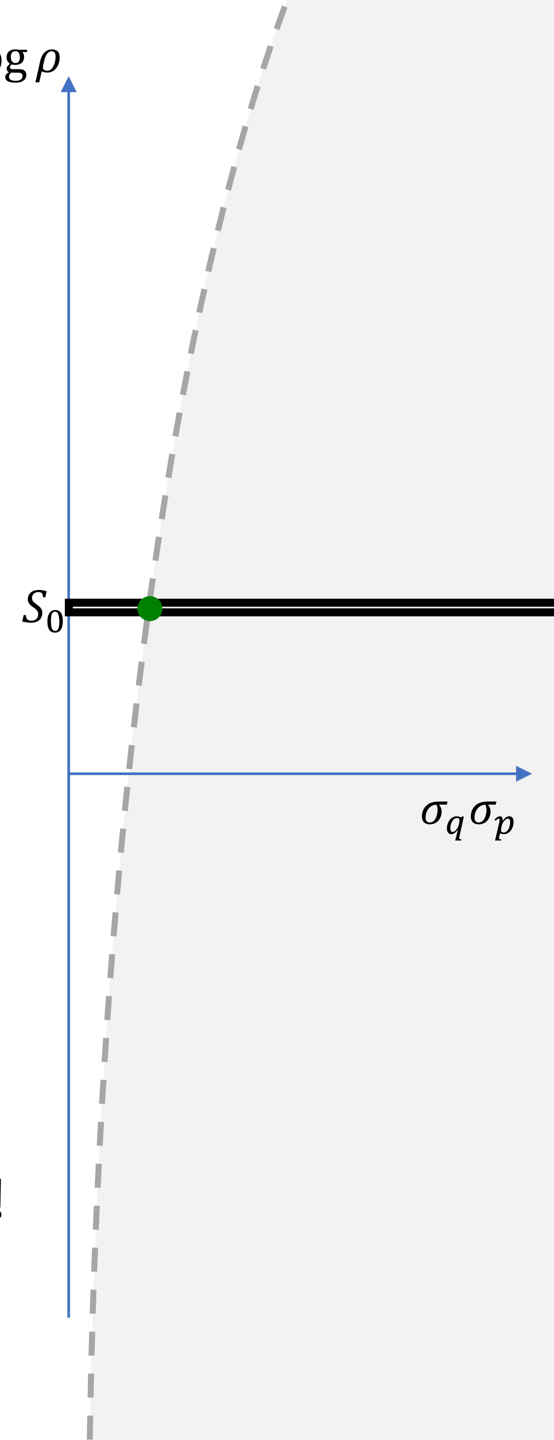
$$\sigma_q \sigma_p \geq \frac{e^{S_0}}{2\pi e}.$$

Equality for independent Gaussians

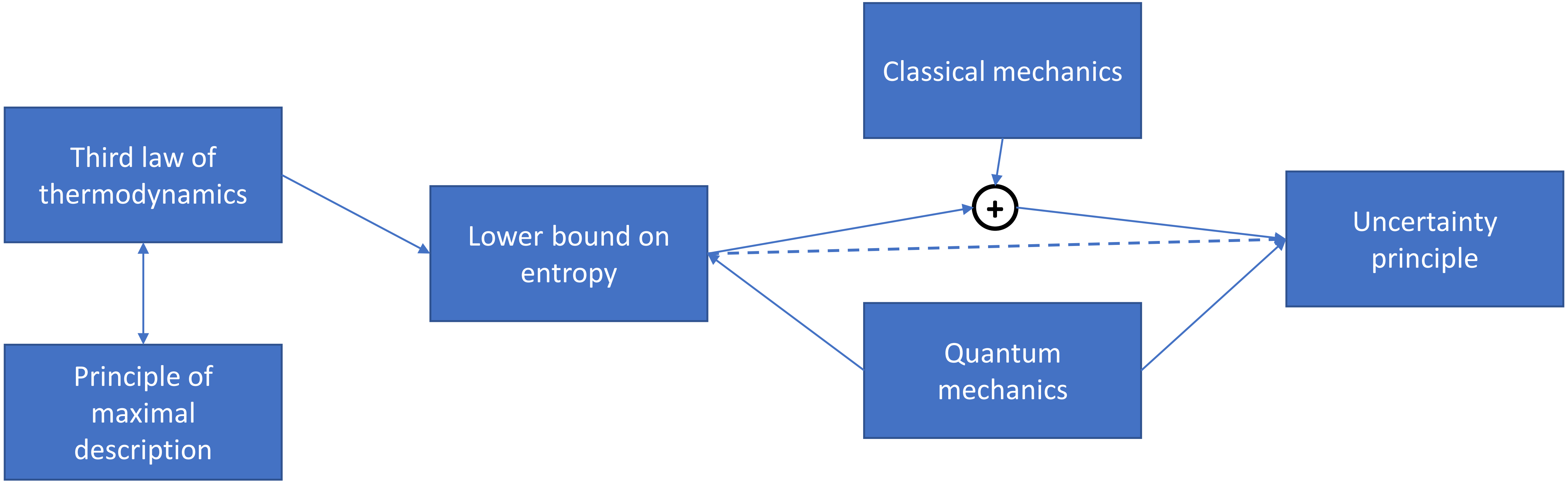
Lower bound on entropy ⇒ lower bound on uncertainty

Don't need the full quantum theory to derive the uncertainty principle: *only the lower bound on entropy*

The difference is that in classical mechanics we can prepare ensembles with arbitrarily low entropy…in contradiction with the third law of thermodynamics!!



3rd law of thermodynamics and uncertainty principle



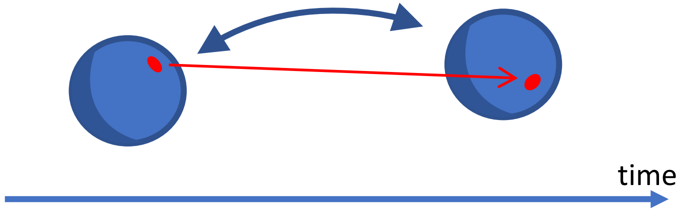
No state can describe a system more accurately than stating the system is not there in the first place

We can understand the uncertainty principle as a consequence of the third law

Can we understand the rest of quantum mechanics in the same way?

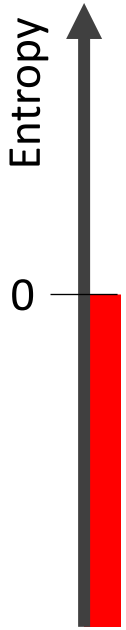
Quantum mechanics as irreducibility

Classical

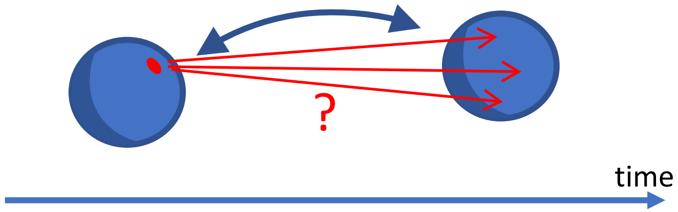


Can prepare ensembles at arbitrarily low entropy: we can study arbitrarily small parts

We always have access to the internal dynamics



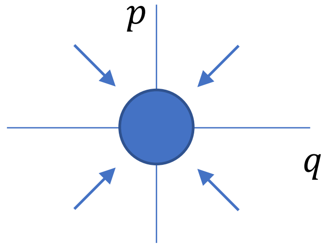
Quantum



Entropy is bounded at zero: we cannot study parts

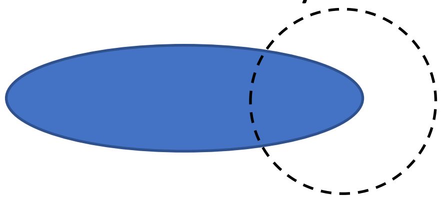
No access to the internal dynamics

Minimum uncertainty



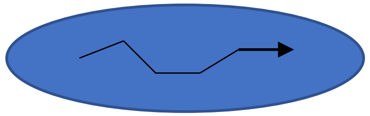
Can't squeeze ensemble arbitrarily

Non-locality



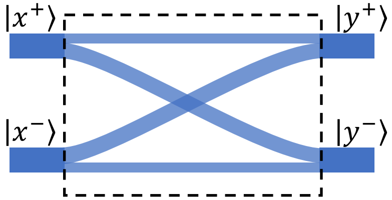
Can't refine ensembles ⇒
Can't interact with parts

Superluminar effects that can't carry information



Can't refine ensembles ⇒
Can't extract information

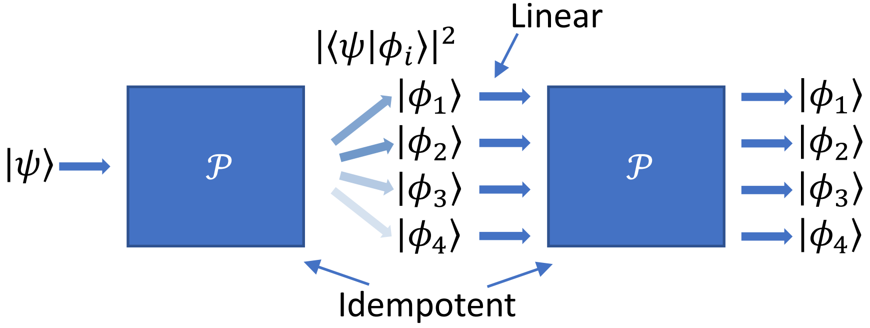
Probability of transition



$p(x^+|y^-) = p(y^-|x^+)$

Symmetry of the inner product

Projections are processes with equilibria (eigenstates)



QM postulates revisited

⇒ Recover mathematical structure of quantum mechanics from properties of ensembles

Postulate		Recovered
State postulate: quantum states are rays of a Hilbert space	⇒	Linearity of Hilbert space can be recovered from rules of ensemble mixing
Measurement postulate: projection measurement and Born rule	⇒	Projections as processes with equilibria Born rule recoverable from entropy of mixing
Composite system postulate: tensor product for composite system	⇒	Derived from other postulates PRL 126, 110402 (2021)
Evolution postulate: unitary evolution (Schrödinger equation)	⇒	Deterministic/reversible evolution

Entropic nature of physical theories

Thermodynamics/Statistical mechanics are not built on top of mechanics

Mechanics is the ideal case of thermodynamics/statistical mechanics

Best preparation ⇒ pure state

Best process ⇒ map between pure states

$\mu(U)$

Symplectic manifold

ρ uniform over U

Standard probability

$\rho(x) = \frac{1}{\mu(U)}$

Information theory

$H(\rho) = \log \mu(U)$

The geometric structure of both classical and quantum mechanics is ultimately an entropic structure

We can never prepare/measure pure states. We can only prepare/measure ensembles.

It makes sense that ensembles can offer a unified way of thinking about both classical and quantum mechanics.

$\langle \psi | \phi \rangle$

Projective Hilbert space

$\rho = \frac{1}{2}\rho_\psi + \frac{1}{2}\rho_\phi$

Quantum probability

$p(\psi|\phi) = |\langle \psi | \phi \rangle|^2$

Quantum information theory

$H(\rho) = H\left(\frac{1+\sqrt{p}}{2}, \frac{1-\sqrt{p}}{2}\right)$

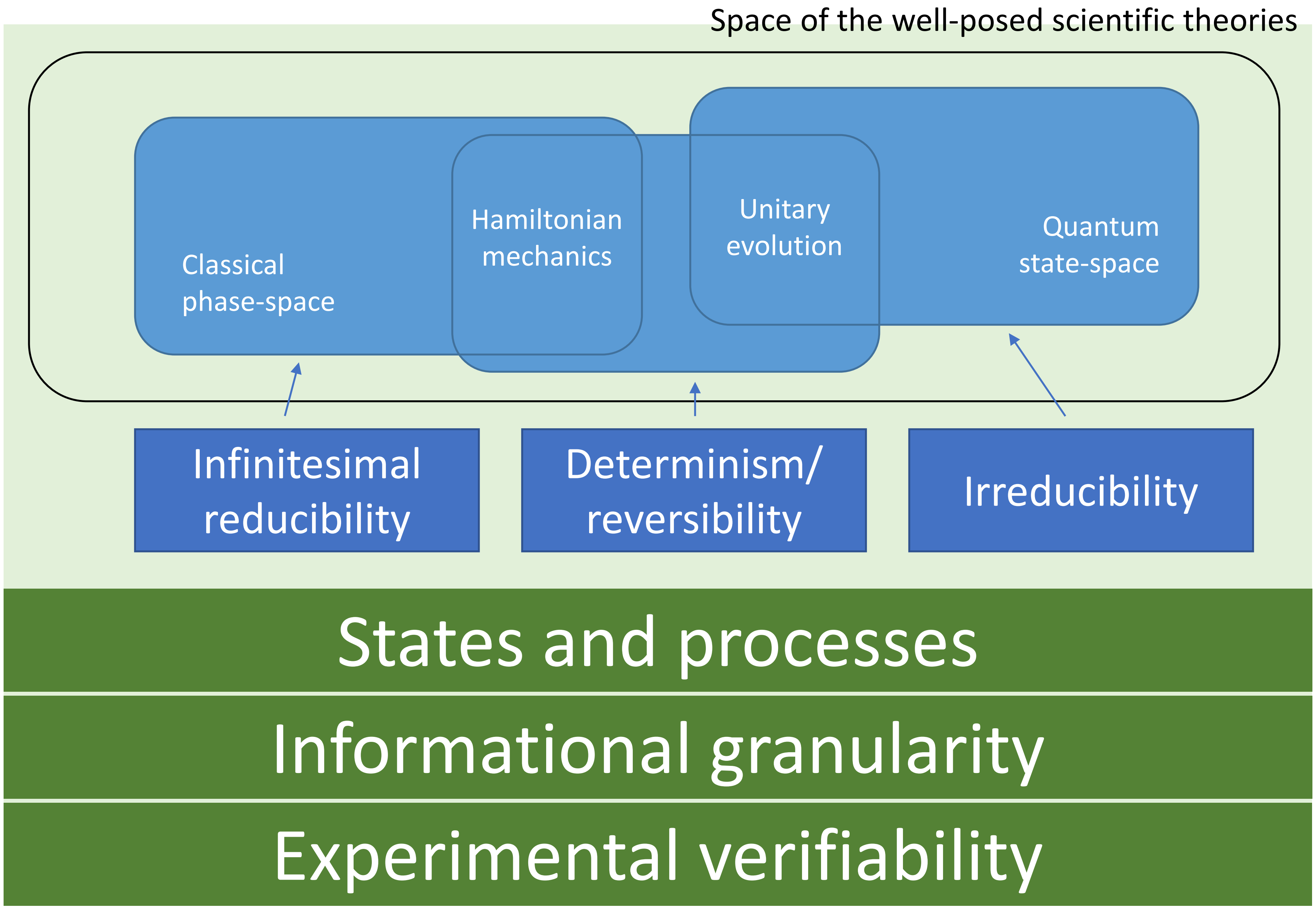
Physical theories

Specializations of the general theory under the different assumptions

Assumptions

General theory

Basic requirements and definitions valid in all theories

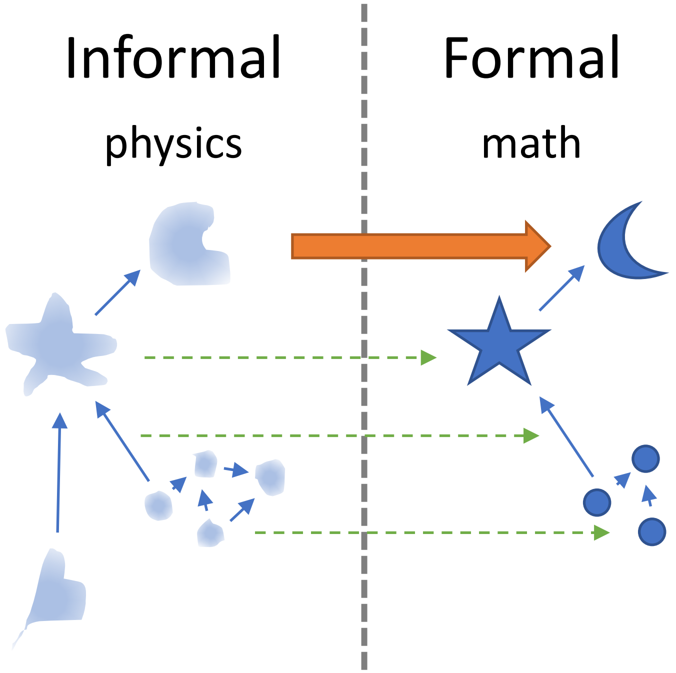


Physical mathematics

Physical mathematics

In modern physics, mathematics is used as the foundation of our physical theories

From Hossenfelder's *Lost in Math*: "[…] finding a neat set of assumptions from which the whole theory can be derived, is often left to our colleagues in mathematical physics […]"



Physics is defined in terms of physical objects and operational definitions

Under assumptions, idealizations and approximations, physical objects and their properties are expressed with a formal system through axioms and definitions.

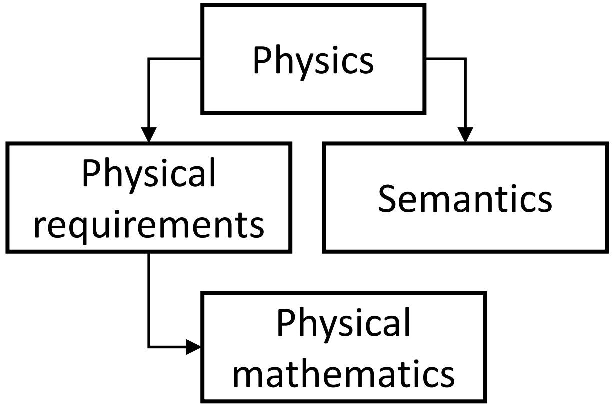
All physical content is captured by the definitions and axioms

Mathematical content of a theory can never tell us the full physical content

David Hilbert: "Mathematics is a game played according to certain simple rules with meaningless marks on paper."

Bertrand Russell: "It is essential not to discuss whether the first proposition is really true, and not to mention what the anything is, of which it is supposed to be true."

The only way we can have a full understanding of a physical theory is if ALL formal structures are strictly justified by physical requirements



We need to identify which parts of mathematics are "correct" to capture physical properties in a specific realm of applicability

Examples: symplectic space and probability spaces

Hamiltonian mechanics

Phase space (symplectic manifold) | Hamiltonian evolution

Differentiable manifold | Symplectic structure

Manifold | Differentiable structure

Topological space | Locally $\mathbb{R}^n$

Determinism/reversibility

Observer independent count of states

Infinitesimal reducibility

Experimentally distinguishable cases with verifiable statements

Identified by independent continuous quantities

Probability space

Set of points | σ-algebra | Measure

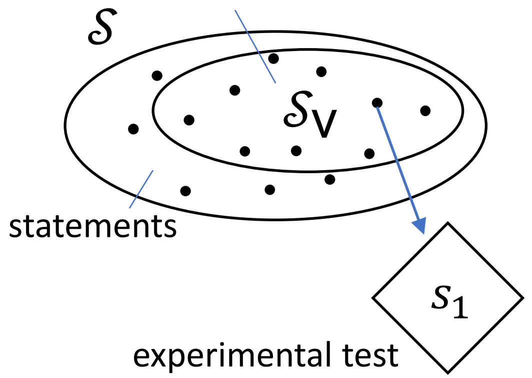
Experimentally distinguishable cases

Statements associated with experimental tests

Probability that a statement is true

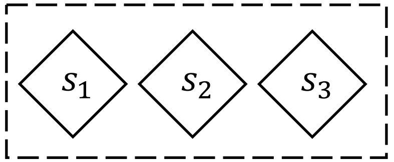
We can see what each additional mathematical layer represents and under what assumptions

Logic of experimental verifiability



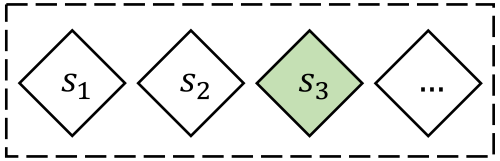
s_1	Test Result
T	SUCCESS (in finite time)
F	FAILURE (in finite time)
	UNDEFINED

Finite conjunction (logical AND) $\bigwedge_{i=i}^{n} s_i$



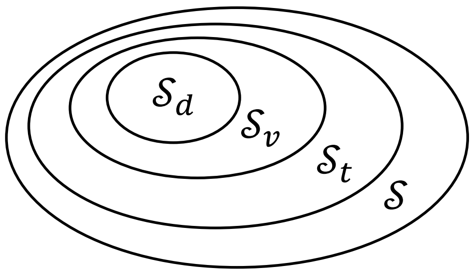
All tests must succeed

Countable disjunction (logical OR) $\bigvee_{i=1}^{\infty} s_i$



One successful test is sufficient

Physical theories (evidence based)
⇒ all theoretical statements associated with tests



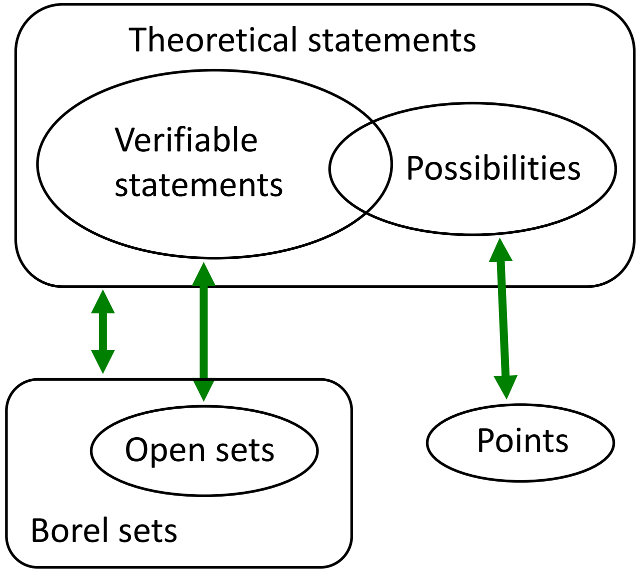
Operator	Gate	Statement	Theoretical Statement	Verifiable Statement	Decidable Statement
Negation	NOT	allowed	allowed	disallowed	allowed
Conjunction	AND	arbitrary	countable	finite	finite
Disjunction	OR	arbitrary	countable	countable	finite

Mathematical theories (formally well-posed)
have “too many statements” to be physically meaningful

Theoretical domain:
set of statements experimentally well-defined
⇒ countably generated countably complete Boolean algebra

Possibilities: experimentally distinguishable cases
⇒ atoms of the algebra
($|\mathbb{R}|$ max cardinality)

Topology and σ-algebra



s_1	Test Result
T	SUCCESS (in finite time)
	UNDEFINED
F	UNDEFINED
	FAILURE (in finite time)

$int(A)$ corresponds to the verifiable part of a statement

∂A corresponds to the undecidable part of a statement

$ext(A)$ corresponds to the falsifiable part of a statement

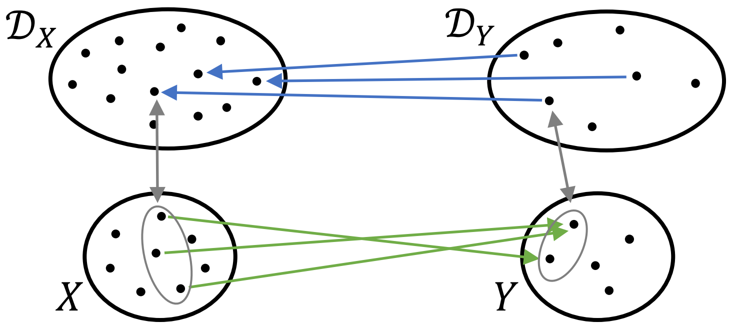
Experimental verifiability $\Rightarrow$ topology and σ-algebras (foundation of geometry, probability, …)

Open set $(509.5, 510.5) \Longleftrightarrow$ Verifiable "the mass of the electron is 510 ± 0.5 KeV"

Closed set $[510] \Longleftrightarrow$ Falsifiable "the mass of the electron is exactly 510 KeV"

Borel set $\mathbb{Q}$ $(int(\mathbb{Q}) \cup ext(\mathbb{Q}) = \emptyset) \Longleftrightarrow$ Theoretical "the mass of the electron in KeV is a rational number" (undecidable)

Inference relationship $r: \mathcal{D}_Y \to \mathcal{D}_X$ such that $r(s) \equiv s$

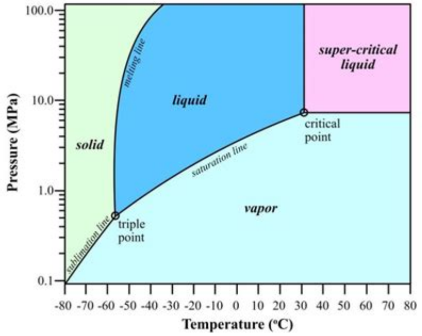


Causal relationship $f: X \to Y$ such that $x \preccurlyeq f(x)$

Inference relationship
$\Longleftrightarrow$
Causal relationship

Relationships must be topologically continuous

Topologically continuous consistent with analytic discontinuity on isolated points



Phase transition $\Longleftrightarrow$ Topologically isolated regions

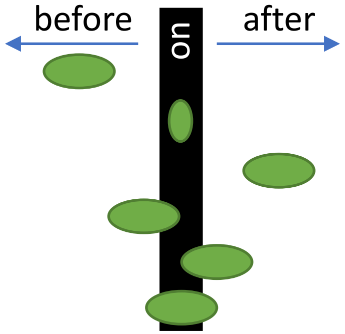
Perfect map between math and physics

NB: in physics, topology and σ-algebra are parts of the **same** logic structure

Quantities and ordering

Goal: deriving the notion of quantities and numbers (i.e. integers, reals, …) from an operational (metrological) model

A **reference** (i.e. a tick of a clock, notch on a ruler, sample weight with a scale) is something that allows us to distinguish between a before and an after



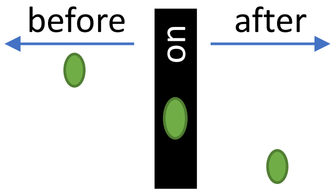
Mathematically, it is a triple (b, o, a) such that:

- b and a are verifiable
- The reference has an extent ($o \not\equiv \bot$)
- If it's not before or after, it is on ($\neg b \wedge \neg a \preccurlyeq o$)
- If it's before and after, it is on ($b \wedge a \preccurlyeq o$)

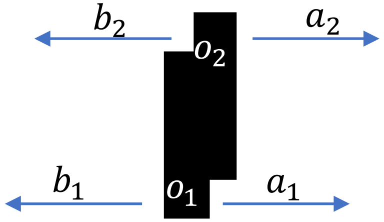
Numbers defined by metrological assumptions, NOT by ontological assumptions

To define an **ordered** sequence of possibilities, the references must be (nec/suff conditions):

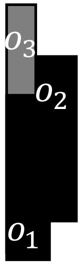
Strict



Aligned



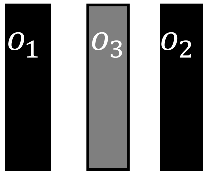
Refinable



$\Rightarrow \ (X, \leq)$

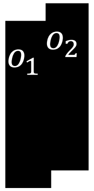
+

Dense



$\Rightarrow \ (X, \leq) \cong (\mathbb{R}, \leq)$

Sparse



$\Rightarrow \ (X, \leq) \cong (\mathbb{Z}, \leq)$

The hard part is to recover ordering. After that, recovering reals and integers is simple.

Assumptions untenable at Planck scale: no consistent **ordering**: no "objective" "before" and "after"

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Differentiability in physics

Mathematicians have developed several, increasingly abstract, definitions for differentials, derivatives, integrations, tangent vectors… which one is best for physics?

E.g. in differential topology/geometry

$$v: C^\infty(X,\mathbb{R}) \to C^\infty(X,\mathbb{R})$$

vector basis

$$v(f) = v^i \partial_i f$$

Vector defined as derivation

$$dx: V \to \mathbb{R} \qquad B: V \times V \to \mathbb{R}$$

$$dx(v) = dx(v^i \partial_i) = v^x \qquad B(v,w) = B_{ij} v^i w^j$$

Differential forms are functions of derivations

$$\int_\gamma dx = \Delta x \qquad \int_\Sigma B = \Phi$$

Integrals defined on top of forms

Does not make sense physically!

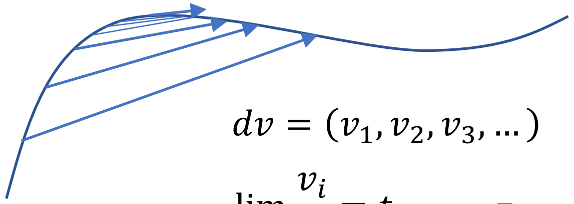
momentum is not a function of a derivation

velocity is not a derivation

derivations ∂_i depend on units: can't be summed

infinitesimal objects are the limit of finite objects, not the other way around

Infinitesimal reducibility ⇒ differentiability



$$dv = (v_1, v_2, v_3, \ldots)$$

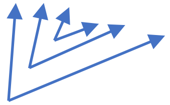
Infinitesimal linear change

$$\lim_{i\to\infty} \frac{v_i}{a_i} = t$$

Tangent vector

Convergence at all points ⇒ differentiability of curve

Infinitesimal surface change



$$d\sigma = ((v_1 \otimes w_1), (v_2 \otimes w_2), \ldots)$$

$$\lim_{i\to\infty} \frac{v_i \otimes w_i}{a_i} = t$$

	Time		Space		Temperature
Quantity	t	$\xrightarrow{x(t)}$	x	$\xrightarrow{T(x)}$	T
	↓		↓		↓
Differential	dt	$\xrightarrow{\frac{dx}{dt}}$	dx	$\xrightarrow{\frac{dT}{dx}}$	dT

Differentiable function: infinitesimal changes map to infinitesimal changes

Differentiable space: infinitesimal changes are well-defined

Derivative: map between differentials

$$dx^i = \frac{dx^i}{dt} dt \qquad dT = \frac{\partial T}{\partial x^i} dx^i$$

velocity (vector)

gradient (covector)

Differentiability: forms and linear functionals

Starting point: finite values defined on finite regions

k-functional, k-surface, k-form, k-vector

$$f_k(\sigma^k) = \int \theta_k(d\sigma^k)$$

Thinking in terms of relationships between finite objects leads to better physical intuition

Physically measurable quantities

zero-form

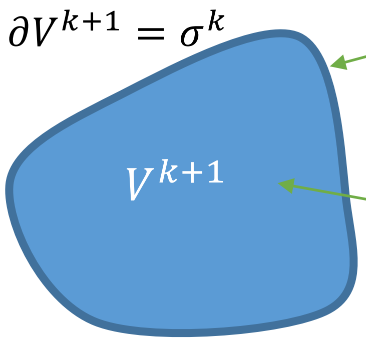
Differential forms: infinitesimal limit

Temperature:	$T(P)$		zero-form
Work:	$W(\gamma) = \sum_i W(\gamma_i) = \int f(d\gamma)$	$f = dW/d\gamma$	one-form
Magnetic flux:	$\Phi(\sigma) = \sum_i \Phi(\sigma_i) = \iint B(d\sigma)$	$B = d\Phi/d\sigma$	two-form
Mass:	$m(V) = \sum_i m(V_i) = \iiint \rho(dV)$	$\rho = dm/dV$	three-form

Assume additivity over disjoint regions

The mathematics is contingent upon the assumption of infinitesimal reducibility (e.g. mass in volumes sums only if boundary effects can be neglected)

We can define functionals that acts on boundaries



f^k Given a functional

Define higher dimensional functional that act on the boundary

$$g^{k+1}(V^{k+1}) \equiv f^k(\sigma^k)$$

Exterior functional

$$\partial\partial f^k(\sigma^{k+2}) = f^k(\partial\partial\sigma^{k+2}) = f^k(\emptyset) = 0$$

Boundary of a boundary is the empty set ⇒ exterior derivative of exterior derivative is zero

$$\begin{tikzcd}
g_{k+1} \arrow[d, shift right, "d/dV^{k+1}"'] & f_k \arrow[l, "\text{exterior functional}"'] \arrow[d, shift right, "d/d\sigma^k"'] \\
\omega_{k+1} \arrow[u, shift right, "\int_{V^{k+1}}"'] & \theta_k \arrow[l, "\partial"', "\text{exterior derivative}"] \arrow[u, shift right, "\int_{\sigma^k}"']
\end{tikzcd}$$

Reversing the exterior derivative is finding a (non-unique) potential

Generalized Stokes theorem

$$\int_{V^{k+1}} \partial\theta_k = \int_{\partial V^{k+1}} \theta_k$$

Abstract mathematical definitions at points, finite from infinitesimal

Physical definitions on finite, infinitesimal as a limit

Information granularity

Logical relationships $\Longleftrightarrow$ Topology/σ-algebra

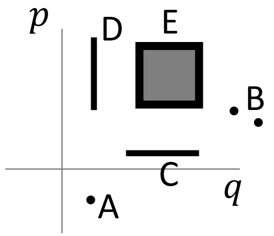
- “The position of the object is between 0 and 1 meters”
  $\preccurlyeq$ “The position of the object is between 0 and 1 kilometers”
- “The fair die landed on 1” $\preccurlyeq$ “The fair die landed on 1 or 2”
- “The first bit is 0 and the second bit is 1” $\preccurlyeq$ “The first bit is 0”

Granularity relationships $\Longleftrightarrow$ Order theory

- “The position of the object is between 0 and 1 meters”
  $\underline{\lessdot}$ “The position of the object is between 2 and 3 kilometers”
- “The fair die landed on 1” $\underline{\lessdot}$ “The fair die landed on 3 or 4”
- “The first bit is 0 and the second bit is 1” $\underline{\lessdot}$ “The third bit is 0”

$\Rightarrow$ Measure theory, geometry, probability theory, information theory, … all quantify the level of granularity of different statements

A partially ordered set allows us to compare size at different level of infinity and to keep track of incommensurable quantities (i.e. physical dimensions)



A $\underline{\lessdot}$ B $\underline{\lessdot}$ C $\underline{\lessdot}$ E

C $\not\underline{\lessdot}$ D

D $\not\underline{\lessdot}$ C

Once a “unit” is chosen, a measure quantifies the granularity of an other statement with respect to the unit

$$\mu_u : \bar{\mathcal{D}} \to \mathbb{R}$$

$$\mu_u(u) = 1$$

$$s_1 \,\underline{\lessdot}\, s_2 \Rightarrow \mu_u(s_1) \le \mu_u(s_2)$$

$\mu_u(s_1 \vee s_2) = \mu_u(s_1) + \mu_u(s_2)$ if s_1 and s_2 are incompatible

However, quantum mechanics requires a “twist” at the measure theoretic level

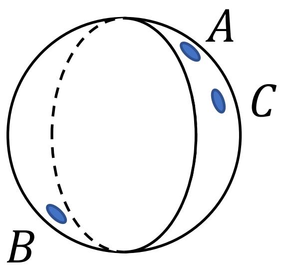
Need for non-additive measure

entropy of uniform distribution

count of states

$$H(\rho_U) = \log \mu(U)$$

Assume usual link between entropy and count of states



$\mu(\{A\}) = 2^0 = 1$

$\mu(\{A, B\}) = 2^1 = 2$

not additive

$\mu(\{A, C\}) < 2 = \mu(\{A\}) + \mu(\{C\})$

$\mu(\{A, B, C\}) < 2 = \mu(\{A, B\})$

not monotonic

In quantum mechanics, literally $1 + 1 \leq 2$

	Single point		Finite continuous range	
	$\mu(U)$	$\log \mu(U)$	$\mu(U)$	$\log \mu(U)$
Counting measure $\mu(U) = \#U$ Number of points	1	0	$+\infty$	$+\infty$
Lebesgue measure $\mu([a, b]) = b - a$ Interval size	0	$-\infty$	$< \infty$	$< \infty$
"Quantized" measure $\mu(U) = 2^{H(\rho_U)}$ Entropy over uniform distribution	1	0	$< \infty$	$< \infty$

Pick two!

1. Single point is a single case (i.e. $\mu(\{\psi\}) = 1$)
2. Finite range carries finite information (i.e. $\mu(U) < \infty$)
3. Measure is additive for disjoint sets (i.e. $\mu(\cup U_i) = \sum \mu(U_i)$)

Physically, we count states all else equal

Contextuality $\Longleftrightarrow$ non-additive measure

Unphysicality of Hilbert spaces

Hilbert space: $\boxed{\text{complete}}$ $\boxed{\text{inner product}}$ $\boxed{\text{vector space}}$

Redundant on finite-dimensional spaces. For infinite-dimensional spaces, it allows us to construct states with infinite expectation values from states with finite expectation values

Exactly captures measurement probability/entropy of mixtures

Physically required

Exactly captures superposition/ statistical mixing

Physically required

⇒ Thus requires us to include unitary transformations (i.e. change of representations and finite time evolution) that change finite expectation values into infinite ones

Extremely physically suspect!!!

Suppose we require all polynomials of position and momentum to have finite expectation

Maybe more physically appropriate?

⇒ Schwartz space

Only space closed under Fourier transforms
Used as starting point of theories of distributions

Conclusion

- We strongly believe that this is a much more fruitful approach to the foundations of physics
  - It helps us better understand what the current physical theories are about; it forces us to spell out all the hidden assumptions we inevitably make when describing physical systems; it forces us to investigate whether the current mathematical structures are the "correct" ones for doing physics
- There is a strong connection between different disciplines within and outside of physics
  - Nature does not care about our academic subdivisions
- The goal is ambitious and requires a wide collaboration
  - Always looking for people to collaborate with in physics, math, philosophy, …

Getting involved

- Ideally, we want to run the project as an open source project
  - Community of people with different backgrounds working toward a common goal
  - Book is the main output, currently preparing v2.0, which adds Reverse Physics for classical mechanics; v2.1 will add Reverse Physics for thermodynamics.
- Many ways to contribute, with different levels of commitment
  - Help us popularize the project
    - Simply advertise it, help us understand how/create material to advertise it
  - “Beta testing”
    - Review the book and other material
  - Small contributions
    - Figures, editing text, literature search, proofs, examples from your field, arguments, …
    - Incorporate ideas into educational materials
  - Bigger contributions
    - Help with part of the research, contribute to research papers …