The Assumptions of Physics 2023-24 status

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https://assumptionsofphysics.org

About the project

Identify a handful of physical starting points from which the basic laws can be rigorously derived

Assumptions of Physics

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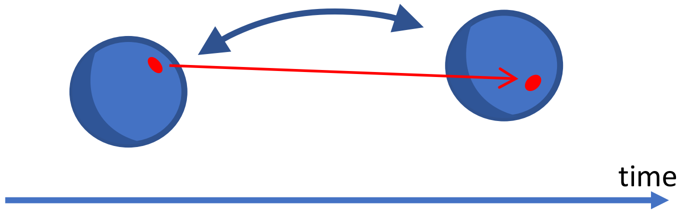
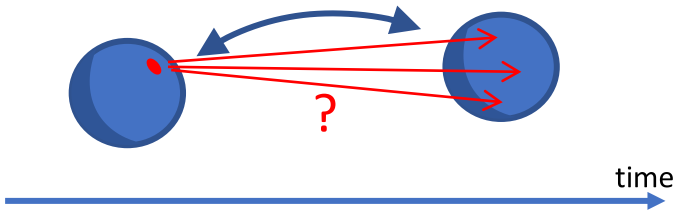
For example:

Infinitesimal reducibility ⇒ Classical state



Irreducibility ⇒ Quantum state



This also requires rederiving all mathematical structures from physical requirements

For example:

Science is evidence based ⇒ scientific theory must be characterized by experimentally verifiable statements ⇒ topology and σ-algebras

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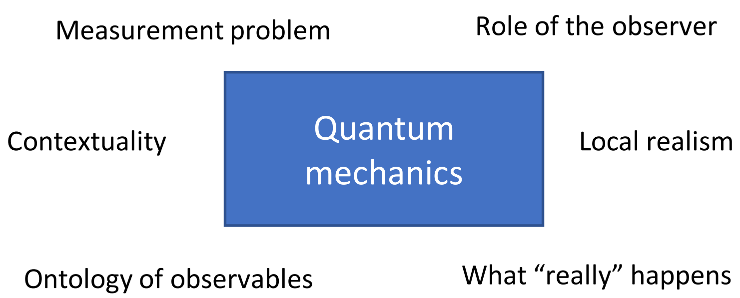
Assumptions of Physics

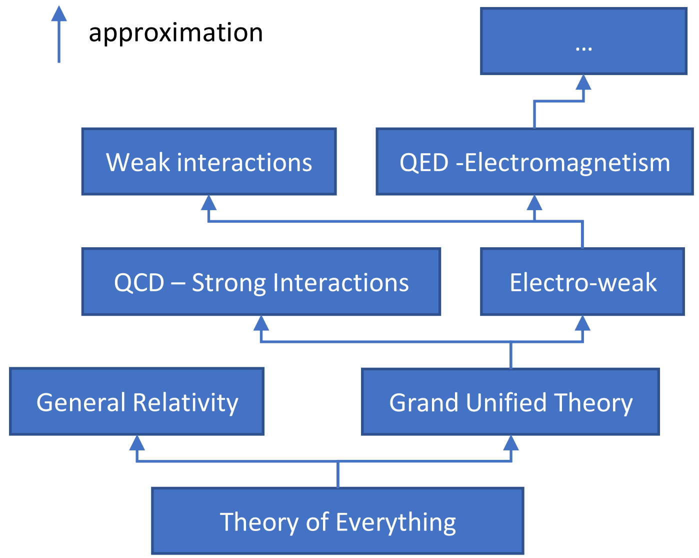
Different approach to the foundations of physics

Typical approaches

Construct interpretations

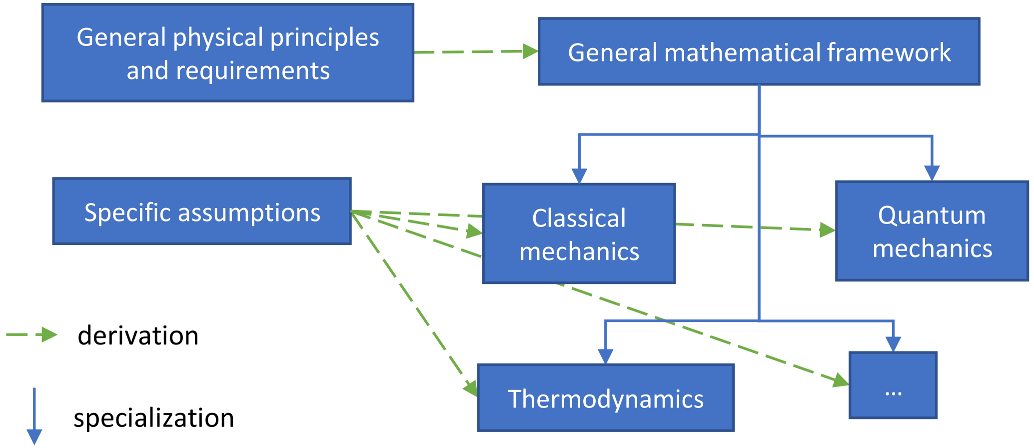


Find ultimate theory



Our approach

Find a minimal set of physical assumptions from which to rigorously rederive the laws



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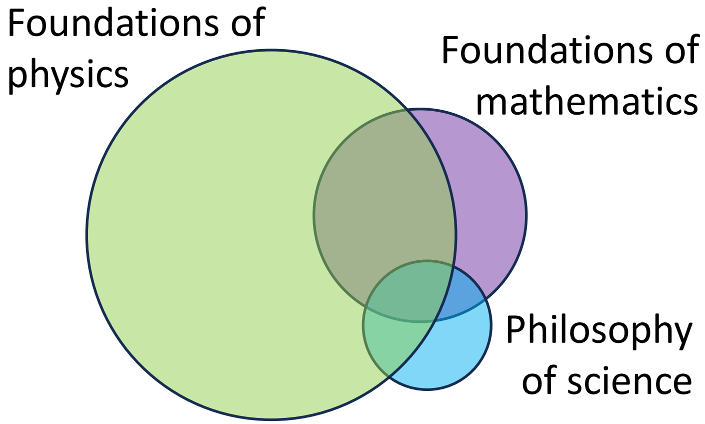
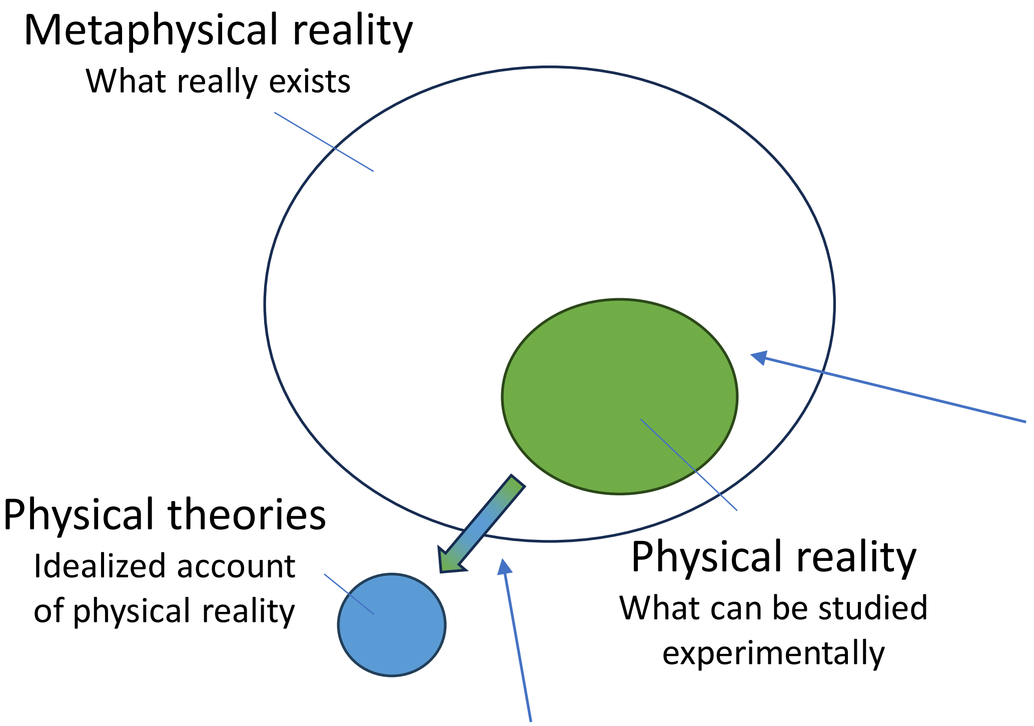
Underlying perspective





How exactly do we draw that boundary?

How exactly does the abstraction/idealization process work?

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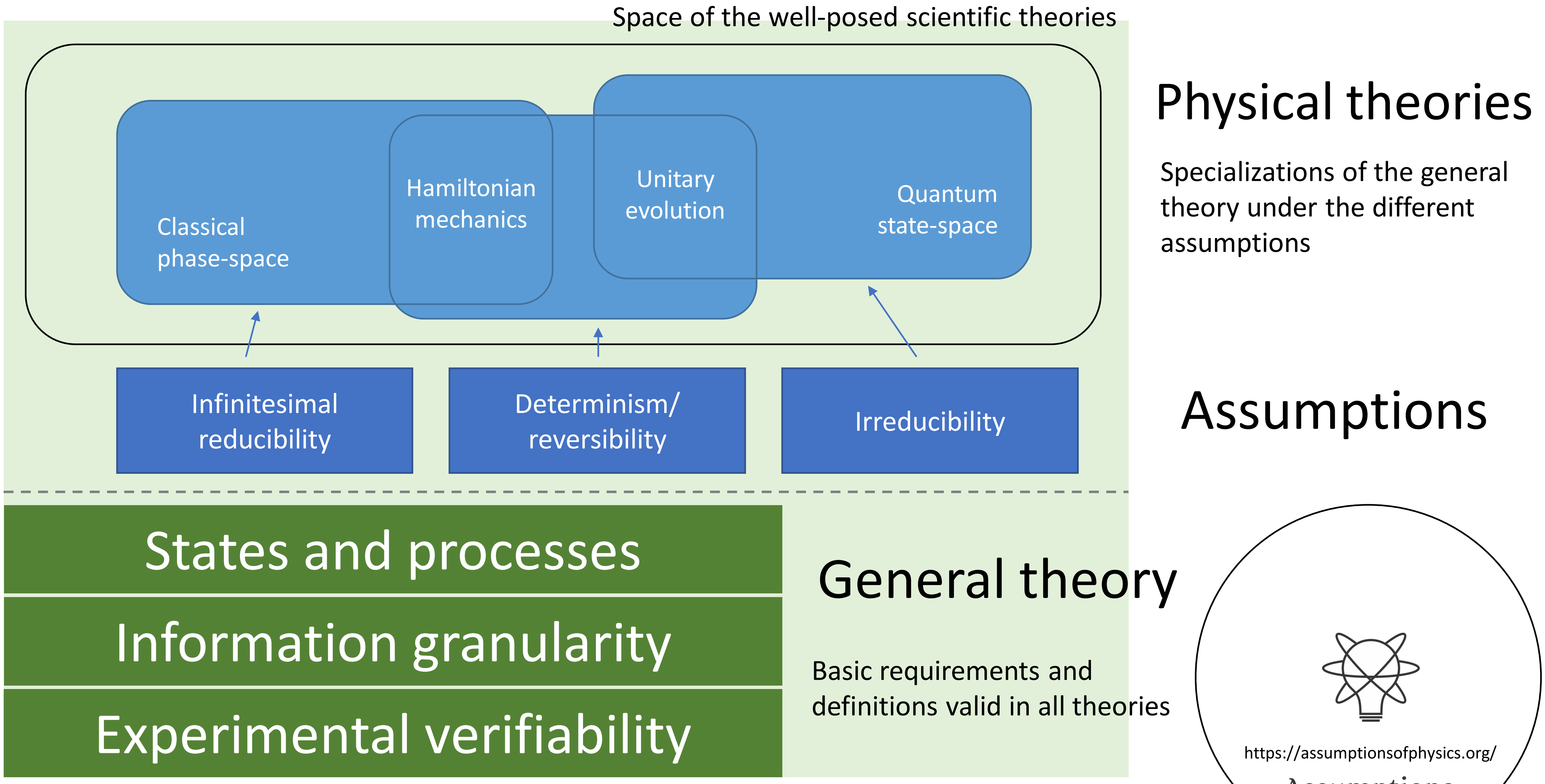

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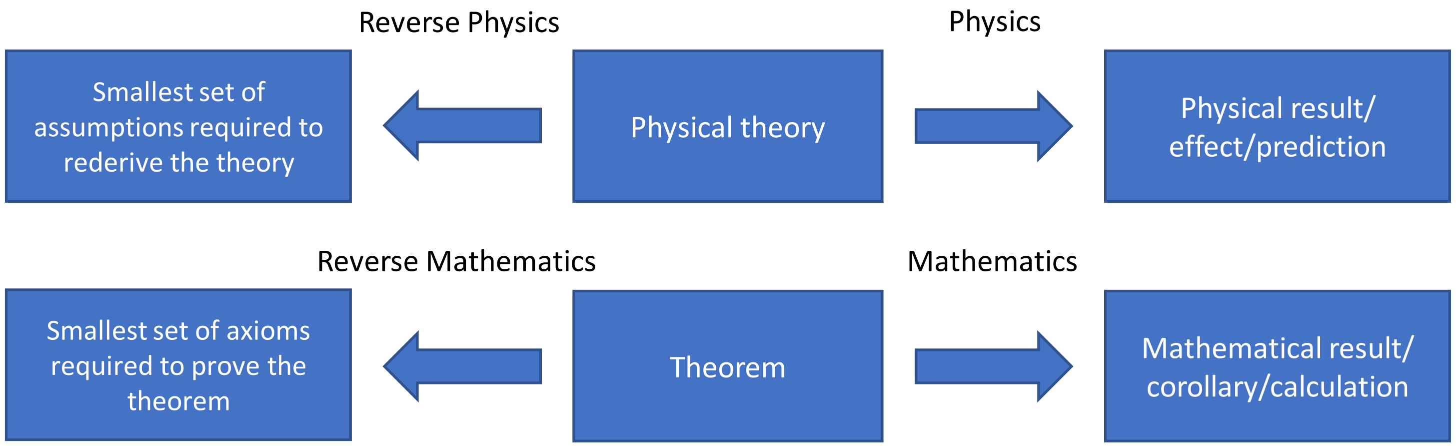
Reverse physics:

Start with the equations, reverse engineer physical assumptions/principles

$$S^q = \frac{dq}{dt} = \frac{\partial H}{\partial p}$$

$$S^p = \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

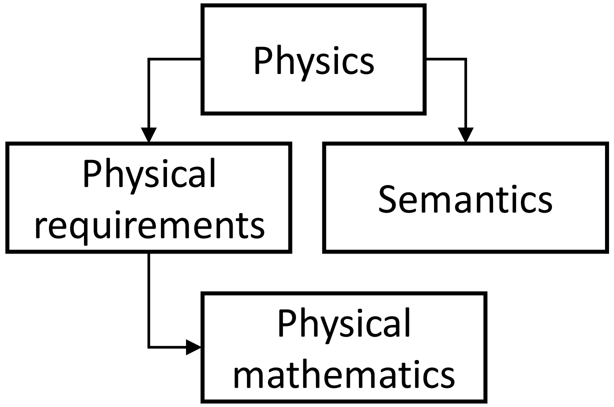
Found Phys **52**, 40 (2022)



Goal: “Elevate” the discussion from mathematical constructs to physical principles, assumptions and requirements

Physical mathematics:

Start from scratch and rederive all mathematical structures from physical requirements



Goal: Construct a perfect one-to-one map between mathematical and physical objects

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Assumptions of Physics

Reverse Physics: Classical mechanics

Assumptions of Physics, *Michigan Publishing* (v2 2023)

J. Phys. Commun. **2** 045026 (2018)

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Assumptions of Physics

7 equivalent characterizations of Hamiltonian mechanics

12 in the book

one DOF

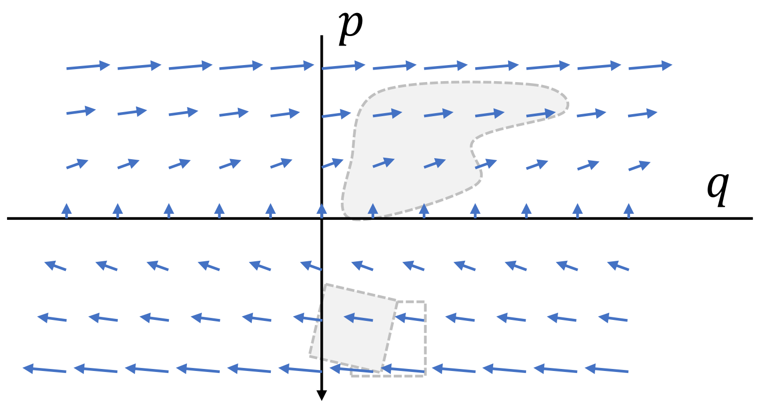
(1) Hamilton's equations

$$S^q = \frac{dq}{dt} = \frac{\partial H}{\partial p}$$

$$S^p = \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

(2) Divergenceless displacement

$$div(S^a) = \frac{\partial S^q}{\partial q} + \frac{\partial S^p}{\partial p} = 0$$



(3) Area conservation ($|J| = 1$)

$$dQdP = |J|dqdp$$

(4) Deterministic and reversible evolution

Area conservation ⟺ state count conservation

⟺ deterministic and reversible evolution

(7) Uncertainty conservation

$$|\Sigma(t + dt)| = |J||\Sigma(t)||J|$$

for peaked distributions

(5) Deterministic and thermodynamically reversible evolution

$$S = k_B \log W$$

Area conservation ⟺ entropy conservation

⟺ thermodynamically reversible evolution

(6) Information conservation

$$I[\rho(t + dt)] = I[\rho(t)] - \int \rho \log|J| \, dqdp$$

A full understanding of classical mechanics means understanding these connections

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Assumptions of Physics

Reversing the principle of least action

DR

$$\nabla \cdot \vec{S} = 0$$

No state is "lost" or
"created" as time evolves

$$\vec{S} = -\nabla \times \vec{\theta}$$

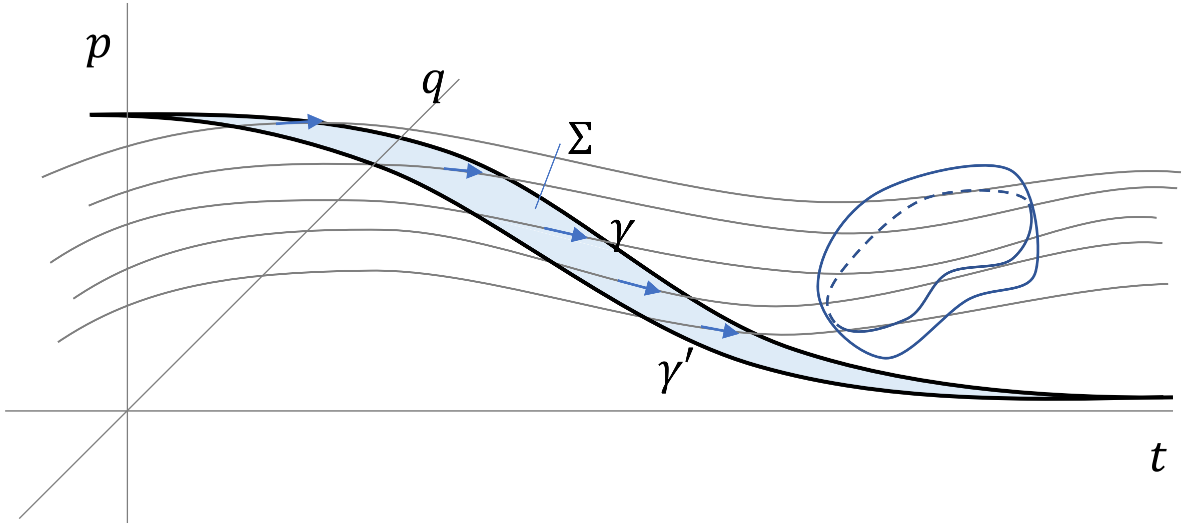
(Minus sign to match convention)

KE

$$\mathcal{S}[\gamma] = \int_\gamma L dt = \int_\gamma \vec{\theta} \cdot d\vec{\gamma}$$

Sci Rep **13**, 12138 (2023)

The action is the line integral of the vector potential (unphysical)



Variation of the action

$$\delta\mathcal{S}[\gamma] = \oint_{\partial\Sigma} \vec{\theta} \cdot d\vec{\gamma}$$
$$= -\iint_\Sigma \vec{S} \cdot d\vec{\Sigma}$$

Gauge independent,
physical!

Variation of the action measures the flow of states (physical).
Variation = 0 ⇒ flow of states tangent to the path.

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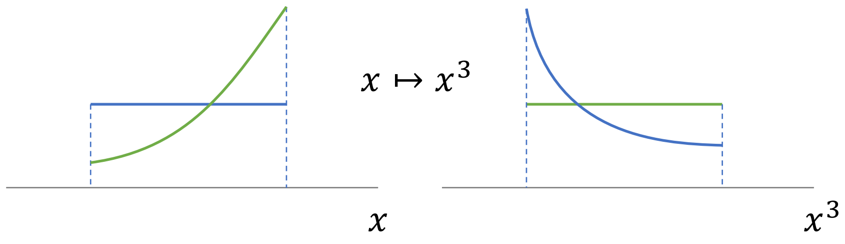
Assumptions of Physics

Hamiltonian Privilege, *Erkenn* (2023)

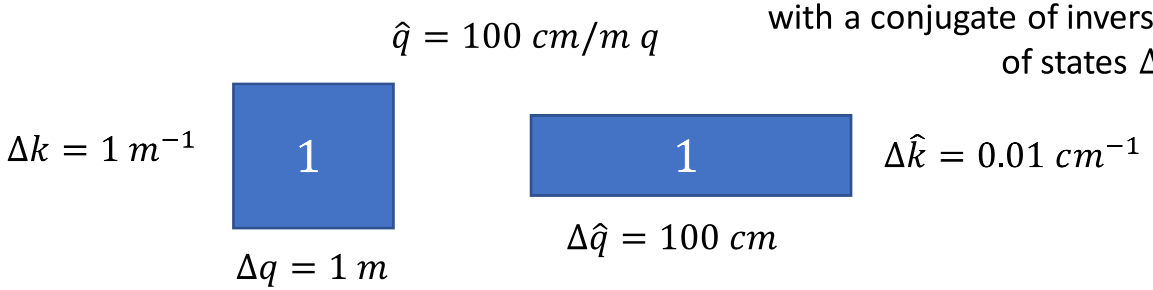
Stud Hist Phil **71**, 082020, 60-71 (2020)

Reversing phase-space



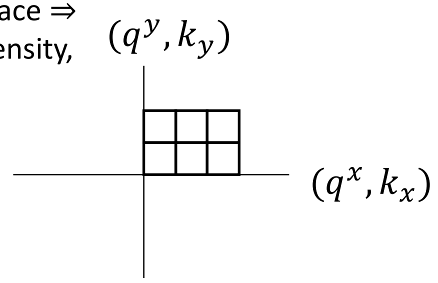
Density, entropy, uniform distributions NOT in general coordinate invariant

Each unit variable (i.e. coordinate) paired with a conjugate of inverse units: number of states $\Delta q \Delta k$ is invariant



Phase space (symplectic) structure is the only one that supports coordinate invariant density, entropy, state count

Independence of DOFs ⇒ independence of units ⇒ orthogonality in phase-space ⇒ invariant marginals (for density, entropy, state count)



Total number of states = product of number of cases in each independent DOF

Hamiltonian mechanics preserves count of states and DOF independence over time

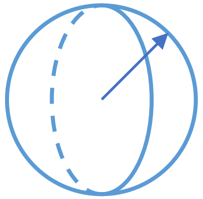
$$\omega_{ab} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \delta^i_j$$

Symplectic form (geometric structure of phase space)

Areas/possibilities in each DOF

Orthogonality/independence across DOFs

Only 3 spatial dimensions are possible



Directional DOF

2-sphere the only symplectic manifold

Invariance at equal time (relativity) gives us the structure of phase space

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Assumptions of Physics

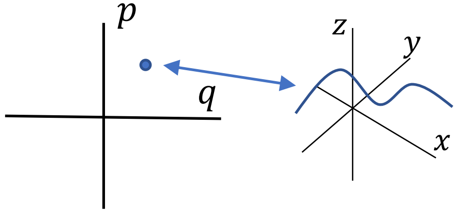
Massive particles under potential forces

Kinematic equivalence assumption: the state can be recovered from space-time trajectories

$$\frac{\partial p_i}{\partial \dot{q}^j} \equiv m g_{ij}$$

Must be a linear transformation in terms of coordinates

Fixes the units



Integration of the previous expression

$$p_i = m g_{ij} \dot{q}^j + q A_i(q^k)$$

$$\dot{q} = \frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} = \frac{1}{m} g^{ij} (p_j - q A_j)$$

$$H = \frac{1}{2m} (p_i - q A_i) g^{ij} (p_j - q A_j) + q V(q^k)$$

Hamiltonian for massive particles under potential forces

Mass quantifies number of states per unit of velocity

Higher mass ⇒ more states to go through ⇒ harder to accelerate

BUT

Zero mass ⇒ zero states within finite range of velocity ⇒ velocity is fixed

The laws themselves are highly constrained by simple assumptions

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Assumptions of Physics

Relativistic mechanics

Relativistic aspects without space-time and in Newtonian mechanics

potential of the displacement

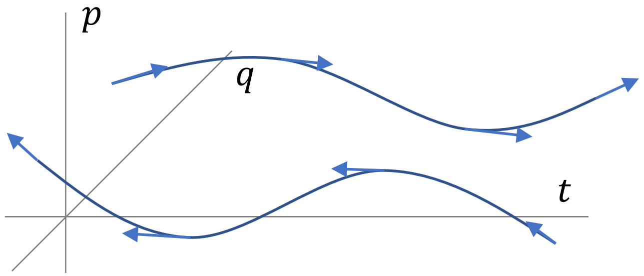
$$\theta = [p^i, -H, 0, 0]$$

energy-momentum co-vector

$$F = \frac{d\hat{t}}{dt}\hat{F} = \frac{d}{dt}\left(\hat{m}\frac{dt}{d\hat{t}}\frac{dx}{dt}\right) = \frac{d}{dt}\left(m\frac{dx}{dt}\right)$$

rest mass scaled by time dilation

Classical antiparticles



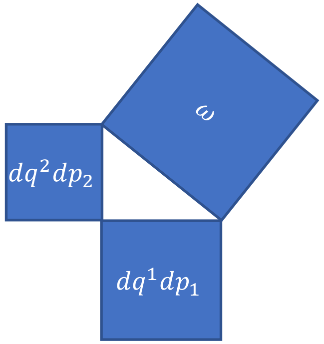
$$\frac{dt}{ds} = \frac{\partial \mathcal{H}}{\partial E}$$

Affine parameter anti-aligned with time: parameterization "goes back" in time

Lorentzian relativity is the only "correct" one

Minkowski signature appears on the extended phase space

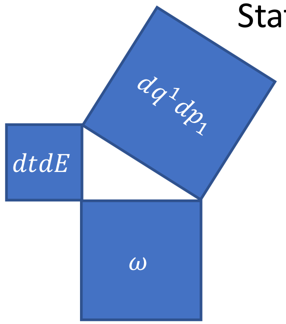
$$\omega = dq^1dp_1 + dq^2dp_2$$



Indep DOF are orthogonal

$$\omega = dq^1dp_1 - dtdE$$

$$dq^1dp_1 = \omega + dtdE$$

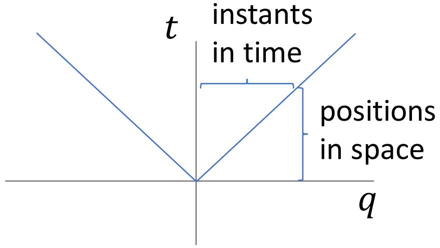


States are counted at equal time: temporal DOF orthogonal to ω

No clear idea what $G_{\alpha\beta\gamma}$ is…
Inertial forces?

$$\omega_{ab} = \begin{bmatrix} -mG_{\alpha\beta\gamma}u^\gamma + qF_{\alpha\beta} & g_{\alpha\beta} \\ -g_{\alpha\beta} & 0 \end{bmatrix}$$

Metric tensor quantifies states charted by position and velocity



Constant c converts state count between space and time

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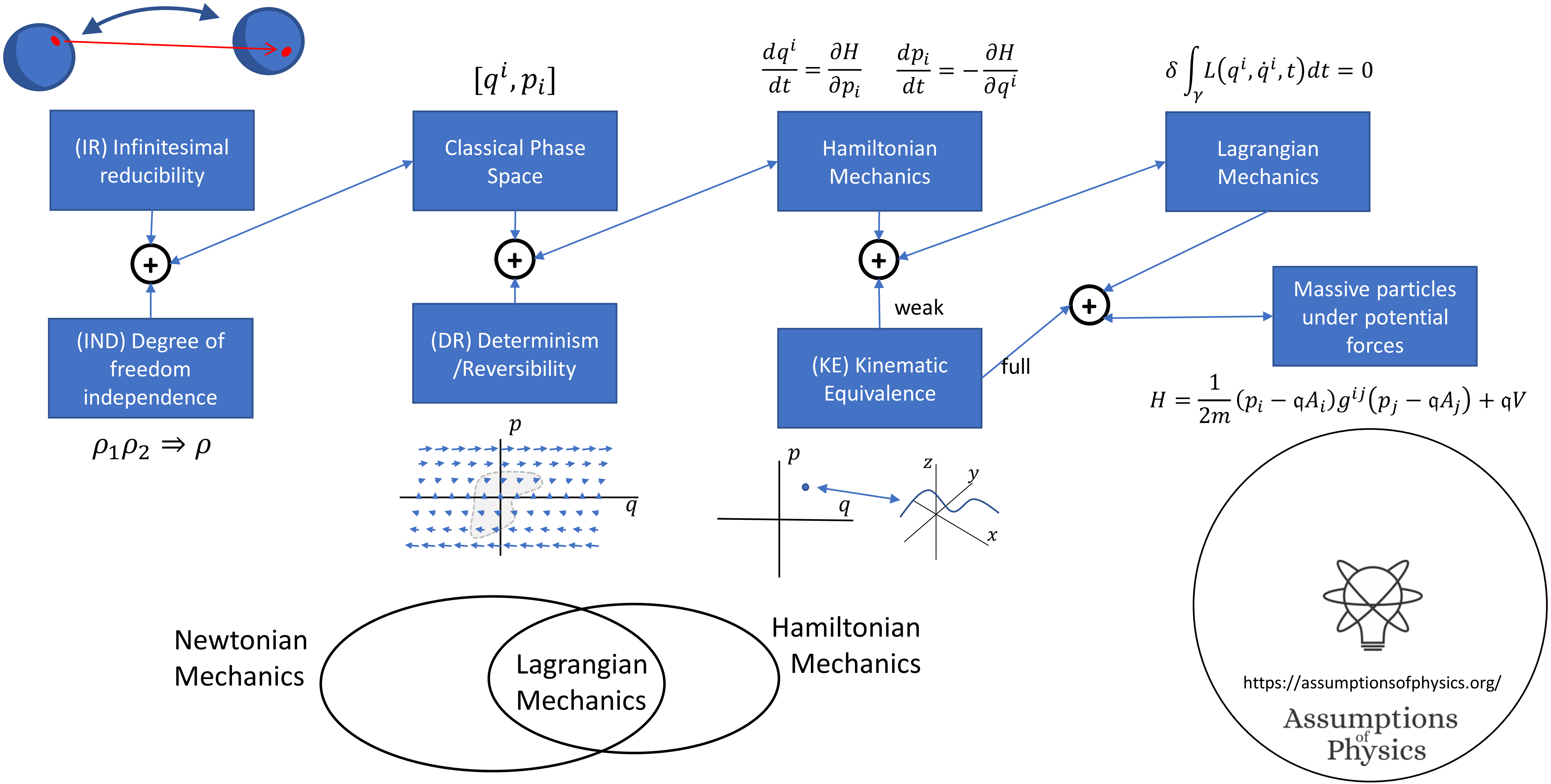
Assumptions of Physics

Assumptions of classical mechanics



Reverse physics gives us links between theories

Deterministic and reversible evolution

$\Rightarrow$ existence and conservation of energy (Hamiltonian)

Why?

Stronger version of the first law of thermodynamics

Deterministic and reversible evolution

$\Rightarrow$ past and future depend only on the state of the system

$\Rightarrow$ the evolution does not depend on anything else

$\Rightarrow$ the system is isolated

$\Rightarrow$ the system conserves energy

First law of thermodynamics!

Reverse Physics: Thermodynamics

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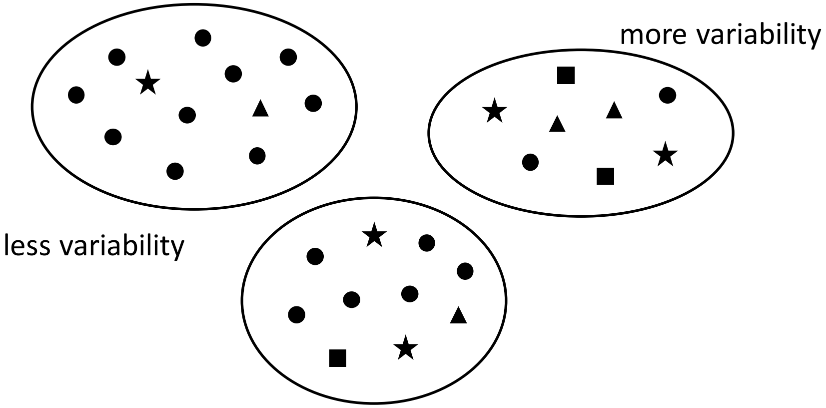
Assumptions of Physics

Shannon entropy as variability

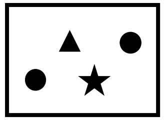
Eur. J. Phys. 42, 045102 (2021)

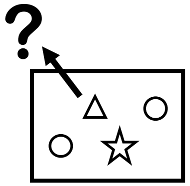


Shannon entropy quantifies the variability
of the elements within a distribution

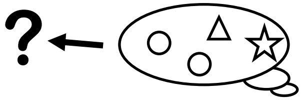
Meaning depends on the type of distribution

Statistical distribution:
variability of what is there





Probability distribution:
variability of what could be there



Credence distribution:
variability of what one believes to be there

$-\sum p_i \log p_i$ only indicator of variability
that satisfies simple requirements

1) Continuous function of p_i only
2) Increases when number of cases increases
3) Linear in p_i

This characterization
works across disciplines

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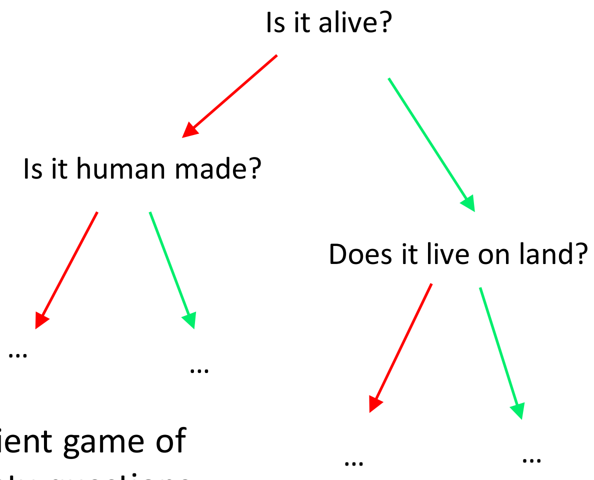
Assumptions of Physics

Shannon entropy as variability

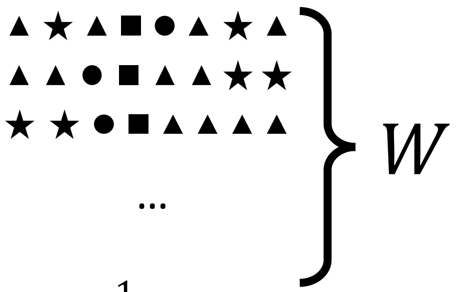
Eur. J. Phys. 42, 045102 (2021)



Efficient game of
twenty questions

More variability, more questions

Variability is quantified by the expected minimum number of questions required to identify an element



$$\frac{1}{N}\log W \approx -\sum p_i \log p_i$$

More variability, more permutations

Variability is also quantified by the logarithm of the number of possible permutations per element

More variability for a distribution at equilibrium, more fluctuations, more physical entropy

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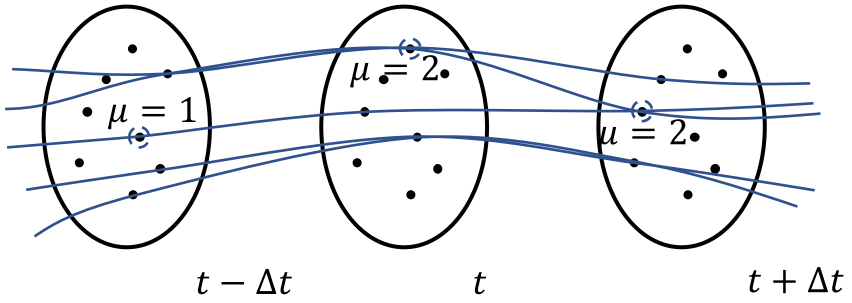
Assumptions of Physics

Entropy as logarithm of evolutions

$\mu(s_t)$: how many evolutions go through s_t?

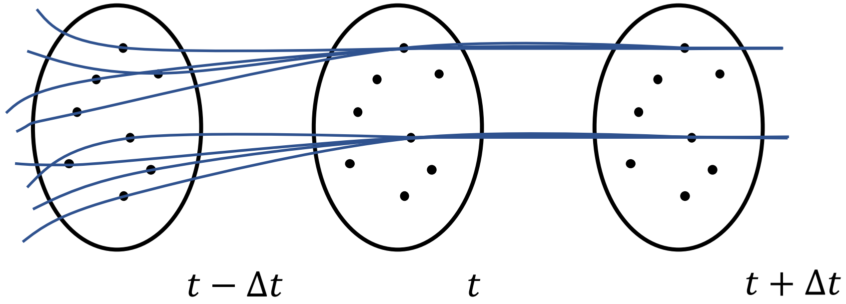
Process entropy: $S = \log \mu$



$$P(s_{t+\Delta t}|s_t) = \frac{\mu(s_t \cap s_{t+\Delta t})}{\mu(s_t)}$$

Determinism: evolutions cannot split $\mu\big(s(t+\Delta t)\big) \geq \mu\big(s(t)\big)$

Reversibility: evolutions cannot merge $\mu\big(s(t+\Delta t)\big) \leq \mu\big(s(t)\big)$



For a deterministic process

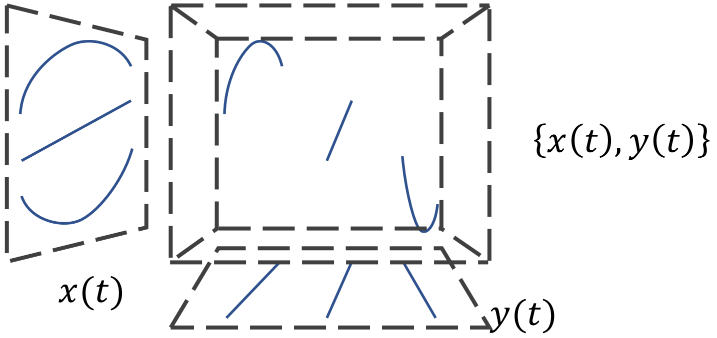
$$\mu\big(s(t+\Delta t)\big) \geq \mu\big(s(t)\big)$$

(equal if reversible)
(maximum at equilibrium)

For a deterministic process

$$S\big(s(t+\Delta t)\big) \geq S\big(s(t)\big)$$

(equal if reversible)
(maximum at equilibrium)



System independence: evolutions of the composite are the product of individual systems: $\mu_{XY} = \mu_X \mu_Y$

Entropy additive for independent systems
$S_{XY} = S_X + S_Y$

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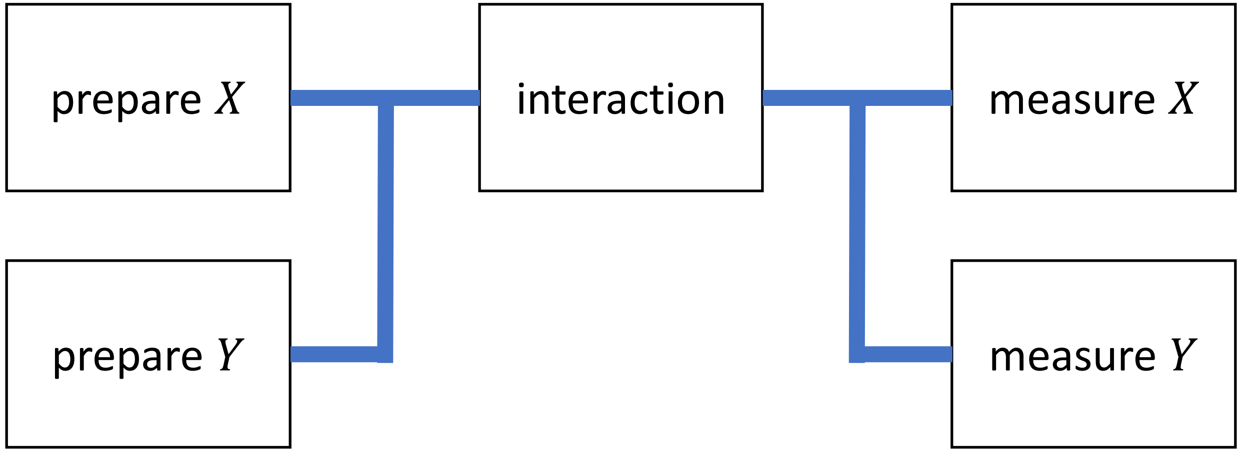
Assumptions of Physics

$\mu(s_t)$: how many evolutions go through s_t?

Process entropy: $S = \log \mu$

Entropy as logarithm of evolutions

Note: defining an evolution count is necessary in physics



We compose processes by connecting inputs and outputs: all evolutions must connect!

Recovers other notions of entropy!

If det/rev, one state per evolution, count of evolutions is count of states
$\Rightarrow$ recover fundamental postulate of statistical mechanics!

If microstate fluctuates according to a distribution ρ, count of evolutions is count of permutations $\Rightarrow$ recover Shannon entropy!

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Assumptions of Physics

"Reversing" thermodynamics

Assume states are equilibria of faster scale processes

Assume states identified by extensive properties

Assume one of these quantities is energy U

$$S(U, x^i)$$

Existence of equation of state

$$\beta = \frac{1}{k_B T} = \frac{\partial S}{\partial U} \quad \text{and} \quad -\beta X_i = \frac{\partial S}{\partial x^i}$$

Define intensive quantities

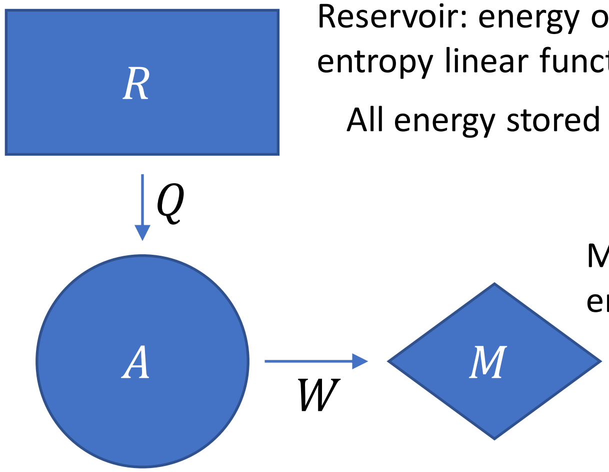
$$dS = \frac{\partial S}{\partial U} dU + \frac{\partial S}{\partial x^i} dx^i = \beta dU - \beta X_i dx^i$$

$$k_B T dS = dU - X_i dx^i$$

$$dU = T(k_B dS) + X_i dx^i$$

Recover usual relationships

Study interplay of changes of energy and entropy



Reservoir: energy only state variable, entropy linear function of energy

All energy stored in entropy

Mechanical system: same entropy for all states

No energy stored in entropy

$$\beta = \frac{1}{k_B T} = 0$$

$$\Delta U = 0 = \Delta U_A + \Delta U_R + \Delta U_M$$
$$= \Delta U_A - Q + W$$

Recover first law

First law recovered from existence and conservation of Hamiltonian

$$0 \leq \Delta S = \Delta S_A + \Delta S_\mathrm{R} + \Delta S_\mathrm{M}$$
$$= \Delta S_A + \beta_R \Delta U_R + 0 = \Delta S_A + \frac{-Q}{k_B T_R}$$

Recover second law

Second law recovered from definition of entropy as count of evolutions

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Assumptions of Physics

3rd law and principle of maximal description

Can be formulated as:

Every substance has a finite positive entropy, but at the absolute zero of temperature the entropy may become zero, and does so become in the case of perfect crystalline substances.

G. N. Lewis and M. Randall

Better “special case” than “crystalline substance”

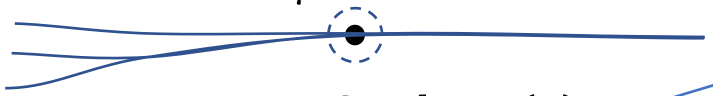
Null state $\emptyset$: system is absent (e.g. gas with zero particles)

$$A = A \cup \emptyset$$

$$S_A = S_{A\cup\emptyset} = S_A + S_\emptyset \Rightarrow S_\emptyset = 0$$

Entropy for the null state of any system must be 0

$\mu = 3$



Count of evolutions

$S = \log \mu(x)$

Count of evolutions can’t be < 1 therefore S can’t be < 0

3rd law can be restated as:

No state can describe a system more accurately than stating the system is not there in the first place.

Principle of maximal description

We can reformulate the 3rd law of thermodynamics as a logical necessity

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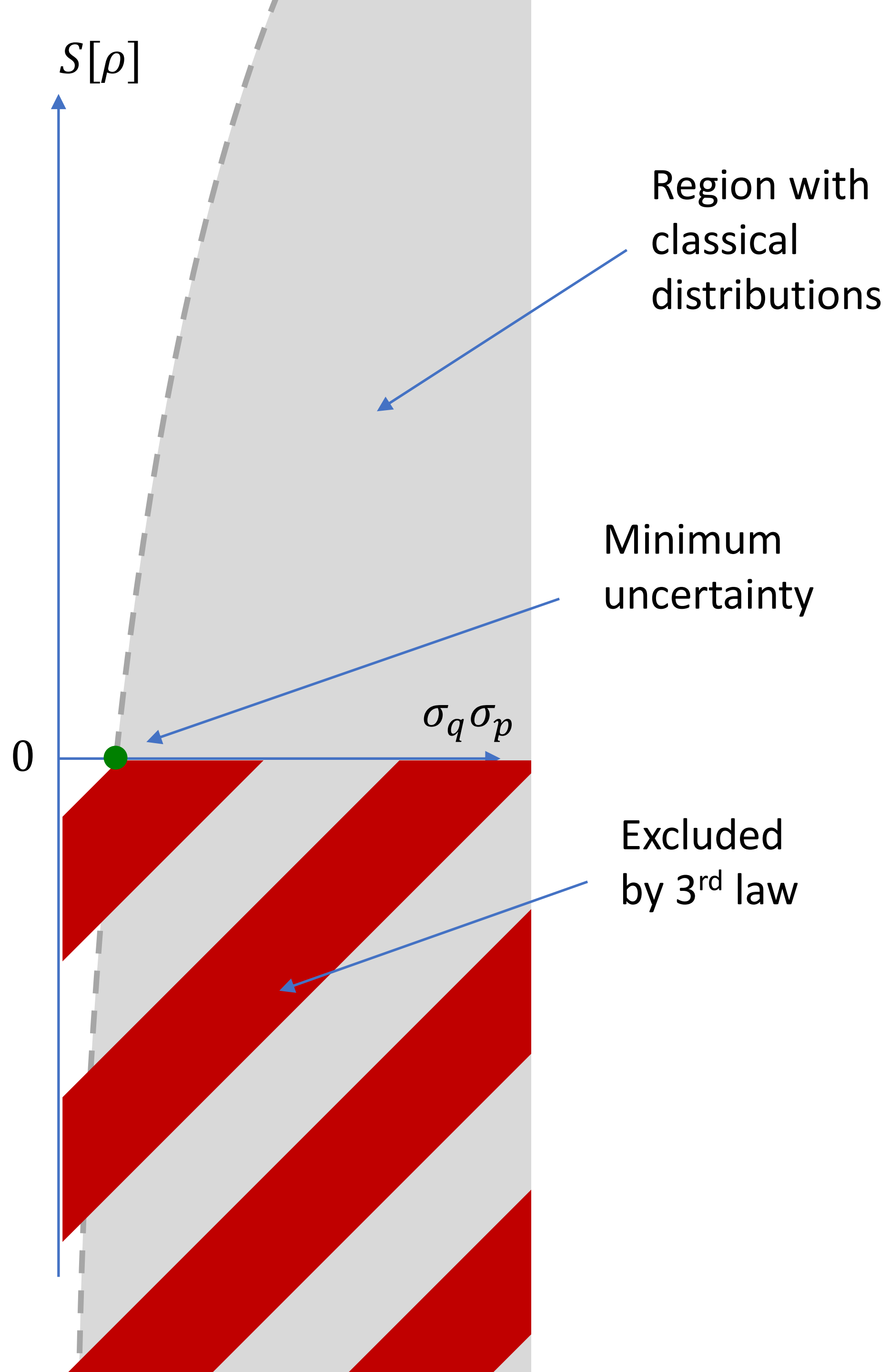

Classical uncertainty principle

Classical mechanics has no lower bound on entropy
⇒ violates third law! What happens if we impose one?

Let W_0 the volume of phase space over which a uniform distribution has zero entropy.

Int J Quant Inf **18**, 01, 1941025 (2020)

$$\sigma_q \sigma_p \geq \frac{W_0}{2\pi e}$$

Equality for independent Gaussians

Lower bound on entropy
⇒ lower bound on uncertainty

Don't need the full quantum theory to derive the uncertainty principle:
only the lower bound on entropy

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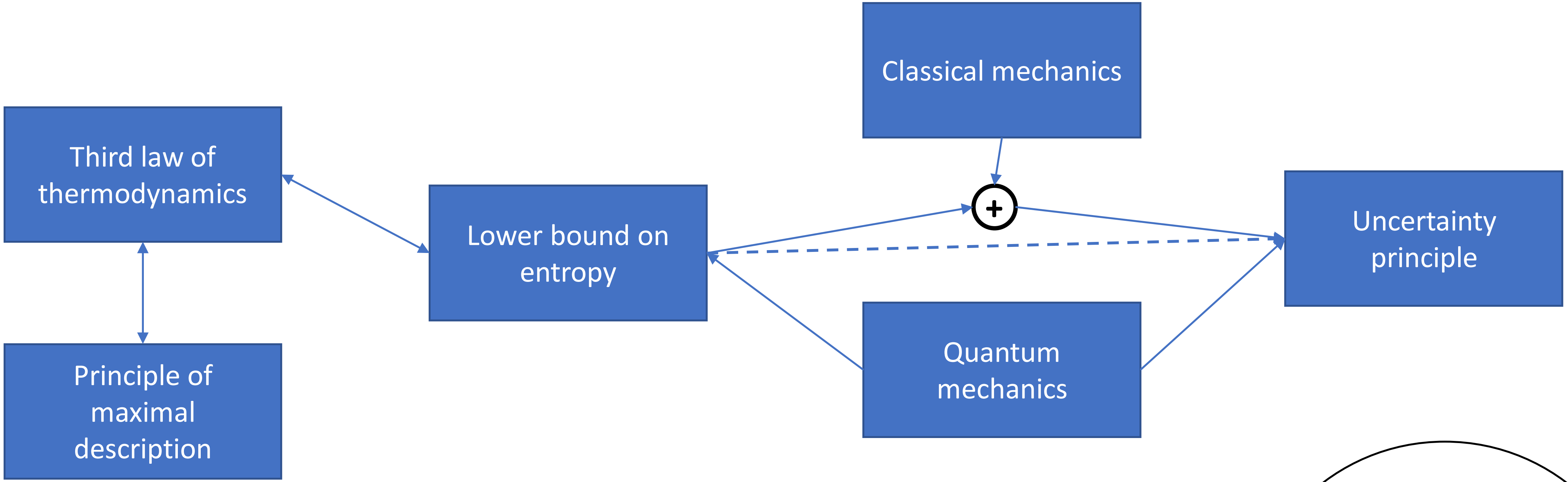
Assumptions of Physics

3rd law of thermodynamics and uncertainty principle



No state can describe a system more accurately than stating the system is not there in the first place

The uncertainty principle is a consequence of the principle of maximal description

Can we understand the rest of quantum mechanics in the same way?

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Assumptions of Physics

Reverse Physics: Quantum mechanics

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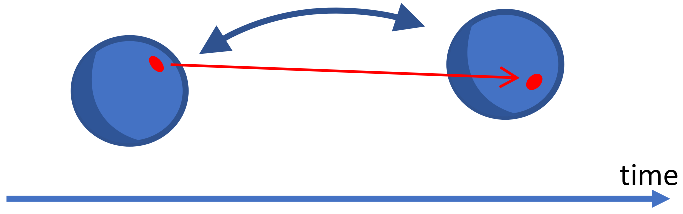
Assumptions of Physics

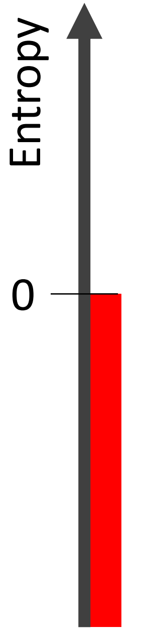
Quantum mechanics as irreducibility

Classical

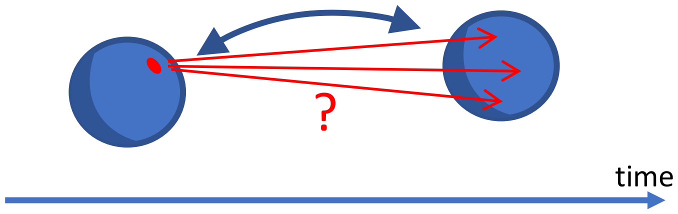


Can prepare ensembles at arbitrarily low entropy: we can study arbitrarily small parts

We always have access to the internal dynamics



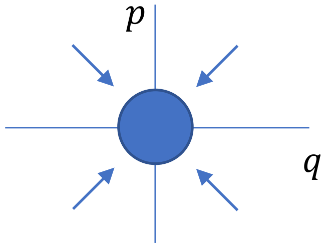
Quantum



Entropy is bounded at zero: we cannot study parts

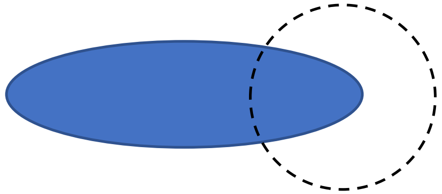
No access to the internal dynamics

Minimum uncertainty



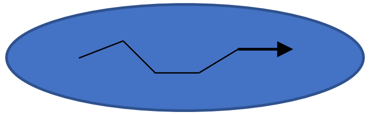
Can't squeeze ensemble arbitrarily

Non-locality



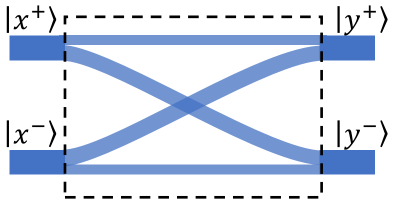
Can't refine ensembles ⇒
Can't interact with parts

Superluminal effects that can't carry information



Can't refine ensembles ⇒
Can't extract information

Probability of transition



$$p(x^+|y^-) = p(y^-|x^+)$$

Symmetry of the inner product

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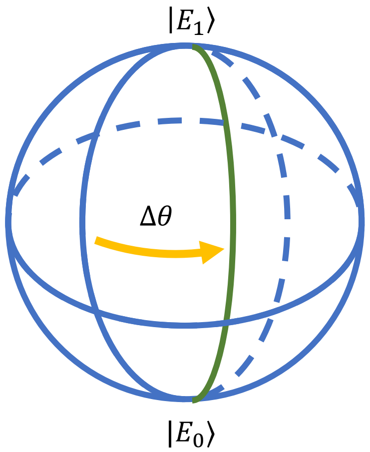
Assumptions of Physics

Time evolution and measurements

$$\rho_I \longrightarrow \boxed{P} \longrightarrow \rho_O = P(\rho_I)$$

Any process (deterministic or stochastic) will take an ensemble as input and return an ensemble as output

$$P(p_1\rho_1 + p_2\rho_2) = p_1 P(\rho_1) + p_2 P(\rho_2)$$



Deterministic and reversible

$$\rho_I \longrightarrow \boxed{P} \xrightarrow{\rho_O} \boxed{P^{-1}} \longrightarrow \rho_I$$

Conserves probability and allows an "inverse"

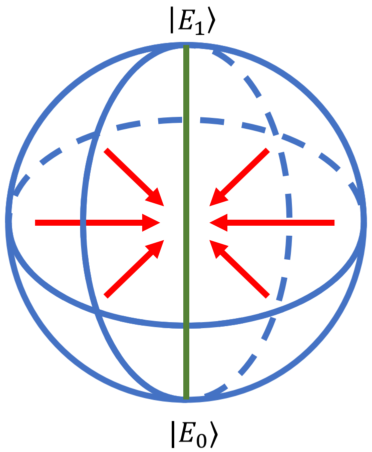
$\Rightarrow$ Unitary operation

Measurement

$$\rho_I \longrightarrow \boxed{P} \xrightarrow{\rho_O} \boxed{P} \longrightarrow \rho_O$$

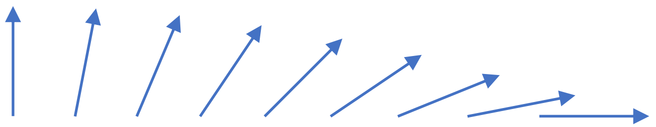
Must be repeatable

$\Rightarrow$ Projection



Measurement problem: unitary $\nRightarrow$ projections … projections $\Rightarrow$ unitary

Unitary evolution $\equiv$ sequence of infinitesimal projections



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Assumptions of Physics

Parallels between QM and thermodynamics

$$U = e^{\frac{O\Delta t}{\imath\hbar}}$$

Eigenstates → states unchanged by the process → equilibria of the process

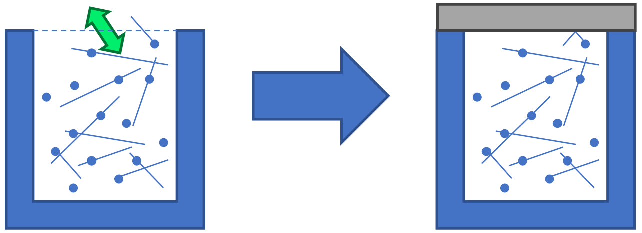
Every state is an eigenstate of some unitary / Hermitian operator → all states are equilibria

Every mixed state commutes with some unitary operator (same eigenstates used calculate entropy)

Quantum contexts

⟺

Boundary conditions between system and environment



$[\mu, V, T]$

$[N_1, V, T]$

$[N_2, V, T]$

$[\dots, V, T]$

Different equilibria, different variables

$|x^+\rangle$

$|z^+\rangle$

$|z^-\rangle$

Different contexts, different variables

Equilibration

Projections ⟺ ~~Measurements~~

Unitary ⟺ Quasi-static

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Assumptions of Physics

Entropic nature of physical theories

Thermodynamics/Statistical mechanics are not built on top of mechanics

Mechanics is the ideal case of thermodynamics/statistical mechanics

Best preparation ⇒ pure state

Best process ⇒ map between pure states

$\mu(U)$

Symplectic manifold

ρ uniform over U

Standard probability

$$\rho(x) = \frac{1}{\mu(U)}$$

Information theory

$$H(\rho) = \log \mu(U)$$

The geometric structure of both classical and quantum mechanics is ultimately an entropic structure

We can only prepare/measure ensembles. Ensembles can offer a unified way of thinking about states.

$\langle \psi | \phi \rangle$

Projective Hilbert space

$$\rho = \frac{1}{2}\rho_\psi + \frac{1}{2}\rho_\phi$$

Quantum probability

$$p(\psi|\phi) = |\langle \psi | \phi \rangle|^2$$

Quantum information theory

$$H(\rho) = H\left(\frac{1+\sqrt{p}}{2}, \frac{1-\sqrt{p}}{2}\right)$$

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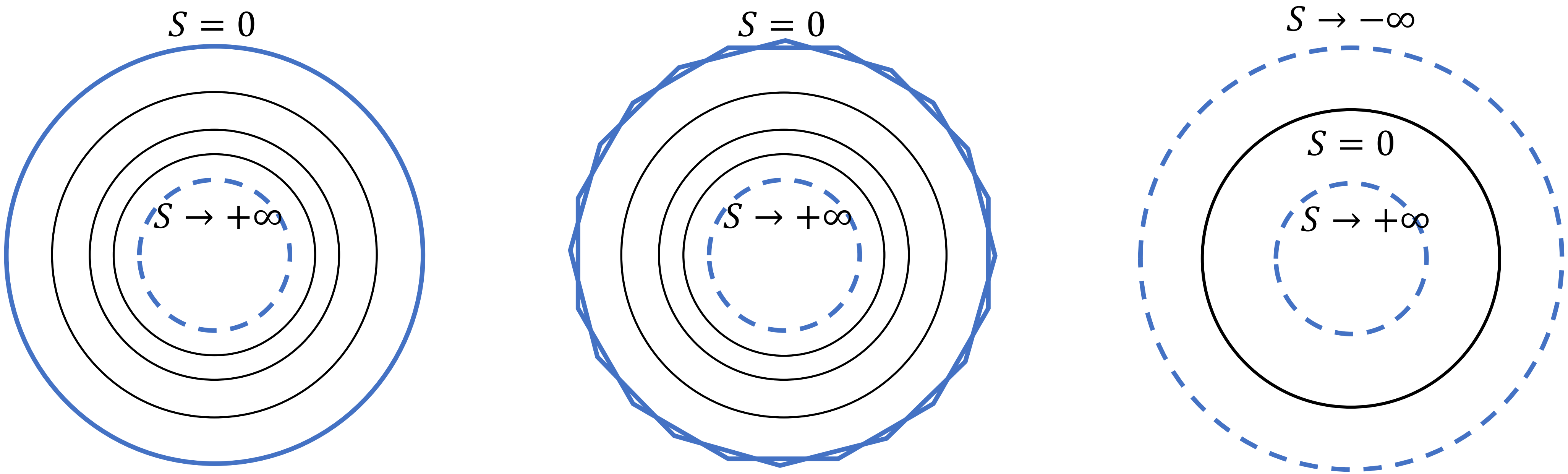
Assumptions of Physics



Quantum | Classical discrete infinite | Classical continuum

Quantum mechanics is a hybrid between discrete and continuum

Quantum pure states form a manifold (like classical continuum) where each state has zero entropy (like classical discrete)

Quantum mixed states have no single decomposition in terms of pure states, classical continuum mixed states have no single decomposition in terms of zero entropy states

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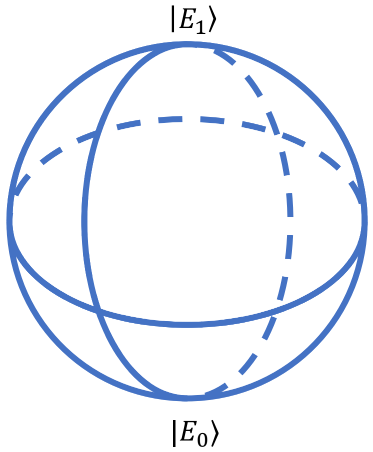
Assumptions of Physics

Recovering QM from assumptions on ensembles



Ensembles can mix ⇒ Form a convex space

Irreducibility ⇒ Extreme points in the convex space

Continuous time ⇒ Extreme points form a manifold (not discrete)

Frame-invariance ⇒ Manifold is symplectic

Homogeneity ⇒ All two dimensional subspaces are spheres

2-sphere only symplectic sphere

Is this enough to recover complex projective spaces?

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Assumptions of Physics

Unphysicality of Hilbert spaces

Hilbert space: complete inner product vector space

Redundant on finite-dimensional spaces. For infinite-dimensional spaces, it allows us to construct states with infinite expectation values from states with finite expectation values

Exactly captures measurement probability/entropy of mixtures and superposition/statistical mixing

Physically required

Extremely physically suspect!!!

⇒ Thus requires us to include unitary transformations (e.g. change of representations and finite time evolution) that change finite expectation values into infinite ones

Suppose we require all polynomials of position and momentum to have finite expectation

⇒ Schwartz space

Maybe more physically appropriate?

Closed under Fourier transforms
Used as starting point for theories of distributions

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Assumptions of Physics

QM postulates revisited

⇒ Recover mathematical structure of quantum mechanics from properties of ensembles

State postulate:
states are rays of a complex vector space

→ Recovered from properties of ensembles and rules of ensemble mixing

Measurement postulate:
projection measurement and Born rule

→ Projections as processes with equilibria

Born rule recoverable from entropy of mixing

Composite system postulate:
tensor product for composite system

→ Derived from other postulates

PRL 126, 110402 (2021)

Evolution postulate:
unitary evolution (Schrödinger equation)

→ Deterministic/reversible evolution

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Assumptions of Physics

What about field theories?

Classical field theory (EM fields, general relativity, …)

Quantum field theory (QED, QCD, Electroweak, …)

We lack the “correct math” to generalize

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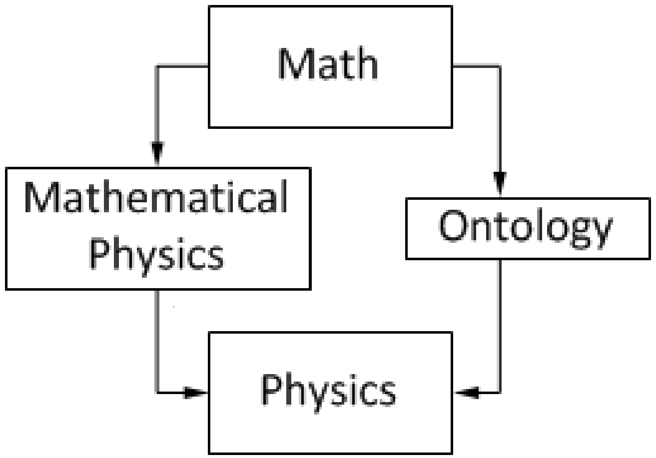
Assumptions of Physics

Physical mathematics

In modern physics, mathematics is used as the foundation of our physical theories

From Hossenfelder's *Lost in Math*: "[…] finding a neat set of assumptions from which the whole theory can be derived, is often left to our colleagues in mathematical physics […]"



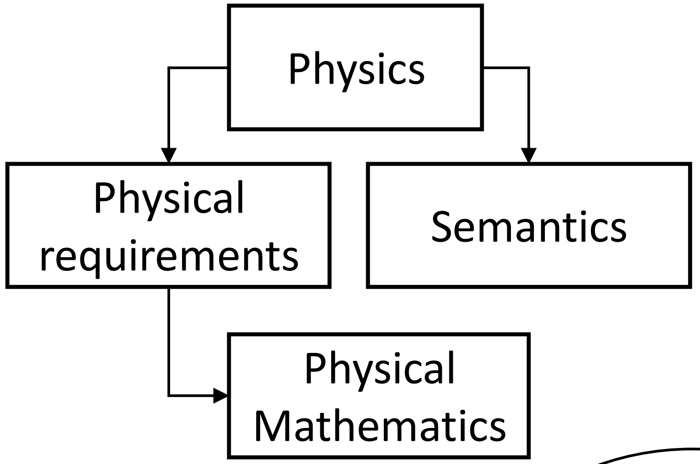
From Wikipedia "Mathematical Physics"

Mathematical content of a theory can never tell us the full physical content

David Hilbert: "Mathematics is a game played according to certain simple rules with meaningless marks on paper."

Bertrand Russell: "It is essential not to discuss whether the first proposition is really true, and not to mention what the anything is, of which it is supposed to be true."

We need to identify which parts of mathematics are "correct" to capture physical properties in a specific realm of applicability



Mathematical structures must be justified by physical requirements

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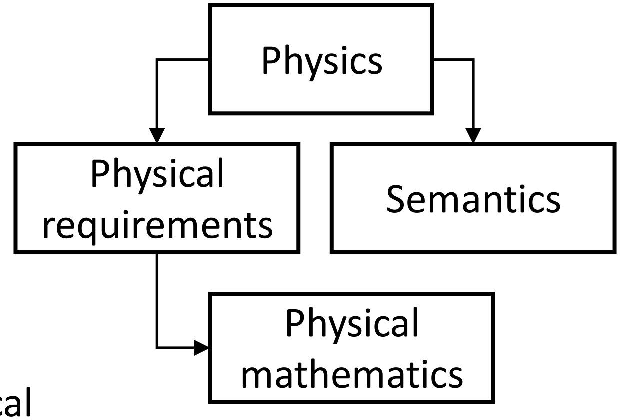
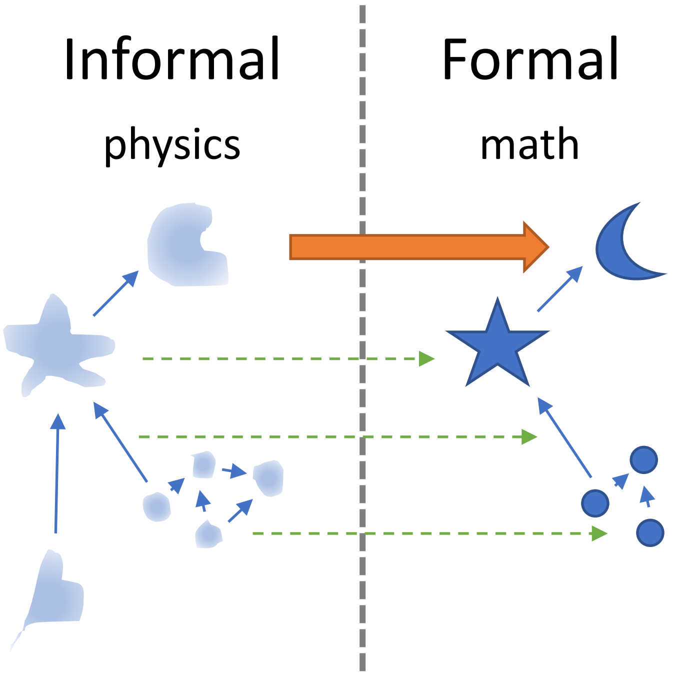
Assumptions of Physics

Physical mathematics



Physics is defined in terms of physical objects and operational definitions



Under assumptions, idealizations and approximations, physical objects and their properties are expressed with a formal system through axioms and definitions.

All physical content is captured by the definitions and axioms

The map between informal and formal is the most delicate and important step, and it is also the least studied!!!

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Assumptions of Physics

Examples: symplectic space and probability spaces

Hamiltonian mechanics

Phase space (symplectic manifold) | Hamiltonian evolution

Differentiable manifold | Symplectic structure

Manifold | Differentiable structure

Topological space | Locally $\mathbb{R}^n$

Determinism/reversibility

Observer independent count of states

Infinitesimal reducibility

Experimentally distinguishable cases with verifiable statements

Identified by independent continuous quantities

We can see what each additional mathematical layer represents and under what assumptions

Probability space

Set of points | σ-algebra | Measure

Experimentally distinguishable cases

Statements associated with experimental tests

Probability that a statement is true

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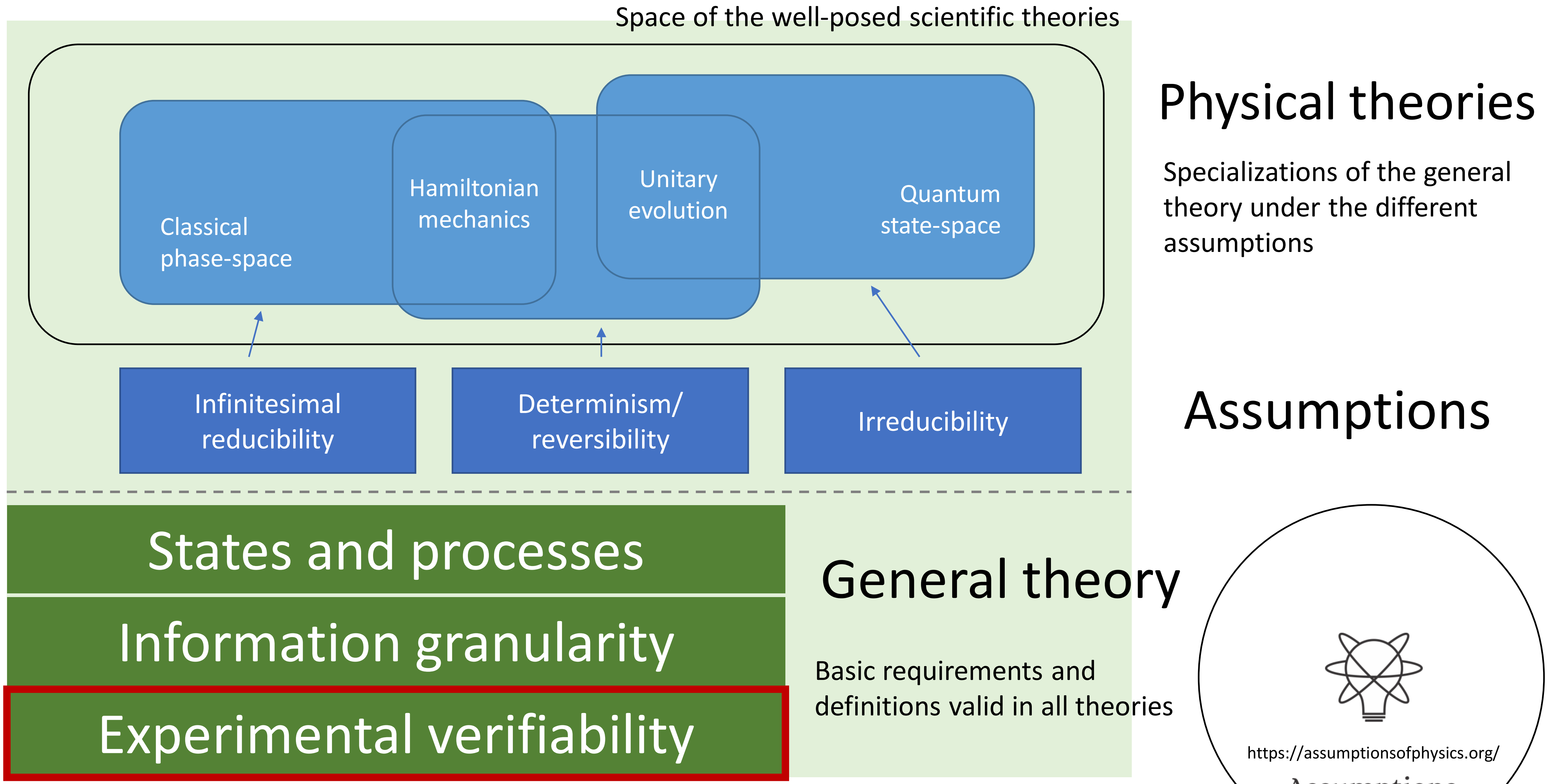
Assumptions of Physics



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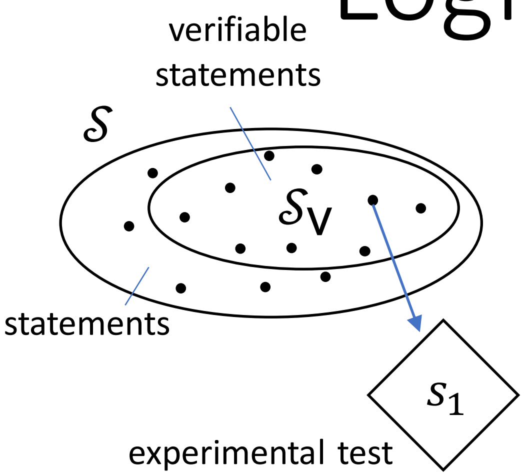
Assumptions of Physics

Logic of experimental verifiability

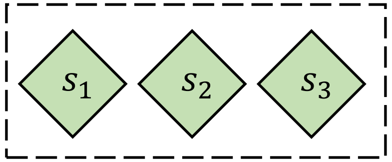
Top. Proc. **54**
pp. 271-282 (2019)



s_1	Test Result
T	SUCCESS (in finite time)
F	FAILURE (in finite time)
	UNDEFINED

Finite conjunction
(logical AND)

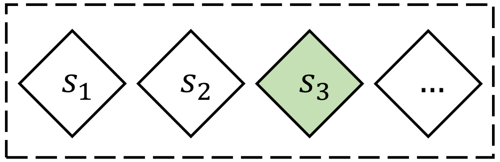
$$\bigwedge_{i=i}^{n} s_i$$



All tests must succeed

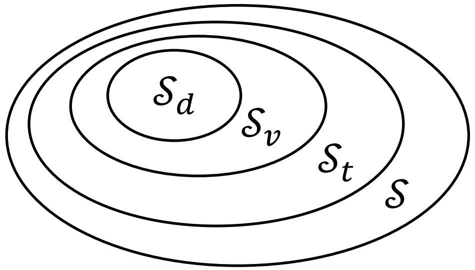
Countable disjunction
(logical OR)

$$\bigvee_{i=1}^{\infty} s_i$$



One successful test is sufficient

Physical theories (evidence based)
⇒ all theoretical statements associated with tests



Operator	Gate	Statement	Theoretical Statement	Verifiable Statement	Decidable Statement
Negation	NOT	allowed	allowed	disallowed	allowed
Conjunction	AND	arbitrary	countable	finite	finite
Disjunction	OR	arbitrary	countable	countable	finite

Some mathematical theories (formally well-posed)
have “too many statements” to be physically meaningful

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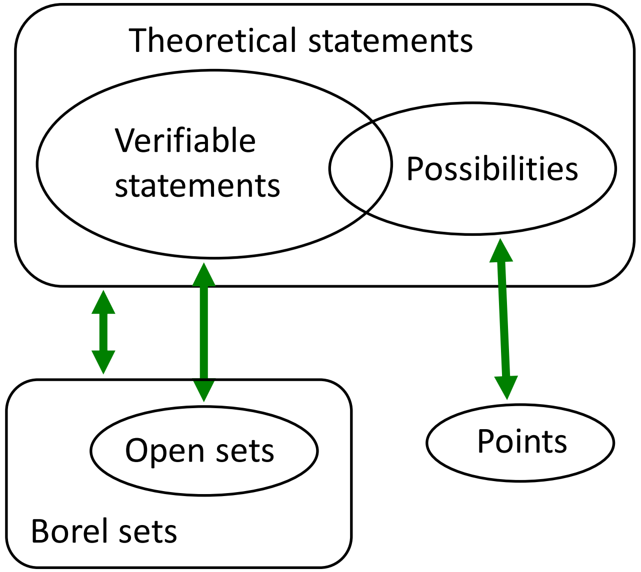
Assumptions of Physics

Topology and σ-algebra

Experimental verifiability $\Rightarrow$ topology and σ-algebras (foundation of geometry, probability, …)

Perfect map between math and physics



s_1	Test Result
T	SUCCESS (in finite time)
	UNDEFINED
F	UNDEFINED
	FAILURE (in finite time)

$int(A)$ corresponds to the verifiable part of a statement

∂A corresponds to the undecidable part of a statement

$ext(A)$ corresponds to the falsifiable part of a statement

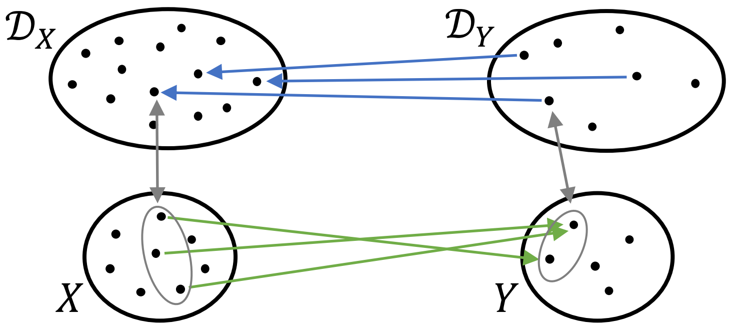
NB: in physics, topology and σ-algebra are parts of the **same** logic structure

Open set $(509.5, 510.5) \Leftrightarrow$ Verifiable "the mass of the electron is 510 ± 0.5 KeV"

Closed set $[510] \Leftrightarrow$ Falsifiable "the mass of the electron is exactly 510 KeV"

Borel set $\mathbb{Q}$ ($int(\mathbb{Q}) \cup ext(\mathbb{Q}) = \emptyset$) $\Leftrightarrow$ Theoretical "the mass of the electron in KeV is a rational number" (undecidable)

Inference relationship $r: \mathcal{D}_Y \to \mathcal{D}_X$ such that $r(s) \equiv s$

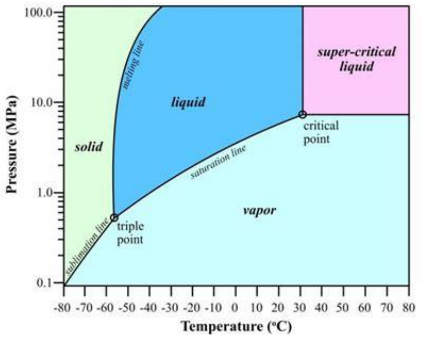


Causal relationship $f: X \to Y$ such that $x \preccurlyeq f(x)$

Inference relationship
$\Leftrightarrow$
Causal relationship

Relationships must be topologically continuous

Topologically continuous consistent with analytic discontinuity on isolated points



Phase transition $\Leftrightarrow$ Topologically isolated regions

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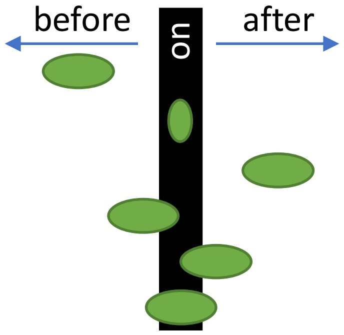
Assumptions of Physics

Quantities and ordering

Phys. Scr. **95** 084003 (2020)

Goal: deriving the notion of quantities and numbers (i.e. integers, reals, …) from an operational (metrological) model

A **reference** (i.e. a tick of a clock, notch on a ruler, sample weight with a scale) is something that allows us to distinguish between a before and an after



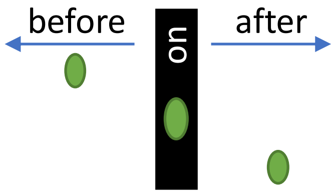
Mathematically, it is a triple (b, o, a) such that:

- b and a are verifiable
- The reference has an extent ($o \not\equiv \bot$)
- If it's not before or after, it is on ($\neg b \wedge \neg a \preccurlyeq o$)
- If it's before and after, it is on ($b \wedge a \preccurlyeq o$)

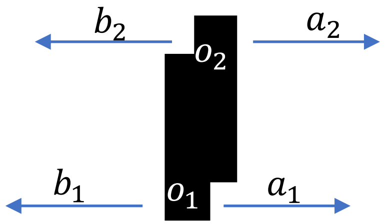
Numbers defined by metrological assumptions, NOT by ontological assumptions

To define an **ordered** sequence of possibilities, the references must be (nec/suff conditions):

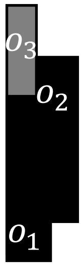
Strict



Aligned



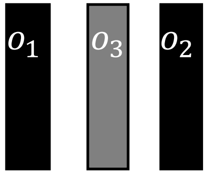
Refinable



$\Rightarrow (X, \leq)$

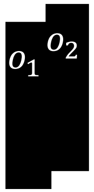
+

Dense



$\Rightarrow (X, \leq) \cong (\mathbb{R}, \leq)$

Sparse



$\Rightarrow (X, \leq) \cong (\mathbb{Z}, \leq)$

The hard part is to recover ordering. After that, recovering reals and integers is simple.

Assumptions untenable at Planck scale:
no consistent **ordering**: no "objective" "before" and "after"

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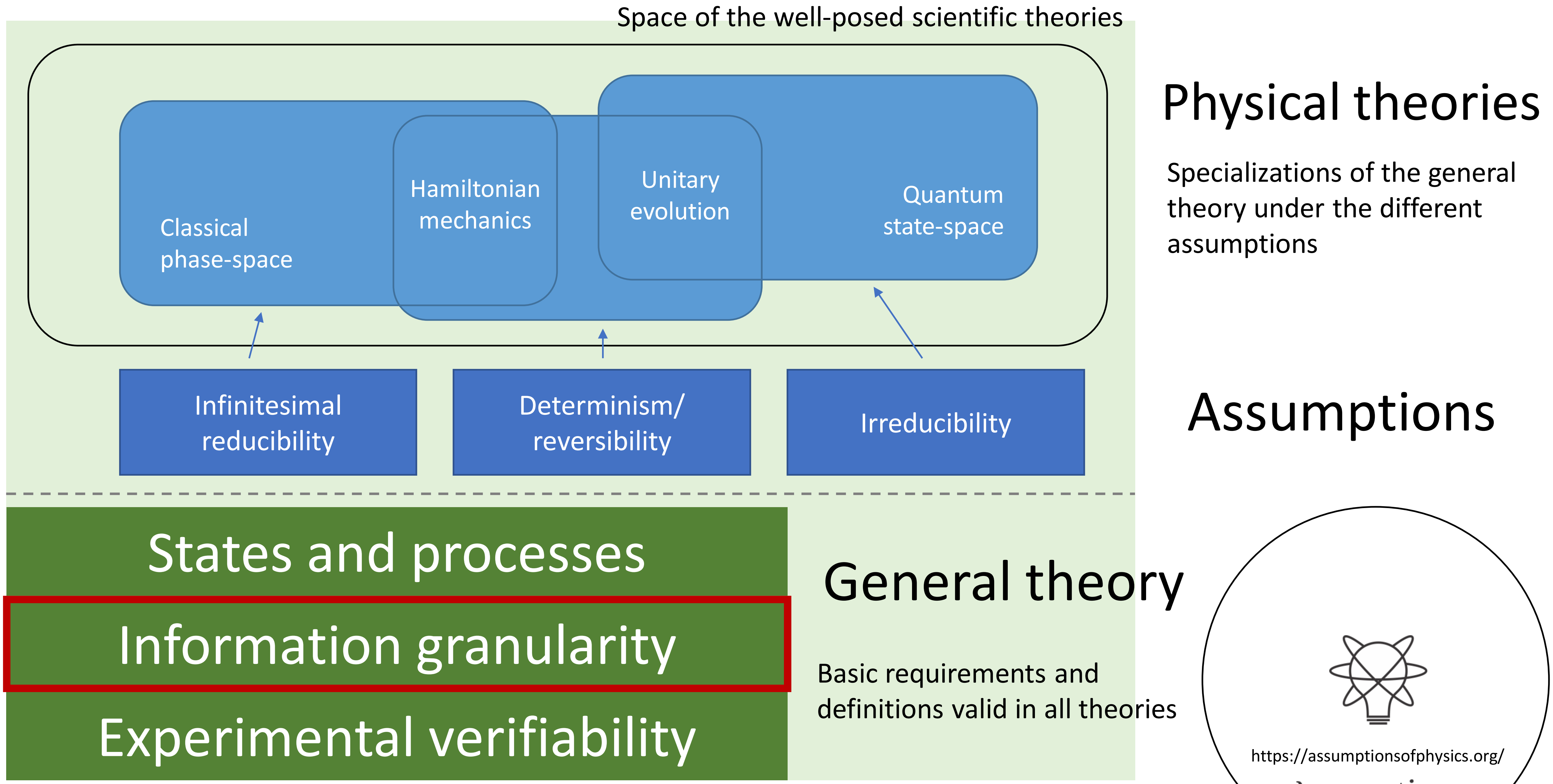
Assumptions of Physics



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Assumptions of Physics

Information granularity

Logical relationships $\Leftrightarrow$ Topology/σ-algebra

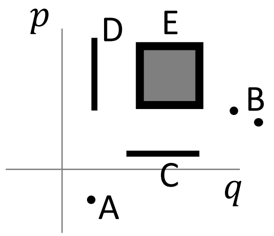
- "The position of the object is between 0 and 1 meters" $\preccurlyeq$ "The position of the object is between 0 and 1 kilometers"
- "The fair die landed on 1" $\preccurlyeq$ "The fair die landed on 1 or 2"
- "The first bit is 0 and the second bit is 1" $\preccurlyeq$ "The first bit is 0"

Granularity relationships $\Leftrightarrow$ Geometry/Probability/Information

- "The position of the object is between 0 and 1 meters" $\lesssim$ "The position of the object is between 2 and 3 kilometers"
- "The fair die landed on 1" $\lesssim$ "The fair die landed on 3 or 4"
- "The first bit is 0 and the second bit is 1" $\lesssim$ "The third bit is 0"

$\Rightarrow$ Measure theory, geometry, probability theory, information theory, … all quantify the level of granularity of different statements

A partially ordered set allows us to compare size at different level of infinity and to keep track of incommensurable quantities (i.e. physical dimensions)



A $\lesssim$ B $\lesssim$ C $\lesssim$ E

C $\not\lesssim$ D

D $\not\lesssim$ C

Once a "unit" is chosen, a measure quantifies the granularity of another statement with respect to the unit

$\mu_u : \bar{\mathcal{D}} \to \mathbb{R}$

$\mu_u(u) = 1$

$s_1 \lesssim s_2 \Rightarrow \mu_u(s_1) \leq \mu_u(s_2)$

$\mu_u(s_1 \vee s_2) = \mu_u(s_1) + \mu_u(s_2)$ if s_1 and s_2 are incompatible

However, quantum mechanics requires a "twist" at the measure theoretic level

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Assumptions of Physics

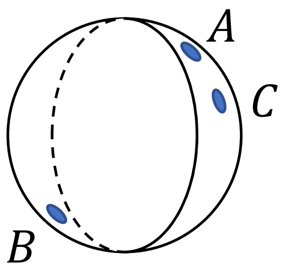
Need for non-additive measure

entropy of uniform distribution

count of states

$$H(\rho_U) = \log \mu(U)$$

Assume usual link between entropy and count of states



$\mu(\{A\}) = 2^0 = 1$

$\mu(\{A, B\}) = 2^1 = 2$

not additive

$\mu(\{A, C\}) < 2 = \mu(\{A\}) + \mu(\{C\})$

$\mu(\{A, B, C\}) < 2 = \mu(\{A, B\})$

not monotonic

In quantum mechanics, literally $1 + 1 \leq 2$

	Single point		Finite continuous range	
	$\mu(U)$	$\log \mu(U)$	$\mu(U)$	$\log \mu(U)$
Counting measure $\mu(U) = \#U$ Number of points	1	0	$+\infty$	$+\infty$
Lebesgue measure $\mu([a, b]) = b - a$ Interval size	0	$-\infty$	$< \infty$	$< \infty$
"Quantized" measure $\mu(U) = 2^{H(\rho_U)}$ Entropy over uniform distribution	1	0	$< \infty$	$< \infty$

Pick two!

1. Single point is a single case (i.e. $\mu(\{\psi\}) = 1$)
2. Finite range carries finite information (i.e. $\mu(U) < \infty$)
3. Measure is additive for disjoint sets (i.e. $\mu(\cup U_i) = \sum \mu(U_i)$)

Physically, we count states all else equal

Contextuality $\Longleftrightarrow$ non-additive measure

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Assumptions of Physics

Differentiability in math

Differentiable manifold

Manifold | Differentiable structure

Mathematicians have developed several, increasingly abstract, definitions for differentials, derivatives, integrations, tangent vectors… are they suitable for physics?

Changes of coordinates are differentiable

Defined on top of Fréchet derivative

Vector defined as derivation of a scalar function

$$v: C^\infty(X, \mathbb{R}) \to C^\infty(X, \mathbb{R})$$

vector basis

$$v(f) = v^i \partial_i f$$

Does not make sense physically!

- velocity is not a derivation
- momentum is not a function of a derivation
- derivations ∂_i depend on units and can't be summed (e.g. $\partial_r + \partial_\theta$)
- Two mathematical notions of differentials (the new one and the one hidden in the Fréchet derivative)
- Infinitesimal objects are limits of finite objects, not the other way around

Differentials defined as linear functions of vectors

$$dx: V \to \mathbb{R}$$

$$dx(v) = dx(v^i \partial_i) = v^x$$

So are convectors, like momentum

Integrals defined on top of differential forms

$$\int_\gamma dx = \Delta x$$

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Assumptions of Physics

Differentiability in physics

Infinitesimal reducibility ⇒ differentiability

General notion of differential as an infinitesimal change in ANY vector space

	Time		Space		Temperature

$$
\begin{tikzcd}
\text{Quantity} & t \arrow[r, "x(t)"] \arrow[d] & x \arrow[r, "T(x)"] \arrow[d] & T \arrow[d] \\
\text{Differential} & dt \arrow[r, "\frac{dx}{dt}"] & dx \arrow[r, "\frac{dT}{dx}"] & dT
\end{tikzcd}
$$

Derivative: map between differentials

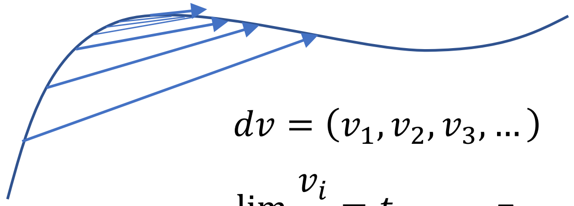
$$dx^i = \frac{dx^i}{dt} dt \qquad dT = \frac{\partial T}{\partial x^i} dx^i$$

velocity (vector)

gradient (covector)

Differentiable function: infinitesimal changes map to infinitesimal changes

Differentiable space: infinitesimal changes are well-defined



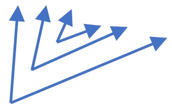
$$dv = (v_1, v_2, v_3, \ldots)$$

$$\lim_{i\to\infty} \frac{v_i}{a_i} = t$$

Tangent vector

Convergence at all points ⇒ differentiability of curve

Infinitesimal surface change



$$d\sigma = ((v_1 \otimes w_1), (v_2 \otimes w_2), \ldots)$$

$$\lim_{i\to\infty} \frac{v_i \otimes w_i}{a_i} = t$$

$$dx^i e_i = dP$$

Manifold displacement (unit free)

Map between the two

Coordinate displacement (units of x^i)

Goal: one notion of derivative

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Assumptions of Physics

Differentiability: forms and linear functionals

Starting point: finite values defined on finite regions

k-functional $\quad$ k-surface $\quad$ k-form $\quad$ k-vector

$$f_k(\sigma^k) = \int \theta_k(d\sigma^k)$$

Thinking in terms of relationships between finite objects leads to better physical intuition

Physically measurable quantities $\quad$ zero-form $\quad$ Differential forms: infinitesimal limit

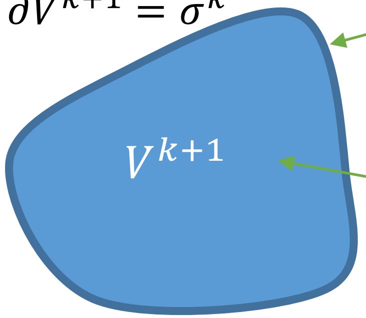
Temperature:	$T(P)$		
Work:	$W(\gamma) = \sum_i W(\gamma_i) = \int f(d\gamma)$	$f = dW/d\gamma$	one-form
Magnetic flux:	$\Phi(\sigma) = \sum_i \Phi(\sigma_i) = \iint B(d\sigma)$	$B = d\Phi/d\sigma$	two-form
Mass:	$m(V) = \sum_i m(V_i) = \iiint \rho(dV)$	$\rho = dm/dV$	three-form

Assume additivity over disjoint regions

The mathematics is contingent upon the assumption of infinitesimal reducibility (e.g. mass in volumes sums only if boundary effects can be neglected)

We can define functionals that act on boundaries

$\partial V^{k+1} = \sigma^k$



f^k $\quad$ Given a functional

Define higher dimensional functional that acts on the boundary

$$g^{k+1}(V^{k+1}) \equiv f^k(\sigma^k)$$

Exterior functional

$$\begin{tikzcd}
g_{k+1} \arrow[d, shift right, "d/dV^{k+1}"'] & f_k \arrow[l, "\text{exterior functional}"'] \arrow[d, shift right, "d/d\sigma^k"'] \\
\omega_{k+1} \arrow[u, shift right, "\int_{V^{k+1}}"'] & \theta_k \arrow[l, "\partial"', "\text{exterior derivative}"] \arrow[u, shift right, "\int_{\sigma^k}"']
\end{tikzcd}$$

Reversing the exterior derivative is finding a (non-unique) potential

$$\partial\partial f^k(\sigma^{k+2}) = f^k(\partial\partial\sigma^{k+2}) = f^k(\emptyset) = 0$$

Boundary of a boundary is the empty set ⇒ exterior derivative of exterior derivative is zero

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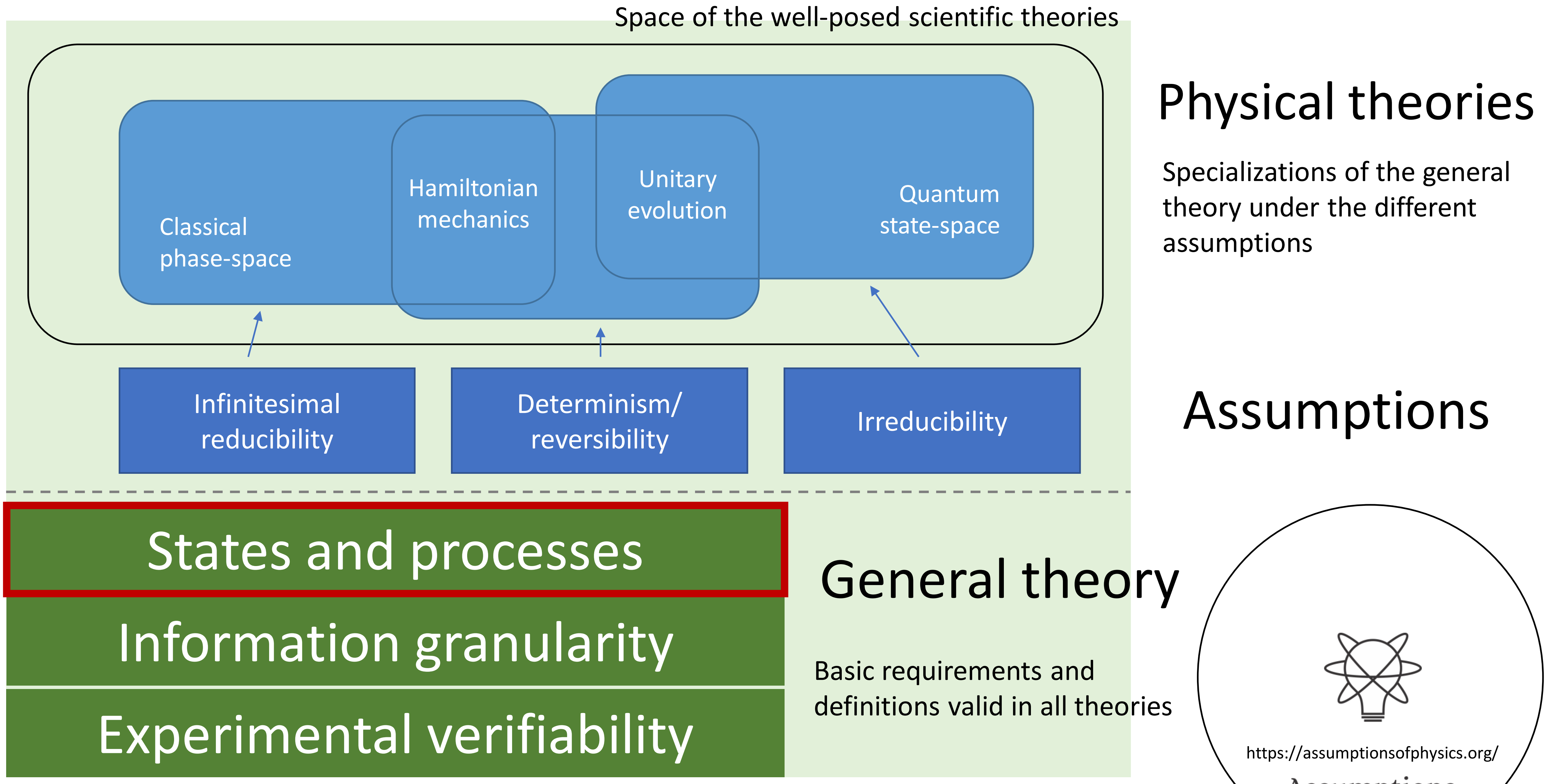
Assumptions of Physics



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Assumptions of Physics

States and processes

Good ideas on how to proceed
Slides would get old fast

Base the notion of states on ensembles

Ensembles form a convex space

$$\sum_i p_i \rho_i$$

Identity: $1\rho_1 + 0\rho_2 = \rho_1$
Idempotence: $p_1\rho_1 + p_2\rho_1 = \rho_1$
Commutativity: $p_1\rho_1 + p_2\rho_2 = p_2\rho_2 + p_1\rho_1$
Associativity: $p_1\rho_1 + \bar{p}_1\left(\frac{p_2}{\bar{p}_1}\rho_2 + \frac{p_3}{\bar{p}_1}\rho_3\right) = \bar{p}_3\left(\frac{p_1}{\bar{p}_3}\rho_1 + \frac{p_2}{\bar{p}_3}\rho_2\right) + p_3\rho_3$

And require an entropy defined

Strictly concave: $S(p_1\rho_1 + p_2\rho_2) \geq p_1 S(\rho_1) + p_2 S(\rho_2)$
Bounded increase: $S(p_1\rho_1 + p_2\rho_2) \leq I(p_1, p_2) + p_1 S(\rho_1) + p_2 S(\rho_2)$

Shannon entropy,
increase due to mixing

Is it a vector space?

$$p\rho_1 + \bar{p}\rho_2 = p\rho_1 + \bar{p}\rho_3 \quad \Longrightarrow \quad \rho_2 = \rho_3$$

May not be necessary

Jensen-Shannon divergence

$$0 \leq S\left(\frac{1}{2}\rho_1 + \frac{1}{2}\rho_2\right) - \frac{1}{2}\left(S(\rho_1) + S(\rho_2)\right) \leq 1$$

Square of a distance function
Related to the Fisher-Rao metric
Defines the geometry of the space

How much the entropy increases during mixing

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Assumptions of Physics

Wrapping it up

- Different approach to the foundations of physics
  - No interpretations, no theories of everything: physically meaningful starting points from which we can rederive the laws and the mathematical frameworks they need
- Reverse physics (reverse engineer principles from the known laws)
  - Classical mechanics is "completed"; very good ideas for both thermodynamics and quantum mechanics; still do not know how to generalize to field theories
- Physical mathematics (rederive the mathematical structures from scratch)
  - Topology and σ-algebras are derived from experimental verifiability; measure theory still needs major work; differentiability we have a good idea; started to formalize states/processes
- The goal is ambitious and requires a wide collaboration
  - Always looking for people to collaborate with in physics, math, philosophy, …

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