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A kernel search heuristic for a fair facility location problem

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ABSTRACT

We consider an uncapacitated location problem where p facilities have to be located in order to serve a given set of customers, and we assume that a customer requesting for a service has to reach a facility at his/her own cost. In this setting, a central issue is that of fairness among customers for the accessibility to the services provided. Every choice regarding the location of facilities corresponds to a distance distribution of customers to reach an open facility. Minimizing the average of this distribution would lead to a p-median problem, where system efficiency is optimized but the fair treatment of users is neglected. Minimizing the maximum (worst-case) of the distribution would lead to a p-center problem, where the unfair treatment of users is mitigated but system efficiency is neglected. To compromise between these two extremes, we minimize the conditional β-mean, i.e., the average distance traveled by the $100 \times \beta\%$ of customers farther from a facility. We call Fair Facility Location Problem (FFLP(β)) the resulting optimization problem, which is formulated as a Mixed-Integer linear Program (MIP) with a proven integer-friendly property. We propose a heuristic framework to produce a set of representative solutions to the FFLP(β). The framework is based on Kernel Search, a heuristic scheme that has been shown to obtain high-quality solutions for a number of MIPs. Computational experiments are reported to validate the quality of the solutions found by the proposed solution algorithm, and to provide some general guidelines regarding the trade-off between average and worst-case optimization. Finally, we report on a case study stemming from the screening activities related to the pandemic triggered by the SARS-CoV-2 virus. The case study regards the optimal location of a number of drive-thru temporary testing sites for collecting swab specimens.

1. Introduction

In all situations in which a decision-maker has to open a set of facilities that will be reached by customers at their own cost, the fair treatment of the customers assumes a crucial role. Such situations are common in location problems encountered in the public sector, but often also in the private one. Some examples are the location of health facilities (e.g., public and private hospitals, community health clinics, primary care centers, or specialized clinics, see Güneş and Nickel, 2015), and public services (e.g., stores, bank branches, public offices, see Filippi et al. (2021)). A special application where the fair treatment of the customers is of paramount importance is connected with the SARS-CoV-2 pandemic. To systematically collect swab specimens, an effective policy is to set up drive-thru temporary testing sites, where citizens remain in their own car while they get tested. Clearly, citizens have to reach the sites at their own cost, and this fact poses an issue of fairness. Other public sector applications involve emergency service units (e.g., fire and police stations, see Calik et al., 2015). In this case, facilities are usually not reached by users at their own cost, but it is of paramount importance to ensure that the most distant demand nodes can be reached rapidly.

Fairness is also relevant in other situations, for example related to customer satisfaction. Among the various elements that can impact on customer satisfaction, performance and fairness are those that can be affected by location decisions. In general terms, *performance* is related to the quality of the product or service purchased, but also to the ease and cost to obtain it. *Fairness* refers to the perception of customers that the ratio of their outcomes to inputs is comparable to the one of the exchange partners (e.g., see Oliver and DeSarbo, 1988). In the context of location problems, the ease and cost to obtain a product or service can be modeled as a function of the distance traveled by the customers to reach the facility providing it. On the other side, fairness can be modeled as a function of the variability of such distance among the customers involved (Filippi et al., 2021).

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In traditional discrete location theory, two prototypical problems have been introduced in the seminal work of Hakimi (1964): the p-median and the p-center problems, both introduced. The p-median problem calls for a location of p facilities that minimizes the average distance between each customer and the nearest of the selected facilities (Daskin and Maass, 2015). On the other side, the p-center problem calls for a location of p facilities that minimizes the maximum distance between a customer and the closest selected facility (Calik et al., 2015). One can improve the overall system efficiency by minimizing the average (or total) distance, at the expense of losing control of the variability among the individual cases. On the other side, one can improve the cost distribution variability (i.e., the fairness) by minimizing the worst distance, but without any attention to the overall system efficiency as this approach focuses only on the worst-case, neglecting all the others. Halpern (1978, 1980) introduces the "cent-dian" concept, where both the above-mentioned objectives are simultaneously incorporated in the same optimization model by minimizing a convex combination of the p-median and the p-center objective functions. By varying the parameter of the convex combination, it is possible to get insights on the trade-off between efficiency and fairness.

Ogryczak and Zawadzki (2002) propose the use of the *conditional β-mean* as a measure of fairness. In location problems, the conditional β-mean can be defined as the average distance traveled by the $(100 \times \beta)$ % of the customers that travel the longest distances to reach the assigned facilities. Hence, parameter $\beta \in (0, 1]$ defines the fraction β of the most poorly served customers that are considered. When $\beta = 1$, the conditional β-mean is computed over the entire set of customers, thereby corresponding to the p-median objective. When β approaches 0, the minimization of the conditional β-mean corresponds to a minimax approach (e.g., see Filippi et al., 2019). Conditional β-mean optimization generalizes the concept of k-sum optimization (Gupta and Punnen, 1990) and, in location theory, the concept of k-centrum (Slater, 1978). Additionally, the conditional β-mean is strictly related to the Conditional Value-at-Risk (CVaR), a risk measure widely used in financial applications (as well as in other application areas, see Filippi et al., 2020) to determine the conditional expectation of losses above a given threshold based on a confidence level (Rockafellar and Uryasev, 2000). Roughly speaking, if we look at the distribution of the distances as a discrete probability distribution, then the conditional β-mean corresponds to the CVaR (with parameter β) of such distribution.

Some properties of the conditional β-mean make it particularly attractive for a location problem, in particular: (*a*) by construction, it is an upper bound on the total average distance; (*b*) it can be embedded into a Mixed-Integer linear Program (MIP) by adding a limited number of continuous variables and linear constraints; (*c*) it is more effective in modeling a compromise between the p-median and the p-center problems than the Halpern's cent-dian approach. The latter property is highlighted by Ogryczak and Zawadzki (2002), who show empirically that the conditional β-mean minimization is able to identify more compromise solutions than the cent-dian approach. A more detailed discussion of the above properties can be found in Filippi et al. (2019), where the authors study the analogies between k-sum and CVaR optimization.

Contributions. Based on the optimization paradigm developed in Filippi et al. (2019), we study a location problem in which a decision-maker has to locate p (uncapacitated) facilities such that performance and fairness are both relevant objectives. To this aim, the conditional β-mean is adopted as optimization criterion. The resulting problem is named the *Fair Facility Location Problem* (FFLP(β)). The specific contributions of this paper are the following. Firstly, a mean-fairness objective function is proposed that includes the average distance of the worst served customers and the average distance traveled by all. Secondly, we prove an integer friendly property allowing to relax most of the integrality requirements. As a proof of concept, Filippi et al. (2019) applied their optimization paradigm, among others, to the classical p-median problem, obtaining the p-median(β) formulation recalled in the following, which is then solved by means of a general-purpose MIP solver. Although the application of the conditional β-mean requires only the addition of continuous variables and linear inequalities, the experiments conducted in Filippi et al. (2019) show empirically that an optimization model incorporating such measure is considerably harder to solve for a MIP solver than its counterpart without it. These findings motivated the third contribution. We develop a heuristic framework able to generate a set of representative solutions to the FFLP(β) that can allow a decision-maker to assess the trade-off between average and worst-case minimization. The framework is based on *Kernel Search* (KS), a heuristic scheme that has been shown to find high-quality solutions for a number of MIPs. Besides a tailored implementation of a KS, the framework includes a generation scheme for the values of β and a mechanism to pass the information gathered on the structure of good solutions from one iteration of the KS to the following one. By using benchmark instances originally introduced for the p-median problem, we validate the performance of the algorithm against a commercial solver. We also investigate the role of the β parameter in the trade-off between the two objectives of minimizing the average distance and minimizing the worst-case distance, providing some general guidelines on the role of the β value. Finally, inspired by the screening activities related to the pandemic triggered by the SARS-CoV-2 virus, we analyze the problem of deciding the optimal location of p drive-thru temporary testing sites for collecting swab specimens.

Structure of the paper. The paper is organized as follows. In Section 2, the literature most closely related to our research is reviewed. In Section 3, we formally define the FFLP(β) and provide its mathematical formulation. In Section 4, we detail the proposed heuristic framework. In Section 5 computational experiments are described and the results are discussed. Finally, some concluding remarks and future research directions are outlined in Section 6.

2. Literature review

Since the early seventies, a large number of "fairness" or "equity" objectives have been developed in the literature on location problems. It is not within the scope of the present paper to provide a thorough review of all these objectives. The interested reader is referred to Marsh and Schilling (1994), Eiselt and Laporte (1995), and Barbati and Piccolo (2016) for an overview of equity measures in location problems; Farahani et al. (2010) for a survey on multiple criteria facility location problems; Farahani et al. (2019) for a review of facility location problems; Karsu and Morton (2015) for an overview of the operational research literature on inequity averse problems, with a particular focus on those cases showing a trade-off between efficiency and fairness. In the following, we review the research most closely related to our work.

Marsh and Schilling (1994) classify 20 equity measures for facility location problems introduced in the literature prior to 1994. Among these measures, it is worth citing the maximum distance, the variance, the mean absolute deviation, the range, and the Gini coefficient. Since then, a stream of research has focused on the computational aspects arising with the use of such measures. Along this line of research, we mention López-de-los-Mozos and Mesa (2001) who analyze the maximum absolute deviation and its properties, and Mesa et al. (2003) who consider several equity measures, including the variance, the sum of weighted absolute deviations, the maximum weighted absolute deviation, the sum of absolute weighted differences, the range, and the Lorenz measure. Further, Drezner and Drezner (2007) investigate location models which incorporate the variance and, separately, the range of the distances as equity measures. Drezner et al. (2009) analyze the minimization of the Gini coefficient, whereas López-de-los-Mozos et al. (2008) apply the concept of the ordered weighted averaging operator to define a model which unifies and generalizes several inequality measures. Finally, we mention the paper of Kalcsics et al. (2015) who develop some algorithms for the minimization of the variance, the mean of absolute weighted deviations, the maximum weighted absolute deviation, the sum of absolute weighted differences, and the range.

Since the review of Marsh and Schilling (1994), a few additional equity measures have been introduced. Espejo et al. (2009), Chanta et al. (2014), and Rey et al. (2018) study location problems where the concept of envy is applied as a measure of fairness. In general terms, a solution to a decision process involving a number of agents is envy-free, if every agent likes its own situation at least as much as that of any other agent. Drezner et al. (2014) analyze the application of the Quintile Share Ratio as an objective in location problems. In such a context, the Quintile Share Ratio represents the ratio of the total distance traveled by the 20% of the customers with the shortest distance (lowest quintile) over the total distance traveled by the 20% of the customers with the longest distance (highest quintile). Notice that the denominator of the objective function used in Drezner et al. (2014) is related to what, in the context of the present paper, we would call "conditional 0.20-mean".

Ogryczak (2000) suggests a representation of location problems as multiple criteria models, obtained by mapping each customer to an individual objective function, which measures how much a location pattern impacts customer satisfaction (for instance, it can measure the distance or the travel time between the customer and the assigned facility). Such representation leads to a multiple criteria model that considers the entire distribution of individual effects (e.g., distances), and poses the basis for the introduction of the concept of equitable efficiency, which links location problems with theories of inequality measures. This analysis enabled (Ogryczak and Zawadzki, 2002) to propose the conditional β-mean of the distances from customers to facilities as a fairness measure in facility location. As mentioned above, the conditional β-mean generalizes the concept of k-centrum (Slater, 1978) which, in turn, generalizes the p-median and p-center problems by minimizing the sum of the k largest service distances (e.g., see Tamir, 2001). Nevertheless, the k-centrum is not universally considered to be a fairness measure, and in fact is not included in Marsh and Shilling's classification. Filippi et al. (2021) analyze a single-source capacitated facility location problem with two objectives: the minimization of the total cost of the system and the minimization of the conditional β-mean. The resulting formulation is a bi-objective MIP, which is solved with a weighted sum method combined with a Benders decomposition approach.

As already mentioned, the minimization of the conditional β-mean is strictly connected with the optimization of CVaR for a discrete distribution. In the context of location theory, CVaR has been applied as a risk measure to disaster management applications, often modeling the corresponding problem as a two-stage stochastic program. This approach has been adopted in Noyan (2012), Elçi and Noyan (2018), Hu et al. (2016, 2017). Additionally, in the context of humanitarian logistics, Chapman et al. (2018) apply to a deterministic location problem a CVaR-like measure that is, in fact, the conditional β-mean.

From an algorithmic viewpoint, the heuristic proposed in this paper is based on the KS framework (or simply KS) that was initially proposed by Angelelli et al. (2010) for the solution of the multi-dimensional knapsack problem. The KS has been applied to several specific problems, including a portfolio optimization problem (see Angelelli et al., 2012), the index tracking problem (see Guastaroba and and Speranza, 2012), and facility location problems (see Guastaroba and Speranza, 2014). More recently, Tran et al. (2018) adapt the KS to solve the alternative-fuel station location problem, Labbé et al. (2019) apply the KS to the feature selection problem in support vector machine, whereas Santos-Peñate et al. (2020) use the KS to solve the leader problem in a discrete leader–follower location problem. Finally, the KS framework has been extended to the solution of general MIPs in Guastaroba et al. (2017).

3. A MIP model

In this section, after introducing the basic notation, we define the reference formulations for the p-median and p-center problems. Subsequently, we formally define the conditional β-mean and recall the p-median(β) formulation proposed by Filippi et al. (2019). Then, we highlight the drawback of the latter formulation and present the adopted mathematical model for the FFLP(β).

3.1. Definitions and preliminaries

Let $I = \{1, 2, \dots, n\}$ denote a set of customers (or demand points), each with unit demand. Let $J = \{1, 2, \dots, m\}$ be a set of potential locations where one facility could be open. For the sake of simplicity, in the following we refer to each potential location j simply as facility j. Each of such facilities has unlimited capacity, and exactly p of them must be located. Let $c_{ij} \geqslant 0$ denote the distance between customer $i \in I$ and facility $j \in J$. The optimization models detailed below use the following decision variables. For each $j \in J$, variable y_j takes value 1 if facility j is opened, and 0 otherwise. For each pair $i \in I$ and $j \in J$, variable x_{ij} takes value 1 if customer i is assigned to facility j, and 0 otherwise. The p-median problem calls for the selection of p facilities to open from set J such that the average distance traveled to fully serve all the customers is minimized. The feasible domain of the p-median problem, denoted hereafter as $\mathscr{XY}$, can be formulated as follows:

$$\mathscr{XY} := \{\mathbf{x} \in \{0,1\}^{n \times m}, \mathbf{y} \in \{0,1\}^m \mid \sum_{j \in J} x_{ij} = 1 \;\; (i \in I); \quad \sum_{j \in J} y_j = p; \quad x_{ij} \leqslant y_j \;\; (i \in I; j \in J)\}.$$

The *p-median* can be cast as the following MIP model:

$$[p-median] \quad \min\left\{ \frac{1}{n} \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \;\middle|\; (\mathbf{x}, \mathbf{y}) \in \mathscr{XY} \right\}. \tag{1}$$

The *p-center problem* aims at minimizing the maximum distance between a customer and the assigned facility, and can be cast as the following MIP model:

$$[p-center] \quad \min\left\{ u \;\middle|\; u \geqslant \sum_{j \in J} c_{ij} x_{ij} \;\; (i \in I); (\mathbf{x}, \mathbf{y}) \in \mathscr{XY} \right\}.$$

For a given solution $(\mathbf{x}, \mathbf{y}) \in \mathscr{XY}$, the quantity $C_i(\mathbf{x}) = \sum_{j \in J} c_{ij} x_{ij}$ represents the distance traveled by customer i, with $i \in I$. Once $C_i(\mathbf{x})$ has been computed for each customer i, one can construct a discrete distribution of travel distances. This leads to the following definition of *conditional β-mean*, denoted as $M_\beta(\mathbf{x})$.

Definition 1. Given a value of $\beta \in (0, 1]$ and a solution $(\mathbf{x}, \mathbf{y}) \in \mathscr{XY}$, the conditional β-mean $M_\beta(\mathbf{x})$ is the average distance traveled by the $(100 \times \beta)$% of the customers that travel the longest distances $C_i(\mathbf{x})$.

In other words, given a discrete distribution of travel distances, $M_\beta(\mathbf{x})$ is the average of the β-quantile of the customers traveling the longest distances. Hence, the smaller the $M_\beta(\mathbf{x})$, the shorter the right tail of the distribution of the travel distances $C_i(\mathbf{x})$, reducing the dispersion of travel distances, especially the longest ones. The latter remark is related to the observation that $M_\beta(\mathbf{x})$ is a Schur-convex function of the travel distance (see Filippi et al., 2021 for further details).

Let $k_\beta = \lceil \beta \times n \rceil$ denote the (integer) number of customers corresponding to β. If $\beta \to 0$, then $k_\beta = 1$ and the conditional β-mean is simply $\max_i\{C_i(\mathbf{x})\}$. Therefore, it corresponds to a minimax approach, as for the p-center problem. Conversely, when $\beta = 1, k_\beta = n$, and the conditional β-mean is computed over the entire travel distance distribution and corresponds to the average travel distance, as for the p-median problem. Since $M_\beta(\mathbf{x})$ is an average computed over $(100 \times \beta)$% of the customers that travel the longest distances, for any given $\mathbf{x}$ the conditional β-mean is a non-increasing function of β.

The minimization of the conditional β-mean requires the solution of the following optimization problem:

$$\min\{M_\beta(\mathbf{x}) | (\mathbf{x}, \mathbf{y}) \in \mathscr{XY}\},$$

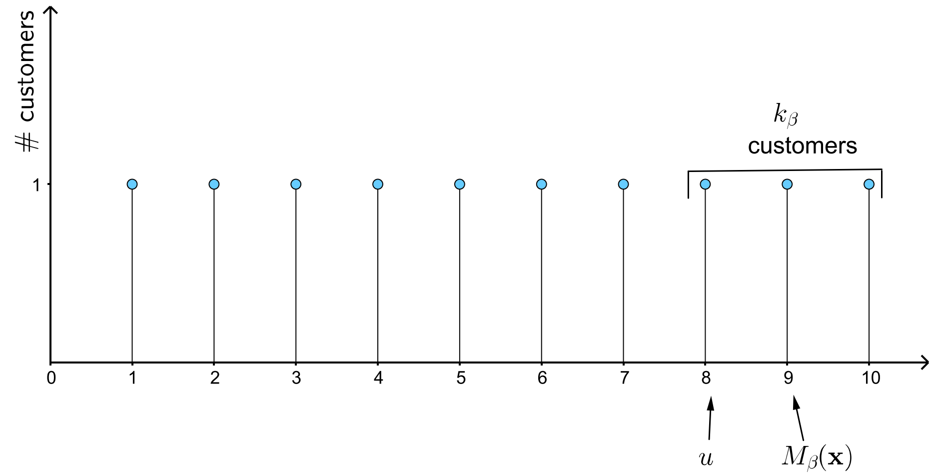


Fig. 1. An illustrative example of the concept of conditional β-mean.

where, following Definition 1, the conditional β-mean can be expressed as:

$$M_\beta(\mathbf{x}) = \max\left\{\frac{1}{k_\beta}\sum_{i\in I}C_i(\mathbf{x})z_i \,\middle|\, \sum_{i\in I}z_i = k_\beta, \quad 0\leqslant z_i\leqslant 1 \text{ and integer } (i\in I)\right\}. \tag{2}$$

As the coefficient matrix in optimization problem (2) is Totally UniModular (TUM), we can relax the integrality constraints on variables z_i $(i \in I)$ and construct the dual problem, obtaining:

$$M_\beta(\mathbf{x}) = \min\left\{k_\beta u + \sum_{i\in I}v_i \,\middle|\, u + v_i \geqslant \frac{1}{k_\beta}C_i(\mathbf{x}), v_i\geqslant 0 \ (i\in I)\right\}, \tag{3}$$

where u is a free variable representing (at the optimum) the β-quantile, i.e., the shortest travel distance associated with the chosen value of β. It is worth noting that the objective function and all the constraints in optimization problem (3) are linear. Expression (3) can be embedded into the p-median model (1) to obtain the following optimization model, named the p-median(β) model (see Filippi et al., 2019):

$$[p-median(\beta)] \quad \min\left\{k_\beta u + \sum_{i\in I}v_i \,\middle|\, u + v_i \geqslant \frac{1}{k_\beta}\sum_{j\in J}c_{ij}x_{ij}, v_i\geqslant 0 \ (i\in I); (\mathbf{x},\mathbf{y}) \in \mathscr{X}\mathscr{Y}\right\}. \tag{4}$$

The general concept of the conditional β-mean is illustrated in the following example.

Example 1. Consider an instance with $n = 10$ customers and a solution $(\mathbf{x},\mathbf{y}) \in \mathscr{X}\mathscr{Y}$ such that the distance traveled by customer 1 is 1, by customer 2 is 2, and similarly for the remaining customers. The associated discrete distribution of travel distances is shown in Fig. 1. In this figure, the horizontal axis displays the travel distance, whereas the vertical axis shows the number of customers associated with each value. Consider $\beta = 0.3$. Then, k_β is equal to 3 (i.e., $M_\beta(\mathbf{x})$ is computed considering the three customers traveling the longest distances). The associated value of u is equal to 8 (the shortest travel distance associated with $\beta = 0.3$), whereas the value of the conditional β-mean $M_\beta(\mathbf{x})$ is 9 (i.e., the average distance traveled by the 3 customers considered).

3.2. Mathematical formulation

A drawback of the conditional β-mean, as well as of the minimax and CVaR, is that it depends only on a (possibly) small portion of the relevant distribution, neglecting the rest of it. In our case, the conditional β-mean is a function of only a $(100\times\beta)$% of the customers, neglecting the remaining $100\times(1-\beta)$%. A remedy is to apply a post-processing procedure to an optimal solution, i.e., once the y variables are fixed, the procedure assigns every customer to the closest open facility. A better approach is to replace the objective function of model (4) with a mean-fairness objective cast as a convex combination of the conditional β-mean and the average distance, where the weight of the latter term is very small. In this way, a lexicographic optimization is implicitly obtained since, among all solutions optimizing the average distance for the k_β most poorly served customers, one that optimizes the average distance of all customers is selected. Hence, we modify the p-median(β) formulation in the following MIP program, that we refer to as the FFLP (β):

$$[FFLP(\beta)] \quad \min \lambda(k_\beta u + \sum_{i\in I}v_i) + (1-\lambda)\sum_{i\in I}\sum_{j\in J}\frac{c_{ij}x_{ij}}{n} \tag{5}$$

$$s.t.\ u + v_i \geqslant \frac{1}{k_\beta}\sum_{j\in J}c_{ij}x_{ij} \qquad i\in I \tag{6}$$

$$\sum_{j\in J}x_{ij} = 1 \quad i\in I \tag{7}$$

$$\sum_{j\in J}y_j = p \tag{8}$$

$$x_{ij}\leqslant y_j \quad i\in I;\ j\in J \tag{9}$$

$$v_i\geqslant 0 \quad i\in I \tag{10}$$

$$y_j\in\{0,1\} \quad j\in J \tag{11}$$

$$x_{ij}\in\{0,1\} \quad i\in I; \quad j\in J. \tag{12}$$

Objective function (5) minimizes the weighted sum of the average distance traveled by the k_β customers with the longest distances, and the average distance traveled by all the customers. Parameter λ is the weight of the convex combination: the larger the λ, the larger the weight on the conditional β-mean. As our analysis is more focused on the first term in the objective function, we have conducted the computational experiments detailed in the following by setting a very large weight on the first term (namely, $\lambda = 0.99$). Constraints (6) along with constraints (10) define the non-negative variables v_i as $\max\{0, \frac{1}{k_\beta}\sum_{j\in J}c_{ij}x_{ij} - u\}$. Thus, each variable v_i measures the deviation of the travel distance $C_i(\mathbf{x})$, divided by k_β, from u when $\frac{1}{k_\beta}\sum_{j\in J}c_{ij}x_{ij} > u$, whereas it is equal to zero in all the other

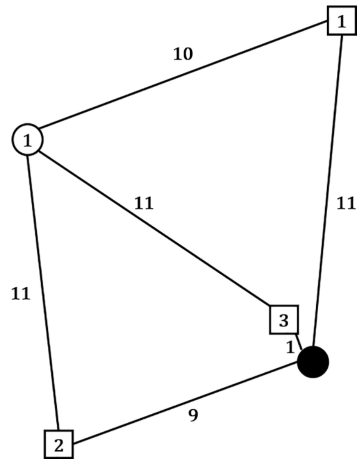


Fig. 2. An illustrative example of the difference between models P-median(β) and FFLP(β). The potential facilities are depicted as squares and the customers are depicted as circles. The black circle represents customers 2, 3, …, 11, assumed to be very close.

cases. Constraints (7) state that each customer i must be fully served, whereas constraint (8) imposes that p facilities must be opened. Constraints (9) establish that customers must be served by an open facility. Finally, constraints (10)–(12) define the variable domains.

Formally speaking, the difference between our model FFLP(β) and model p-median(β) proposed in Filippi et al. (2019) is not large. However, this difference may be relevant in applications. To illustrate this issue, consider a case where $I = \{1,\ldots,11\}, J = \{1,2,3\}, p = 2$, and $\beta = 0.05$. Assume that $c_{11} = 10, c_{12} = c_{13} = 11$, while $c_{i1} = 11, c_{i2} = 9, c_{i3} = 1$ for all $i \in I\setminus\{1\}$, see Fig. 2. Since $k_\beta = 1$, we have that $M_\beta(\mathbf{x})$ is the maximum c_{ij} with $x_{ij} = 1$. It is then easy to verify that the conditional β-mean is minimized for any vector $(\mathbf{x},\mathbf{y}) \in \mathscr{XY}$ such that $x_{11} = 1$ and $x_{i1} = 0$ for all $i \in I\setminus\{1\}$. In particular, two optimal solutions to the p-median(β) formulation are $(\mathbf{x}',\mathbf{y}') \in \mathscr{XY}$ with $x_{11}' = 1$ and $x_{i2}' = 1$ for all $i \in I\setminus\{1\}$ (and $y_1' = y_2' = 1, y_3' = 0$), and $(\mathbf{x}'',\mathbf{y}'') \in \mathscr{XY}$ with $x_{11}'' = 1$ and $x_{i3}'' = 1$ for all $i \in I\setminus\{1\}$ (and $y_1'' = y_3'' = 1, y_2'' = 0$). In terms of the p-median(β) objective function, both solutions have a value equal to 10. Nevertheless, choosing solution $(\mathbf{x}'',\mathbf{y}'')$ corresponds to a total distance traveled by the customers equal to 20 (i.e.,$c_{11}x_{11}'' + \sum_{i\in I} c_{i3}x_{i3}''$), whereas choosing solution $(\mathbf{x}',\mathbf{y}')$ corresponds to a total distance of 100 (i.e., $c_{11}x_{11}' + \sum_{i\in I} c_{i2}x_{i2}'$). Despite the two solutions are equivalent to the p-median(β) model, they are not for the FFLP(β) formulation when $\lambda < 1$. Setting $\lambda = 0.99$, as in our computational experiments, in the latter model the objective function values of $(\mathbf{x}',\mathbf{y}')$ and $(\mathbf{x}'',\mathbf{y}'')$ are approximately equal to 9.99 and 9.92, respectively. Hence, the optimal solution to FFLP(β) is $(\mathbf{x}'',\mathbf{y}'')$. Conversely, model p-median(β) could return $(\mathbf{x}',\mathbf{y}')$ as an optimal solution, with a 5 times larger total distance traveled compared to $(\mathbf{x}'',\mathbf{y}'')$.

3.3. An integer friendly property

The constraint coefficient matrix of model FFLP(β) is not TUM because of the coefficients of variables x_{ij} in constraints (6). Despite that, it is possible to relax the integrality constraints on the x variables, solve the relaxed model and round arbitrarily the obtained solution to get an equivalent optimal solution to the original model. Let $\overline{\text{FFLP}}(\beta)$ denote model FFLP(β) where constraints (12) are replaced by:

$$x_{ij} \geqslant 0 \qquad i \in I; j \in J. \tag{13}$$

The following theorem implies that to optimize model FFLP(β) it is sufficient to solve the relaxed model $\overline{\text{FFLP}}(\beta)$.

Theorem 1. *Let $(u^\star,\mathbf{v}^\star,\mathbf{x}^\star,\mathbf{y}^\star)$ be an optimal solution to $\overline{\text{FFLP}}(\beta)$. If $\mathbf{x}^\star$ is integer, let $\overline{\mathbf{x}} = \mathbf{x}^\star$. Otherwise, for all $i \in I$, choose any $\bar{j}$ such that $x^\star_{i\bar{j}} > 0$ and set $\overline{x}_{i\bar{j}} = 1, \overline{x}_{ij} = 0$ for all $j \neq \bar{j}$. Then, $(u^\star,\mathbf{v}^\star,\overline{\mathbf{x}},\mathbf{y}^\star)$ is an optimal solution to FFLP(β).*

To prove the above result, we need the following lemma.

Lemma 1. *Let $(u^\star,\mathbf{v}^\star,\mathbf{x}^\star,\mathbf{y}^\star)$ be an optimal solution to $\overline{\text{FFLP}}(\beta)$. If there is an index $\bar{\imath}$ and two distinct indices j' and j'' such that $x^\star_{\bar{\imath}j'} > 0$ and $x^\star_{\bar{\imath}j''} > 0$, then $c_{\bar{\imath}j'} = c_{\bar{\imath}j''}$.*

Proof. By contradiction. Assume that $0 < x^\star_{\bar{\imath}j'} < 1$ for some $\bar{\imath}$ and j'. Since from constraint (7) $\sum_{j\in J} x^\star_{\bar{\imath}j} = 1$, there is at least another j'' such that $0 < x^\star_{\bar{\imath}j''} < 1$. Notice that $x^\star_{\bar{\imath}j'} + x^\star_{\bar{\imath}j''} \leqslant 1$. Suppose that $c_{\bar{\imath}j'} < c_{\bar{\imath}j''}$, and set $\overline{x}_{\bar{\imath}j'} = x^\star_{\bar{\imath}j'} + x^\star_{\bar{\imath}j''}, \overline{x}_{\bar{\imath}j''} = 0, \overline{x}_{\bar{\imath}j} = x^\star_{\bar{\imath}j}$ for all $j \neq j', j''$, and $\overline{x}_{ij} = x^\star_{ij}$ for all $i \neq \bar{\imath}$ and all j. Then, we have:

$$\delta = \frac{1}{k_\beta}\sum_{j\in J} c_{\bar{\imath}j}x^\star_{\bar{\imath}j} - \frac{1}{k_\beta}\sum_{j\in J} c_{\bar{\imath}j}\overline{x}_{\bar{\imath}j} = \frac{1}{k_\beta}(c_{\bar{\imath}j''} - c_{\bar{\imath}j'})x^\star_{\bar{\imath}j''} > 0.$$

Notice that:

$$\sum_{i\in I}\sum_{j\in J}\frac{c_{ij}\overline{x}_{ij}}{n} = \sum_{i\in I}\sum_{j\in J}\frac{c_{ij}x^\star_{ij}}{n} - \frac{k_\beta\delta}{n} < \sum_{i\in I}\sum_{j\in J}\frac{c_{ij}x^\star_{ij}}{n}. \tag{14}$$

Inequality (14) guarantees that the change in the values of the $\mathbf{x}$ variables reduces the value of the second term in the objective function of $\overline{\text{FFLP}}(\beta)$. It remains to consider the effect on the u and $\mathbf{v}$ variables, and thus on the first term of the objective function. To do so, we need to distinguish three cases:

(a) $u^\star > (1/k_\beta)\sum_{j\in J} c_{\bar{\imath}j}x^\star_{\bar{\imath}j}$: in this case, the u and the $\mathbf{v}$ variables are not affected;
(b) $u^\star \leqslant (1/k_\beta)\sum_{j\in J} c_{\bar{\imath}j}x^\star_{\bar{\imath}j}$ and $v^\star_{\bar{\imath}} \geqslant \delta$: in this case, the value of $v_{\bar{\imath}}$ must be reduced, but the u variable and all the remaining $\mathbf{v}$ variables need no change;
(c) $u^\star \leqslant (1/k_\beta)\sum_{j\in J} c_{\bar{\imath}j}x^\star_{\bar{\imath}j}$ and $v^\star_{\bar{\imath}} < \delta$: in this case, the u variable and at least k_β variables $\mathbf{v}$ need an adjustment.

In case (a), $(u^\star,\mathbf{v}^\star,\overline{\mathbf{x}},\mathbf{y}^\star)$ is feasible for $\overline{\text{FFLP}}(\beta)$, so that inequality (14) contradicts the optimality of $(u^\star,\mathbf{v}^\star,\mathbf{x}^\star,\mathbf{y}^\star)$.

In case (b), set $\overline{u} = u^\star, \overline{v}_{\bar{\imath}} = v^\star_{\bar{\imath}} - \delta$, and $\overline{v}_i = v^\star_i$ for all $i \neq \bar{\imath}$. Then, $(\overline{u}, \overline{\mathbf{v}},\overline{\mathbf{x}},\mathbf{y}^\star)$ is feasible for $\overline{\text{FFLP}}(\beta)$ and:

$$k_\beta\overline{u} + \sum_{i\in I}\overline{v}_i = k_\beta u^\star + \sum_{i\in I} v^\star_i - \delta < k_\beta u^\star + \sum_{i\in I} v^\star_i.$$

The above inequality along with inequality (14) contradict the optimality of $(u^\star,\mathbf{v}^\star,\mathbf{x}^\star,\mathbf{y}^\star)$.

In case (c), let $I^\star_+ = \{i : v^\star_i > 0\}$ and $I^\star_0 = \{i : u^\star - (1/k_\beta)\sum_{j\in J} c_{ij}x^\star_{ij} = 0\}$. Notice that $I^\star_+ \cup I^\star_0$ includes all indices $i \in I$ such that $(1/k_\beta)\sum_{j\in J} c_{ij}x^\star_{ij} \geqslant u^\star$. Hence, $|I^\star_+| + |I^\star_0| \geqslant k_\beta$, where a strict inequality may apply if $(1/k_\beta)\sum_{j\in J} c_{ij}x^\star_{ij} = u^\star$ for more than one index i. Moreover, in the current case $\bar{\imath} \in I^\star_+ \cup I^\star_0$. Finally, notice that $v^\star_{\bar{\imath}} < \delta$ implies that $u^\star > (1/k_\beta)\sum_{j\in J} c_{\bar{\imath}j}\overline{x}_{\bar{\imath}j}$ due to the definition of δ.

We distinguish two further sub-cases: (c.1) $|I^\star_+| + |I^\star_0| = k_\beta$; ($c$.2) $|I^\star_+| + |I^\star_0| > k_\beta$.

In sub-case (c.1), the indices in $I^\star_+ \cup I^\star_0$ identify exactly the k_β customers with the highest values of $\sum_{j=1}^{m} c_{ij}x^\star_{ij}$, i.e., the customers on which the conditional β-mean $M_\beta(\mathbf{x}^\star)$ is computed. Let:

$$\hat{\imath} = \arg\max\left\{\sum_{j\in J} c_{ij}\bar{x}_{ij} : \frac{1}{k_\beta}\sum_{j\in J} c_{ij}\bar{x}_{ij} < u^\star\right\}$$
$$= \arg\max\left\{\max_{i\neq\bar{\imath}}\left\{\sum_{j\in J} c_{ij}\bar{x}_{ij} : \frac{1}{k_\beta}\sum_{j\in J} c_{ij}\bar{x}_{ij} < u^\star\right\}; \sum_{j\in J} c_{\bar{\imath}j}\bar{x}_{\bar{\imath}j}\right\}.$$

Set $\bar{u} = \frac{1}{k_\beta}\sum_{j\in J} c_{\hat{\imath}j}\bar{x}_{\hat{\imath}j}, \bar{v}_i = v_i^\star + (u^\star - \bar{u})$ for all $i \in (I_+^\star \cup I_0^\star)\setminus\{\bar{\imath}\}, \bar{v}_i = 0$ otherwise. Then, $(\bar{u}, \bar{\mathbf{v}}, \bar{\mathbf{x}}, \mathbf{y}^\star)$ is feasible for $\overline{\text{FFLP}}(\beta)$ and:

$$\begin{aligned} k_\beta\bar{u} + \sum_{i\in I}\bar{v}_i &= k_\beta u^\star - k_\beta(u^\star - \bar{u}) + \sum\nolimits_{i\in(I_+^\star\cup I_0^\star)\setminus\{\bar{\imath}\}}(v_i^\star + (u^\star - \bar{u})) \\ &= k_\beta u^\star - k_\beta(u^\star - \bar{u}) + \sum_{i\in I} v_i^\star + (k_\beta - 1)(u^\star - \bar{u}) - v_{\bar{\imath}}^\star \\ &= k_\beta u^\star + \sum_{i\in I} v_i^\star - (u^\star - \bar{u}) - v_{\bar{\imath}}^\star < k_\beta u^\star + \sum_{i\in I} v_i^\star. \end{aligned}$$

The above inequality along with inequality (14) contradict the optimality of $(u^\star, \mathbf{v}^\star, \mathbf{x}^\star, \mathbf{y}^\star)$.

In sub-case $(c.2)$, we have $|(I_+^\star \cup I_0^\star)\setminus\{\bar{\imath}\}| \geqslant k_\beta$. Set $\bar{u} = u^\star, \bar{v}_{\bar{\imath}} = 0, \bar{v}_i = v_i^\star$ for all $i \neq \bar{\imath}$. Then, $(\bar{u}, \bar{\mathbf{v}}, \bar{\mathbf{x}}, \mathbf{y}^\star)$ is feasible for $\overline{\text{FFLP}}(\beta)$ and:

$$k_\beta\bar{u} + \sum_{i\in I}\bar{v}_i = k_\beta u^\star + \sum_{i\in I} v_i^\star - v_{\bar{\imath}}^\star < k_\beta u^\star + \sum_{i\in I} v_i^\star.$$

The above inequality along with inequality (14) contradict the optimality of $(u^\star, \mathbf{v}^\star, \mathbf{x}^\star, \mathbf{y}^\star)$. This concludes the proof. □

Proof of Theorem 1. If $\mathbf{x}^\star$ is integer, the result is trivial. If $\mathbf{x}^\star$ is not integer, then $\bar{\mathbf{x}}$ satisfies by construction constraints (7), (9), and (12). As a consequence of Lemma 1, $\sum_{j\in J} c_{ij}\bar{x}_{ij} = \sum_{j\in J} c_{ij}x_{ij}^\star$ for all $i \in I$. Hence, the right-hand side of (6) does not change, so that $(u^\star, \mathbf{v}^\star, \bar{\mathbf{x}}, \mathbf{y}^\star)$ is feasible for FFLP(β). Moreover, the objective function value corresponding to $(u^\star, \mathbf{v}^\star, \bar{\mathbf{x}}, \mathbf{y}^\star)$ coincide with the optimal value of the relaxed model $\overline{\text{FFLP}}(\beta)$. This implies the optimality. □

4. A parametric approach

Model FFLP(β) is a parametric problem, in the sense that its optimal solution depends on the chosen value of β. This value captures the trade-off between fairness and efficiency: the smaller the value of β, the higher the importance given to fairness. However, it is not clear how to relate the value of β with the preferences of a decision-maker, and preferences themselves are, in general, hard to elicit. For these reasons, instead of solving model FFLP(β) for a given value of β, we develop a parametric approach to describe how the solution to FFLP(β) changes when β varies from 1 to 0. This description may support a decision-maker in finding the best compromise solution according to his/her own preferences.

As producing the complete set of solutions for any significant value of β would require a huge computational effort and produce an overwhelming amount of information for the decision-maker, we aim at generating a set of representative solutions to FFLP(β). In view of Theorem 1, the solution method developed hereafter is based on the relaxed model $\overline{\text{FFLP}}(\beta)$, where we assume the x variables are continuous.

In the following, we first introduce some basic notations and then detail a heuristic, called KS-FFLP, for the generation of representative solutions to $\overline{\text{FFLP}}(\beta)$.

4.1. Basic notation and definitions

We denote as β_ℓ the value of β used in the ℓ-th iteration of KS-FFLP. Moreover, we denote as $\text{MIP}_{\beta_\ell}(U), U\subseteq J$, the formulation $\overline{\text{FFLP}}(\beta)$ with $\beta = \beta_\ell$ and where the binary variables with an index not in U are fixed to zero, i.e., $y_j = 0$ for all $j \in J\setminus U$. We call $\text{MIP}_{\beta_\ell}(U)$ *restricted* when $U\subset J$.

Unless otherwise stated, any restricted problem includes all continuous variables u and v_j, and a subset of variables x_{ij} selected as detailed below. At each iteration of KS-FFLP, the binary variables are partitioned into a *kernel* $\mathscr{K}\subset J$ and a sequence $\{B_h\}_{h=1,\dots,N_b}$ of N_b *buckets*. Each bucket (except possibly the last one) contains a fixed number of L_b indices. Finally, $\overline{N}_b \leqslant N_b$ is a parameter set by the user which determines the maximum number of buckets to analyze.

4.2. KS for the solution of the $\overline{\text{FFLP}}(\beta)$

A first choice in the design of KS-FFLP is how to generate the representative values of β. If $\beta = 1$ then $k_\beta = n$, and $M_\beta(\mathbf{x})$ –the main term in the $\overline{\text{FFLP}}(\beta)$ objective– is the average of the entire travel distance distribution. When $\beta < 1, M_\beta(\mathbf{x})$ is the average of the k_β longest distances. If β approaches 0 then $k_\beta = 1$ and $M_\beta(\mathbf{x})$ is simply the maximum distance value. Intuition suggests, and preliminary tests confirm, that the sensitivity of the $\overline{\text{FFLP}}(\beta)$ solution to the value of β decreases with β. For this reason, we generate the representative values of β using a geometric progression with common ratio $\Delta \in (0,1)$.

KS-FFLP has two main phases: the initialization and the iteration phases. In the initialization phase, $\beta_1 = 1$. Therefore, $k_1 = n$ and the mathematical formulation for $\overline{\text{FFLP}}(\beta)$ is equivalent to the p-median model. Based on the solution to the LP relaxation of problem $\text{MIP}_{\beta_1}(J)$, the initial kernel $\mathscr{K}$ and the sequence of buckets $\{B_h\}_{h=1,\dots,N_b}$ are created. Then, the algorithm proceeds as presented in Section 4.2.1. The iteration phase is carried out as long as the value of k_β is not equal to 1 –that is, the second phase stops when the number of customers considered is equal to one (i.e., equivalent to the p-center problem). In each iteration of the second phase, the initial kernel and the sequence of buckets are taken from the end of the previous iteration, as detailed in Section 4.2.2. The general scheme of KS-FFLP is outlined in Algorithm 1.

Algorithm 1. The KS-FFLP algorithm.

Require:
Initial value of $\beta_1 = 1$; number of buckets to analyze $\overline{N}_b$; value of $\Delta \in (0,1)$.
Phase 1: Initialization
1: Set $k_\beta \leftarrow n$, and $\ell \leftarrow 1$.
2: Construct an initial kernel $\mathscr{K}$ and the sequence of buckets $\{B_h\}_{h=1,\dots,N_b}$ based on the LP solution to $\text{MIP}_{\beta_1}(J)$.
3: Invoke Algorithm 2.
Phase 2: Iteration
4: $\beta_{\ell+1} \leftarrow \beta_l \times \Delta$, and $\ell \leftarrow \ell + 1$.
5: **repeat**
6: $k_\beta = \lceil \beta_\ell \times n \rceil$.
7: Construct $\mathscr{K}$ and $\{B_h\}_{h=1,\dots,N_b}$ based on their composition at the end of the previous iteration.
8: Invoke Algorithm 2.
9: $\beta_{\ell+1} \leftarrow \beta_\ell \times \Delta$, and $\ell \leftarrow \ell + 1$.
10: **until** $k_\beta = 1$
Ensure:
$\{s_\ell^{UB}\}_{\ell=1,\dots,L}$: Set of best solutions found; $\{z_\ell^{UB}\}_{\ell=1,\dots,L}$: Set of best upper bounds.

4.2.1. Phase 1: Initialization

First, the algorithm solves the LP relaxation of problem $\text{MIP}_1(J)$. Based on this solution, the indices in J are sorted in non-increasing order

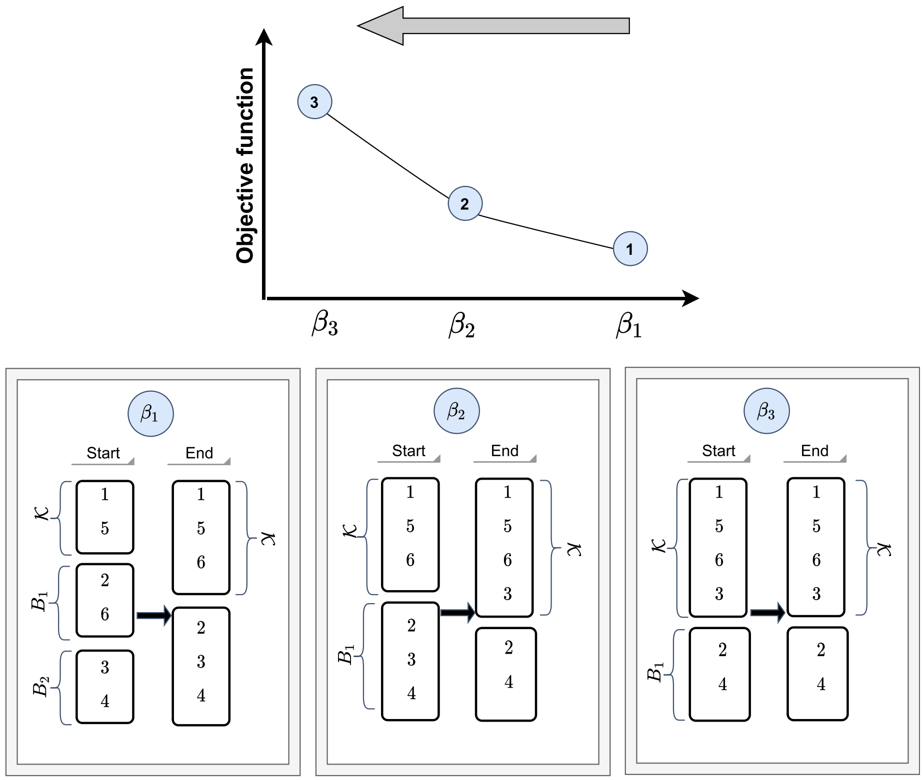


Fig. 3. An example of the construction of the kernel and buckets for three iterations of KS-FFLP.

of the total demand served by the corresponding facility (i.e., $\sum_{i\in I}x_{ij}$) and, if the served demand is null, in non-decreasing order of the reduced cost. Then, for each index $j \in J$, the associated x_{ij} variables are sorted in non-increasing order of their value, first, and for those with value equal to 0, in non-decreasing order of the associated reduced cost. The initial kernel is constructed (Line 2 in Algorithm 1) by selecting all the indices j such that $y_j \geqslant \gamma$, i.e., $\mathscr{K} = \{j \in J | y_j \geqslant \gamma\}$, where $\gamma \in (0,1)$ may be tuned on the characteristics of the instances considered. All the indices in $J\backslash\mathscr{K}$ are partitioned into N_b buckets, each comprising $L_b = |\mathscr{K}|$ elements. To speed up the solution process, for each $j \in J$, we fix to zero some of the x_{ij} that are not promising. More precisely, we fix to zero the $100 \times \delta\%$ of the x_{ij} variables with largest and strictly positive reduced costs. The value of the fraction $\delta \in (0,1)$ may also be tuned on the characteristics of the instances.

Algorithm 2. Solution steps.

Require:
Value of β_ℓ; initial kernel $\mathscr{K}$; sequence of buckets $\{B_h\}_{h=1,\dots,N_b}$; number of buckets to analyze $\overline{N}_b$.
1: Solve $\text{MIP}_{\beta_\ell}(\mathscr{K})$.
2: Initialize solution s_ℓ^{UB} and its value z_ℓ^{UB} with the best feasible solution found.
3: **while** $h \leqslant \min\{N_b, \overline{N}_b\}$ **and** time limit is not reached **do**
4: $s_\ell \leftarrow$ solve modified$\text{MIP}_{\beta_\ell}(\mathscr{K} \cup B_h)$.
{**Learn and adjust**}
5: **if** s_ℓ is feasible **then**
6: Add to $\mathscr{K}$ all indices in B_h that correspond to a variable with a positive value in s_ℓ.
7: Update $s_\ell^{UB} \leftarrow s_\ell$ and $z_\ell^{UB} \leftarrow z_\ell$.
8: **end if**
9: **end while**
Ensure:
s_ℓ^{UB}: Best solution found; z_ℓ^{UB}: Best upper bound.

Subsequently, the restricted problem is solved (Line 1 in Algorithm 2), and the best feasible solution found along with its value are initialized (Line 2 in Algorithm 2). The algorithm continues by iteratively solving a sequence of restricted problems (Lines 3–9 in Algorithm 2). In each iteration h the restricted problem $\text{MIP}_{\beta_1}(\mathscr{K} \cup B_h)$ is solved after the addition of the following two constraints: *(i)* z_1^{UB} is used as a cutoff value to the objective function, and *(ii)* the solution to the restricted problem must include at least one facility from those not selected in the best solution found. More precisely, constraint *(ii)* takes one of the following two forms: (a) if the best solution s_1^{UB} found in the previous iterations is *proven optimal*, at least one variable y_j with an index j in the current bucket B_h must be selected, i.e., $\sum_{j\in B_h} y_j \geqslant 1$; (b) if the best solution found is *feasible but not proven optimal*, at least one variable y_j with an index j among those not selected in the current kernel $\mathscr{K}$ or bucket B_h must be selected, i.e., $\sum_{j\in\mathscr{K}\backslash s_1^{UB}} y_j + \sum_{j\in B_h} y_j \geqslant 1$.

If the modified $\text{MIP}_{\beta_1}(\mathscr{K} \cup B_h)$ is feasible, any of its feasible solutions improves upon the incumbent solution s_1^{UB}. The new best solution, if any, is used to adjust the kernel composition (Line 6 in Algorithm 2) by employing a "learn and adjust" mechanism: all indices j not belonging to $\mathscr{K}$ and such that $y_j > 0$ in the best solution to $\text{MIP}_{\beta_1}(\mathscr{K} \cup B_h)$ are added to kernel $\mathscr{K}$.

Since the value of $\overline{N}_b$ –the maximum number of MIPs solved in each of the main iterations of KS-FFLP– is kept fixed during the execution, while the value of N_b may fluctuate when β changes, the condition $\overline{N}_b \leqslant N_b$ is not guaranteed. For this reason, Algorithm 2 stops when $\min\{N_b, \overline{N}_b\}$ restricted problems have been solved or a maximum time limit has elapsed.

4.2.2. Phase 2: Iteration phase

The iteration phase (Lines 5–10 in Algorithm 1) is repeated as long as the number of customers corresponding to β_ℓ is greater than or equal to 1. At each iteration of Phase 2, a new representative solution is produced. At iteration $\ell + 1$, the value of β is reduced by multiplying the value used in iteration ℓ by Δ (Line 9 in Algorithm 1). Notice that Δ controls the number of solutions produced by KS-FFLP: the smaller the

Table 1
The main characteristics of the tested instances.

n	Name	p	n	Name	p	n	Name	p	n	Name	p
100	pmed1	5	300	pmed11	5	500	pmed21	5	700	pmed31	5
	pmed2	10		pmed12	10		pmed22	10		pmed32	10
	pmed3	10		pmed13	30		pmed23	50		pmed33	70
	pmed4	20		pmed14	60		pmed24	100		pmed34	140
	pmed5	33		pmed15	100		pmed25	167	800	pmed35	5
200	pmed6	5	400	pmed16	5	600	pmed26	5		pmed36	10
	pmed7	10		pmed17	10		pmed27	10		pmed37	80
	pmed8	20		pmed18	40		pmed28	60	900	pmed38	5
	pmed9	40		pmed19	80		pmed29	120		pmed39	10
	pmed10	67		pmed20	133		pmed30	200		pmed40	90

value of Δ, the smaller the number of iterations performed in Phase 2, and, hence, the smaller the number of solutions produced.

At iteration ℓ, KS-FFLP follows the general lines of the Initialization phase. The main difference is related to how the initial kernel $\mathscr{K}$ and the sequence of buckets $\{B_h\}_{h=1,\dots,N_b}$ are identified. The basic idea is to use the previous iteration to identify the initial kernel and the buckets for the next iteration. The kernel at the end of iteration $\ell-1$, denoted as $\mathscr{K}_{\ell-1}$, becomes the initial kernel $\mathscr{K}$ at iteration ℓ. The remaining indices are sorted according to their position in the ranked list available at the beginning of iteration $\ell-1$ and are partitioned to create the sequence of buckets for iteration ℓ, each containing $L_b=|\mathscr{K}_{\ell-1}|$ indices. The value of N_b is updated consequently. The subset of x_{ij} variables associated with each index j identified in the Initialization phase remains unchanged in Phase 2. Fig. 3 illustrates an example of how the initial kernel and the buckets are determined in each iteration of Phase 2.

In this example, at iteration $\ell=1$ the initial kernel $\mathscr{K}$ contains indices 1 and 5. Consequently, two buckets are constructed, each comprising 2 indices. Let us assume that the kernel at the end of iteration 1 includes also index 6. Then, the final kernel at iteration 1 becomes the initial kernel of iteration 2 and the 3 remaining indices are listed according to their position in the ranked list at the beginning of iteration 1 and create one bucket. KS-FFLP proceeds similarly for the following iteration.

5. Computational experiments

This section is devoted to the presentation and discussion of the computational experiments. They were conducted on a Workstation HP Intel(R)-Xeon(R), equipped with a 3.5 GHz 64-bit processor, 64 GB RAM, and Win 10 Pro. The processor is equipped with 6 cores, but all the tests were performed using one thread. The KS-FFLP was implemented in C++ and by using the ILOG Concert Technology API (CPLEX version 12.8).

As anticipated above, Theorem 1 implies that an optimal solution to model FFLP(β) can be obtained from an optimal solution to the relaxed model $\overline{\text{FFLP}}(\beta)$ by applying an arbitrary rounding. As a consequence, we have designed the KS-FFLP for the solution of model $\overline{\text{FFLP}}(\beta)$, and we focus here on instances of the same relaxed model.

This section is organized as follows. In Section 5.1, we describe the set of instances used in the computational experiments and the methodology adopted to assess the performance of the KS-FFLP. The results of the computational experiments are discussed in Section 5.2. Finally, Section 5.3 provides a trade-off analysis between efficiency and fairness.

5.1. Testing environment and assessment methodology

Our experiments have been conducted using a set of 40 benchmark instances originally introduced for the p-median problem. This set of instances is part of the OR library and, currently, is publicly available at http://people.brunel.ac.uk/ mastjjb/jeb/orlib/pmedinfo.html. In each of these instances, the set of customers coincides with the set of facilities, i.e., $I=J$ and $n=m$ in all the instances of the test bed. Hence, and to the sake of brevity, we will hereafter refer only to the value of n to define the size of an instance. The main characteristics of the test bed are summarized in Table 1, where n is the number of customers (and potential facilities), *Name* is the instance name, whereas p is the number of facilities to open, as indicated in the original instance.

All buckets are analyzed (i.e., we set $\overline{N}_b=+\infty$). For each instance and each value of β_ℓ, the solution steps detailed in Algorithm 2 are carried out with a time limit equal to 7,200 s, allocated among the restricted MIPs as follows. The time limit is initially equally allotted among the $(1+N_b)$ restricted MIPs. If the solution to a restricted MIP takes less time than what initially allocated, the time saved is equally reallocated to all the remaining restricted MIPs. After preliminary experiments, we decided to set $\gamma=0.1$, i.e., after the LP relaxation of MIP$_{\beta_1}(J)$ is solved, the first kernel is defined as $\mathscr{K}=\{j\in J \mid y_j\geqslant 0.1\}$. Additionally, we set $\delta=0.25$, i.e., for each $j\in J$, we fix to zero the 25% of the x_{ij} variables with largest and strictly positive reduced costs. Finally, the parameter Δ controlling the number of iterations performed in Phase 2 is set equal to 0.5.

At each iteration, KS-FFLP exploits information gathered up to that point, by using as initial kernel the final kernel of the previous iteration. To computationally gauge the benefits achievable by using this information, we compare the results obtained by KS-FFLP with those of a straightforward iterative application of the classical Kernel Search to model $\overline{\text{FFLP}}(\beta)$ for different values of β. More precisely, we implemented a variant of KS-FFLP in which, at each iteration ℓ of Algorithm 2, the initial kernel and the sequence of buckets are constructed from scratch based on the optimal solution to the LP relaxation of MIP$_{\beta_\ell}(J)$. We call such a variant KSLP-FFLP.

Notice also that, in the beginning of iteration ℓ, KS-FFLP neglects the information that can be obtained from the LP relaxation of MIP$_{\beta_\ell}(J)$. To measure the importance of such an information, we consider another variant of KS-FFLP, called KSC-FFLP, where we add to the initial kernel of KS-FFLP any index j such that $y_j\geqslant\gamma$ in the solution to the LP relaxation of MIP$_{\beta_\ell}(J)$. Regarding the construction of the initial kernel, KSC-FFLP combines KS-FFLP and KSLP-FFLP. The sequence of buckets is constructed with the remaining indices as in KS-FFLP. Finally, the quality of the solutions obtained by KS-FFLP is evaluated compared to those of an exact method.

We thus benchmark the performance of the KS-FFLP against the following three methods:

- CPLEX: $\overline{\text{FFLP}}(\beta)$ model solved by means of CPLEX with a time limit of 7,200 s;
- KSLP-FFLP: variant of KS-FFLP in which, at each iteration ℓ of Algorithm 2, the initial kernel and the sequence of buckets are constructed from scratch based on the optimal solution to the LP relaxation of MIP$_{\beta_\ell}(J)$;
- KSC-FFLP: variant of KS-FFLP where, at each iteration ℓ of Algorithm 2, the initial kernel is the union of the indices belonging to the final kernel of iteration $\ell-1$ (as in KS-FFLP) and the indices $j\in J$

Table 2
Computational results: Average performance of each method.

n	#Inst	#β	ARPD (Worst)				#Best (#Optimal)				CPU (sec.)			
			CPLEX	KS-FFLP	KSLP-FFLP	KSC-FFLP	CPLEX	KS-FFLP	KSLP-FFLP	KSC-FFLP	CPLEX	KS-FFLP	KSLP-FFLP	KSC-FFLP
100	5	8	0.00 (0.11)	0.50 (2.76)	0.60 (5.39)	0.49 (4.03)	39 (15)	22 (12)	24 (13)	22 (12)	4,778.6	808.6	2,685.5	2,332.4
200	5	9	0.01 (0.34)	1.75 (5.89)	1.71 (10.75)	1.08 (5.98)	42 (9)	13 (6)	11 (7)	16 (6)	6,246.2	1,872.7	3,349.8	3,287.3
300	5	10	0.99 (8.09)	0.68 (4.68)	2.16 (16.60)	2.18 (15.29)	25 (9)	24 (6)	15 (6)	24 (7)	6,210.1	2,595.4	3,039.2	3,074.5
400	5	10	5.05 (28.83)	0.69 (5.25)	1.95 (15.70)	1.88 (11.48)	18 (6)	23 (5)	22 (5)	20 (5)	6,508.0	3,187.0	3,151.2	3,473.3
500	5	10	5.90 (21.51)	1.05 (5.54)	1.73 (12.48)	2.00 (16.61)	12 (5)	27 (5)	21 (5)	27 (5)	6,599.4	3,442.0	3,596.7	3,723.3
600	5	11	6.37 (39.29)	0.85 (9.72)	2.03 (15.71)	2.26 (10.46)	9 (5)	30 (5)	24 (5)	30 (5)	6,763.5	3,593.4	3,478.4	4,126.0
700	4	11	14.53 (72.12)	0.46 (8.30)	4.42 (66.41)	1.21 (14.76)	5 (3)	36 (3)	18 (3)	23 (3)	6,913.4	3,511.2	4,008.5	4,035.5
800	3	11	11.24 (41.19)	0.49 (5.85)	7.89 (93.72)	0.92 (8.40)	2 (1)	21 (1)	16 (1)	19 (1)	7,077.3	4,154.1	4,888.8	4,446.2
900	3	11	7.14 (43.07)	0.02 (0.25)	2.51 (27.93)	1.39 (13.74)	1 (0)	29 (0)	14 (0)	20 (0)	7,200.7	5,047.0	5,137.3	5,423.0
	Average/total		**5.22 (28.28)**	**0.76 (5.36)**	**2.61 (29.41)**	**1.56 (11.19)**	**153 (53)**	**225 (43)**	**165 (45)**	**201 (44)**	**6,463.6**	**3,084.0**	**3,615.2**	**3,712.4**

with $y_j \geqslant 0.1$ in the optimal solution to the LP relaxation of $\text{MIP}_{\beta_\ell}(J)$ (similar to KSLP-FFLP).

Notice that in KSC-FFLP the threshold 0.1 for the acceptance of indices in the initial kernel is the unique sensitive parameter in the proposed procedures. Indeed, this value influences the size of the initial kernel and thus the partition of the remaining indices into buckets. Clearly, a threshold closer to zero would imply a larger initial kernel and fewer initial buckets, and hence fewer iterations with larger problems to solve. On the contrary, a larger threshold would imply an opposite effect. In cascade, this initial effect may influence kernels and buckets in all subsequent iterations. As noted above, we chose the value 0.1 after extensive preliminary tests. To evaluate the performance of each method above, for each instance r we have computed the *Average Relative Percentage Deviation* (ARPD_r), defined as follows:

$$\text{ARPD}_r = 100 \times \left(\frac{1}{L_r} \sum_{\ell=1}^{L_r} \frac{z_{r,\ell}^{UB} - z_{r,\ell}^{best}}{z_{r,\ell}^{best}} \right), \tag{15}$$

where $z_{r,\ell}^{UB}$ is the objective function value of the best solution found by the evaluated method while solving the r-th instance for a given value β_ℓ, and $z_{r,\ell}^{best}$ is the best objective function value found over all the tested methods for a given instance r and a given value β_ℓ. The total number of different values of parameter β tested while solving instance r is denoted by L_r.

5.2. Evaluating the performance of KS-FFLP

In this section, we assess the performance of KS-FFLP. The results of the computational experiments are summarized in Table 2. (The complete and detailed computational results are available at http://or-brescia.unibs.it/instances.) To simplify the discussion, we have clustered the 40 instances of the test bed into subsets according to their size. Each row in Table 2 refers to one size. The first three columns of this table report the instance size (n), the number of instances composing the subset (*#Inst*), and the number of different values of β evaluated for each instance (#β), respectively. Note that the number of values of β evaluated depends on the instance size: for a given value of Δ, the larger the number of customers, the larger the number of values evaluated. The remaining columns are separated into three parts. The first part, labeled *ARPD* (*Worst*), provides the values of two statistics measuring the quality of the solutions produced by each method. For each subset of instances and each method, statistic ARPD is the average value of ARPD_r –see Section 5.3– computed over all the instances composing the subset, i.e., over $\#Inst \times \#\beta$ optimizations. Between parentheses we report the worst error, computed as $\max_{r,\ell} \left\{ \frac{z_{r,\ell}^{UB} - z_{r,\ell}^{best}}{z_{r,\ell}^{best}} \right\}$, generated by each method over all the instances belonging to a given subset and all the values of β evaluated. The second part, labeled *#Best* (*#Optimal*), shows the number of times each method found the best solution. Then, for CPLEX, the figures between parentheses indicate the number of instances solved to proven optimality within the time limit. For the three KS methods, the figures between parentheses show the number of times each method obtained an optimal solution, as compared to the proven optimal solution produced by CPLEX. The last part, labeled *CPU (sec.)*, reports the average computing times (in seconds) for each method. Also here, each average is computed over $\#Inst \times \#\beta$ optimizations. It is worth highlighting that Table 2 summarizes the results of, for each method, 400 optimizations.

The main insights that we can gain from this table are as follows:

- KS-FFLP outperforms, on average, the other methods in terms of all the statistics considered: ARPD, Worst, and #Best.
- The three KS methods found a number of optimal solutions only slightly smaller than CPLEX.

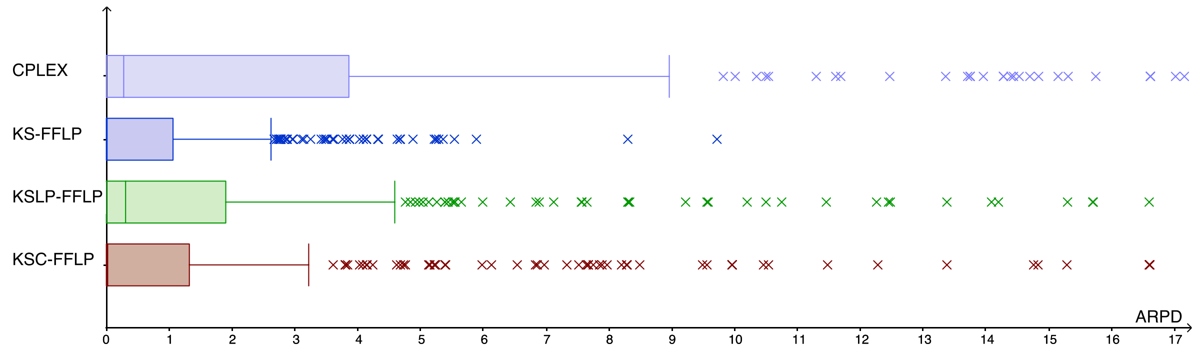


Fig. 4. Box-and-whisker plots of ARPD values.

- KS-FFLP is the fastest method, closely followed by the other two KS methods. The average computing time required by each of the three methods is considerably smaller than that of CPLEX.
- As the instance size increases, the performance of CPLEX deteriorates, both in terms of solution quality and computing time. On the contrary, the values of ARPD achieved by the three KS methods remain relatively steady.
- The performance deterioration shown by CPLEX is confirmed by the figures regarding the number of times this method found the best solution, which decreases considerably as the instance size grows. Conversely, the number of best solutions found by the KS methods increases significantly with the instance size.

The box-and-whisker plot depicted in Fig. 4 provides a graphical representation of the ARPD values reported in Table 2.

The first and third quartiles (Q_1 and Q_3) of the solutions are represented by the left and right box limits; the second quartile (Q_2) corresponds to the median and is represented by the vertical line inside the box. The *whiskers* (lines extending horizontally to the right from each box) indicate the variability outside Q_3. As conventional, we consider as *outliers*, represented by crosses, all the values that are $1.5 \times (Q_3 - Q_1)$ far from Q_3.

From Fig. 4, we can observe that the width of the box corresponding to KS-FFLP, as well as the number of outliers, indicate that this method outperforms all the others in terms of quality of the solutions found. It is worth highlighting that all outliers having a value larger than 17.5 have not been plotted, and that such values concern all methods other than KS-FFLP.

Summarizing the analysis above, we can gain three main insights: (1) the developed Kernel Search framework is able to generate quickly a representative set of high-quality solutions to FFLP(β); (2) the computational effort scales well with the size of the instances; (3) KS-FFLP performs better than the two variants, both in terms of solution quality and computing times.

Going more into the details of the third insight, from Table 2 one can notice that KS-FFLP significantly outperforms both variants in terms of solution quality: the average ARPD deteriorates from 0.76% to 1.56% (for KSC-FFLP), and even to 2.61% (for KSLP-FFLP). Similar conclusions can be drawn observing the worst errors and the number of times each method found the best solution. For the latter statistic, KS-FFLP found 225 times the best solution, against 165 times for KSLP-FFLP and 201 for KSC-FFLP. KS-FFLP outperforms both variants also in terms of computing times: by employing KS-FFLP one can achieve an average time reduction of roughly 530 s compared to KSLP-FFLP, and of approximately 628 s w.r.t. KSC-FFLP. Concluding, simply using the final kernel of an iteration as initial kernel of the next one seems the best choice, and trying to build a better tailored initial kernel does not worth the effort.

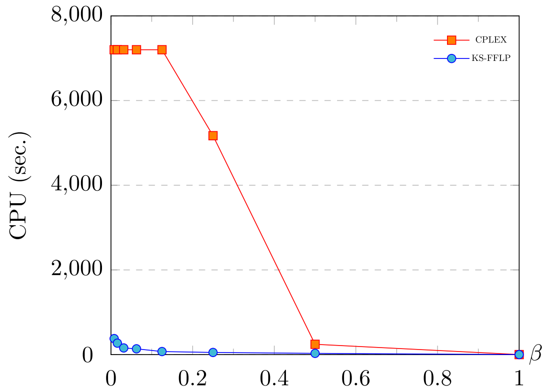
To gain some further insights into the performance of CPLEX and KS-FFLP, we have chosen three instances of different sizes and compared in Fig. 5 the corresponding computing times for the different values of β evaluated. The behavior on these instances is observed also for most of the remaining instances which, to the sake of brevity, are not reported here. From Fig. 5, we can gain the following insights:

- Computing times spent by KS-FFLP are smaller than those required by CPLEX.
- Computing times required by CPLEX increase when the value of β becomes smaller. This is clearly shown in instance pmed2 (Fig. 5(a)) and, to a certain extent, also in instance pmed22 (Fig. 5(b)). Such an impact is also observed for KS-FFLP, but with a much smaller magnitude.

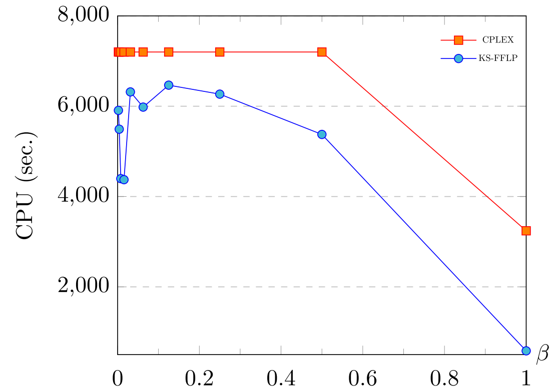
We now evaluate the impact of parameter $\overline{N}_b$ (i.e., the maximum number of buckets analyzed) on the performance of KS-FFLP. To this aim, we have run KS-FFLP with the following different parameter

Table 3
Computational results: The impact of the number of buckets analyzed.

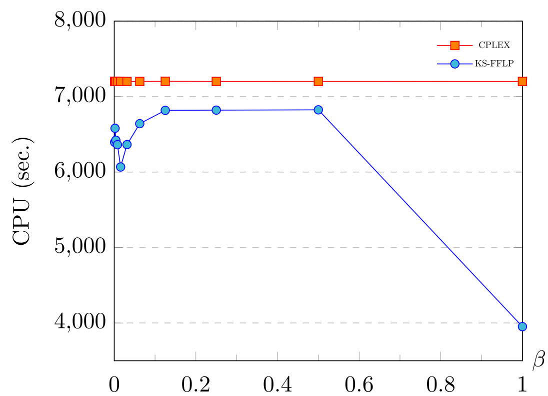
	ARPD*				CPU (sec.)			
n	$\overline{N}_b = 0$	$\overline{N}_b = 5$	$\overline{N}_b = 10$	$\overline{N}_b = N_b$	$\overline{N}_b = 0$	$\overline{N}_b = 5$	$\overline{N}_b = 10$	$\overline{N}_b = N_b$
100	1.50	0.57	0.51	0.49	1.8	808.2	808.4	808.6
200	2.78	1.89	1.73	1.73	76.5	1,840.3	1,870.2	1,872.7
300	0.91	−0.12	−0.27	−0.28	336.7	2,402.6	2,560.4	2,595.4
400	−2.78	3.74	−3.80	−3.82	347.6	2,592.0	3,044.3	3,187.0
500	−2.86	−3.93	−4.17	−4.23	192.7	2,478.7	2,992.3	3,442.0
600	−3.60	−4.55	−4.59	−4.70	308.3	2,337.8	2,874.6	3,593.4
700	−9.15	−9.90	−9.96	−10.09	226.6	2,101.7	2,563.8	3,511.2
800	−7.74	−7.85	−8.04	−8.20	289.6	1,795.4	2,527.0	4,154.0
900	−4.91	−5.08	−5.32	5.64	257.1	1,751.0	2,611.2	5,047.0
Avg.	**−2.45**	**−3.29**	**−3.41**	**−3.48**	**230.8**	**2,067.2**	**2,467.1**	**3,084.0**



(a) Computing times (PMED2, $n = 100$).



(b) Computing times (PMED22, $n = 500$).



(c) Computing times (PMED39, $n = 900$).

Fig. 5. KS-FFLP vs. CPLEX: Computing times comparison for different values of β.

values: $\overline{N}_b = 0$ (it considers only the indices in the initial kernel $\mathscr{K}$); $\overline{N}_b = 5$ (it analyzes 5 buckets); $\overline{N}_b = 10$ (10 buckets); and $\overline{N}_b = N_b$ (all buckets created). The quality of the solutions produced by each variant is measured by statistic ARPD*, which is computed by averaging the values calculated for each instance as in Eq. (15), where $z_{r,l}^{best}$ is replaced by the value of the best solution found by CPLEX. The results are summarized in Table 3.

From Table 3 we can gain the following insights:

- Considering only the indices in the initial kernel (i.e., $\overline{N}_b = 0$) is sufficient to improve, on average, the solutions produced by CPLEX. Only for the small-size instances (i.e., $n \leqslant 300$) CPLEX obtains, on average, better solutions.
- Increasing the number of buckets evaluated, the average quality of the solutions found tends to improve at the expense of increased computing times.
- Computing times for each variant are remarkably shorter than those required by CPLEX (the latter figures can be found in Table 2).

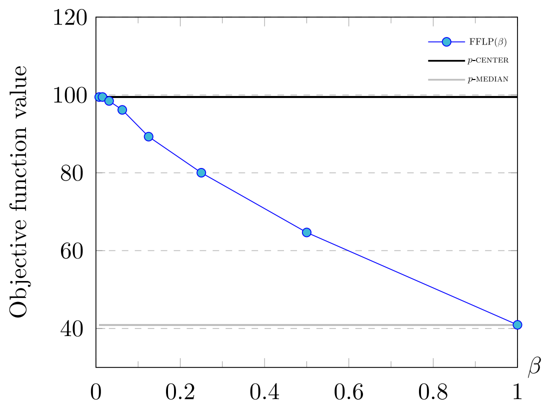
5.3. Trade-off analysis

This section aims at showing empirically the relations between the FFLP(β), the p-median, and the p-center problems. To this aim, we depict in Fig. 6 the results obtained for the same three instances considered above: instances pmed2, pmed22, and pmed39. Recalling that for $\beta = 1$, the FFLP(β) corresponds to the p-median problem, and for $\beta \to 0$ the FFLP (β) tends to the p-center problem, the relations among these three problems are evident from Fig. 6, where the vertical axis corresponds to the FFLP(β) objective function value: the p-median and p-center are antagonist, and the FFLP(β) is able to provide a set of compromise solutions between their two objectives. These findings can also be observed analyzing most of the remaining instances which, for the sake of brevity, are not reported here.

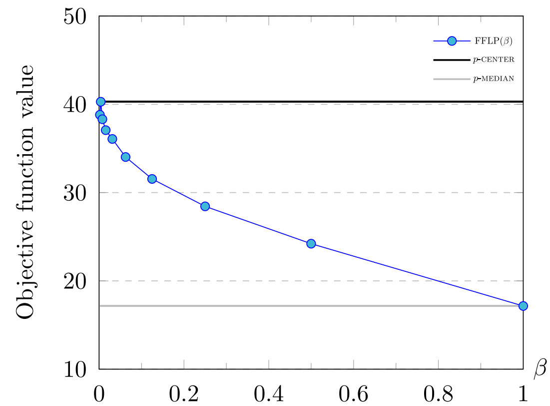
To gauge in more details the impact of parameter β for each solution produced by KS-FFLP, we have computed the following three statistics: the skewness and the semi-kurtosis of the distance distribution induced by the best solution found by KS-FFLP, and the Price Of Fairness (POF). Fig. 7 visualizes the dynamics of each of the statistics above with respect to the value of β considered. To facilitate the comparison, in Fig. 7 we show only the values of β evaluated in all the instances (recall that the number of values of β evaluated varies according to the size of the instance). Hence, for each of the values of β reported, each statistic is averaged over all the instances tested.

The *skewness* is a measure of the asymmetry of a distribution. Recall that a symmetric distribution has, by definition, skewness equal to zero, whereas a positive skewness commonly indicates that the right tail of the distribution is longer than the left tail, and that the mass of the distribution is concentrated on its left. Applied to our context, we can notice from Fig. 7(a) that the solutions with the largest values of β are those showing, on average, the highest skewness: the long right tail of the distance distribution indicates that some customers are associated with long travel distances, whereas the mass of the distribution concentrated on the left suggests that the average distance is small. The average skewness tends to reduce as the value of β becomes smaller. On one side, this indicates that the average distance tends to increase. On the other side, this suggests that the length of the right tail of the distribution becomes shorter: the number of customers associated with long travel distances becomes smaller.

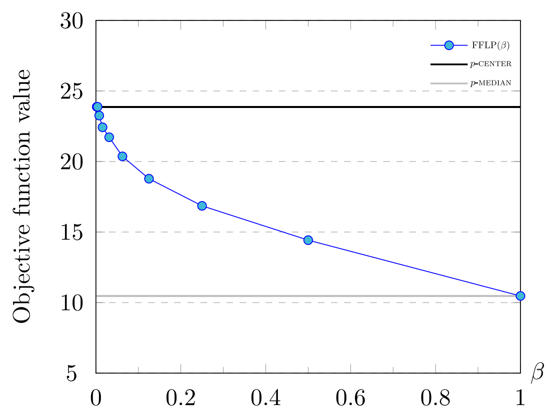
The *kurtosis* is a measure of how heavily the tails of a distribution differ from those of a normal distribution. Intuitively speaking, the



(a) Instance PMED2 with $n = 100$.



(b) Instance PMED22 with $n = 500$.



(c) Instance PMED39 with $n = 900$.

Fig. 6. The trade-off between the p-MEDIAN and the p-CENTER provided by the FFLP(β).

kurtosis describes the presence of extreme values in a distribution. As the standard kurtosis equally weighs the values in the left and right tails, and the conditional β-mean has only an impact on the worst-case distances (i.e., the extreme values lying in the right tail of the distance distribution), we report for each value of β the average value of the *semi-kurtosis*. For a single instance and given a solution $\widehat{\mathbf{x}}$, the latter statistic is computed as follows:

$$\text{SEMI-KURTOSIS} = \frac{\frac{1}{n}\times\sum_{i\in I}\left(\sum_{j\in J}c_{ij}\widehat{x}_{ij} - \frac{1}{n}\sum_{i\in I}\sum_{j\in J}c_{ij}\widehat{x}_{ij}\right)_+^4}{\left(\frac{1}{n}\times\sum_{i\in I}\left(\sum_{j\in J}c_{ij}\widehat{x}_{ij} - \frac{1}{n}\sum_{i\in I}\sum_{j\in J}c_{ij}\widehat{x}_{ij}\right)_+^2\right)^2},$$

where $(q)_+$ denotes the positive part of a scalar q (that is, $(q)_+ = \max\{0, q\}$). From Fig. 7(b), it is evident that the larger the value of β, the larger the semi-kurtosis, and, hence, the larger the worst-case distances. The average value of the semi-kurtosis tends to decrease as the value of β reduces, indicating that the average worst-case distances take smaller values.

Finally, we report the *POF*. This measure has been introduced by Bertsimas et al. (2011) to measure the relative system efficiency loss under a fair allocation of utilities compared to the one that maximizes the sum of utilities. The authors propose to compute the POF of a problem involving the utility set U and the fairness scheme $\mathscr{S}$ as:

$$\mathrm{POF}(U,\mathscr{S}) = \frac{SYSTEM(U) - FAIR(U,\mathscr{S})}{SYSTEM(U)}.$$

In the above formula, $SYSTEM(U)$ denotes the value of the so-called utilitarian solution (i.e., the solution that maximizes the sum of the utilities of the players), and $FAIR(U,\mathscr{S})$ is the sum of such utilities under the fairness scheme $\mathscr{S}$. For non-negative utilities, the POF ranges between 0 and 1. We applied the approach of Bertsimas et al. (2011) to our problem as follows. The utility function $f_i : \mathscr{XY}\rightarrow\mathbb{R}_+$ of each customer $i\in I$ is defined as $f_i(\mathbf{x},\mathbf{y}) = C_i - \sum_{j\in J}c_{ij}x_{ij}$, where $C_i = \max_{j\in J}\{c_{ij}\}$ is the maximum distance between a customer and a potential facility. The utility set is thus $U = \{\mathbf{u}\in\mathbb{R}_+^n \mid \exists(\mathbf{x},\mathbf{y})\in\mathscr{XY} : f_i(\mathbf{x},\mathbf{y}) = u_i \quad (i\in I)\}$. Notice that:

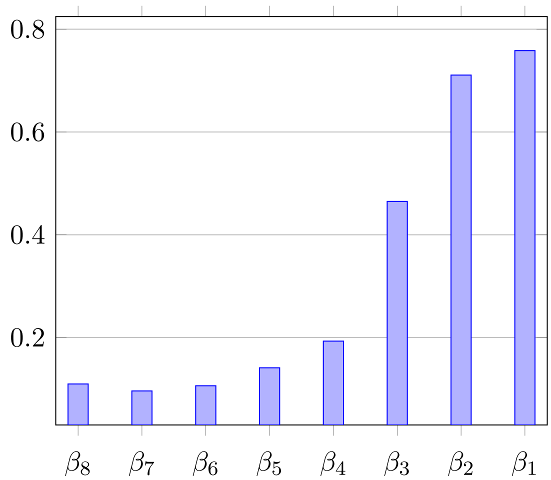
$$\sum_{i\in I}u_i = \sum_{i\in I}C_i - \sum_{i\in I}\sum_{j\in J}c_{ij}x_{ij},$$

where the first term on the right-hand side is a constant. Hence, the utilitarian solution is equivalent to the solution that minimizes the total distance, equivalent to setting $\beta = 1$ in FFLP(β). In our context, the fairness scheme $\mathscr{S}$ has to be intended as the application of the conditional β-mean with a given value of parameter β. Thus, a different fairness scheme is deployed by changing the value of β. For a given instance and value of β, $FAIR(U,\mathscr{S})$ is computed as the sum of the utilities $f_i(\mathbf{x},\mathbf{y})$ in the best solution found by KS-FFLP.

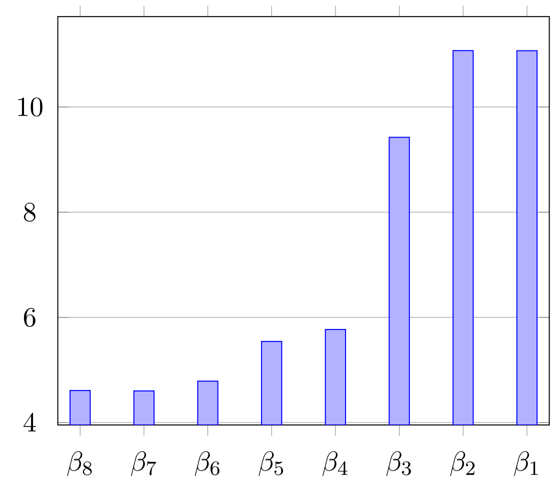
Fig. 7(c) illustrates the average values of POF. By construction, it attains its minimum value for $\beta_1 = 1$, and, as expected, the POF tends to increase as the degree of fairness introduced by the decision-maker increases, that is the value of β decreases.

5.4. Optimal location of testing sites for collecting swab specimens

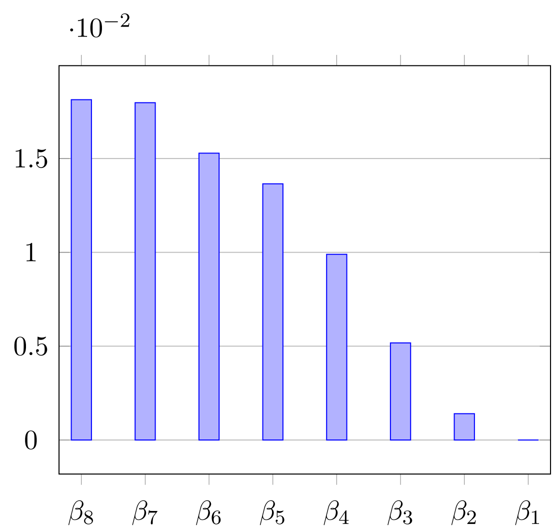
In year 2020, the pandemic triggered by the SARS-CoV-2 virus had a



(a) Skewness.



(b) Semi-kurtosis.



(c) Price of fairness.

Fig. 7. Average values of skewness, semi-kurtosis, and price of fairness.

Fig. 8. Cars queuing for testing (left) and health workers in protective gears collecting swabs (right).

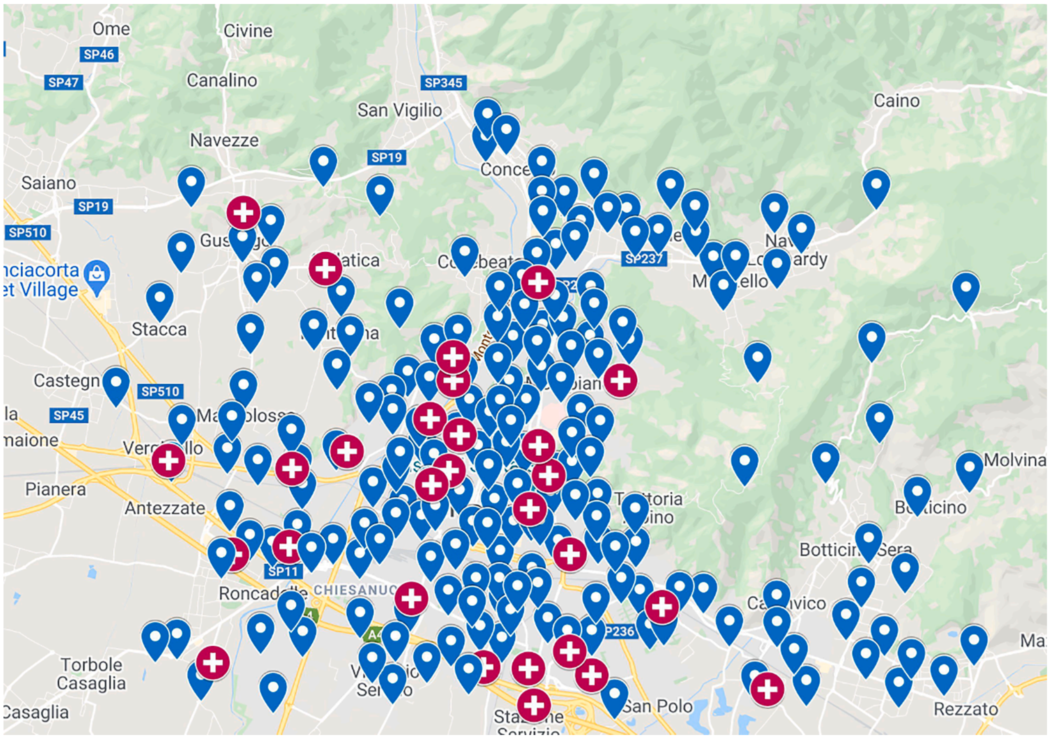


Fig. 9. The geographical area covered in the case study

devastating impact in terms of human being losses, had radically modified our social and work relationships, and had also imposed a number of additional obligations to prevent the spreading of the virus. In this section, we describe the application of the $\overline{\text{FFLP}}(\beta)$ model to an optimization problem that decision-makers in the health sector were facing during the pandemic. The description of the application derives from the personal experience of the authors regarding how the activities illustrated were conducted in Italy.

We consider the problem of determining an optimal location of p temporary drive-thru testing sites, where citizens can stay in their cars while they get screened for the virus (see Fig. 8). In this application, the potential locations correspond to the geographical positions where a temporary testing site can be located (usually corresponding to medium- and large-size parking lots). The set of customers is represented by the citizens (or groups of them, see below) that have to reach a testing site with their own car. Each customer has a unit demand (corresponding to one person that has to be screened). More generally, the citizens that have to be screened can be clustered in groups having the same cardinality, so that the demand of each customer (in this case corresponding to a cluster of citizens) is identical. Although in this application there is a daily capacity, given by the maximum number of citizens that can be screened in one day, in their nature these location decisions are at a strategic level so that on the long-term the capacities can be neglected (that is, the testing sites will stay open until all the citizens that need to do so have been tested). Analogously, the opening cost of these testing sites can be neglected. On one side, the opening cost is relatively small: it involves buying, if not available already, and installing a temporary tent and the necessary equipment to collect the swab specimens. On the other side, the opening cost is identical for all the potential locations.

The geographical area considered in the case study covers the city of Brescia and some of the neighboring towns. During the first half of year 2020, this was one of the most severely hit areas in Italy with several thousand of positive cases. In this study, the citizens that need a screening have been grouped in 200 clusters of identical size. These clusters represents the set of customers $I = \{1, 2, \ldots, 200\}$. The centroid of each cluster is shown in Fig. 9 as a blue pin. We have identified 28 potential locations where a temporary testing site can be opened, that is set J. They correspond to medium- and large-size public parking lots spread over the geographical area considered in the case study. Each of these potential locations is shown in Fig. 9 as a white cross over a red background. The distance c_{ij} between cluster i and potential location j is the driving distance from the centroid of cluster i to the potential location j. The problem sketched above can be modeled as a FFLP(β) where the decision-maker has to locate p testing sites such that the distance traveled by the β fraction of the most distant clusters of citizens is minimized.

We have chosen $p = 3$ and solved the case study by means of KS-FFLP. The computational time required to solve the case study is negligible, in total 4.31 s. Similar to the trade-off analysis reported above, we have computed the POF also for each solution produced by KS-FFLP. Nevertheless, in this case the value of POF for each of the solution found is very close to zero and, therefore, is not reported here. This highlights that, for this case study, the relative system efficiency loss due to an increase of the degree of fairness is substantially negligible. Fig. 10 shows the cumulative frequency distribution of travel distances $C_i(\mathbf{x})$, with $i \in I$, for four solutions produced by KS-FFLP. The upper panel displays the complete distributions, whereas the lower panel focuses on the longest distances traveled: those larger than or equal to 8 km. Looking at the complete distribution, one can notice that the percentage of clusters traveling a distance smaller than 6 km is larger in the p-MEDIAN compared to the other solutions. While a comparison between the FFLP(0.25) and the p-CENTER solutions is difficult, as their

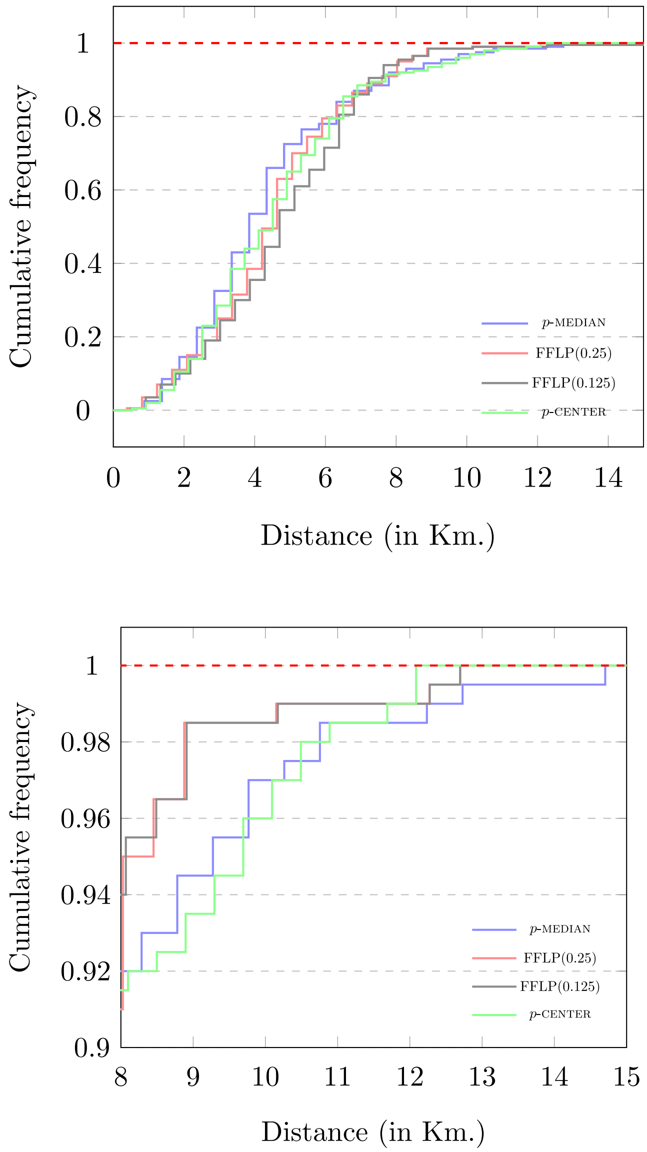


Fig. 10. Cumulative frequency plots for the case study ($p = 3$). Complete distribution (above) and a focus on the worst distances (below).

distributions crosses each other several times, one can notice that the smallest percentage of clusters traveling a distance smaller than 7 km corresponds to the FFLP(0.125) solution. As expected, the situation dramatically changes if we turn our attention to the worst distances (lower panel). In this case, the percentage of clusters traveling a distance between 8 and, roughly, 11.5 km is the largest for the FFLP(0.25) and FFLP(0.125) solutions, with very small differences between these latter two. The p-CENTER solution, which we recall optimizes the worst-case, is the one where the most poorly served cluster travels the shortest distance. Conversely, the p-MEDIAN solution is where the most poorly served cluster travels the longest distance. In the FFLP(0.25) and FFLP(0.125) solutions the most poorly served cluster travels a distance that is between the above two cases, confirming that the FFLP(β) is able to produce a set of compromise solutions between the former two objectives.

Finally, Fig. 11 shows on the map the four solutions discussed above. The three locations selected in each solution are represented as a white cross over a red, yellow, and blue background. The clusters of citizens assigned to each of them are represented with the corresponding light color. From Fig. 11 one can notice that although in all the four solutions the blue location selected is the same, the other two locations change, as well as the assignment of the clusters. Fig. 11 must be read in combination with the plots shown in Fig. 10. They highlight that by simply changing the value of β in the FFLP(β) model, a decision-maker can generate different solutions where system efficiency and fairness are differently balanced. Choosing among these solutions, he/she may find the best trade-off, taking into account elements not directly captured by the optimization model.

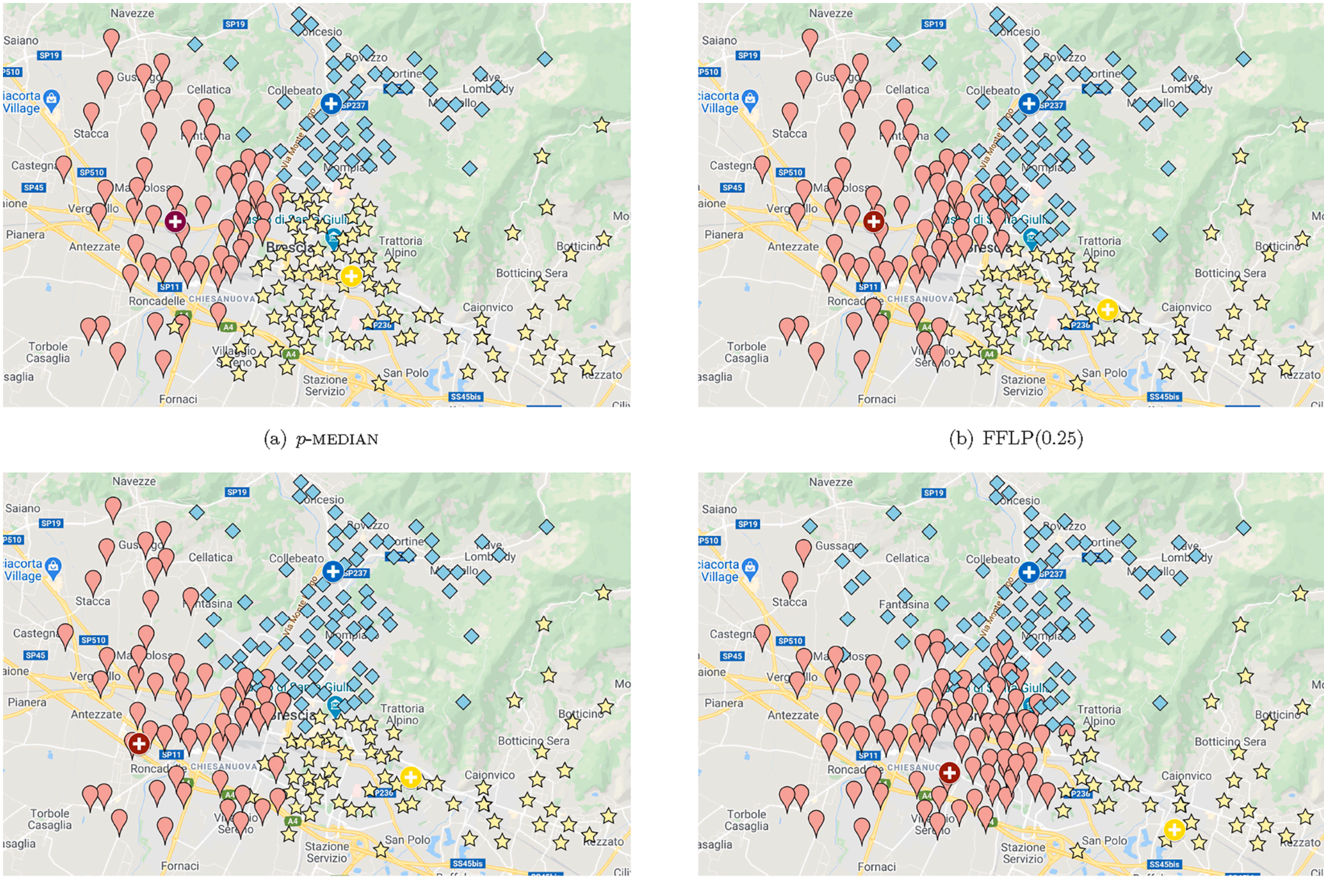
6. Conclusions and future research directions

In this paper, we have studied a location problem in which a decision-maker has to locate p facilities such that the average distance the most poorly served customers have to travel is minimized. We have applied the conditional β-mean and introduced the Fair Facility Location Problem (FFLP(β)), which is formulated as a mean-fairness Mixed-Integer linear Program (MIP). We have shown that by changing the value of β, the decision-maker controls the trade-off between maximizing the efficiency (measured by the average distance, as for the p-median problem) and maximizing the fairness (measured by the worst distance, as for the p-center). We have also proven that in the proposed FFLP(β) formulation it is possible to relax the integrality of the assignment variables and, once an optimal solution is found, restore integrality with a straightforward rounding procedure.

We have developed a Kernel Search (KS) heuristic to produce a set of representative solutions to the FFLP(β) that can be employed to assess the trade-off between efficiency and fairness. Computational experiments conducted on 40 instances have shown that the heuristic, denoted as KS-FFLP, can obtain, within a given time limit, better solutions than CPLEX. In terms of computing times, KS-FFLP requires significantly smaller computational times than CPLEX. Finally, inspired by the screening activities related to the pandemic triggered by the SARS-CoV-2 virus, we have analyzed the problem of deciding the optimal location of p drive-thru temporary testing sites for collecting swab specimens.

From an algorithmic point of view, in KS-FFLP the kernel is created by considering only the binary variables y_j. Studying the impact of reducing also the number of variables v_i in each restricted MIP, rather than including them all, is also an interesting research avenue. The application of the methodology to other classes of problems where efficiency and fairness are both relevant is another important research direction.

From a modeling viewpoint, the proposed FFLP(β) formulation is based on the assumptions that a unit demand originates from each customer, the facilities are uncapacitated, and their opening costs are negligible. This is motivated by the reported case study, where such assumptions are compelling. An interesting research development would be the extension to a more general case, where customers have different demands and facilities have different operating costs and finite capacities. These extensions are not straightforward. Particularly, Theorem 1 does not extend directly to the capacitated case, even under special conditions, such as single-source policy, unit demand, and integer capacities of the facilities. Consequently, without Theorem 1, the problem becomes significantly more complicated to solve (we can not relax the integrality restriction of the assignment variables), and the KS-



(a) p-MEDIAN

(b) FFLP(0.25)

(c) FFLP(0.125)

(d) p-CENTER

Fig. 11. Case study: A visual comparison of four solutions ($p = 3$).

FFLP must be redesigned.

CRediT authorship contribution statement

C. Filippi: Conceptualization, Methodology, Writing - review & editing, Formal analysis, Validation. **G. Guastaroba:** Conceptualization, Methodology, Validation, Formal analysis, Writing - review & editing, Investigation, Data curation, Visualization. **D.L. Huerta-Muñoz:** Conceptualization, Software, Validation, Formal analysis, Investigation, Writing - original draft, Data curation, Visualization. **M.G. Speranza:** Conceptualization, Methodology, Validation, Formal analysis, Resources, Writing - review & editing, Supervision, Project administration.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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