Linear and Bilinear Functions

LA3⋄1. Find the number of all
(a) linear maps $f\colon \mathbb{F}_q^n \to \mathbb{F}_q^k$.
(b) linear injective maps $f\colon \mathbb{F}_q^n \to \mathbb{F}_q^k$.
(c) linear functions $f\colon \mathbb{F}_q^n \to \mathbb{F}_q$.

LA3⋄2. Suppose that a linear map $A\colon V \to W$ in the bases (v_1, v_2, v_3) of V and (w_1, w_2) of W has the matrix

$$\begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix}.$$

Find the matrix of A in bases $(v_1, v_1 + v_2, v_1 + v_2 + v_3)$ and $(w_1, w_1 + w_2)$.

LA3⋄3. Suppose $A, B\colon V \to W$ are linear maps and $\dim(\operatorname{Im} A) \le \dim(\operatorname{Im} B)$. Prove that there exist such linear operators $C\colon V \to V$ and $D\colon W \to W$ that $A = DBC$ and C (or D) is non-degenerate.

LA3⋄4. Suppose f is a nonzero linear function on some vector space V and $U = \ker f$. Prove that $V = U \oplus \langle a \rangle$ for any $a \notin U$.

LA3⋄5. Which of the following functions are bilinear and which are symmetric?
(a) $f(X, Y) = X^T Y$
(b) $f(A, B) = \operatorname{tr}(AB)$
(c) $f(A, B) = \operatorname{tr}(AB - BA)$
(d) $f(A, B) = \operatorname{tr}(A + B)$
(e) $f(A, B) = \det(AB)$
(f) $f(A, B) = (AB)_{ij}$