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A New Method for the Determination of the Epoch of Minimum of Binary Star Systems

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New mathematical expressions are derived for the epoch of minimum of a light curve and the uncertainty in the epoch for binary star systems. Certain limitations associated with the older Hertzsprung and the Kwee and van Woerden approaches to the computation of the epoch of minimum are discussed. The new method has a somewhat better mathematical foundation and is simpler to execute. Since the orbital period and epoch of minimum of a binary system is of such great importance in studies, it is recommended that the uncertainties in both of these quantities be specified when the period and epoch are designated.

DETERMINATION OF EPOCH OF MINIMUM

A COMMON approach to determining the epoch of minimum in a light curve of an eclipsing binary is the method originally proposed by Hertzsprung (1928). Modifications, additions, and formalizations followed with Oosterhoff (1929), deKort (1941), Hertzsprung (1941), and Kwee and van Woerden (1956). The purpose of this manuscript is to specify a similar approach which differs in some basic senses. Certain limitations and errors associated with the former approaches are also discussed.

The true intensity of a light curve as a function of time $l(t)$ is assumed to be symmetrical about the epoch of minimum $t=T_0$. If U is a certain time interval, then

$$l(T_0+U)=l(T_0-U). \tag{1}$$

If the observed intensity of the curve is designated by $m(t)$, the observational error at time t is

$$\mu(t)=m(t)-l(t). \tag{2}$$

Consider z independent values of m, spaced at time intervals Δt on either side of some arbitrary epoch T designated by $m(T\pm k\Delta t)$, where $k=1, \cdots, z$. Since observations will not generally occur at the values $T\pm k\Delta t$, interpolations between the actually observed data points should be considered. Thus, Eq. (2) can be rewritten for some epoch $T+k\Delta t$ as

$$l(T+k\Delta t)=m(T+k\Delta t)-\mu_{+k}. \tag{3}$$

If $\tau\equiv T-T_0$ is small, the Taylor series expansion of

$$l(T+k\Delta t)=l(T_0+k\Delta t+\tau)$$

is

$$l(T_0+k\Delta t)+\frac{\tau}{1!}\left(\frac{dl}{dt}\right)_{T_0+k\Delta t}+\frac{\tau^2}{2!}\left(\frac{d^2l}{dt^2}\right)_{T_0+k\Delta t}+\frac{\tau^3}{3!}\left(\frac{d^3l}{dt^3}\right)_{T_0+k\Delta t}+\cdots. \tag{4}$$

With the aid of Eqs. (1) and (3), and line (4), the quantity

$$m(T+k\Delta t)+m(T-k\Delta t)\equiv S_k=(\mu_{+k}+\mu_{-k})+2l(T_0+k\Delta t)+\tau^2(d^2l/dt^2)_{T_0+k\Delta t}+\cdots. \tag{5}$$

If Eq. (5) is truncated at the second power of τ,

$$\sum_{k=1}^{z} S_k$$

is in the form of a parabola and possesses a minimum at $t=T_0$ since a solution of

$$\frac{d}{d\tau}\sum_{k=1}^{z} S_k=0$$

is $\tau=0$. Thus the method of determining the epoch of minimum is essentially determining the minimum of the parabola designated by

$$\sum_{k=1}^{z} S_k.$$

Let the value of

$$\sum_{k=1}^{z} S_k$$

be computed at three different epochs designated by $T-\Delta T$, T, $T+\Delta T$ and be specified by Y_-, Y, and Y_+, respectively. It can be shown that the minimum of

$$\sum_{k=1}^{z} S_k$$

is given by $t=T_0=T+\frac{1}{2}[(Y_--Y_+)/(Y_--2Y+Y)]\Delta T$. Thus, T can be thought of as a guess to the epoch of minimum. It is only necessary to compute $\frac{1}{2}[(Y_--Y_+)/(Y_--2Y+Y)]\Delta T$ and add it to T to obtain the epoch of minimum.

A number of the parameters cited above are depicted in Fig. 1.

FORM OF $l(\tau)$

An important epoch in a cycle of revolution of a double star system is the one where the projection of the line between the centers of the two components of minimum length in the line of sight of the observer is at a minimum. The epoch of minimum light is often assumed to be identical with the epoch of minimum projected separation. This, of course, is not necessarily the case in general. Nevertheless, if the system does not evolve rapidly, intervals between minimum points on the light curve may be used to obtain the period of the system.

Ideally, the epoch of minimum of $l(t)$ should be obtained from $dl/dt=0$. This would require the explicit mathematical expression for $l(t)$, which in general is extremely complicated. Consequently the higher-order terms in a series expansion of $l(t)$ may be important.

By expressing

$$\sum_{k=1}^{z} S_k$$

as a second-order expansion in τ it was effectively necessary to assume a third-order expansion in τ for $l(T+k\Delta t)$. In the Kwee and van Woerden (1956) approach, the quantity

$$\sum_{k=1}^{z} [m(T+k\Delta t)-m(T-k\Delta t)]^2 \equiv \sum_{k=1}^{z} \Delta m_k^2$$

is considered to be parabolic and instead of the quantity

$$\sum_{k=1}^{z} [m(T+k\Delta t)+m(T-k\Delta t)] \equiv \sum_{k=1}^{z} S_k.$$

Thus the form of $l(t)$ in their approach was tacitly assumed to be a linear function in τ. This is certainly not a good approximation to a curved minimum, in general, whereas a cubic function has nonlinear terms to account for the curvature.

In the case of an eclipsing binary the loss of light is given by a well-known expression proportional to $\phi_1+(r_s^2/r_g^2)\phi_2-(r_s/r_g)\sin(\phi_1+\phi_2)$, where r_s and r_g are

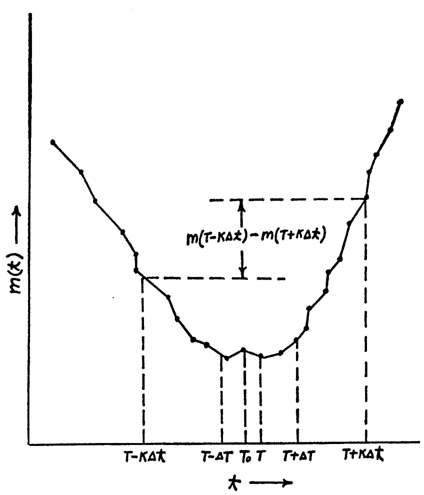
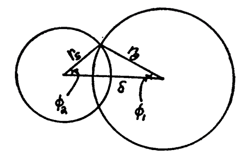


FIG. 1. Light curve variation in the neighborhood of the minimum value with parameters designated.

FIG. 2. Components during eclipse.



the radii of the two components and ϕ_1, and ϕ_2 are designated in Fig. 2. With the aid of the dynamical expression $\delta^2=\cos^2 i+\sin^2 i\,\sin^2\omega t$, where i is the orbital inclination and ω the angular velocity of the system, it can be shown that $l(t)$ is quite complicated in general. Under the very special conditions $\phi_1 \doteq \frac{1}{2}\pi$ and $\phi_2 \doteq \frac{1}{2}\pi$, and $i \doteq \frac{1}{2}\pi$, the form of $l(t)$ can be approximated by a cubic.

A more complete description of the expressions and equations used in the above paragraph may be found in the literature (e.g., Binnendijk 1960). The conclusion drawn from this section is that there are some classes of physical situations where $l(t)$ may be approximated by a second- or third-order polynomial in t. In general, powers of two and higher are needed to express $l(t)$.

ERROR ANALYSIS

Theoretically, T_0 is exactly the minimum of

$$\sum_{k=1}^{z} S_k$$

while T_0 is only an approximation to the minimum of

$$\sum_{k=1}^{z} \Delta m_k^2,$$

which is used in the Kwee and van Woerden (1956) approach. Note that

$$\frac{d}{d\tau}\sum_{k=1}^{z} S_k=0$$

implies $t=T_0$ while

$$\frac{d}{d\tau}\sum_{k=1}^{z} \Delta m_k^2=0$$

implies

$$t=T_0-\frac{\sum_{k=1}^{z}(\mu_{+k}-\mu_{-k})\left(\frac{dl}{dt}\right)_{T_0+k\Delta t}}{2\sum_{k=1}^{z}\left(\frac{d^2l}{dt^2}\right)_{T_0+k\Delta t}} \equiv T_0-R. \qquad (6)$$

The fact that the epoch of minimum here is a function of the random quantities μ, and does not occur at T_0 causes

$$\sum_{k=1}^{z} \Delta m_k^2$$

to have a random rate of approach to the minimum value. The rate of approach of

$$\sum_{k=1}^{z} S_k$$

to its minimum value is zero. Thus both

$$\frac{d}{d\tau}\sum_{k=1}^{z} \Delta m_k^2$$

and

$$\frac{d}{d\tau}\sum_{k=1}^{z} S_k$$

do not have desirable characteristics near $\tau=0$.

It can be seen from Eq. (6) that the errors in T_0 as computed from the Hertzsprung–Kwee and van Woerden approaches are due to the errors in $\frac{1}{2}[(Y_- - Y_+)/(Y_- - 2Y + Y_+)]\Delta T$ as well as the term R. Since each of the terms on the right-hand side of (6) have statistical variations, the standard deviation in the epoch of minimum must be evaluated separately each and every time from a convolution of the form $p_t(t) = \int_{T_0} p_R(T_0 - t) p_{T_0}(T_0) dT_0$, where p represents a probability density. The approach for determining the standard deviation in T_0 is quite different from the approach given by Kwee and van Woerden (1956) since they have not accounted for both terms.

The error in T_0 as determined from the new approach is assumed to be due entirely to the uncertainty in the factor $(Y_- - Y_+)/(Y_- - 2Y + Y_+)$ since T and ΔT are assumed to be known accurately. A fairly good approximation to the error can be obtained by applying a delta process to $(Y_- - Y_+)/(Y_- - 2Y + Y_+)$ and then letting $Y_- \approx Y_+$ (If Y_- is not approximately equal to Y_+, another value of T should be chosen.) This yields $(Y_- - Y_+)/2(Y_+ - Y_-)$. Since the uncertainty in Y_- and Y_+ is due to observational errors only, $\Delta(Y_- - Y_+)$ is formed by the four sums of the form

$$\sum_{k=1}^{z} \mu_k.$$

Since μ_k is a random variable,

$$\sum_{k=1}^{z} \mu_k$$

will also be a random sum whose mean value is zero and whose standard deviation is $\sqrt{(z)}\sigma_\mu$. This result is obtained by repeated use of elementary convolutions of the form $p_{\mu'}(\mu') = \int_\mu p_\mu(\mu' - \mu) p_\mu(\mu) d\mu$, where $\mu' = \mu + \mu$ and p represents a probability density. The quantity σ_μ is the standard deviation in the scatter of the observed data in the ordinate direction. Thus, with repeated convolutions of the type already used, the standard deviation in $\frac{1}{2}[(Y_- - Y_+)/(Y_- - 2Y + Y_+)]\Delta T$, becomes $[\sqrt{(z)}\sigma_\mu/2(Y_+ - Y)]\Delta T = \sigma_{T_0}$. In general the values of $m(T + k\Delta t)$ will not be actually observed data points but will be obtained from linear interpolation of the observed data. Thus the interpolated values of $m(T + k\Delta t)$ are not really independent as was previously assumed. The use of interpolated points tends to make σ_{T_0} too small since interpolated points tend to lie closer to the true curve. In order to compensate for this linear interpolation, it has been suggested by Kwee and van Woerden (1956) that the total weight of the values making up σ_{T_0} be reduced by a factor of $\frac{1}{4}\pi = 0.79$. Thus σ_{T_0} is increased by a factor of $(0.79)^{-\frac{1}{2}} = 1.11$. Linear interpolation is justified only by virtue of its simplicity. More sophisticated interpolation is not warranted in view of the fact that

$$\sum_{k=1}^{z} S_k$$

is an arbitrary means for obtaining T_0.

When both T_0 and σ_{T_0} are to be computed, it is necessary to have a value of z. It is recommended by Kwee and van Woerden (1956) that z be approximately equal to half the number of observed points of the minimum.

An interesting drawback for both the

$$\sum_{k=1}^{z} \Delta m_k^z$$

and

$$\sum_{k=1}^{z} S_k$$

functions is that they should not be evaluated such that $\Delta T = \Delta t$. In both cases the factor becomes a function of only the seven quantities $m[T + (z+1)\Delta t]$, $m(T + z\Delta t)$, $m(T + \Delta t)$, $m(T)$, $m(T - \Delta t)$, $m(T - z\Delta t)$, and $m[T - (z+1)\Delta t]$. This implies that T_0 is determined from these seven observed values, only. Some loss of data will occur in a similar way whenever ΔT and Δt are highly commensurate. Consequently, values of ΔT and Δt which are not highly commensurate are suggested for use. There is a tendency to use $\Delta T = \Delta t$ since this would involve the least number of interpolations of the observed light curve.

It is a well-known fact that the determination of many of the parameters of a binary star system are dependent on the period. The way in which a system evolves is also somewhat indicated by changes in period. Some systems appear to change their period continuously. Among the recent discussions of period changes in binary systems are two publications by Plavec (1966), and one by Wood (1950) and Huang (1963). Thus, the uncertainty in the period is extremely important. An uncertainty in the period can be expressed simply if σ_{T_0} is known for both extremes of the interval over which the period P is computed. If the uncertainties in T_0 at each end are specified by $\sigma_{T_{01}}$, and $\sigma_{T_{02}}$, respectively, and there are N revolutions of the binary components between T_{01}, and T_{02} the standard deviation of the period is given with a simple convolution as

$$\sigma_P=(\sigma_{T_{01}}{}^2+\sigma_{T_{02}}{}^2)^{\frac{1}{2}}/N.$$

Note that the probable error in a normal variable is approximately 0.68σ. The uncertainty in T_0 and P appears too infrequently in the literature where T_0 and P are specified. As a result when T_0 and P are specified it is recommended that they should be specified as

$$HJD\ T_0(\pm\text{p.e. in } T_0)+PE(\pm\text{p.e. in } P).$$

SPECTROSCOPIC BINARY APPLICATIONS

The proposed method for determining the epoch of minimum in a light curve can readily be applied to the determination of the epoch of minimum of a radial velocity curve of a spectroscopic binary. In the case of a spectroscopic curve, the form of $l(t)$ is given by $K_1+K_2\sin[v(t)+K_3]$. This is a well-known expression where K_1, K_2 and K_3 are constants, $\tan[\frac{1}{2}v(t)]=[(1+e)/(1-e)]^{\frac{1}{2}}\tan[\frac{1}{2}E(t)]$, v is the true anomaly, e is the eccentricity of the orbit, and E is the eccentric anomaly. When e is small, v is proportional to t and l can be approximated by a cosine curve in the neighborhood of the minimum. A shifted cosine curve which is approximated by the first two terms of its Maclaurin series can be approximated as a second-order polynomial in t. Consequently the new approach for the determination of the epoch of minimum more accurately represents the curve of the radial velocity curve in the neighborhood of the minimum value.

SUMMARY

The epoch of minimum, its uncertainty, the period, and its uncertainty should be specified for a binary star system. Formulas for these quantities are as follows.

$$T_0=T+\tfrac{1}{2}[(Y_- - Y_+)/(Y_- -2Y+Y_+)]\Delta T,$$

where T_0 represents the epoch of minimum, T represents the guessed epoch of minimum,

$$Y_-=\sum_{k=1}^{z}[m(T-\Delta T+k\Delta t)+m(T+\Delta T-k\Delta t)],$$

$$Y=\sum_{k=1}^{z}[m(T+k\Delta t)+m(T-k\Delta t)],$$

$$Y_+=\sum_{k=1}^{z}[m(T+\Delta T+k\Delta t)+m(T-\Delta T-k\Delta t)],$$

ΔT represents the interval on both sides of T at the end of which computations are made, $m(t)$ represents the observed intensity of the light curve at epoch t, Δt represents a small time interval such that $2z\Delta t$ approximately covers all the observed points; further, $\Delta t\neq\Delta T$ explicitly, $2z$ represents the number of data points approximately. Note that ΔT is large enough so that T_0 is essentially guaranteed to fall in the interval $T-\Delta T$ to $T+\Delta T$ but small enough so that the Taylor series expansion of l holds.

$$\sigma_{T_0}=[\surd(z)\sigma_\mu/2(Y_+-Y)]\Delta T,$$

where σ_{T_0} represents the standard deviation of the distribution of the errors in T_0, σ_μ represents the standard deviation of the distribution of the errors in l,

$$P=(T_{01}-T_{02})/N,$$

where P represents the period of the binary system, N represents the number of cycles between the epochs T_{01} and T_{02},

$$\sigma_P=(\sigma_{T_{01}}{}^2+\sigma_{T_{02}}{}^2)^{\frac{1}{2}}/N,$$

where σ_P represents the standard deviation of the distribution of the errors in P. This method of determining the epoch of minimum and its uncertainty has a better mathematical foundation than previous methods, and is simpler to execute.

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