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COMMUNICATIONS FROM THE OBSERVATORY AT LEIDEN

A METHOD FOR COMPUTING ACCURATELY THE EPOCH OF MINIMUM OF AN ECLIPSING VARIABLE

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A method is given for the accurate computation of the minimum epoch for eclipsing variables, and of its mean error. The method is presented in section 2 and analysed mathematically in section 3. The importance of using well-defined methods is stressed.

1. *Introduction.*

The derivation of times of minimum for eclipsing variables is a problem which ought to have more attention from variable star observers. In papers communicating results of photometric investigations of eclipsing binaries, very little is usually said about the methods used for the derivation of the minimum epochs. However, this is a matter of not only methodical interest. The time of minimum observed brightness being unacceptable as a definition of the epoch, already on account of observational errors, one will use both the descending and ascending branches of the minimum for the determination. But with asymmetrical minima – and slight asymmetry is to be considered the most usual case – different methods of using the observations on the branches will lead to different results, and the epoch of minimum is only exactly defined by the way of determining it. These differences will often be not negligible, since present observational techniques enable us to get series of many accurate observations. Combination of epochs derived by different methods may lead to spurious discrepancies and erroneous conclusions, and should therefore be avoided.

Although it is realized that atmospheric and instrumental troubles will often reduce the effect of exact definitions, the authors feel that full profit of the accuracy obtainable with photoelectric equipment will only be drawn if unnecessary inexactness is avoided. With this purpose in mind, a method to compute accurately the epoch of a minimum observed photoelectrically and its mean error is proposed in this article.

Although the basic ideas of the method have already been introduced by HERTZSPRUNG in 1928[1]), and later on worked out by OOSTERHOFF[2]), HERTZSPRUNG[3]) and DE KORT[4]), a new presentation of the method is given here. Firstly, because these earlier discussions have been rather hidden in papers giving results about a great number of variable stars, and so probably did not receive the attention they deserved. Secondly, because in this paper an extension of the method is given to cases where the equations of condition are not independent of each other.

2. *The method.*

The observations of only one minimum should be used to determine its epoch. Fitting the observations to a "normal light-curve" determined previously, and deriving the epoch from a "normal minimum" and the time interval found by this fitting process, is dangerous as soon as there are only slight variations of the light-curve.

Before starting the computations, one must first decide about the total phase interval to be used. Preferably all phases of the eclipse should be included in the determination, but if, for any reason, this is impossible, care should be taken that on both branches corresponding phase intervals are used. In the case of asymmetric minima, it is of great importance that for the same star always the same phase interval is used, for the epoch is only exactly defined by this interval.

Let N be the total number of observations in the

1) *B.A.N.* **4**, 179 (No. 147), 1928.
2) *B.A.N.* **5**, 39 (No. 166), 1929.
3) *B.A.N.* **9**, 209 (No. 340), 1941.
4) *B.A.N.* **9**, 252 (No. 345), 1941.

CONTENTS

phase interval. Form $(2n+1)$ magnitudes spaced by equal time intervals Δt, by linear interpolation between consecutive observations. To prevent a too unequal use of weights of the observed magnitudes, it is recommended to take $(2n+1)$ about equal to N. One of the equidistant times, T_1 (say), should represent a preliminary time of minimum. A simple estimate made from a plot of the observations will usually suffice to determine T_1. The computer will find it convenient to choose the equidistant times at rounded-off fractions of minutes or of Julian Days.

Take the time T_1 as reflection axis and reflect the interpolated magnitudes of one branch upon the other, giving two magnitudes for every equidistant time on the latter branch. Take the differences of these magnitude pairs Δm_k ($k = 1, \ldots n$), and compute the sum of their squares $s(T_1) \equiv \sum_{k=1}^{n} (\Delta m_k)^2$. Then shift the symmetry axis to $(T_1 + \frac{1}{2}\Delta t)$ and $(T_1 - \frac{1}{2}\Delta t)$ successively, and proceed as above to compute the sums $s(T_1 + \frac{1}{2}\Delta t)$ and $s(T_1 - \frac{1}{2}\Delta t)$.

If T_1 was properly chosen, $s(T_1)$ will be smaller than both other sums. If this is not the case, e.g. if $s(T_1 + \frac{1}{2}\Delta t) < s(T_1)$, the subsequent sum $s(T_1 + \Delta t)$ must be computed; care should be taken that n is the same in all the summations.

The function $s(T)$ is represented by a quadratic formula:

$$s(T) = aT^2 + bT + c. \qquad (1)$$

The constants a, b and c can be computed using the three s-values derived above. The parabola represented by $s(T)$ has a minimum value

$$s(T_\circ) = c - \frac{b^2}{4a} \qquad (2)$$

at

$$T_\circ = -\frac{b}{2a}\,; \qquad (3)$$

$T_\circ$ is the time of minimum sought.

The mean error of the epoch is given by:

$$\sigma^2_{T_\circ} = \frac{4ac - b^2}{4a^2(Z-1)}. \qquad (4)$$

Here Z is the maximum number of independent magnitude pairs. In the case of linear interpolation recommended above, $Z = \frac{1}{4}N$ (section 3). If the observed magnitudes were already equidistant in time, making interpolation unnecessary, $Z = \frac{1}{2}N$.

3. *Analysis of the method.*

In this section we will give an analytical treatment of the method described in section 2.

Let $l(t)$ be the true magnitude as a function of time, freed from observational errors. We assume an axis of symmetry at $t = T_\circ$, and define $T_\circ$ to be the minimum epoch. If u is a certain interval of time, we have:

$$l(T_\circ + u) = l(T_\circ - u). \qquad (5)$$

Suppose we have $2Z$ measured magnitudes at constant time intervals Δt, independent of each other; denote them by $m(T \pm k\Delta t)$, $k = 1, \ldots Z$, then we have the relation:

$$l(T + k\Delta t) = m(T + k\Delta t) - \mu_{+k}, \qquad (6)$$

where μ_{+k} is the deviation in magnitude at the time $(T + k\Delta t)$, arising from observational errors. We now choose T so that $T - T_\circ \equiv \tau$ is a small quantity not larger than Δt. Then by (5) and (6) we may write:

$$\Delta m_k \equiv m(T + k\Delta t) - m(T - k\Delta t) = \mu_{+k} - \mu_{-k} + $$
$$+ l(T_\circ + \tau + k\Delta t) - l(T_\circ - \tau + k\Delta t),$$

and expanding $l(T_\circ + k\Delta t \pm \tau)$ in Taylor series, we get:

$$\Delta m_k = (\mu_{+k} - \mu_{-k}) + 2\tau\left(\frac{\partial l}{\partial t}\right)_{T_\circ + k\Delta t} + $$
$$+ \frac{1}{3}\tau^3\left(\frac{\partial^3 l}{\partial t^3}\right)_{T_\circ + k\Delta t} + \ldots . \qquad (7)$$

For $T = T_\circ$ or $\tau = 0$, Δm_k should vanish, except for observational errors, and we get Z equations of condition:

$$\Delta m_k \equiv m(T + k\Delta t) - m(T - k\Delta t) = 0, \qquad (8)$$

from which $T_\circ$ can be solved so as to be the value of T, satisfying these equations as well as possible. This solution can be easily made by the method of least squares. Its principle states that $S(T) \equiv \sum_{k=1}^{Z} (\Delta m_k)^2$ should be a minimum. With (7) we have, disregarding terms of higher order in τ than the second:

$$S(T) = \sum_{k=1}^{Z} (\mu_{+k} - \mu_{-k})^2 + 4\tau \sum_{k=1}^{Z} \Big[(\mu_{+k} - \mu_{-k}) \times $$
$$\times \left(\frac{\partial l}{\partial t}\right)_{T_\circ + k\Delta t} \Big] + 4\tau^2 \sum_{k=1}^{Z} \left(\frac{\partial l}{\partial t}\right)^2_{T_\circ + k\Delta t}. \qquad (9)$$

As the $\mu_{\pm k}$'s only depend on accidental errors of observation, the second term on the right-hand side in (9) will, for not too small values of Z, be small compared with the sum of the other two terms, and for the first approximation this term may therefore be neglected. So, setting $\tau = T - T_\circ$, (9) transforms into:

$$S(T) = \sum_{k=1}^{Z} (\mu_{+k} - \mu_{-k})^2 + 4\,T_\circ^2 \sum_{k=1}^{Z} \left(\frac{\partial l}{\partial t}\right)^2_{T_\circ + k\Delta t} - $$
$$- 8\,T_\circ T \sum_{k=1}^{Z} \left(\frac{\partial l}{\partial t}\right)^2_{T_\circ + k\Delta t} + 4\,T^2 \sum_{k=1}^{Z} \left(\frac{\partial l}{\partial t}\right)^2_{T_\circ + k\Delta t}, \qquad (10)$$

which we may also write as a quadratic function of T, with coefficients A, B and C, standing for:

$$\left.\begin{aligned}A &= +4\sum_{k=1}^{Z}\left(\frac{\partial l}{\partial t}\right)^2_{T_\circ+k\Delta t},\\ B &= -8\,T_\circ\sum_{k=1}^{Z}\left(\frac{\partial l}{\partial t}\right)^2_{T_\circ+k\Delta t},\\ C &= \sum_{k=1}^{Z}(\mu_{+k}-\mu_{-k})^2+4\,T_\circ^2\sum_{k=1}^{Z}\left(\frac{\partial l}{\partial t}\right)^2_{T_\circ+k\Delta t}.\end{aligned}\right\}\quad(11)$$

The condition for minimum of $S(T)$ leads to the equation (differentiating formula (10)):

$$A\,T_\circ+\tfrac{1}{2}B=0,\qquad(12)$$

from which $T_\circ$ can be solved. The minimum value of $S(T)$ itself may be derived by substituting $T=T_\circ$ in (10), resulting in:

$$S(T_\circ)=\sum_{k=1}^{Z}(\mu_{+k}-\mu_{-k})^2.\qquad(13)$$

The mean error of $T_\circ$, $\sigma_{T_\circ}$, can be derived as follows: we note that our procedure is equal to a least-squares solution with one unknown, formulae (8) serving as equations of condition, and expression (12) as normal equation. The weight of the determination of $T_\circ$ will then be A, and the mean error of a determination with unit weight $\sigma=\sqrt{\dfrac{S(T_\circ)}{Z-1}}$.

So we have:

$$\sigma^2_{T_\circ}=\frac{S(T_\circ)}{A(Z-1)},$$

or with (11) and (13),

$$\sigma^2_{T_\circ}=\frac{\sum\limits_{k=1}^{Z}(\mu_{+k}-\mu_{-k})^2}{4\sum\limits_{k=1}^{Z}\left(\frac{\partial l}{\partial t}\right)^2_{T_\circ+k\Delta t}}\times\frac{1}{Z-1},\qquad(14)$$

or

$$\sigma^2_{T_\circ}=\frac{4AC-B^2}{4A^2}\times\frac{1}{Z-1}.$$

In deriving the formulae above, we have assumed that Z is the number of independent equations of condition (8). Only in this case formula (14) holds. We see, however, that the first factor on the right-hand side of (14) is independent of Z, except for statistical fluctuations and provided that the summations take place over the same interval on the branches. This means that if we had made $2n$ equidistant magnitudes dependent of each other, e.g. by interpolation, out of our N observations – which may be either equidistant or not – and if we define, similar to formulae (11):

$$\left.\begin{aligned}a &= +4\sum_{k=1}^{n}\left(\frac{\partial l}{\partial t}\right)^2_{T_\circ+k\Delta t},\\ b &= -8\,T_\circ\sum_{k=1}^{n}\left(\frac{\partial l}{\partial t}\right)^2_{T_\circ+k\Delta t},\\ c &= \sum_{k=1}^{n}(\mu_{+k}-\mu_{-k})^2+4\,T_\circ^2\sum_{k=1}^{n}\left(\frac{\partial l}{\partial t}\right)^2_{T_\circ+k\Delta t},\end{aligned}\right\}\quad(15)$$

we have, except for statistical fluctuations:

$$\frac{\sum\limits_{k=1}^{n}(\mu_{+k}-\mu_{-k})^2}{\sum\limits_{k=1}^{Z}(\mu_{+k}-\mu_{-k})^2}=\frac{\sum\limits_{k=1}^{n}\left(\frac{\partial l}{\partial t}\right)^2_{T_\circ+k\Delta t}}{\sum\limits_{k=1}^{Z}\left(\frac{\partial l}{\partial t}\right)^2_{T_\circ+k\Delta t}},$$

or

$$\frac{b}{2a}=\frac{B}{2A}\quad\text{and}\quad\frac{4ac-b^2}{4a^2}=\frac{4AC-B^2}{4A^2},$$

and we may write for (12) and (14) respectively:

$$T_\circ=-\frac{B}{2A}=-\frac{b}{2a},$$

$$\sigma^2_{T_\circ}=\frac{4AC-B^2}{4A^2}\times\frac{1}{Z-1}=\frac{4ac-b^2}{4a^2}\times\frac{1}{Z-1},$$

giving (3) and (4) respectively, and in which Z remains the number of independent equations of condition (8) which can be formed out of the N observed magnitudes.

In principle it is useful to make n large, since then the statistical fluctuations of $\dfrac{4ac-b^2}{4a^2}$ will be smaller. On the other hand, a large n will make the computations long and tiresome. The authors have found a compromise by taking $n\approx 2Z$.

When the original observations are not equidistant in time, as is generally the case, interpolation is necessary for making equidistant magnitudes. Any interpolation, however, is then coupled by a loss of total weight. When the interpolation is made linearly, as the authors have proposed in section 2, this loss of total weight can be computed statistically, and it is found that the total weight is reduced by a factor $\frac{\pi}{4}$. Although other methods of interpolation might conserve more of the total weight, the simplicity of linear interpolation is considered to be a decisive advantage.

4. *Discussion.*

The method has been analysed in section 3 on the assumption that the true light-curve is symmetric. Analytical treatment of the case of asymmetry is difficult. There is, however, no objection at all against application of the method described in section 2 to asymmetric minima. In that case, $S(T_\circ)$ will consist of observational errors and of asymmetry terms. Consequently, $S(T_\circ)$ will be larger than in the case of symmetry, and the mean error of the resulting epoch higher; both increase with the amount of asymmetry. Moreover, the resulting $T_\circ$ will depend on the total phase interval on the branches used; it is therefore necessary always to use the same phase interval for one star. In case of serious asymmetry, the computed epoch may be sensibly shifted from the time of minimum brightness. Even so, the epoch will be sharply defined; in some other methods this important feature is lacking in case of asymmetry.

Other advantages of the method are the following. Firstly, since it is completely quantitative, unambiguous results are obtained, not depending on subjective judgment as is often the case with graphical methods. Secondly, a quantitative measure of the accuracy of the epoch is obtained from the expression for the mean error. This is important for the combination of epochs, especially of those obtained photoelectrically which are often accurate to about 10 seconds. Thirdly, curvature does no harm, as it sometimes does when the branches are approximated by straight lines.

A disadvantage is that the method involves a fair amount of calculation. The risk of computing errors can, however, be removed by suitable checks. Furthermore, the writers feel that the reliability of the result does well repay the time and effort.

PHOTOELECTRIC OBSERVATIONS OF THE SHORT-PERIOD ECLIPSING VARIABLE W URSAE MAJORIS

BY K. K. KWEE

The eclipsing variable star W Ursae Majoris was observed photoelectrically at the Leiden Observatory during several nights in 1951, 1953 and 1954. Principally, times of primary minima were observed with accuracies of five to ten seconds. There seems to have been a rather abrupt change in the period from $0^d.33363699$ to $0^d.33363779$. A complete light-curve was observed in one single night in order to derive orbital elements. Evidence is present that the light-curve may be slightly variable from cycle to cycle. The secondary minimum appeared to be shifted notably from phase 0.5, but the displacement may be caused by the observed unequal maxima in the light-curve. The rectification for the sine terms in the representation of the uneclipsed light-curve was performed by division. To represent the rectified eclipse-curve it was necessary to assume an amount of additional light of about 25% of the total light of the system.

In the last section the results are discussed briefly in connection with the simplified methods of computation used.

1. *Introduction.*

The short-period eclipsing variable W Ursae Majoris has been observed photometrically many times and by several observers, since its discovery by MÜLLER and KEMPF in 1903 [1]). A complete compilation of literature up till 1942 of this star system has been given by WOODWARD [2]). After this date only some epochs of minima have been observed by KAPKO [3]) and BAV observers [4]). Of all these photometric data HUFFER's [5]) photoelectric observations of 1934 were still the most interesting and reliable ones. The final orbital elements derived from these observations have been given by PLAUT [6]). Of the spectroscopic observations those of POPPER [7]), and of STRUVE and HORAK [8]) are most up to date. From these observations a mass ratio of 2.0 can be derived, the heavier component being in front at the time of primary minimum.

2. *Observations and reductions.*

Photoelectric observations of this star have been made at the Leiden Observatory during several nights in 1951, 1953 and 1954. The original aim was to investigate changes in the period, so mainly primary minima were observed. In the night following January 23, 1954, a one-run complete light-curve was obtained, from which a set of orbital elements can be derived.

The observations were made with the 18″ Zunderman reflector. The telescope was equipped with the same photometer as A. B. MULLER used for his second series of photoelectric observations of XZ Cygni [9]). In order to get as many points as possible on the descending and ascending branches of the minima, no filters were used. The effective wave length is therefore approximately 4650 Å.

The comparison star used was HD 84142. Some data about the variable and the comparison star are given in Table 1. The spectral type of the variable was taken from STRUVE and HORAK's spectroscopic observations, while that of the comparison star was taken from the Henry Draper Catalogue. The photographic magnitudes were taken from WOODWARD's article. As the range in wave length is rather wide, because no colour filters were used, the advantage of similar spectral type was felt to be more important than the disadvantage of the rather large separation of the two stars.

The measurements were performed by pointing the telescope alternately to the comparison star and the

1) *Ap. J.* **17**, 201, 1903.
2) *Harvard Circ.* No 446, 1942.
3) *Astron. Circ. USSR.* No 88, 7, 1949.
4) *A.N.* **279**, 178, 1950.
5) *Ap. J.* **79**, 369, 1934.
6) *Pub. Kapteyn Astron. Lab. Groningen* No. 54, 1950 and No. 55, 1953.
7) *P.A.S.P.* **62**, 115, 1950.
8) *Ap. J.* **112**, 178, 1950.
9) *B.A.N.* **12**, 11 (No. 443), 1953.