Computational Content of the Classical Axiom of Countable Choice

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1 The Computational System

Definition 1 (Lambda Terms). *Let* Var *be a countably infinite set of variable names. The notions of free variable and closed term are defined as usual. We define the set* Λ_{open} *of not necessarily closed lambda terms, and take* Λ *to be the set of closed terms of* Λ_{open}*. When we say lambda term, we mean closed lambda term.* Λ_{open} *is defined as follows*

$$
\begin{aligned}
\Lambda_{open} := &\mid x \in \mathsf{Var} \\
&\mid \Lambda_{open}\ \Lambda_{open} \\
&\mid \lambda x.\Lambda_{open} && \textit{where } x \in \mathsf{Var} \\
&\mid 0 \mid \mathsf{succ} \mid \mathsf{rec}_{\mathbb{N}} && \textit{constructors and the recursor for } \mathbb{N} \\
&\mid \mathsf{true} \mid \mathsf{false} \mid \mathsf{rec}_{\mathsf{bool}} && \textit{constructors and the recursor for } \mathsf{bool} \\
&\mid \Phi && \textit{the bar recursion operator} \\
&\mid \lambda\!\!\lambda n.t_n \textit{ where } (t_n)_{n\in\mathbb{N}} \subseteq \mathbb{N} && \textit{when applied to } \mathsf{succ}^n 0\textit{, reduces to } t_n \\
&\mid \mathsf{cc} && \textit{call/cc} \\
&\mid \mathsf{k}_\pi && \textit{continuation, where } \pi \textit{ is a stack}
\end{aligned}
$$

Definition 2 (Prooflike Term). *A lambda term is prooflike if it does not contain* $\lambda\!\!\lambda.$*.*

Definition 3 (Stacks). *A stack is a finite list of closed lambda terms. We let* Π *be the set of all stacks. We write* $t \cdot \pi$ *for prepending a lambda term to a stack and* $\pi \cdot \pi'$ *for concatenating two stacks. We write* π_{empty} *for the empty stack.*

Notation 1. *Let* $\vec{t} \subseteq \Lambda$ *and* $\vec{\pi} \subseteq \Pi$*. We write* $\vec{t} \cdot \vec{\pi}$ *for* $\{t \cdot \pi \mid t \in \vec{t}, \pi \in \vec{\pi}\}$*. For* $t \in \Lambda$ *and* $\pi \in \Pi$*, we write* $t \cdot \vec{\pi}$ *for* $\{t\} \cdot \vec{\pi}$ *and* $\vec{t} \cdot \pi$ *for* $\vec{t} \cdot \{\pi\}$*.*

Definition 4 (Process). *A process is a pair* $\langle t \mid \pi \rangle$ *of a lambda term and a stack. We write* $\Lambda \times \Pi$ *for the set of all processes.*

Definition 5 (Reduction Relation)**.** *The big step reduction relation $\succ$ is the smallost transitive and reflexive relation which staisfies:*

$$\begin{aligned}
\langle tu \mid \pi\rangle &\succ \langle t \mid u\pi\rangle \\
\langle \lambda x.t \mid u\pi\rangle &\succ \langle t[x := u] \mid \pi\rangle \\
\langle \mathsf{rec}_{\mathbb{N}} \mid t_0 \cdot t_{\mathsf{succ}} \cdot 0 \cdot \pi\rangle &\succ \langle t_0 \mid \pi\rangle \\
\langle \mathsf{rec}_{\mathbb{N}} \mid t_0 \cdot t_{\mathsf{succ}} \cdot \mathsf{succ}\ n \cdot \pi\rangle &\succ \langle \mathsf{rec}_{\mathbb{N}}\ (t_{\mathsf{succ}}\ t_0)\ t_{\mathsf{succ}}\ n \mid \pi\rangle \\
\langle \mathsf{rec}_{\mathsf{bool}} \mid t_{\mathsf{true}} \cdot t_{\mathsf{false}} \cdot \mathsf{true} \cdot \pi\rangle &\succ \langle t_{\mathsf{true}} \mid \pi\rangle \\
\langle \mathsf{rec}_{\mathsf{bool}} \mid t_{\mathsf{true}} \cdot t_{\mathsf{false}} \cdot \mathsf{false} \cdot \pi\rangle &\succ \langle t_{\mathsf{false}} \mid \pi\rangle \\
\langle \Phi \mid H \cdot P \cdot C \cdot \overline{k} \cdot \pi\rangle &\succ \langle P\ (C; \geq \overline{k} \mapsto H\ (\lambda z.\ \Phi\ H\ P\ \overline{k+1}\ (C; \geq \overline{k} \mapsto z))) \mid \pi\rangle \\
\langle \mathsf{cc} \mid t \cdot \pi\rangle &\succ \langle t \mid \mathsf{k}_\pi \cdot \pi\rangle \\
\langle \mathsf{k}_\pi \mid t \cdot \pi'\rangle &\succ \langle t \mid \pi\rangle
\end{aligned}$$

All the syntactic sugar used in the rule for Φ will be defined in the following subsection.

For $t, u \in \Lambda$, we write $t \succ u$ for $\forall \pi \in \Pi. \langle t \mid \pi\rangle \succ \langle u \mid \pi\rangle$.

1.1 Syntactic Sugar and Special Terms

naturals For $n \in \mathbb{N}$, we write $\overline{k}$ for $\mathsf{succ}^n\ 0$.

if then else We write $\mathsf{if}\ t\ \mathsf{then}\ u\ \mathsf{else}\ v$ for $\mathsf{rec}_{\mathsf{bool}}\ u\ v\ t$.

comparison For $t, u \in \Lambda$, we use $\mathsf{rec}_{\mathbb{N}}$ to define a term $t \leq u$ such that for all $n, m \in \Lambda$

$$\begin{aligned}
\overline{n} \leq \overline{m} &\succ \mathsf{true} && \text{if } n \leq m \\
\overline{n} \leq \overline{m} &\succ \mathsf{false} && \text{otherwise}
\end{aligned}$$

function cons Let $f, t, n \in \Lambda$. Define

$$(f; \geq n \mapsto t) \in \Lambda := \lambda k.\ \mathsf{if}\ n\ \leq\ k\ \mathsf{then}\ t\ \mathsf{else}\ f\ k$$

This term satisfies, for all $f, t \in \Lambda, n, m \in \mathbb{N}$,

$$\begin{aligned}
(f; \geq \overline{n} \mapsto t)\ \overline{m} &\succ f\ \overline{m} && \text{if } m < n \\
(f; \geq \overline{n} \mapsto t)\ \overline{m} &\succ t && \text{if } m \geq n
\end{aligned}$$

We omit parentheses and write $(f; \geq k_1 \mapsto t_1; \geq k_2 \mapsto t_2; \ldots; \geq k_n \mapsto t_n)$ for $((\ldots((f; \geq k_1 \mapsto t_1); \geq k_2 \mapsto t_2))\ldots)); \geq k_n \mapsto t_n)$

2 Realizability

2.1 Logic

We define higher order logic in this section.

types Types are syntactically defined as $\tau, \sigma, \dots := \mathsf{nat} \mid \mathsf{prop} \mid \tau \to \sigma$

variables For each type τ, take a countably infinite set of variables of this type denoted $x^\tau, y^\tau, \dots$ or $x, y, \dots$.

terms Terms are tied with a type and defined inductively as follows

- **variable** If x^τ is a variable of type τ, then x^τ is a term of type τ.
- **abstraction** If x^τ is a variable of type τ and M is a term of type σ, then $\lambda x^\tau.M$ is a term of type $\tau \to \sigma$.
- **application** If M is a term of type $\tau \to \sigma$ and N is a term of type τ, then MN is a term of type σ.
- **zero** 0 is a term of type nat.
- **successor** succ is a term of type $\mathsf{nat} \to \mathsf{nat}$.
- **recursor for naturals** For every type τ, recnat_τ is a term of type $\tau \to (\mathsf{nat} \to \tau \to \tau) \to \mathsf{nat} \to \tau$.
- **implication** If M and N are terms of type prop, then $M \Rightarrow N$ is a term of type prop.
- **universal quantification** If x^τ is a variable of type τ and M is a term of type prop, then $\forall x^\tau.M$ is a term of type prop.
- **dependent universal quantification** If x^{nat} is a variable of type nat and M is a formula of type prop, then $\forall^{\mathbb{N}} x.M$ is a formula of type prop.
- **equality** If M and N are terms of type nat, then $M = N$ is a term of type prop (should I define equality for any type?).
- **top and bottom** $\top$ and $\bot$ are formulas of type prop.

As usual, we write $\neg M$ for $M \to \bot$.

2.2 The Realizability Relation

Definition 6 (interpretation of types)**.** *The interpretation $[\![\tau]\!]$ of a type τ is a set defined by induction over the syntax of τ by*

$$[\![\mathsf{nat}]\!] := \mathbb{N}$$
$$[\![\mathsf{prop}]\!] := \mathcal{P}(\Pi)$$
$$[\![\tau \to \sigma]\!] := [\![\sigma]\!]^{[\![\tau]\!]}$$

Definition 7 (valuation)**.** *A valuation ρ is a partial function from the set of variables which to each variable of type τ associates an element of the set $[\![\tau]\!]$. We furthermore require that ρ be defined at at most finitely many points.*

For a valuation ρ, variable x of type τ, and $y \in [\![\tau]\!]$, we write $\rho; x \mapsto y$ for $x' \mapsto \begin{cases} y & \text{if } x' = x \\ \rho(x') & \text{otherwise} \end{cases}$.

The empty valuation ρ_{empty} is the valuation defined nowhere.

Definition 8. *For $\vec{\pi} \subseteq \Pi$, let*

$$\vec{\pi}^{\perp\!\!\perp} \subseteq \Lambda = \{t \in \Lambda \mid \forall \pi \in \vec{\pi}. \langle t \mid \pi \rangle \in \perp\!\!\perp\}$$

Definition 9 (intetrpretation of terms)**.** *Let ρ be a valuation. For a term M of type τ such that ρ is defined at all the free variables of M, we define an interpretation $\|M\|_\rho$ by syntactic induction over M as follows*

$$\begin{aligned}
\|x\|_\rho &:= \rho(x) \\
\|\lambda x^\tau.M\|_\rho &:= v \in [\![\tau]\!] \mapsto \|M\|_{\rho;x\mapsto v} \\
\|MN\|_\rho &:= \|M\|_\rho(\|N\|_\rho) \\
\|0\|_\rho &:= 0 \\
\|\mathsf{succ}\|_\rho &:= n \mapsto n+1 \\
\|\mathsf{recnat}_\tau\|_\rho &:= P_0 \mapsto P_{\mathsf{succ}} \mapsto n \mapsto \begin{cases} P_0 & \text{if } n = 0 \\ P_{\mathsf{succ}}(\|\mathsf{recnat}_\tau\|_\rho(P_0)(P_{\mathsf{succ}})(n-1)) & \text{otherwise} \end{cases} \\
\|M = N\|_\rho &:= \begin{cases} \emptyset & \text{if } \|M\|_\rho = \|N\|_\rho \text{ in the standard model of } \mathbb{N} \\ \mathcal{P}(\Pi) & \text{otherwise} \end{cases} \\
\|M \Rightarrow N\|_\rho &:= \|M\|_\rho^{\perp\!\!\perp} \cdot \|N\|_\rho \\
\|\forall x^\tau.M\|_\rho &:= \bigcup_{v \in [\![\tau]\!]} \|M\|_{\rho,x\mapsto v} \\
\|\forall^{\mathbb{N}} x.M\|_\rho &:= \bigcup_{n \in \mathbb{N}} \overline{n} \cdot \|M\|_{\rho;x\mapsto n} \\
\|\top\|_\rho &:= \emptyset \\
\|\bot\|_\rho &:= \mathcal{P}(\Pi)
\end{aligned}$$

Notation 2. *Let M be a formula, t be a term, and $\vec{\pi} \subset \Pi$. We write $|M|_\rho$ for $\|M\|_\rho^{\perp\!\!\perp}$, $t \Vdash_\rho M$ for $t \in |M|_\rho$, and $t \Vdash \vec{\pi}$ for $t \in \vec{\pi}^{\perp\!\!\perp}$.*

We write $\|\cdot\|$ for $\|\cdot\|_{\rho_{\mathsf{empty}}}$, $|\cdot|$ for $|\cdot|_{\rho_{\mathsf{empty}}}$, and $\Vdash$ for $\Vdash_{\rho_{\mathsf{empty}}}$.

2.3 Properties

Propetry 1. *If $t \succ u$ and $u \Vdash_\rho M$, then $t \Vdash_\rho M$.*

Proof. Suppose $t \succ u$ and $u \Vdash_\rho M$, that is, $\forall \pi \in \|M\|_\rho . \langle u \mid \pi \rangle \in \perp\!\!\!\perp$. Let $\pi \in \|M\|_\rho$, we need to show that $\langle t \mid \pi \rangle \in \perp\!\!\!\perp$. This is true because $\langle t \mid \pi \rangle \succ \langle u \mid \pi \rangle$. $\square$

Propetry 2. *$t \Vdash_\rho M \to N$ if and only if $\forall u \in \Lambda . u \Vdash_\rho M \Rightarrow tu \Vdash_\rho N$.*

Proof.

$\Rightarrow$ **direction** Suppose $t \Vdash_\rho M \to N$ and let $u \in \Lambda$ be such that $u \Vdash_\rho M$. Let $\pi \in \|N\|_\rho$. We need to show that $\langle tu \mid \pi \rangle \in \perp\!\!\!\perp$. But $\langle tu \mid \pi \rangle \succ \langle t \mid u\pi \rangle$ and $\langle t \mid u\pi \rangle \in \perp\!\!\!\perp$ since $u\pi \in \|M \to N\|_\rho$.

$\Leftarrow$ **direction** Suppose $\forall u \in \Lambda . u \Vdash_\rho M \Rightarrow tu \Vdash_\rho N$. Let $\pi \in \|M \to N\|_\rho$, that is, $\pi = u \cdot \pi'$ for some $u \in |M|_\rho$ and $\pi' \in \|N\|_\rho$. We need to show that $\langle t \mid u \cdot \pi' \rangle \in \perp\!\!\!\perp$. It suffices to show that $\langle tu \mid \pi' \rangle \in \perp\!\!\!\perp$, which is the case since $tu \Vdash_\rho N$ and $\pi' \in \|N\|_\rho$.

$\square$

Propetry 3 (Consistency)**.** *There is no t, ρ such that $t \Vdash_\rho \bot$.*

Proof. Suppose for the sake of contradiction that $t \Vdash_\rho \bot$. Then, for all $\pi \in \|\bot\|_\rho = \mathcal{P}(Pi)$, $\langle t \mid \pi \rangle \in \perp\!\!\!\perp$. We have to take a pole for which this yields to a contradiction. $\square$

Propetry 4. *If $t \Vdash \bot$ then $t \Vdash F$ for all formula F.*

Proof. By consistency, there is no such t, ρ. $\square$

Propetry 5. *$t \Vdash_\rho \forall^{\mathbb{N}} x.M$ if and only if for all $n \in \mathbb{N}$, $t\overline{n} \Vdash_{\rho; x \mapsto \overline{n}} M$.*

Proof.

$\Rightarrow$ **direction** Suppose $t \Vdash_\rho \forall^{\mathbb{N}} x.M$. Let $n \in \mathbb{N}$ and $\pi \in \|M\|_{\rho; x \mapsto \overline{n}}$. We need to show that $\langle t\overline{n} \mid \pi \rangle \in \perp\!\!\!\perp$. It suffices to show that $\langle t \mid \overline{n} \cdot \pi \rangle \in \perp\!\!\!\perp$, which is the case since $\overline{n} \cdot \pi \in \overline{n} \cdot \|M\|_{\rho; m \mapsto \overline{n}} \subseteq \|M\|_\rho$.

$\Leftarrow$ **direction** Suppose that for all n, $t\overline{n} \Vdash_{\rho; x \mapsto \overline{n}} M$. Let $n \in \mathbb{N}$ and $\pi \in \|M\|_{\rho; x \mapsto \overline{n}}$. We need to show that $\langle t \mid \overline{n} \cdot \pi \rangle \in \perp\!\!\!\perp$. ???

$\square$

Propetry 6 (Continuity)**.** *Let t be a term with one free variable x. Let M be a formula. Let $u_0, u_1, \cdots \in \Lambda$. Suppose that $t[x := \lambda\!\!\lambda\, n.u_n] \Vdash M$. Then there exists $N \in \mathbb{N}$ such that for all $f \in \Lambda$ such that $\forall n < N.\ f\ \overline{n} \succ u_n$ we have $t[x := f] \Vdash M$.*

3 The Induced Evidenced Frame

Definition 10. *Define an evidenced frame by taking*

propositions Φ *is* $\mathcal{P}(\Pi)$

evidence E *is the set of prooflike lambda terms.*

evidence relation *For* $\phi_1, \phi_2 \in \Phi$ *and* $e \in E$, $\phi_1 \xrightarrow{e} \phi_2$ *if and only if* $\forall t \in \Lambda. t \Vdash \phi_1 \Rightarrow et \Vdash \phi_2$.

top $\top = \emptyset$.

conjunction *For* $\phi_1, \phi_2 \in \Phi$, $\phi_1 \wedge \phi_2$ *is* $\bigcup_{\phi \in \Phi} (\phi_1^{\perp\!\!\perp} \cdot \phi_2^{\perp\!\!\perp} \cdot \phi)^{\perp\!\!\perp} \cdot \phi$.

universal implication *For* $\phi_1 \in \Phi$ *and* $\vec{\phi} \subseteq \Phi$, $\phi_1 \supset \vec{\phi}$ *is* $\phi_1^{\perp\!\!\perp} \cdot \bigcup \vec{\phi}$.

Theorem 1. *The evidenced frame defined above is an evidenced frame.*

Proof.

reflexivity Take $e_{\mathsf{id}} = \lambda x.x$. If $t \Vdash F$ then $e_{\mathsf{id}} t \Vdash F$ since $e_{\mathsf{id}} t \succ t$.

top Take $e_{\mathsf{top}} = \lambda x.x$. If $t \Vdash F$ then $e_{\mathsf{top}} t \Vdash \top$ since every lambda term realizes $\top$, which can be seen by unfolding the definitions of $\top$ and $\Vdash$.

conjunction elimination Take $e_{\mathsf{fst}} = \lambda t.t(\lambda x.\lambda y.x)$. $\lambda x.\lambda y.x \Vdash \phi_1 \to \phi_2 \to \phi_1$ so $\lambda x.\lambda y.x \Vdash \phi_1^{\top} \cdot \phi_2^{\top} \cdot \phi_1$. Thus if $t \Vdash \phi_1 \wedge \phi_2$ we have $t \Vdash (\phi_1 \to \phi_2 \to \phi_1) \to \phi_1$ so $t(\lambda x.\lambda y.x) \Vdash \phi_1$ and thus $e_{\mathsf{fst}} t \Vdash \phi_1$. We do the same thing for e_{snd}.

conjunction introduction Take $< |e_1, e_2| >= \lambda t.te_1e_2$. Suppose $e_1 \Vdash \phi_1$ and $e_2 \Vdash \phi_2$. We need to show $< |e_1, e_2| >\Vdash \phi_1 \wedge \phi_2$. Let $\phi \in \Phi$ and $t \Vdash \phi_1 \to \phi_2 \to \phi$. It suffices to show $te_1e_2 \Vdash \phi$, which is the case.

universal implication introduction Take $\lambda e = \lambda t.\lambda u.e < |t, u| >$. Let $\phi_1, \phi_2 \in \Phi$, $\vec{\phi} \subseteq \Phi$, and $e \in E$ such that $\forall \phi \in \vec{\phi}. \phi_1 \wedge \phi_2 \xrightarrow{e} \phi$. We need to show that for all $t \Vdash \phi_1$, $(\lambda e)t \Vdash \phi_1 \supset \vec{\phi}$, that is, for all $u \Vdash \phi_2$ and $\phi \in \vec{\phi}$, $(\lambda e)tu \Vdash \phi$. But $(\lambda e)tu \succ e < |t, u| >$, which concludes since $e \Vdash \phi_1 \wedge \phi_2 \to \phi$ and $< |t, u| >\Vdash \phi_1 \wedge \phi_2$.

universal quantification elimination Take $e_{\mathsf{eval}} = \lambda t.(e_{\mathsf{fst}} t)(e_{\mathsf{snd}} t)$. Suppose $t \Vdash (\phi_1 \supset \vec{\phi}) \wedge \phi_1$ and let $\phi \in \vec{\phi}$, we need to show $e_{\mathsf{eval}} t \Vdash \phi$. It suffices to show $(e_{\mathsf{fst}} t)(e_{\mathsf{snd}} t) \Vdash \phi$, which is the case since $e_{\mathsf{fst}} t \Vdash \phi_1^{\top} \phi$ and $e_{\mathsf{snd}} \Vdash \phi_1$.

□

4 Realizability of Countable Choice

Throughout this section, let τ be a type. We define

$$AC_{\mathbb{N},\tau} := \forall R^{\mathsf{nat}\to\tau\to\mathsf{prop}}.(\forall n^{\mathsf{nat}}.\neg\forall i^{\tau}.\neg R(n,i)) \to \neg\forall f^{\mathsf{nat}\to\tau}.\neg\forall n^{\mathsf{nat}}.R(n,f(n))$$

Theorem 2. *We have*

$$\lambda H.\ \lambda P.\ \Phi\ H\ P\ \overline{0}\ \overline{0} \Vdash AC_{\mathbb{N},\tau}$$

Proof. Let $R_0 \in \mathcal{P}(\Pi)^{\mathbb{N}\times[\![\tau]\!]}$. Let $\rho = (\rho_{\mathsf{empty}}; R \mapsto R_0)$.

Let

$$H \Vdash_\rho \forall n.\neg\forall i.\neg R(n,i)$$
$$P \Vdash_\rho \forall f.\neg\forall^{\mathbb{N}} n.R(n,f(n))$$

By propetry 2, it is necessarily and sufficient to show that

$$\Phi\ H\ P\ \overline{0}\ \overline{0} \Vdash_\rho \bot$$

Definition 11 ($< k$ -cache)**.** *Let $k \in \mathbb{N}$. A $< k$ -cache is a term $C \in \Lambda$ such that $\forall n < k.\exists i.C\ \overline{n} \Vdash_\rho R(n,i)$.*

Lemma 1. *Let $k \in \mathbb{N}$. Let C be a $< k$ -cache. Suppose that $\Phi\ H\ P\ C\ \overline{k} \not\Vdash_\rho \bot$. Then, there exists i_k and $r_k \Vdash_\rho R(k,i_k)$ such that $\Phi\ H\ P\ (C; \geq \overline{k} \mapsto r_k)\ \overline{k+1} \not\Vdash_\rho \bot$.*

Note that $(C; \geq \overline{k} \mapsto r_k)$ is then a $< k+1$ -cache.

Proof. There exist $r_k \in \Lambda, i_k \in I$ such that $r_k \Vdash_\rho R(k,i_k)$. Indeed, if none did exists, then for all i, any term would realize $\neg R(k,i)$, thus, any term would realize $\forall i.\neg R(k,i)$. Thus, H applied to any term would realize $\bot$, which contradicts consistency.

Now, suppose that for all r_k, i_k such that $r_k \Vdash_\rho R(k,i_k)$,

$$\Phi\ H\ P\ (C; \geq \overline{k} \mapsto r_k)\ \overline{k+1} \Vdash_\rho \bot$$

it suffices to find a contradiction. Then, for all i,

$$\lambda z.\ \Phi\ H\ P\ (C; \geq \overline{k} \mapsto z)\ \overline{k+1} \Vdash_\rho \neg R(k,i)$$

and so by the hypothesis on H and then by ???,

$$\begin{aligned} H\ (\lambda z.\ \Phi\ H\ P\ (C; \geq \overline{k} \mapsto z)\ \overline{k+1} \Vdash_\rho \neg R(k,i)) &\Vdash_\rho \bot \\ &\Vdash_\rho R(n,i) \quad \text{for all } n \text{ and } i \end{aligned}$$

Now, by definition of a cache, for all $n < k$, $C\ \overline{n} \Vdash_\rho R(n,i)$ for some i. Let $f(n)$ be such an i for $n < k$ and be arbitrary for $n \geq k$. Then,

$$(C; \geq \overline{k} \mapsto H\ (\lambda z.\ \Phi\ H\ P\ (C; \geq \overline{k} \mapsto z)\ \overline{k+1} \Vdash_\rho \neg R(k,i))) \Vdash_\rho \forall^{\mathbb{N}} n.R(n,f(n))$$

since this term applied to $n < k$ realizes $R(n, f(n))$ by the definition of a cache and this term applied to $n \geq k$ realizes $R(n, f(n))$ by the previous discussion. Thus,

$$P\,(C; \geq \overline{k} \mapsto H\,(\lambda z.\, \Phi\; H\; P\,(C; \geq \overline{k} \mapsto z)\; \overline{k+1} \Vdash_\rho \neg R(k, i))) \Vdash_\rho \forall^{\mathbb{N}} n.R(n, f(n)) \Vdash_\rho \bot$$

But

$$\Phi\;\; H\;\; P\;\; C\;\; \overline{k} \succ P\;\; (C; \geq \overline{k} \mapsto H\;\; (\lambda z.\;\, \Phi\;\; H\;\; P\;\; (C; \geq \overline{k} \mapsto z)\;\; \overline{k+1}))$$

So

$$\Phi\;\; H\;\; P\;\; C\;\; \overline{k} \Vdash_\rho \bot$$

which was supposed to not hold. □

Now, suppose

$$\Phi\;\; H\;\; P\;\; \overline{0}\;\; \overline{0} \nVdash_\rho \bot$$

it is necessary and sufficient to find a contradiction.

By applying lemma 1 repeatedly, we get a sequence $r_0, r_1, \ldots$ such that each r_n realizes $R(n, i)$ for some i. For each n, take $f_0(n)$ to be such an i.

Then, for all n, $(\lambda\!\!\lambda n.r_n)\overline{n} \Vdash_\rho R(n, f_0(n))$. Thus, $\lambda\!\!\lambda n.r_n \Vdash_\rho \forall^{\mathbb{N}} n.R(n, f_0(n))$ and so

$$P\;\; (\lambda\!\!\lambda n.r_n) \Vdash_\rho \bot$$

But then, by continuity, there exists N such that for all C, if $\forall n < N.\; C\;\overline{n} \succ r_n$ then $P\;\; C \Vdash_\rho \bot$. Thus, taking $C = (\overline{0}; \geq \overline{0} \mapsto r_0; \geq \overline{1} \mapsto r_1; \ldots; \geq \overline{N-1} \mapsto r_{N-1}; \geq \overline{N} \mapsto H\;\; (\lambda z.\; \Phi\;\; H\;\; P\;\; \overline{N+1}\;\; (C; \geq \overline{k} \mapsto z)))$, we have $P\;\; C \Vdash_\rho \bot$ since $\forall n < N.C\;\; \overline{n} \succ r_n$. But $\Phi\;\; H\;\; P\;\; \overline{0}\;\; \overline{0} \succ P\;\; C$, so $\Phi\;\; H\;\; P\;\; \overline{0}\;\; \overline{0} \Vdash_\rho \bot$, which was supposed to not hold.

□