Computational Content of the Classical Axiom of Countable Choice

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1 The Computational System

Definition 1 (Lambda Terms). *Let* Var *be a countably infinite set of variable names. The notions of free variable and closed term are defined as usual. We define the set* Λ_{open} *of not necessarily closed lambda terms, and take* Λ *to be the set of closed terms of* Λ_{open}*. When we say lambda term, we mean closed lambda term.* Λ_{open} *is defined as follows*

$$
\begin{aligned}
\Lambda_{open} := &\mid x \in \mathsf{Var} \\
&\mid \Lambda_{open}\ \Lambda_{open} \\
&\mid \lambda x.\Lambda_{open} && \textit{where } x \in \mathsf{Var} \\
&\mid 0 \mid \mathsf{succ} \mid \mathsf{rec}_{\mathbb{N}} && \textit{constructors and the recursor for naturals} \\
&\mid \mathsf{cons} \mid \mathsf{nil} \mid \mathsf{rec}_{\mathsf{list}} && \textit{the constructors and recursor for lists} \\
&\mid \Phi && \textit{the bar recursion operator} \\
&\mid \lambda\!\!\lambda n.t_n \textit{ where } (t_n)_{n\in\mathbb{N}} \subseteq \Lambda && \textit{when applied to } \mathsf{succ}^n 0\textit{, reduces to } t_n \\
&\mid \mathsf{cc} && \textit{call/cc} \\
&\mid \mathsf{k}_\pi && \textit{continuation, where } \pi \textit{ is a stack}
\end{aligned}
$$

cons should be thought of as appending an element at the end of a list and not at the beginning. Note that this implies that the head of a list is a list and the tail of a list is an element, contrary to what's usual.

Definition 2 (Stacks). *A stack is a finite list of lambda terms. We write* Π *for the set of all stacks. We write* $t \cdot \pi$ *for prepending a lambda term to a stack and* $\pi \cdot \pi'$ *for concatenating two stacks. We write* π_{empty} *for the empty stack.*

Definition 3 (Prooflike Term). *A lambda term is prooflike if it does not contain* k *or* $\lambda\!\!\lambda$*.*

Notation 1. *We write* $\overline{n}$ *for* $\mathsf{succ}^n\ 0$*.*

We write $[]$ *for* nil *and* $\ell_{\mathsf{head}} :: x_{\mathsf{tail}}$ *for* $\mathsf{cons}\ \ell_{\mathsf{head}}\ x_{\mathsf{tail}}$*, the associativity of* $\ell :: x_1 :: x_2 :: \cdots :: x_n$ *is* $((\ldots((\ell :: x_1) :: x_2)\ldots) :: x_n$*. We write* $[x_1, x_2, \ldots, x_n]$ *for* $\mathsf{nil} :: x_1 :: x_2 :: \cdots :: x_n$*.*

Notation 2. *Let $\vec{t} \subseteq \Lambda$ and $\vec{\pi} \subseteq \Pi$. We write $\vec{t} \cdot \vec{\pi}$ for $\{t \cdot \pi \mid t \in \vec{t}, \pi \in \vec{\pi}\}$. For $t \in \Lambda$ and $\pi \in \Pi$, we write $t \cdot \vec{\pi}$ for $\{t\} \cdot \vec{\pi}$ and $\vec{t} \cdot \pi$ for $\vec{t} \cdot \{\pi\}$.*

Definition 4 (Process)**.** *A process is a pair $\langle t \mid \pi \rangle$ of a lambda term and a stack. We write $\Lambda \times \Pi$ for the set of all processes.*

Definition 5 (Reduction Relation)**.** *The big step reduction relation $\succ$ is the smallest transitive and reflexive relation which staisfies the following. The term* nth *used in the rule for Φ will be defined right after.*

$$\begin{aligned}
\langle tu \mid \pi \rangle &\succ \langle t \mid u \cdot \pi \rangle \\
\langle \lambda x.t \mid u \cdot \pi \rangle &\succ \langle t[x := u] \mid \pi \rangle \\
\langle \mathsf{rec}_{\mathbb{N}} \mid t_0 \cdot t_{\mathsf{succ}} \cdot 0 \cdot \pi \rangle &\succ \langle t_0 \mid \pi \rangle \\
\langle \mathsf{rec}_{\mathbb{N}} \mid t_0 \cdot t_{\mathsf{succ}} \cdot \mathsf{succ}\ n \cdot \pi \rangle &\succ \langle t_{\mathsf{succ}}\ n\ (\mathsf{rec}_{\mathbb{N}}\ t_0\ t_{\mathsf{succ}}\ n) \mid \pi \rangle \\
\langle \mathsf{rec}_{\mathsf{list}} \mid t_{\mathsf{nil}} \cdot t_{\mathsf{cons}} \cdot [] \cdot \pi \rangle &\succ \langle t_{\mathsf{nil}} \mid \pi \rangle \\
\langle \mathsf{rec}_{\mathsf{list}} \mid t_{\mathsf{nil}} \cdot t_{\mathsf{cons}} \cdot (\ell_{\mathsf{head}} :: x_{\mathsf{tail}}) \cdot \pi \rangle &\succ \langle t_{\mathsf{cons}}\ x_{\mathsf{tail}}\ \ell_{\mathsf{head}}\ (\mathsf{rec}_{\mathsf{list}}\ t_{\mathsf{nil}}\ t_{\mathsf{cons}}\ \ell_{\mathsf{head}}) \mid \pi \rangle \\
\langle \Phi \mid H \cdot P \cdot \ell \cdot \pi \rangle &\succ \langle P\ (\lambda m.\ \mathsf{nth}\ m\ (\ell :: H\ (\lambda z.\ \Phi\ H\ P\ (\ell :: z)))) \mid \pi \rangle \\
\langle \lambda\!\!\lambda n.t_n \mid \overline{n} \cdot \pi \rangle &\succ \langle t_n \mid \pi \rangle \\
\langle \mathsf{cc} \mid t \cdot \pi \rangle &\succ \langle t \mid \mathsf{k}_\pi \cdot \pi \rangle \\
\langle \mathsf{k}_{\pi'} \mid t \cdot \pi \rangle &\succ \langle t \mid \pi' \rangle
\end{aligned}$$

Notation 3. *For $t, u \in \Lambda$, we write $t \succ u$ for $\forall \pi \in \Pi. \langle t \mid \pi \rangle \succ \langle u \mid \pi \rangle$.*

Definition 6 (nth)**.** *We use $\mathsf{rec}_{\mathsf{list}}$ and $\mathsf{rec}_{\mathbb{N}}$ to define a (prooflike) lambda term* nth *such that for all $n \in \mathbb{N}$ and $x_0, \dots, x_{k-1} \in \Lambda$,*

$$\begin{aligned}
\mathsf{nth}\ \overline{n}\ [x_0, \dots, x_{k-1}] &\succ x_n && \text{if } n < k \\
\mathsf{nth}\ \overline{n}\ [x_0, \dots, x_{k-1}] &\succ x_{k-1} && \text{if } n \geq k \text{ and } k > 0
\end{aligned}$$

Definition 7 (Length of a List)**.** *For a term ℓ of the form $[x_0, \dots, x_{k-1}]$, we define, in the metatheory, $|\ell|$ to be k.*

2 Realizability

2.1 Logic

We define higher order logic in this section.

types Types are syntactically defined as $\tau, \sigma, \dots := \mathsf{nat} \mid \mathsf{prop} \mid \tau \to \sigma$

variables For each type τ, take a countably infinite set of variables of this type denoted $x^\tau, y^\tau, \dots$ or $x, y, \dots$, where x and x^τ is the same variable name.

Formulas Formulas are tied with a type and defined inductively as follows

variable If x^τ is a variable of type τ, then x^τ is a formula of type τ.

abstraction If x^τ is a variable of type τ and M is a formula of type σ, then $\lambda x^\tau.M$ is a term of type $\tau \to \sigma$.

application If M is a formula of type $\tau \to \sigma$ and N is a formula of type τ, then MN is a formula of type σ.

zero 0 is a formula of type nat.

successor succ is a formula of type $\mathsf{nat} \to \mathsf{nat}$.

recursor for naturals For every type τ, recnat_τ is a formula of type $\tau \to (\mathsf{nat} \to \tau \to \tau) \to \mathsf{nat} \to \tau$.

implication If M and N are formulas of type prop, then $M \Rightarrow N$ is a formula of type prop.

universal quantification If x^τ is a variable of type τ and M is a formula of type prop, then $\forall x^\tau.M$ is a formula of type prop.

dependent universal quantification If x^{nat} is a variable of type nat and M is a formula of type prop, then $\forall^{\mathbb{N}} x.M$ is a formula of type prop.

equality If M and N are formulas of type nat, then $M = N$ is a formula of type prop (should I define equality for any type?).

top and bottom $\top$ and $\bot$ are formulas of type prop.

As usual, we write $\neg M$ for $M \to \bot$.

Note that since cc realizes Peirce's law, we do not need to define existential quantification since it can be replaced by $\neg\forall\neg$.

2.2 The Realizability Relation

Definition 8 ($\perp\!\!\!\perp$)**.** *We define the pole $\perp\!\!\!\perp \subseteq \Lambda \times \Pi$ to be TO DO.*

Note that $\perp\!\!\!\perp$ is closed by anti-reduction, that is, if $p \succ q$ and $q \in \perp\!\!\!\perp$, then $p \in \perp\!\!\!\perp$.

Definition 9 (interpretation of types)**.** *The interpretation $[\![\tau]\!]$ of a type τ is a set defined by induction over the syntax of τ by*

$$
\begin{aligned}
[\![\mathsf{nat}]\!] &:= \mathbb{N} \\
[\![\mathsf{prop}]\!] &:= \mathcal{P}(\Pi) \\
[\![\tau \to \sigma]\!] &:= [\![\sigma]\!]^{[\![\tau]\!]}
\end{aligned}
$$

Definition 10 (valuation)**.** *A valuation ρ is a partial function from the set of variables which to each variable of type τ associates an element of the set $[\![\tau]\!]$. We furthermore require that ρ be defined at at most finitely many points.*

For a valuation ρ, a variable x of type τ, and $y \in [\![\tau]\!]$, we write $\rho; x \mapsto y$ for the valuation which maps x to y and every $x' \neq x$ to $\rho(x')$.

The empty valuation ρ_{empty} is the valuation which is defined nowhere.

Definition 11. *For $\vec{\pi} \subseteq \Pi$, let*

$$\vec{\pi}^{\perp\!\!\!\perp} \subseteq \Lambda = \{t \in \Lambda \mid \forall \pi \in \vec{\pi}.\langle t \mid \pi \rangle \in \perp\!\!\!\perp\}$$

Definition 12 (intetrpretation of terms)**.** *Let ρ be a valuation. For a term M of type τ such that ρ is defined at all the free variables of M, we define the falsity interpretation $\|M\|_\rho \in [\![\tau]\!]$ by syntactic induction over M as follows*

$$\begin{aligned}
\|x\|_\rho &:= \rho(x) \\
\|\lambda x^\tau.M\|_\rho &:= v \in [\![\tau]\!] \mapsto \|M\|_{\rho;x\mapsto v} \\
\|MN\|_\rho &:= \|M\|_\rho(\|N\|_\rho) \\
\|0\|_\rho &:= 0 \\
\|\mathsf{succ}\|_\rho &:= n \mapsto n+1 \\
\|\mathsf{recnat}_\tau\|_\rho &:= P_0 \mapsto P_{\mathsf{succ}} \mapsto n \mapsto \begin{cases} P_0 & \textit{if } n = 0 \\ P_{\mathsf{succ}}(n-1)(\|\mathsf{recnat}_\tau\|_\rho(P_0)(P_{\mathsf{succ}})(n-1)) & \textit{otherwise} \end{cases} \\
\|M = N\|_\rho &:= \begin{cases} \emptyset & \textit{if } \|M\|_\rho = \|N\|_\rho \textit{ in the standard model of } \mathbb{N} \\ \mathcal{P}(\Pi) & \textit{otherwise} \end{cases} \\
\|M \Rightarrow N\|_\rho &:= \|M\|_\rho^{\perp\!\!\!\perp} \cdot \|N\|_\rho \\
\|\forall x^\tau.M\|_\rho &:= \bigcup_{v \in [\![\tau]\!]} \|M\|_{\rho,x\mapsto v} \\
\|\forall^{\mathbb{N}} x.M\|_\rho &:= \bigcup_{n\in\mathbb{N}} \overline{n} \cdot \|M\|_{\rho;x\mapsto\overline{n}} \\
\|\top\|_\rho &:= \emptyset \\
\|\bot\|_\rho &:= \mathcal{P}(\Pi)
\end{aligned}$$

Notation 4. *Let M be a formula, t be a term, and $\vec{\pi} \subset \Pi$. We write $|M|_\rho$ for $\|M\|_\rho^{\perp\!\!\!\perp}$, $t \Vdash_\rho M$ for $t \in |M|_\rho$, and $t \Vdash \vec{\pi}$ for $t \in \vec{\pi}^{\perp\!\!\!\perp}$.*

We write $\|\cdot\|$ for $\|\cdot\|_{\rho_{\mathsf{empty}}}$, $|\cdot|$ for $|\cdot|_{\rho_{\mathsf{empty}}}$, and $\Vdash$ for $\Vdash_{\rho_{\mathsf{empty}}}$.

2.3 Properties

Propetry 1. *If $t \succ u$ and $u \Vdash_\rho M$, then $t \Vdash_\rho M$.*

This property will be used a lot without being explicitly mentioned.

Proof. Suppose $t \succ u$ and $u \Vdash_\rho M$, that is, $\forall \pi \in \|M\|_\rho.\langle u \mid \pi\rangle \in \perp\!\!\!\perp$. Let $\pi \in \|M\|_\rho$, we need to show that $\langle t \mid \pi\rangle \in \perp\!\!\!\perp$. This is true because $\langle t \mid \pi\rangle \succ \langle u \mid \pi\rangle$ and the pole is closed by anti-reduction. □

Propetry 2. *If $t \Vdash_\rho M \Rightarrow N$ and $u \Vdash_\rho M$ then $tu \Vdash_\rho N$.*

Proof. Suppose $t \Vdash_\rho M \Rightarrow N$ and let $u \in \Lambda$ be such that $u \Vdash_\rho M$. Let $\pi \in \|N\|_\rho$. We need to show that $\langle tu \mid \pi\rangle \in \perp\!\!\!\perp$. But $\langle tu \mid \pi\rangle \succ \langle t \mid u \cdot \pi\rangle$ and $\langle t \mid u\cdot\pi\rangle \in \perp\!\!\!\perp$ since $u \cdot \pi \in \|M \Rightarrow N\|_\rho$. □

Propetry 3. *Suppose that for all lambda term u, if $u \Vdash_\rho M$, then $t[x := u] \Vdash_\rho N$. Then $\lambda x.t \Vdash_\rho M \Rightarrow N$.*

Proof. Let $\pi \in \|M \Rightarrow N\|_\rho$, that is, $\pi = u \cdot \pi'$ for some $v \in |M|_\rho$ and $\pi' \in \|N\|_\rho$. We need to show that $\langle \lambda x.t \mid u \cdot \pi' \rangle \in \perp\!\!\!\perp$, but this reduces to $\langle t[x := u] \mid \pi' \rangle$, which is in the pole by assumption. □

Propetry 4 (Consistency)**.** *There is no ρ and prooflike t such that $t \Vdash_\rho \bot$.*

Proof. TO DO □

Propetry 5. *If $t \Vdash_\rho \bot$ then $t \Vdash_\rho M$ for all formula M.*

Proof. Realizing $\bot$ is a universal quantification over a bigger set than realizing M. □

Propetry 6. *If $t \Vdash_\rho \forall^{\mathbb{N}} x.M$ then for all $n \in \mathbb{N}$, $t\overline{n} \Vdash_{\rho; x \mapsto \overline{n}} M$.*

Proof. Suppose $t \Vdash_\rho \forall^{\mathbb{N}} x.M$. Let $n \in \mathbb{N}$ and $\pi \in \|M\|_{\rho; x \mapsto \overline{n}}$. We need to show that $\langle t\overline{n} \mid \pi \rangle \in \perp\!\!\!\perp$. It suffices to show that $\langle t \mid \overline{n} \cdot \pi \rangle \in \perp\!\!\!\perp$, which is the case since $\overline{n} \cdot \pi \in \overline{n} \cdot \|M\|_{\rho; x \mapsto \overline{n}} \subseteq \|\forall^{\mathbb{N}} x.M\|_\rho$. □

Propetry 7. *Suppose that for all $n \in \mathbb{N}$, $t[x := \overline{n}] \Vdash_{\rho; x \mapsto \overline{n}} M$. Then $\lambda x.t \Vdash_\rho \forall^{\mathbb{N}} x.M$.*

Proof. Let $\pi \in \|\forall^{\mathbb{N}} x.M\|_\rho$, that is, $\pi = \overline{n} \cdot \pi'$ for some $n \in \mathbb{N}$ and $\pi' \in \|M\|_{\rho; x \mapsto \overline{n}}$. We need to show $\langle \lambda x.t \mid \overline{n} \cdot \pi' \rangle \in \perp\!\!\!\perp$. But this process reduces to $\langle t[x := \overline{n}] \mid \pi' \rangle$, which is in the pole by assumption. □

Propetry 8 (Continuity)**.** *Let t be a term with one free variable x. Let M be a formula. Let $u_0, u_1, \cdots \in \Lambda$. Suppose that $t[x := \lambda\!\!\lambda n.u_n] \Vdash_\rho M$. Then there exists $N \in \mathbb{N}$ such that for all $f \in \Lambda$ such that $\forall n < N.\ f\ \overline{n} \succ u_n$ we have $t[x := f] \Vdash_\rho M$.*

Proof. TO DO □

Propetry 9. *For all type τ, the set $[\![\tau]\!]$ is nonempty.*

Proof. An element of $[\![\tau]\!]$ can be defined by induction over the syntax of τ as follows:

$$\begin{aligned} e(\mathsf{nat}) &:= 0 \\ e(\mathsf{prop}) &:= \emptyset \\ e(\tau \to \sigma) &:= x \mapsto e(\sigma) \end{aligned}$$

□

3 The Induced Evidenced Frame

Definition 13. *Define an evidenced frame by taking*

propositions Φ *is* $\mathcal{P}(\Pi)$

evidence E *is the set of prooflike lambda terms.*

evidence relation *For* $\phi_1, \phi_2 \in \Phi$ *and* $e \in E$, $\phi_1 \xrightarrow{e} \phi_2$ *if and only if* $e \Vdash \phi_1^{\perp\!\!\perp} \cdot \phi_2$.

top $\top = \emptyset$.

conjunction *For* $\phi_1, \phi_2 \in \Phi$, $\phi_1 \wedge \phi_2$ *is* $\bigcup_{\phi \in \Phi} (\phi_1^{\perp\!\!\perp} \cdot \phi_2^{\perp\!\!\perp} \cdot \phi)^{\perp\!\!\perp} \cdot \phi$.

universal implication *For* $\phi_1 \in \Phi$ *and* $\vec{\phi} \subseteq \Phi$, $\phi_1 \supset \vec{\phi}$ *is* $\phi_1^{\perp\!\!\perp} \cdot \bigcup \vec{\phi}$.

We now prove that it satisfies all the axioms of an evidenced frame.

Proof. We will be using the fact that if for all term u such that $u \Vdash \phi_1$ we have $t[x := u] \Vdash \phi_2$, then $\phi_1 \xrightarrow{\lambda x.t} \phi_2$. This can be proved the same way as propetry 3.

reflexivity Take $e_{\mathsf{id}} = \lambda x.x$. If $t \Vdash \phi$ then $x[x := t] = t \Vdash \phi$.

top Take $e_{\mathsf{top}} = \lambda x.x$. If $t \Vdash F$ then $x[x := t] = t \Vdash \top$ since every lambda term realizes $\top$, which can be seen by unfolding the definitions of $\top$ and $\Vdash$.

conjunction elimination Take $e_{\mathsf{fst}} = \lambda t.t(\lambda x.\lambda y.x)$. $\lambda x.\lambda y.x \Vdash \phi_1^{\perp\!\!\perp} \cdot \phi_2^{\perp\!\!\perp} \cdot \phi_1$ since if $t \Vdash \phi_1$ and $u \Vdash \phi_2$ then $x[x := t, y := u] = t \Vdash \phi_1$. Thus if $t \Vdash \phi_1 \wedge \phi_2$ we have $t \Vdash (\phi_1^{\perp\!\!\perp} \cdot \phi_2^{\perp\!\!\perp} \cdot \phi_1)^{\perp\!\!\perp} \cdot \phi_1$ so $t(\lambda x.\lambda y.x) \Vdash \phi_1$, which concludes. We do the same thing for e_{snd}.

conjunction introduction Take $< |e_1, e_2| >= \lambda t.\lambda u.u(e_1 t)(e_2 t)$. Suppose $e_1 \Vdash \phi^{\perp\!\!\perp} \cdot \phi_1$, $e_2 \Vdash \phi^{\perp\!\!\perp} \cdot \phi_2$, and $t \Vdash \phi$. We need to show $\lambda u.u(e_1 t)(e_2 t) \Vdash \phi_1 \wedge \phi_2$. Let $\phi' \in \Phi$ and $u \Vdash \phi_1^{\perp\!\!\perp} \cdot \phi_2^{\perp\!\!\perp} \cdot \phi'$. We need to show $u(e_1 t)(e_2 t) \Vdash \phi'$, which is the case since $e_1 t \Vdash \phi_1$ and $e_2 t \Vdash \phi_2$.

universal implication introduction Take $\lambda e = \lambda t.\lambda u.e(\lambda v.vtu)$. Let $\phi_1, \phi_2 \in \Phi$, $\vec{\phi} \subseteq \Phi$, and $e \in E$ such that $\forall \phi \in \vec{\phi}.\phi_1 \wedge \phi_2 \xrightarrow{e} \phi$. We need to show that for all $t \Vdash \phi_1$, $\lambda u.e(\lambda v.vtu) \Vdash \phi_1 \supset \vec{\phi}$, that is, for all $u \Vdash \phi_2$ and $\phi \in \vec{\phi}$, $e(\lambda v.vtu) \Vdash \phi$, which is true since $e \Vdash (\phi_1 \wedge \phi_2)^{\perp\!\!\perp} \cdot \phi$ and $\lambda v.vtu \Vdash \phi_1 \wedge \phi_2$, as seen for conjunction introduction.

universal quantification elimination Take $e_{\mathsf{eval}} = \lambda t.(t(\lambda x.\lambda y.x))(t(\lambda x.\lambda y.y))$. Suppose $t \Vdash (\phi_1 \supset \vec{\phi}) \wedge \phi_1$ and let $\phi \in \vec{\phi}$, we need to show $(t(\lambda x.\lambda y.x))(t(\lambda x.\lambda y.y)) \Vdash \phi$, which is the case since $t(\lambda x.\lambda y.x) \Vdash \phi_1^{\perp\!\!\perp} \phi$ and $t(\lambda x.\lambda y.y) \Vdash \phi_1$, as seen for conjunction elimination.

$\square$

4 Realizability of Countable Choice

Throughout this section, let τ be a type.

Definition 14 (Axiom of Countable Choice)**.** *We define*

$$AC_{\mathbb{N},\tau} := \forall R^{\mathsf{nat}\to\tau\to\mathsf{prop}}.(\forall n^{\mathsf{nat}}.\neg\forall i^{\tau}.\neg R(n,i)) \to \neg\forall f^{\mathsf{nat}\to\tau}.\neg\forall^{\mathbb{N}} n.R(n,f(n)))$$

Theorem 1. *We have*

$$\lambda H.\ \lambda P.\ \Phi\ H\ P\ []\Vdash AC_{\mathbb{N},\tau}$$

Proof. Let $R_0 \in (\mathcal{P}(\Pi)^{[\![\tau]\!]})^{\mathbb{N}}$. Let $\rho = (\rho_{\mathsf{empty}}; R \mapsto R_0)$.

Let

$$H \Vdash_\rho \forall n.\neg\forall i.\neg R(n,i)$$
$$P \Vdash_\rho \forall f.\neg\forall^{\mathbb{N}} n.R(n,f(n))$$

By propetry 3, it suffices to show that

$$\Phi\ H\ P\ []\Vdash_\rho \bot$$

Definition 15 (Cache)**.** *$[r_0, r_1, \ldots, r_n]$ is a cache if for all $k \le n$, there exists some $i_k \in [\![\tau]\!]$ such that $r_k \Vdash_{\rho;i\mapsto i_k} R(\overline{k}, i)$.*

Lemma 1. *Let ℓ be a cache. Suppose that $\Phi\ H\ P\ \ell \nVdash_\rho \bot$. Then, there exists $i_{|\ell|} \in [\![\tau]\!]$ and $r_{|\ell|} \in \Lambda$ such that $\ell :: r_{|\ell|}$ is a cache and $\Phi\ H\ P\ (\ell :: r_{|\ell|}) \nVdash_\rho \bot$.*

Proof. Since ℓ is a cache, for all $k < |\ell|$, there exists an $i_k \in [\![\tau]\!]$ such that $\mathsf{nth}\ \overline{k}\ \ell \Vdash_{\rho;i\mapsto i_k} R(\overline{k}, i)$. Define $f_0(k)$ to be such an i for $k < |\ell|$ and an arbitrary element of $[\![\tau]\!]$ for $k \ge |\ell|$ (this set is nonempty by propetry 9).

There exist $r_{|\ell|} \in \Lambda, i_{|\ell|} \in [\![\tau]\!]$ such that $r_{|\ell|} \Vdash_{\rho;i\mapsto i_{|\ell|}} R(\overline{|\ell|}, i)$. Indeed, if none did exists, then for all i, any term would realize $\neg R(\overline{|\ell|}, i)$, thus, any term would realize $\forall i.\neg R(\overline{|\ell|}, i)$. Thus, H applied to any term would realize $\bot$, namely,

$$H\ (\lambda z.\ H\ P\ (l :: z)) \Vdash_\rho \bot$$

. Thus

$$\lambda m.\mathsf{nth}\ m\ (\ell :: H\ (\lambda z.\ H\ P\ (l :: z))) \Vdash_\rho \forall^{\mathbb{N}} n.R(n, f_0(n))$$

In effect, the body applied to $\overline{k}$ for $k < |\ell|$ reduces to the k^{th} element of the cache and thus realizes $R(k, f_0(k))$ and the body applied to $\overline{k}$ for $k \ge |\ell|$ reduces to $H\ (\lambda z.\ H\ P\ (l :: z))$, which realizes $\bot$ and therefore by propetry 5 any realizes formula. Thus,

$$P\ (\lambda m.\mathsf{nth}\ m\ (\ell :: H\ (\lambda z.\ H\ P\ (l :: z)))) \Vdash_\rho \bot$$

but $\Phi\ H\ P\ \ell$ reduces to this term and was supposed to not realize $\bot$, which is a contradiction.

Now, suppose that for all $r_{|\ell|} \in \Lambda, i_{|\ell|} \in [\![\tau]\!]$ such that $r_{|\ell|} \Vdash_{\rho; i \mapsto i_{|\ell|}} R(\overline{|\ell|}, i)$,

$$\Phi\ H\ P\ (\ell :: r_{|\ell|}) \Vdash_{\rho} \bot$$

it suffices to find a contradiction. Then by propetry 2 and propetry 5, for all $i_{|\ell|} \in [\![\tau]\!]$,

$$\lambda z.\ \Phi\ H\ P\ (\ell :: z) \Vdash_{\rho; i \mapsto i_{|\ell|}} \neg R(\overline{k}, i)$$

and so by the hypothesis on H,

$$H\ (\lambda z.\ \Phi\ H\ P\ (\ell :: z)) \Vdash_{\rho} \bot$$

which has already been shown to lead to a contradiction. □

Now, suppose

$$\Phi\ H\ P\ [] \not\Vdash_{\rho} \bot$$

it suffices to find a contradiction.

By applying lemma 1 repeatedly, we get a sequence $r_0, r_1, \ldots$ such that for all k there exists $i_k \in [\![i]\!]$ such that $r_k \Vdash_{i \mapsto i_k} R(\overline{k}, i)$. For each k, take $f_0(k)$ to be such an i_k.

Then, for all n, $r_n \Vdash_{\rho} R(\overline{n}, f_0(\overline{n}))$. Thus, $\lambda\!\!\lambda n.r_n \Vdash_{\rho} \forall^{\mathbb{N}} n.R(n, f_0(n))$ and so

$$P\ (\lambda\!\!\lambda n.r_n) \Vdash_{\rho} \bot$$

But then, by continuity, there exists K such that for all f, if $\forall k < K.\ f\ \overline{k} \succ r_k$ then $P\ f \Vdash_{\rho} \bot$. Now, take $\ell = [r_0, \ldots, r_{K-1}]$ and $f = \lambda m.\mathsf{nth}\ m\ (\ell :: H\ (\lambda z.\ \Phi\ H\ P\ (\ell :: z)))$. On the one hand, $\Phi\ H\ P\ \ell \succ P\ f$ and $\phi\ H\ P\ \ell \not\Vdash_{\rho} \bot$ by construction, so $P\ f \not\Vdash_{\rho} \bot$. On the other hard, for $k < K$, $f\ \overline{k} \succ r_k$, thus $P\ f \Vdash_{\rho} \bot$. We thus obtain the sought contradiction.

□