Dynamic Time Warping Example

Consider the following two sequences:

$$\mathbf{x} = (1, 3, 4, 9)$$

$$\mathbf{y} = (1, 2, 3, 4, 8)$$

Step 1: Construct the Distance Matrix

The distance matrix D where each element $D(i, j)$ is the absolute difference between x_i and y_j:

$$D = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 8 \\ \hline 1 & 0 & 1 & 2 & 3 & 7 \\ 3 & 2 & 1 & 0 & 1 & 5 \\ 4 & 3 & 2 & 1 & 0 & 4 \\ 9 & 8 & 7 & 6 & 5 & 1 \end{array}$$

Step 2: Initialize the Cumulative Cost Matrix

Initialize the cumulative cost matrix C:

$$C = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 8 \\ \hline 1 & 0 & \infty & \infty & \infty & \infty \\ 3 & \infty & \infty & \infty & \infty & \infty \\ 4 & \infty & \infty & \infty & \infty & \infty \\ 9 & \infty & \infty & \infty & \infty & \infty \end{array}$$

Step 3: Fill the Cumulative Cost Matrix

Using the recurrence relation:

$$C(i, j) = D(i, j) + \min \begin{cases} C(i-1, j) \\ C(i, j-1) \\ C(i-1, j-1) \end{cases}$$

Filling in C:
First row and first column initialization:

$$\begin{aligned}
C(0, 1) &= D(0, 1) + C(0, 0) = 1 + 0 = 1 \\
C(0, 2) &= D(0, 2) + C(0, 1) = 2 + 1 = 3 \\
C(0, 3) &= D(0, 3) + C(0, 2) = 3 + 3 = 6 \\
C(0, 4) &= D(0, 4) + C(0, 3) = 7 + 6 = 13 \\
C(1, 0) &= D(1, 0) + C(0, 0) = 2 + 0 = 2 \\
C(2, 0) &= D(2, 0) + C(1, 0) = 3 + 2 = 5 \\
C(3, 0) &= D(3, 0) + C(2, 0) = 8 + 5 = 13
\end{aligned}$$

Filling the rest of the matrix:

$$\begin{aligned}
C(1,1) &= D(1,1) + \min\{C(0,1), C(1,0), C(0,0)\} = 1 + \min\{1,2,0\} = 1 \\
C(1,2) &= D(1,2) + \min\{C(0,2), C(1,1), C(0,1)\} = 0 + \min\{3,1,1\} = 1 \\
C(1,3) &= D(1,3) + \min\{C(0,3), C(1,2), C(0,2)\} = 1 + \min\{6,1,3\} = 2 \\
C(1,4) &= D(1,4) + \min\{C(0,4), C(1,3), C(0,3)\} = 5 + \min\{13,2,6\} = 7 \\
C(2,1) &= D(2,1) + \min\{C(1,1), C(2,0), C(1,0)\} = 2 + \min\{1,5,2\} = 3 \\
C(2,2) &= D(2,2) + \min\{C(1,2), C(2,1), C(1,1)\} = 1 + \min\{1,3,1\} = 2 \\
C(2,3) &= D(2,3) + \min\{C(1,3), C(2,2), C(1,2)\} = 0 + \min\{2,2,1\} = 1 \\
C(2,4) &= D(2,4) + \min\{C(1,4), C(2,3), C(1,3)\} = 4 + \min\{7,1,2\} = 5 \\
C(3,1) &= D(3,1) + \min\{C(2,1), C(3,0), C(2,0)\} = 7 + \min\{3,13,5\} = 10 \\
C(3,2) &= D(3,2) + \min\{C(2,2), C(3,1), C(2,1)\} = 6 + \min\{2,10,3\} = 8 \\
C(3,3) &= D(3,3) + \min\{C(2,3), C(3,2), C(2,2)\} = 5 + \min\{1,8,2\} = 6 \\
C(3,4) &= D(3,4) + \min\{C(2,4), C(3,3), C(2,3)\} = 1 + \min\{5,6,1\} = 2
\end{aligned}$$

The final cumulative cost matrix C is:

$$C = \begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 8 \\
\hline
1 & 0 & 1 & 3 & 6 & 13 \\
3 & 2 & 1 & 1 & 2 & 7 \\
4 & 5 & 3 & 2 & 1 & 5 \\
9 & 13 & 10 & 8 & 6 & 2
\end{array}$$

Step 4: Extract the Warping Path

The warping path W is found by tracing back from $C(N, M)$ to $C(0, 0)$ following the minimum cost. Starting from $C(3, 4)$:

$$\begin{aligned}
(3,4) &\to (2,3) \\
(2,3) &\to (1,2) \\
(1,2) &\to (0,1) \\
(0,1) &\to (0,0)
\end{aligned}$$

The warping path is:

$$W = \{(3,4), (2,3), (1,2), (0,1), (0,0)\}$$

Step 5: DTW Distance

The DTW distance is the value at $C(N, M)$:

$$\text{DTW}(\mathbf{x}, \mathbf{y}) = C(3,4) = 2$$