A Cute Trick for Calculating Saturated Sets

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Saturated Sets

Saturated sets are saturated in both directions, e.g.:

$$p \rightarrow q \in H \qquad \leftrightarrow \qquad p \in H \text{ IMPLIES } q \in H \qquad (*)$$

Membership equals satisfiability, so we can prove completeness:

- Build Maximal Consistent Sets (MCSs) using Lindenbaum's lemma
- Prove that any MCS is saturated
- Any non-derivable formula is then falsifiable (its negation is consistent)

But how exactly did we arrive at condition (*)

Semantics

Take propositional logic as a running example.

Syntax: falsity, propositional symbols, implication.

Semantic brackets lift the interpretation $\mathtt{I}$ to formulas $\mathtt{p}$, $\mathtt{q}$:

$$\llbracket \_ \rrbracket \ \bot \quad \longleftrightarrow \quad \mathtt{False}$$

$$\llbracket \mathtt{I} \rrbracket \ (\ddagger \mathtt{P}) \quad \longleftrightarrow \quad \mathtt{I} \ \mathtt{P}$$

$$\llbracket \mathtt{I} \rrbracket \ (\mathtt{p} \longrightarrow \mathtt{q}) \quad \longleftrightarrow \quad \llbracket \mathtt{I} \rrbracket \ \mathtt{p} \longrightarrow \llbracket \mathtt{I} \rrbracket \ \mathtt{q}$$

(code from the Isabelle/HOL formalization)

Semics

Punch a hole in the sem[ant]ics, replacing the recursive call with `rel`:

```
semics _ _ ⊥              ↔    False
semics I _ (‡P)           ↔    I P
semics I rel (p ⟶ q)      ↔    rel I p ⟶ rel I q
```

Now we can express other properties based on subformulas.

Saturated Sets Redux

Saturated sets are saturated in both directions, e.g.:

$$p \longrightarrow q \in H \qquad \longleftrightarrow \qquad p \in H \text{ IMPLIES } q \in H$$

The connection between object-logical ⟶ and meta-logical IMPLIES?

It is exactly the *semics*:

```
semics (hmodel H) (rel H) p ↔ rel H (hmodel H) p
```

Under the model induced by `H`, namely `hmodel H`, the relation `rel H` holds for the subformulas of `p` exactly when it holds for `p`.

Example

Take the usual term model:

```
hmodel H ≡ λP. ‡P ∈ H     and     rel H _ p = (p ∈ H)
```

and the equation from before:

```
semics (hmodel H) (rel H) p ↔ rel H (hmodel H) p
```

For each syntactic constructor, it reduces to:

```
False              ↔      ⊥ ∈ H
‡P ∈ H             ↔      ‡P ∈ H
(p ∈ H → q ∈ H)    ↔      (p → q ∈ H)
```

First-Order Logic

Evaluate universal quantifiers by extending (⨾) the variable denotation E

```
semics (E, F, G) rel (∀p)    ↔   ∀x. rel (x ⨾ E, F, G) p
```

The term model again:

```
hmodel H ≡ (#, †, λP ts. ‡P ts ∈ H)
```

We need to apply E as a substitution to account for p's context:

```
rel H (E, _, _) p = (sub-fm E p ∈ H)
```

The resulting saturation condition:

```
(∀x. ⟨x⟩p ∈ H)    ↔        ∀p ∈ H
```

Hybrid Logic I

Abridged semics:

```
semics (_, g, w) _ (·i)           ↔  w = g i
semics (M, g, w) rel (◇p)         ↔  ∃v ∈ R M w. rel (M, g, v) p
semics (M, g, _) rel (@i p)  ↔  rel (M, g, g i) p
```

We account for the context by labeling the formula `p` with the world `i`:

```
rel H (_, _, i) p = ((i, p) ∈ H)
```

Thus we calculate saturated sets of labeled formulas

Hybrid Logic II

The model is based on equivalence classes `[i]` of nominals where two nominals are equivalent (wrt. `H`) when `([i], ·k) ∈ H`

The saturated set conditions become:

```
[i] = [k]                            ↔    ([i], ·k) ∈ H
([k], p) ∈ H                         ↔    ([i], @k p) ∈ H
(∃v∈reach H [i]. (v, p) ∈ H)         ↔    ([i], ◇p) ∈ H
```

where `reach H i ≡ {[k] | (i, ◇k) ∈ H}`

The Scheme of Things

I have formalized completeness for five logics in Isabelle/HOL

- Propositional sequent calculus and tableau
- First-order and hybrid logic natural deduction
- Modal logic System K (axiomatic system)

The recipe:

- Build MCS with my generic, transfinite formalization of Lindenbaum’s lemma
- Isabelle/HOL calculates the saturated set conditions
- Prove that the MCSs fulfil the conditions
- Profit

Fin

Future work:

- Calculate conditions for *downwards* saturated sets (Hintikka)?
  - Need to consider each syntactic constructor *and their negation*
  - $p \longrightarrow q \in H \quad \longrightarrow \quad \neg p \in H$ OR $q \in H$
  - $\neg (p \longrightarrow q) \in H \quad \longrightarrow \quad p \in H$ AND $\neg q \in H$

References:

- Formalization in the Archive of Formal Proofs:
  - https://devel.isa-afp.org/entries/Synthetic_Completeness.html
- My PhD thesis
  - https://people.compute.dtu.dk/ahfrom/ahfrom-thesis.pdf