Derivation of the time of Sunrise and Sunset

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Contents

Let's call a given point on the surface of the earth P. The hour of sunrise and sunset at P varies according to two factors:

1. The time at which the Sun is highest in the sky (i.e, solar noon)

2. The length of morning and afternoon - that is, the amount of time between sunrise and solar noon, which is (practically) equal to the amount of time between solar noon and sunset.

1 Frame of Reference

We're going to use a geocentric frame of reference. This probably violates your intuition, so take a minute to get comfortable with this mental model.

The origin of our frame of reference is at the center of the Earth, with the Z axis pointing along the Earth's axis of rotation. We'll rotate our frame of reference around the Z axis at a rate of 360° every 365.25 days (i.e., 1 rotation per year). At the equinoxes, the Sun will pass through the same point on the X axis ($Y_{sun} = Z_{sun} = 0$). At the solstices, the Sun will lie in the XZ plane. At the summer solstice $Z_{sun} > 0$ and at the winter solstice $Z_{sun} < 0$.

One way to picture this: imagine that the unit vectors of our coordinate system form a tangible, metal object with the Earth at the origin. Take a very short string. Attach one end to the Sun and tie the other end around the X axis. Now set the entire apparatus on the center of a platform rotating at a rate of one revolution per year. The Earth rotates independently around the Z axis once a day.

1.1 Position of Observer on Earth

Let us call the location of an observer standing on Earth point P. We define a unit vector $\overrightarrow{P}$ from the origin to point P with latitude λ and azimuth angle ϕ which we take to be $0°$ at noon. Over the course of 24 hours, ϕ will uniformly sweep through a range of $360°$ similar to the hour hand of a (24-hour) clock. ϕ can thus be seen as a measure of the local time, and given the location of $\overrightarrow{P}$ at sunrise we can calculate the hour.

We can express the coordinates of P in terms of ϕ and λ.

$$\sin\lambda = \frac{P_z}{1} \tag{1}$$

$$\cos\lambda = \frac{P_{(x,y)}}{1} \tag{2}$$

$$\cos\phi = \frac{P_x}{P_{(x,y)}} \tag{3}$$

$$\sin\phi = \frac{P_y}{P_{(x,y)}} \tag{4}$$

Substituting $\cos\lambda$ for $P_{(x,y)}$ yields the following form of $\overrightarrow{P}$:

$$\overrightarrow{P} = (\cos\lambda\cos\phi, \cos\lambda\sin\phi, \sin\lambda) \tag{5}$$

1.2 Position of Sun

Let us define a unit vector $\overrightarrow{S}$ with latitude β and azimuth ψ which points from the center of the Earth to the Sun. By the same logic as we used to derive the coordinates of point P, the coordinates of the Sun in our rotating frame of reference are:

$$\overrightarrow{S} = (\cos\beta\cos\psi, \cos\beta\sin\psi, \sin\beta) \tag{6}$$

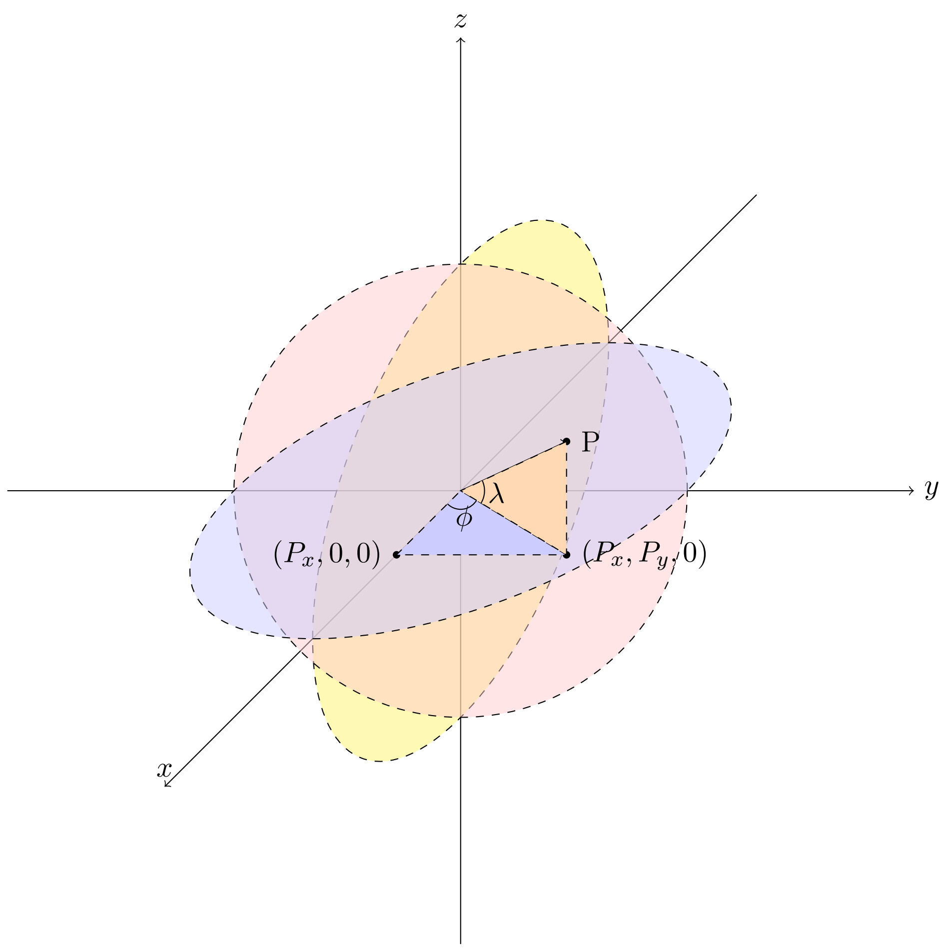

Figure 1: The components of $\overrightarrow{P}$ in our geocentric frame of reference

We're going to make a simplifying assumption: the Sun moves around the Earth in a circle parallel to the equator at uniform speed. The orbit is actually an ellipse, but it's fairly close to a circle and circles are easier to work with. The coordinates of $\overrightarrow{S}$ are $(\cos 2\pi d, \sin 2\pi d, 0)$, where $d = \frac{\text{\# of days since winter solstice}}{365.25}$.

But the Sun's orbit isn't parallel to the Earth; it's actually tilted relative to the equator at an angle that we'll call $\alpha \approx 23.4°$. This path is called the ecliptic.

To compensate for it, we must apply a transformation to rotate our the path of the Sun about the y-axis by α. The form of such a rotation is given below.

$$R_y(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \tag{7}$$

In addition, we must also account for the year-long rotation of our special frame of reference about the Earth. The Sun is moving counter-clockwise around the Earth, but our frame of reference is also rotating counter-clockwise. To account for this, we rotate the Sun "back" into our frame of reference (i.e., clockwise) by $2\pi d$ radians about the Z axis.

$$R_z(2\pi d) = \begin{bmatrix} \cos 2\pi d & \sin 2\pi d & 0 \\ -\sin 2\pi d & \cos 2\pi d & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

[TODO: Insert diagram here]

Composing these transformations of $\overrightarrow{S}$ yields:

$$\begin{aligned}
\overrightarrow{S} &= \begin{bmatrix} \cos 2\pi d & \sin 2\pi d & 0 \\ -\sin 2\pi d & \cos 2\pi d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos 2\pi d \\ \sin 2\pi d \\ 0 \end{bmatrix} && (9) \\
&= \begin{bmatrix} \cos\alpha\cos 2\pi d & \sin 2\pi d & \sin\alpha\cos 2\pi d \\ -\cos\alpha\sin 2\pi d & \cos 2\pi d & -\sin\alpha\sin 2\pi d \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos 2\pi d \\ \sin 2\pi d \\ 0 \end{bmatrix} && (10) \\
&= \begin{bmatrix} \cos\alpha\cos^2 2\pi d + \sin^2 2\pi d \\ \sin 2\pi d \cos 2\pi d\,(1 - \cos\alpha) \\ -\sin\alpha\cos 2\pi d \end{bmatrix} && (11)
\end{aligned}$$

$\cos\alpha \approx .9178$, so

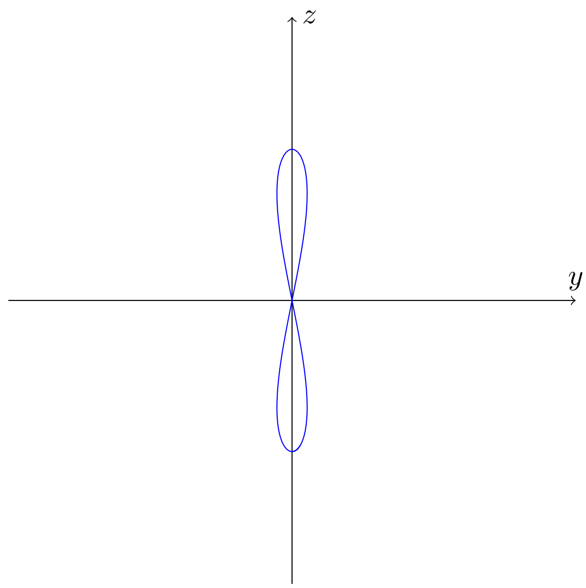

Figure 2: The path of the Sun in the yz plane

$$\overrightarrow{S} \approx \begin{bmatrix} 1 \\ (.08 \sin 2\pi d) \cos 2\pi d \\ (-.4) \cos 2\pi d \end{bmatrix} \tag{12}$$

S_x is nearly constant, and the range of S_y is roughly 1/5 the range of S_z. The resulting shape is a figure-8 about the X axis.

This figure-eight path is called the analemma, and you can observe it from Earth by taking a photograph of the Sun's position in the sky from the same time (i.e., periodically, every 24 hours) and place throughout the year.

2 Hour of Sunrise and Sunset

We're now ready to calculate the two components that determine the hour of sunrise on a given day: symmetric lengths of morning and afternoon and the hour of solar noon.

2.1 Length of morning

Let ϕ_s be the azimuth angle of point P at the time of sunrise.

Figure 3: Long-exposure photograph of the analemma as seen from Budapest. (Source: https://flic.kr/p/j16mbH)

The magnitude of the angle $(\phi_s - \psi)$ represents the (symmetric) length of morning and afternoon at point P. $(\phi - \psi) = 0$ when the Sun reaches its highest elevation in the sky at solar noon.

At sunrise and sunset, $\overrightarrow{P} \perp \overrightarrow{S}$, so $\overrightarrow{P} \cdot \overrightarrow{S} = 0$ (we're making the simplifying assumption that the radius of the Sun is negligible compared to its distance from Earth.)

$$\overrightarrow{P} \cdot \overrightarrow{S} = 0 \tag{13}$$

$$\cos\phi_s \cos\lambda \cos\psi \cos\beta + \sin\phi_s \cos\lambda \sin\psi \cos\beta + \sin\lambda \sin\beta = 0 \tag{14}$$

$$\cos\lambda \cos\beta \left(\cos\phi_s \cos\psi + \sin\phi_s \sin\psi\right) = -\sin\lambda \sin\beta \tag{15}$$

$$\cos\phi_s \cos\psi + \sin\phi_s \sin\psi = -\tan\lambda \tan\beta \tag{16}$$

$$\cos\left(\phi_s - \psi\right) = -\tan\lambda \tan\beta \tag{17}$$

Note: the last step in the above equation makes use of the trigonometric identity

$$\cos(X - Y) = \cos X \cos Y + \sin X \sin Y \tag{18}$$

Now, solving for $\tan\beta$ in terms of α, d, and ψ.

$$\tan\beta = \frac{\sin\beta}{\cos\beta} \tag{19}$$
$$= \frac{S_z \cos\psi}{S_x} \tag{20}$$
$$= \frac{-\sin\alpha\cos 2\pi d\cos\psi}{\cos\alpha\cos^2 2\pi d + \sin^2 2\pi d} \tag{21}$$

Plugging $\tan\beta$ back into our equation for the length of morning yields

$$\phi_s - \psi = \arccos\left(\tan\lambda \cdot \frac{\sin\alpha\cos 2\pi d\cos\psi}{\cos\alpha\cos^2 2\pi d + \sin^2 2\pi d}\right) \tag{22}$$

2.2 Hour of Solar Noon

All that remains in order to calculate the time of sunrise is to solve for the Sun's angle of highest elevation, ψ.

$$\tan\psi = \frac{\sin\psi}{\cos\psi} \tag{23}$$
$$= \frac{S_y}{S_x} \tag{24}$$
$$= \frac{(1-\cos\alpha)\cos 2\pi d\sin 2\pi d}{\cos\alpha\cos^2 2\pi d + \sin^2 2\pi d} \tag{25}$$
$$\psi = \arctan\left(\frac{(1-\cos\alpha)\cos 2\pi d\sin 2\pi d}{\cos\alpha\cos^2 2\pi d + \sin^2 2\pi d}\right) \tag{26}$$

2.3 Full Calculation

Combining our expression for the length of morning and the Sun's angle of highest elevation, we can calculate the hour of sunrise as follows.

$$\text{hour of sunrise, sunset} = 12{:}00 + \frac{24}{2\pi}\left[\psi \pm (\phi_s - \psi)\right] \tag{27}$$
$$\text{hour of sunrise, sunset} = 12{:}00 + \frac{24}{2\pi}\left[\psi \pm \arccos\left(\tan\lambda \cdot \frac{\sin\alpha\cos 2\pi d\cos\psi}{\cos\alpha\cos^2 2\pi d + \sin^2 2\pi d}\right)\right] \tag{28}$$

where

$$\psi = \arctan\left(\frac{(1-\cos\alpha)\cos 2\pi d\sin 2\pi d}{\cos\alpha\cos^2 2\pi d + \sin^2 2\pi d}\right)$$

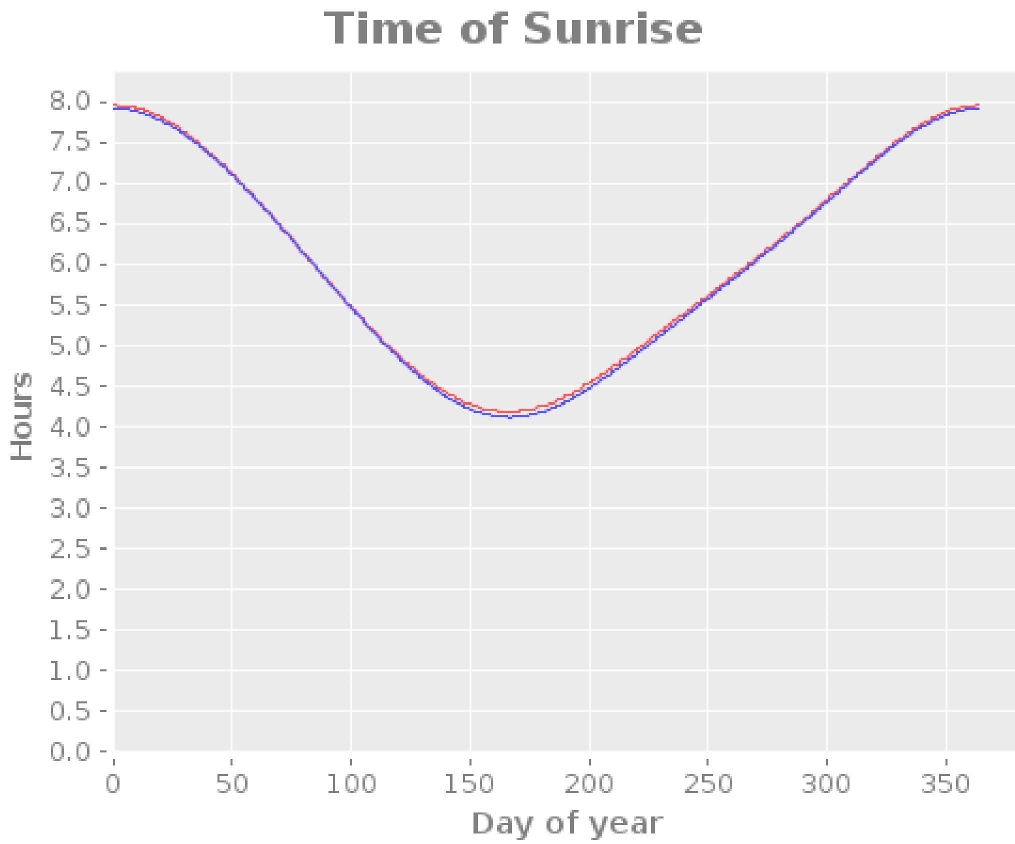

Figure 4: Red: Actual hour of sunrise; Blue: Predicted hour of sunrise

The blue line in the figure below shows the hour of sunrise as predicted by this equation for the calendar year of 2017 in Seattle, Washington ($\lambda = 47.38°$), The red shows the ground truth taken from the tables published by the U.S. Navy Observatory.

The error in our calculation stems from our earlier assumption that the orbit of the Sun is a circle.

3 References

This derivation is based on the one at `http://www.physics.rutgers.edu/~twatts/sunrise/node6.html`.

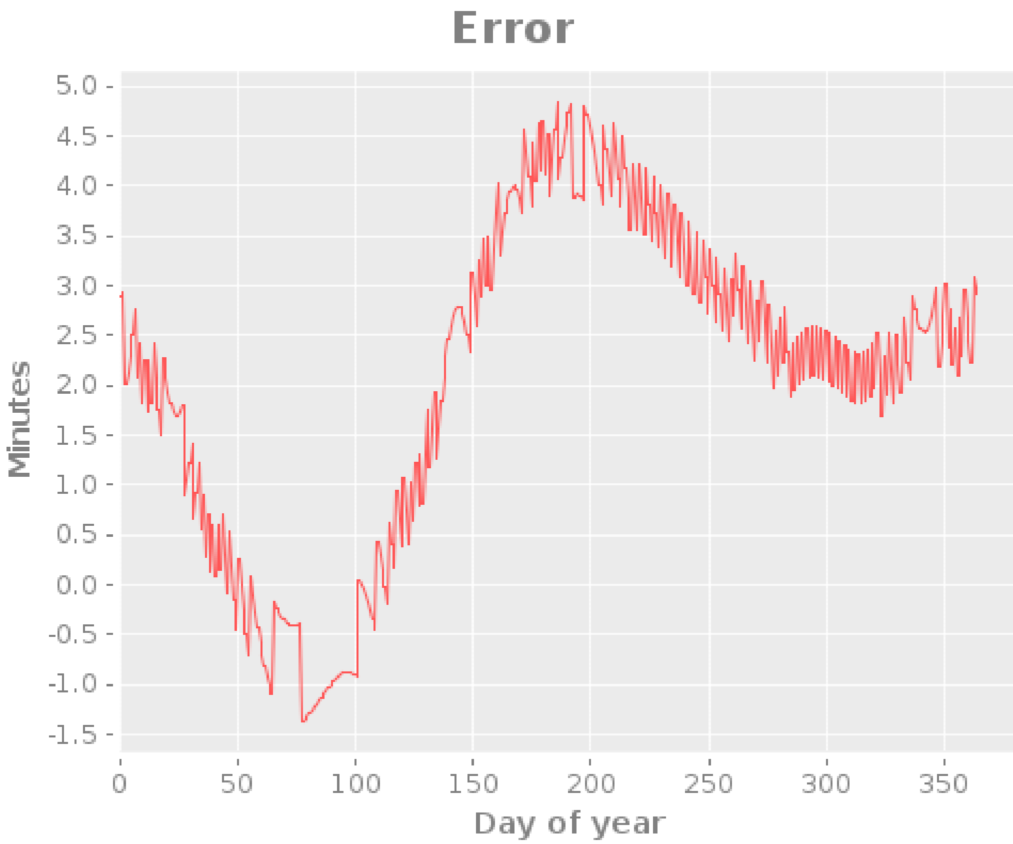

Figure 5: Error in our predicted hour of sunrise, in minutes