Applied Combinatorics, Math 3012 H, Fall 2013
Class Project: Periodic behavior of threshold functions on a Boolean set

Deadline: Nov 1, 2013.

1 Problem Description

Let $K = \{0, 1\}$, $A = (a_{ij})$ be a real-valued symmetric matrix, $\theta = (\theta_1, \ldots, \theta_n)$ a real vector of thresholds, and define the function

$$F = (f_1, \ldots, f_n) : K^n \to K^n, \text{ where}$$

$$f_i(x_1, \ldots, x_n) \begin{cases} 0 & \text{if } \sum_{j=1}^{n} a_{ij} x_j < \theta_i \\ 1 & \text{otherwise} \end{cases}$$

Since K^n is finite, for each $x \in K^n$ there are $m, t \in \mathbb{N}, t > 0$ such that

$$F^{m+t}(x) = F^m(x) \text{ and } F^{m+r}(x) \neq F^m(x)$$

for all $0 < r < t$.

1. Write a program that takes as inputs the matrix A, the vector θ, and an initial vector $x_0 \in K^n$. The program should output the path that is followed from x_0 and the values of m and t in the above description.

2. Can you find an example of A, θ, and x_0 for which $t > 2$?

Example 1.1. *Let $n = 5$,*

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

and $\theta = \{1, 2, 3, 4, 1.5\}$. *If* $x_0 = (0, 1, 0, 1, 0)$. *Then*

$$x_0 = (0,1,0,1,0) \xrightarrow{F} (1,0,0,0,1) \xrightarrow{F} (1,1,0,0,0) \xrightarrow{F} (1,0,0,0,1) \xrightarrow{F}$$

$$(1,1,0,0,0) \xrightarrow{F} (1,0,0,0,1) \xrightarrow{F} (1,1,0,0,0) \xrightarrow{F} (1,0,0,0,1) \xrightarrow{F} \cdots$$

Therefore $m = 1$ *and* $t = 2$.

References

[1] Chris J. Kuhlman, Henning S. Mortveit, David Murrugarra, V. S. Anil Kumar. (2012) Bifurcations in Boolean Networks. Discrete Mathematics and Theoretical Computer Science, proc, AP, 29-46.

[2] E. Goles and J. Olivos. (1981) Comportement periodique des fonctions a seuil binaires et applications. Discrete applied mathematics 3, 93-105.