Quantitative Structuring and Risk Aversion Extraction

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Abstract

The 2007–2008 financial crisis and the following flight to simplicity put most derivative businesses around the world under considerable pressure. We argue that the traditional modeling techniques must be extended to include product design. Quantitative structuring is a rigorous framework proposed from Andrei N. Soklakov for the design of financial products. We show how it incorporates traditional investment ideas while supporting a more accurate expression of clients' views and we touch upon an adjacent topic regarding the safety of financial derivatives.

Introduction
In this part we deliver the motivation of the assignment. Typically, structured products are not 100% consistent with investors beliefs. However many financial products are popular—they either carry large commissions for the sellers or so greatly favor the issuers that they push the products on unsophisticated investors who cannot fathom the complexity (but are assured by the salespeople and the advertising that these are good and often safe products).

Theoretical Framework
Here we provide the theoretical aspects of Quantitative structuring which is a rigorous framework for the design of financial products. Combining Bayesian information processing with rational optimization gives us a simple new tool for product design – the payoff elasticity equation (23). This is the main technical result of the assignment.

Correspondence principle / Toy examples
Although we argue for the greater use of quantitative methods, the historical approach remains enormously relevant. Indeed, it contains a huge amount of valuable experience. In order to preserve this experience, we must follow the correspondence principle and connect quantitative structuring to the more familiar ad hoc approach. Of course, we must expect limits to how well the ad hoc legacy products can be justified, but, at the very least, we must be able to verify some intuitions behind their design. In this paper we examine some of the typical investment structures from the point of view of quantitative structuring by simulating market scenarios from Normal and Skew normal distributions.

Applications with real data
Here, we attempt to apply our framework to real-world data in order to develop a structuring scheme tailored to the investors' beliefs and preferences. After establishing the "market belief", we take two different approaches to address two different problems. In one of them we turn our algorithm "upside-down" and we use the structuring algorithm we introduced and try to examine if it is possible to extract results about an investor's risk profile, given the product he has chosen to buy. Next, we take the straightforward approach and try to construct optimal products given some beliefs of the market and various risk preferences.

Conclusion
In the concluding chapter, we summarize the main findings, discuss the contributions of the study, and suggest avenues for further research.

Contents

1 Introduction

Structuring and trading derivatives have two distinct purposes, namely hedging and leveraged investing, where the latter is our focus. Structured products are composite investment instruments meticulously crafted to meet specific investor needs perceived as unfulfilled by existing securities. Typically bundled as tools for asset allocation to mitigate portfolio risk, these products generally comprise a note and a derivative. The derivative derives its economic value from the price of another asset, often a bond, commodity, currency, or equity, with options being common derivatives. The note pays interest at a predetermined rate and schedule, while the derivative dictates payment at maturity.

Promoted to investors as debt securities due to their derivative components, structured products may offer varying degrees of principal protection. Despite their popularity among retail investors seeking higher yields, empirical research reveals significant drawbacks. Indeed, structured products often result in negative risk-adjusted returns for investors due to high fees and overvaluation. Several studies, including one by Petra Vokata [3] covering 21,000 products from January 2006 to September 2015, reveal concerning findings about structured products. Investors paid an average of 7 percent in annual fees and experienced a 7 percent annual loss relative to risk-adjusted benchmark returns. The underlying securities were often highly volatile stocks with systematic selection for high headline rates and moderate downside protection. Conflicted payments, such as kickbacks to brokers, accounted for nearly half of yield enhancement product fees.

Other studies by Brian Henderson and Neil Pearson or Geng Deng and al. highlight issues such as overpricing, complexity exploitation, and significant premiums in structured products. For instance, absolute return barrier notes were found to be overpriced by approximately 4.5 percent, with yields lower than corresponding corporate yields and sometimes even below the risk-free rate. The overall conclusion is that structured products have consistently under-performed alternative allocations to stocks and bonds due to overvaluation. Despite their apparent appeal, structure products tend to be more advantageous for sellers than buyers.

Hence, we have for goal to rationalize structuring by incorporating more rigorously clients' need into the design of structure product so that they truly are bespoke tailored solutions to investors. Thus, reflecting their knowledge, intuition and risk appetite. We do so by first providing a theoretical framework developed by Soklakov [1][2], which link payoff structure, risk appetite and prior knowledge in a fundamental and elegant relationship.

2 Theoretical Framework

2.1 Growth-optimizing Investor

In Learning Investment and derivatives [2], Soklakov builds a theoretical framework in which investors' belief are converted into tradable payoffs. And in the opposite direction, where investors' belief are inferred from the payoff structure they use. Investing must be distinguished from speculation as it calls for the use of in-depth research as well as a certain attention to safety which together form a belief. The relationship between beliefs and reality is at the core of every investment.

First, three main concepts, all based on the same random variable, need to be introduced: The market view given by the market-implied distribution $m(x)$. The investor's view given by the believed distribution $b(x)$. And finally the realized distribution $p(x)$ which both the market and the investor are trying to predict.

The random variable is observed on the market and could be anything from a stock price to interest rate or index value etc. The information contained in this random variable can be condensed through the utilization of a probability distribution. As the effectiveness of probability distributions in summarizing knowledge about random variables remains independent of the factual accuracy of this knowledge.

Alongside, the reference point of the growth-optimizing investor is considered. Indeed, envision an investor who perceives himself as a specialist in a specific market variable. This investor possesses strong confidence in his expertise and is in search of a buy-and-hold product that exhibits two key features: it must leverage their complete expert knowledge of the variable and provide the maximum expected rate of return. Let our growth-optimizing investor be convinced that the market variable x will evolve in a certain direction. Then the optimal investment strategy regarding x subject to the investor's belief must reward a non-zero payoff in case the investor was right. More realistically, the growth-optimizing investor might have to split his investment between multiple securities in order to maximize his utility. This optimal investment strategy can be formulated as finding which proportions of the total investment should be allocated to each security. For that let N be the number of securities and $\{R_i\}_{i=1}^N$ be the return on each of these securities offered by the market. Then let $\{\beta_i\}_{i=1}^N$ be the proportions in which the investor decides to partition his capital, so that $\sum_i \beta_i = 1$. Furthermore, let b_i be the investor's believed probability that the $\mathrm{i^{th}}$ trade matures in the money. Consequently, the investor can compute his expected rate of return as follow

$$\mathbb{E}_b\left[rate\right] = \sum_i b_i\, ln(\beta_i R_i) \tag{1}$$

Maximizing this expectation over all $\{\beta_i\}_{i=1}^N$ subject to the constraint $\sum_i \beta_i = 1$ result in

$$\beta_i^* := optimal\ \beta_i = b_i \tag{2}$$

Which from the investor point of view can be read as investment proportioned by believed probabilities maximizes the expected rate of return. Whereas the actual expected return of this strategy is expressed as

$$\mathbb{E}_p\left[rate\right] = \sum_i p_i\, ln(\beta_i^* R_i) = \sum_i p_i\, ln(b_i R_i) \tag{3}$$

Where the p_i represent the realized probabilities which are of course unknown at the time of investment. considering the textbook case of "fair odds" when $\sum_i 1/R_i = 1$ allows us to define the market-implied probability as $m_i := 1/R_i$, that the $\mathrm{i^{th}}$ trade matures in the money. Hence, the expected return rate can be rewritten as

$$\mathbb{E}_p\left[rate\right] = \sum_i p_i\, ln\left(\frac{b_i}{m_i}\right) \tag{4}$$

In the above derivation, two simplifying assumptions have been made. First, all of the investor's investment buying power is indeed invested, in other words no cash reserve is kept. Second, that all inverse returns sum up to 1 is used as a logical shortcut. Thus, this derivation is let exempt of any pricing nor econometric model. The market-implied probabilities are only a probabilistic manner of recording the market returns.

2.2 Maximal Payoff of Learning

Here the learning process is describe in a probabilistic way via Baye's rule, where

$$P(x\,|\,research) = \frac{P(research\,|\,x)}{P(research)}\, P(x) \tag{5}$$

Which should be understood as a learning process where the prior knowledge of the random variable $P(x)$ is bettered or updated to $P(x\,|\,research)$ by learning from results of additional research. Since this formula simply satisfies the properties of conditional probabilities it remains valid and consistent independently of which learning process is used.

We have previously define the growth-optimizing return as

$$\beta_i^* R_i = b_i R_i = \frac{b_i}{m_i} \tag{6}$$

Which is indexed by i, the securities that mature in the money. We can consider this index as a function of the same underlying market variable such that $i = i(x)$. Thus, the return to the growth-optimizing investor can be rewritten as a payoff from a single run of the growth-optimizing strategy

$$f(x) = \frac{b(x)}{m(x)} \tag{7}$$

Where we define the believed probability density as $b(x) = b_{i(x)}/\Delta x$, and the market-implied probability density as $m(x) = m_{i(x)}/\Delta x$. Finally, by rearranging the terms, the following expression results

$$b(x) = f(x) \cdot m(x) \tag{8}$$

This formula forms a fundamental link between to the above conditional probability as the underlying learning process and the payoff structure of the optimal growth strategy. Thus, mathematically the market-implied view is taken as a prior by the investor $P(x) = m(x)$. Any logical deviation from this choice of prior implies mistaking market information. Then, the investor can update his prior by either performing, purchasing or conducting additional research. the updated investor view should then follow $b(x) := P(x \,|\, research)$. This derivation can without flaw be generalized to the case of an arbitrary investor. Very important to us, is the observation that the investor's view $b(x)$ and the shape of the growth-optimal payoff $f(x)$ can be computed for each other. Hence, we will be able to provide the growth-optimizing investor with payoff structures that matches his belief. and we will be able to analyze any payoff function $f(x)$ by looking at the respective belief $b(x)$. This can be useful even when looking at the case of an arbitrary investor. Additionally, the growth-optimal payoff $f(x)$ is a likelihood function which is proportional the the conditional probability $(P(research \,|\, x)$.

2.3 Investor Equivalence Principle

In this subsection, we extend the above laid down foundations of the model to account for any investor following the work of Soklakov [1]. Now let us refine our assumptions. First, the market random variable x is now considered as a real-valued financial underlying $x \in \mathbb{R}$. The range of possible values of x is partitioned into non-overlapping intervals using a discrete mesh $(..., x_i, x_{i+1}, ...)$. We now name the traded securities s_i and let each s_i pay 1 when $x \in (x_i, x_{i+1})$ and zero otherwise. Of course, a price is attributed to the purchase of each s_i so that $p_i = quotedprice(s_i)$. Above, we have formulated the problem of investing as finding the best allocation of capital to each s_i. Again, consider β_i the proportions in which the investor split his capital, still subject to $\sum_i \beta_i = 1$. And only one security among s_i can mature in the money, so that the investment payoff F can be calculated as follow

$$F_k = \beta_k \, r_k \tag{9}$$

Where k is the index of the unique security maturing in the money and $r_k = 1/p_k$ is the return on that security. Let us redefine the market-implied distribution by normalizing the prices $m_k = p_k / \sum_i p_i$ and substituting it in the above equation then rearranging the terms gives

$$\beta_k = F_k \, m_k \tag{10}$$

This equation has a very general form since the choice of the underlying variable and the choice of β_k are left free to choose by the investor.

Pursuing the reasoning, b_k now corresponds to the investor-believed probability that securities s_k mature in the money. In the previous equation 10, we were still considering the specific case of the growth-optimizing investor. Where in this special case the proportions β_k coincide with b_k. Thus, equation 10 becomes

$$b_k = f_k \, m_k \tag{11}$$

Where f_k denotes the growth-optimal payoff. Despite forming a polarized case of the investor spectrum, focusing here on growth-optimizing investor does not at all reduce the range of possible investment decisions. Indeed, both β_i and b_i must add up to one, but in the absence of further constraint, both equations 10 and 11 must be solved by the same set of payoffs so that $f = F$. Which by joining together these payoffs, one can observe, considering the generality of these settings that any investor can be viewed as growth-optimizing.

Any investor can be viewed as growth-optimizing.

In other words, that general investors choose the same product, $F = f$, or that they behave similarly as growth-optimizing investors whose belief b_k would happen to coincide with β_k.

2.4 Logical Investors

We will now narrow the scope of this model by assuming rationality from every investor as we want to describe the education process a growth-optimizing investor can undergo in order to act as if he had sophisticated risk preferences. For this, we assume from the general economic literature that rational investors behave by maximizing their expected value of utility. Following an expected utility approach do not reduce the generality of these arguments. As a matter of fact, the goal here is to develop a tool that allows more rationality in investments.

Consider rational investors having utility $u()$ that have a view on x given by the probabilities b_i, each measuring the degree of investors' belief for x maturing between x_i and x_{i+1}. We choose the utility function to depend on both logarithmic rate of return and on x explicitly. Since the logarithm is monotone increasing function we do not amputate any generality to our results. As in equation 3, The optimal investment corresponds to a maximization of the expectation over all possible proportions β_i subject to the budget constraint $\sum_i \beta_i = 1$.

$$\sum_i b_i \, u\left(ln(\beta_i \, r_i), x_i\right) \tag{12}$$

Hence, the Lagrangian optimization is

$$\mathcal{L}\left(\{\beta_i\}, \lambda\right) = \sum_i b_i \, u\left(ln(\beta_i \, r_i), x_i\right) + \lambda \left(\sum_i \beta_i - 1\right) \tag{13}$$

Setting $\partial \mathcal{L} / \partial \beta_k = 0$ we compute

$$b_k u'\left(ln(\beta_k \, r_k), x_k\right) = -\lambda \, \beta_k \tag{14}$$

Where the symbol prime denotes the first derivative with respect to the first argument of the utility function. Which implies that

$$\sum_i b_i u'\left(ln(\beta_i \, r_i), x_i\right) = -\lambda_i \beta_i = -\lambda \tag{15}$$

Substituting this result back into equation 14 gives

$$\beta_k = \frac{u'\left(ln(\beta_k \, r_k), x_k\right) \; b_k}{\sum_i u'\left(ln(\beta_i \, r_i), x_i\right) \; b_i} \tag{16}$$

The monotonicity of the utility function with respect respect to return guarantees that $\beta_k > 0$ where the limiting case of zero β_k is trivial and can be excluded from the optimization. Solving equation 16 for β_k gives the optimal investment strategy. Converting this result in continuous time and substituting equation 9 into 16 gives

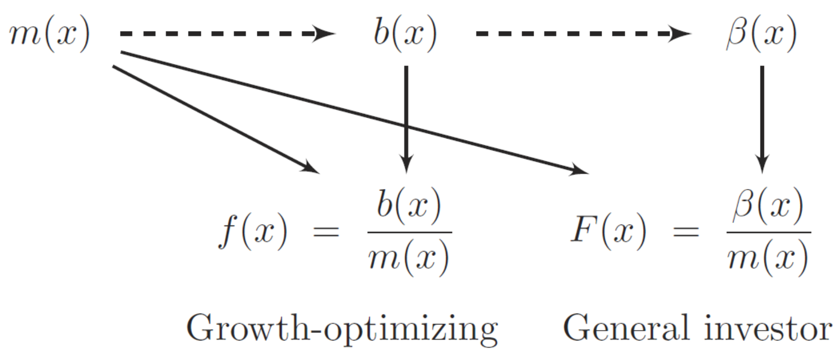
$$\beta(x) = \frac{u'\left(lnF(x), \, x\right)}{\int u'\left(lnF(\tilde{x}), \, \tilde{x}\right) \, b(\tilde{x}) \, d\tilde{x}} \, b(x) \tag{17}$$

We can thus now describe the investors' learning process logic and how does the learning process translate into investment decisions. There are two kinds of information necessary to the investor so that the learning process can take place. On the one hand, as expressed until now, investors have to learn about the market. On the other hand, an investor need to learn about himself, that is to know one's own preferences and goal.

Our starting point is the growth-optimizing investor choosing $m(x)$ as the prior information of the market. Since until now the only available information are the prices p_i of the securities s_i. Then investor undertake their learning process, performing their market analysis, analyzing records. Arriving at that point in time, investors might for example notice differences in drifts suggested by their research as opposed to $m(x)$. With the goal to end up writing their belied distribution $b(x)$.

Considering the case of a pure growth-optimizing investor, the investment strategy would be found out using equation 11 as shown on figure 4.4. However, generally the investor might be in possession of their own personal preferences such as the level of risk desired. Of course, we would like to incorporate such information in addition of the prior on the market. By the investor equivalence principle, we can continue considering the investor as growth-optimizing as long as they can believe $\beta(x)$. It is worth mentioning that this belief of $\beta(x)$ comes in a subsequent manner as an additional information after having realized their research. Such that the growth-optimizing investors will then behave as if they would have different preferences. That is, we want our investor to be able to take their belief $b(x)$ and update it using Baye's rule 5 to arrive at $\beta(x)$. This additional learning step is precisely embodied by equation 17 and figure 4.4 summarize this entire process as well as the derivation of F, the final optimal payoff.

Figure 1: Learning process and investment structuring



Some last observations concerning equation 17 are that first, despite not being obvious by its structure, this equation incorporate information on risk aversion included in product design. Second, the ability the incorporate all information, whether objective (market) or personal (preferences) is fundamental to an investor that wants to act rationally. Such investor must be called a logical investor.

2.5 Payoff Elasticity Equation

Back to equation 17, divinding it on both sides by $m(x)$ and using equations 10 and 11 we can express an integral equation for $F(x)$

$$F(x) = \frac{u'\left(ln\, F(x),\, x\right)}{\int u'\left(ln\, F(\tilde{x}),\, \tilde{x}\right) b(\tilde{x}) d\tilde{x}} f(x) \tag{18}$$

However, this form is yet quite bulky and we would prefer converting it in differential form. To do so, we use the economic principle of elasticity which for a function $\phi(x)$ with respect to its argument x is defined as the derivative $dln\phi(x)/dlnx$. The elasticity measure the percentage change in the function's value with respect to the percentage change in its argument. Let the second argument of u be redundant so that $u(ln(F,x) = u(lnF)$. Now taking log on both side of equation 18 and differentiating, we get

$$d\ lnF = \frac{1}{u'\left(ln\, F\right)} u''(ln\, F)\, dlnF + dlnf \tag{19}$$

We see that x is the only variable quantity and that the denominator on equation 18 does not depend on x so that it drops out from the above differential expression. Then rearranging the terms gives

$$\frac{dlnF}{dlnf} = \frac{u'(lnF)}{u'(lnF) - u''(lnF)} \tag{20}$$

With goal to implement the above equation in structuring we aim at dropping out the extremely theoretical expression of utility. Any linear tranformations of the utiity function has no effect on the investors' preferences. To circumvent this, we use the relative risk aversion measure developed by Arrow and Pratt.

$$R(F) = -F\frac{U''(F)}{U'(F)} \tag{21}$$

Where U stands for the standard definition of utility which has for relationship with our u, $((F(x)) = u(ln(F(x))$. So that,

$$U'(F) = \frac{u'(lnF)}{F}, \, U''(F) = \frac{u''(lnF) - u'(lnF)}{F^2} \tag{22}$$

Using the Arrow-Pratt measure of relative risk-aversion given in equation 21, we can rewrite equation 20 as follow

$$\frac{dlnF}{dlnf} = \frac{1}{R} \tag{23}$$

We have now reach a simple expression linking risk aversion to payoff elasticity. So that the more risk aversion we have the less elastic is the payoff. On the one hand, practically, we can now compute the optimal payoff F from the growth-optimal f and the risk aversion profile R of the investor. On the other hand, we can also extract the investors' risk aversion profiles directly from their open position.

In the following we solve the differential equation 23:

$$\frac{dlnF(x)}{dlnf(x)} = \frac{1}{R(f(x))}$$

Integrate both sides of the equation with respect to $dlnf$

$$\int \frac{dlnF(x)}{dlnf(x)} dlnf(x) = \int \frac{1}{R(f(x))} dlnf(x)$$

$$lnF(x) = \int \frac{1}{R(f(x))} \cdot \frac{f'(x)}{f(x)} dx$$

Take derivatives on both sides

$$\frac{F'(x)}{F(x)} = \frac{1}{R(f(x))} \cdot \frac{f'(x)}{f(x)}$$

$$R(f(x)) = \frac{f'(x)}{f(x)} \cdot \frac{F(x)}{F'(x)}$$

Prerequisite for the above is that in the denominators the values of $f(x)$ and $F'(x)$ have to be non-zero.

2.6 Summary

At the heart of structuring we always have optimization which reflects the goals of the client. This is preceded by preparatory steps defining the problem and followed by pack- aging of the solution into a tradeable product. Together these steps form the backbone of a manufacturing process which we summarized on the above Figure.

The preparatory steps include deciding on the market variable, capturing the relevant market and client views and identifying the goals of the client. This leads us to the key solution stage where our equations come in: the growth-optimal $b = fm$ and the payoff elasticity equation.

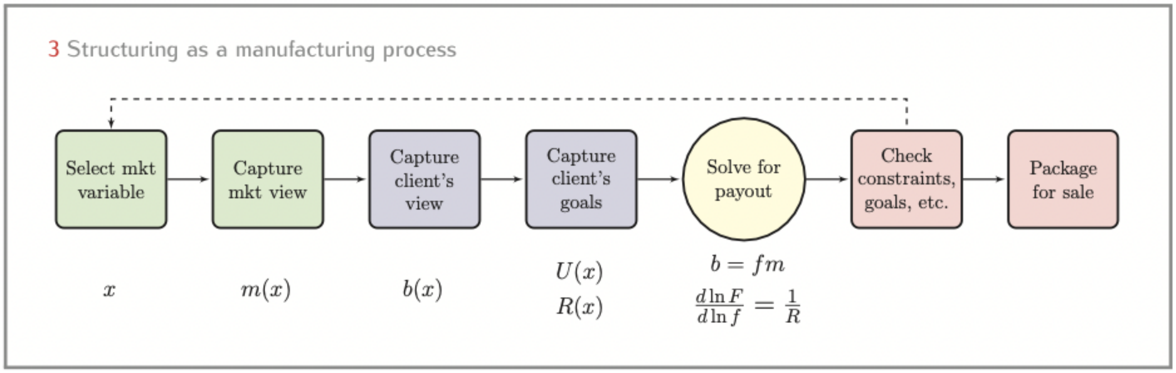


Figure 2: Learning process and investment structuring

After the solution stage we check the quality of the derived product. At this point we might be interested in assessing the expected performance of the product. We might even decide to go back and redefine the underlying variable. Having satisfied ourselves with the solution we proceed to the final stage of packaging it into a tradeable product.

3 Correspondence principle / Toy examples

In quantitative structuring financial products are derived as solutions to clients' needs [1][2]. This is very different from the historical approach where the precise structure of most financial products is best described as ad hoc: intuition-based solutions, inspired, rather than determined, by clients' needs.

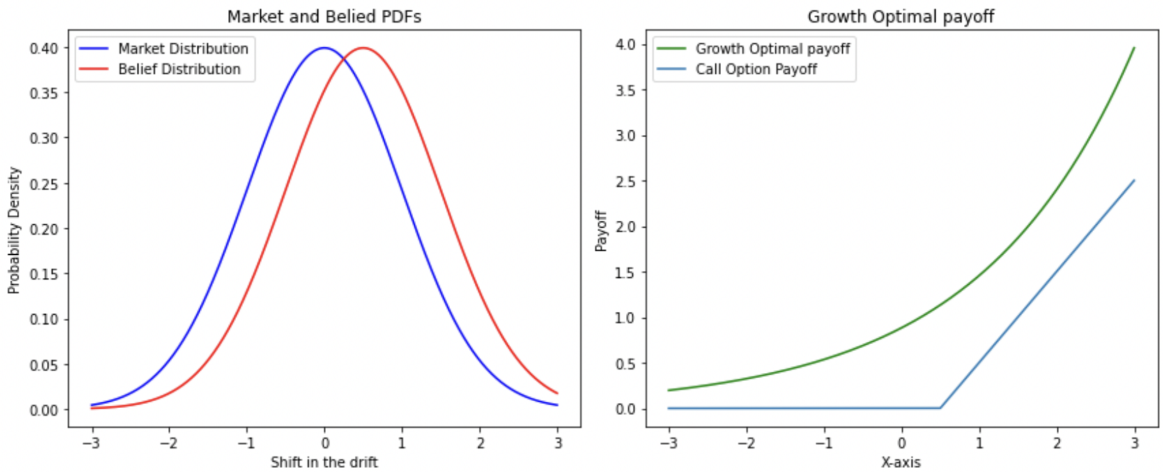
Although we argue for the greater use of quantitative methods, the historical approach remains enormously relevant. Indeed, it contains a huge amount of valuable experience. In order to preserve this experience we must follow the correspondence principle and connect quantitative structuring to the more familiar ad hoc approach. Of course, we must expect limits to how well the ad hoc legacy products can be justified, but, at the very least, we must be able to verify some intuitions behind their design.

Our goal in this section is to shift the beliefs of the client towards the (i) expected value, (ii) volatility and (iii) skew, derive the payoffs through quantitative structuring and provide some intuition behind their design.

3.1 Shift in the beliefs

3.1.1 Expected value shift

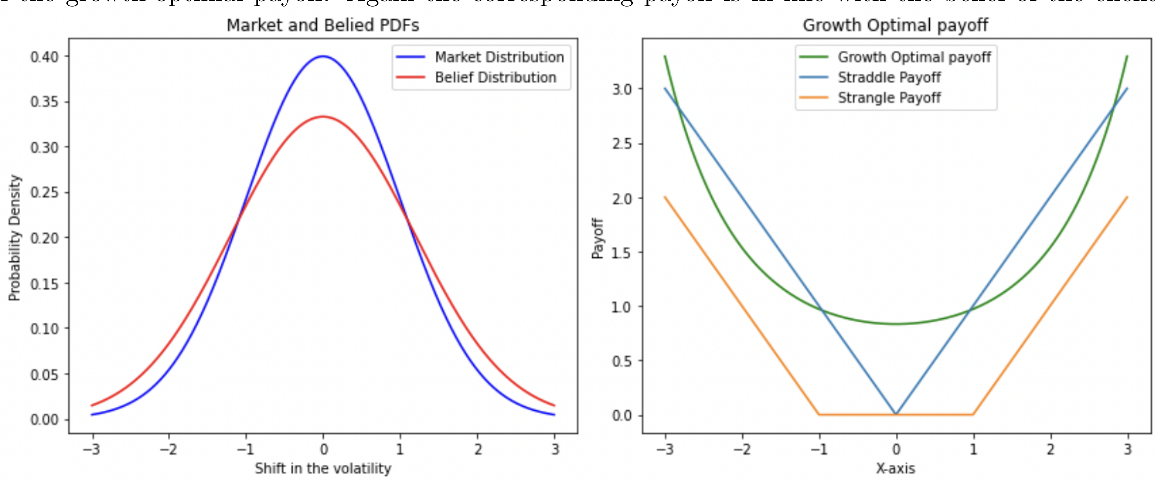
Consider an investor who has a view that the expected value of the index should be higher than suggested by the market corresponds to a believed distribution which is biased towards higher values of the index. The growth-optimal payoff can be easily sketched just by looking at the market-implied and investor-believed distributions. We consider here simplistic scenarios i.e we assume a Normal distribution for the market-implied distribution for the index. Hence, the client's view i.e the shift in the expected value corresponds again to a normal distribution.



We can observe that the corresponding payoff is similar to a call option which is in line with the belief of the client.

3.1.2 Volatility shift

Similarly, a belief that the market underestimates volatility corresponds to a believed distribution which is wider but not as tall as the market-implied distribution. Again, we can sketch the growth-optimal profile and see that it works pretty much the same way as the textbook vanilla combinations – straddles and strangles. These classical vanilla combinations can be considered as crude approximations of the growth-optimal payoff. Again the corresponding payoff is in line with the belief of the client.



3.1.3 Skew shift

Moving on to more complex examples, let us see how we could help investors with views on the skew.

Let $\phi()$ denote the probability density for the standard normal variable and $\Phi()$ denote the corresponding cumulative distribution function:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{-x^2}{2}}, \qquad \Phi(x) = \int_{-\infty}^{x} \phi(t)dt$$

The probability density for a skew-normal random variable is defined as

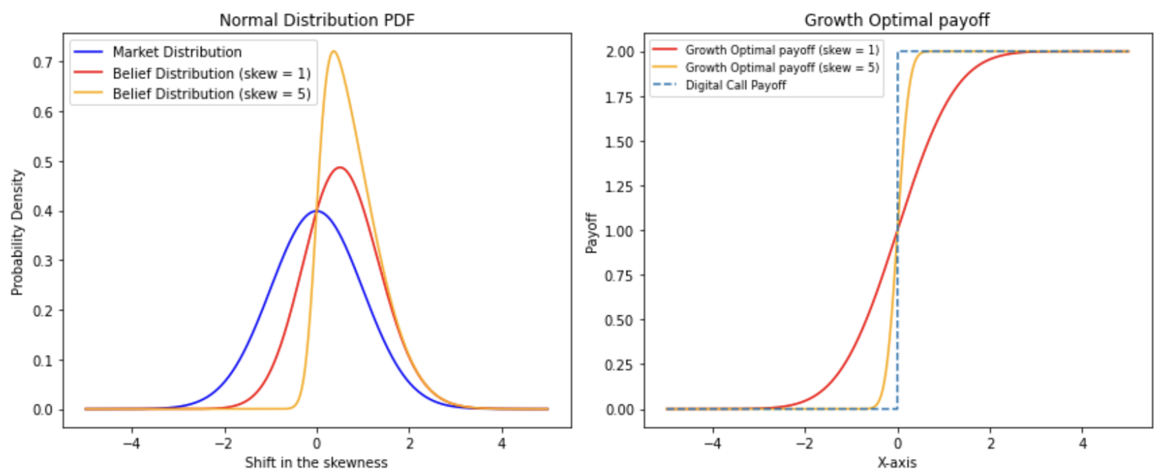
$$SN(x, \xi) = 2\phi(x)\Phi(\xi x)$$

where ξ is a parameter which controls the skew. The skewness (the third standardized moment) of this distribution is limited to the range between -1 and 1, so we should not expect great flexibility from the above analytical formulae. Nevertheless, this is a well-established example of introducing skew into the normal distribution, so we should know what it does in the context of our approach.

To this end, imagine a market which implies a normal distribution, $m(x) = SN(x, 0)$,for some variable x. Now consider an investor which does not agree with the market. The investor believes that, in reality, there is a skew and uses a skew-normal distribution, $b(x) = SN(x, \xi)$, to describe this belief. For the growth-optimal payoff we compute

$$f(x) = \frac{b(x)}{m(x)} = \frac{SN(x, \xi)}{SN(x, 0)} = 2\Phi(\xi x)$$

We immediately recognize the profile of a cumulative distribution function which contains a classic skew product – the digital – as a limiting case.



3.2 Shift in the beliefs incorporated with different risk aversion levels

We have seen before:

$$\frac{dlnF}{dlnf} = \frac{1}{R}$$

where f is the payoff of the Growth-Optimizing Investor and F is the payoff for the General Investor. This simple equation is the central technical result of this paper. It gives us a fundamental link between payoff elasticity and risk aversion. The more risk aversion we have the less elastic is the payoff.

On the practical side, this equation allows us to compute the optimal payoff F from the growth-optimal f and the risk aversion profile R of the client. Conversely, we are now also able to compute risk aversion profiles directly from clients' positions.

We choose two different utility functions for the preferences of the investor : (i) the **power utility function** $\frac{S_T^{1-\gamma} - 1}{1 - \gamma}$ with relative risk aversion (RRA) γ and (ii) the **exponential utility function** $-\frac{e^{-\gamma S_T}}{\gamma}$ with RRA γ·x. We derive the payoffs F,G for the exponential and power utility functions respectively. We have already seen that:

$$R(F(x)) = \frac{f'(x)}{f(x)} \frac{F(x)}{F'(x)}$$

So for the case of power utility function

$$\gamma \frac{F'(x)}{F(x)} = \frac{f'(x)}{f(x)}$$
$$\iff \gamma ln(F(x)) = ln(f(x))$$
$$\iff F(x) = f^{\frac{1}{\gamma}}(x)$$

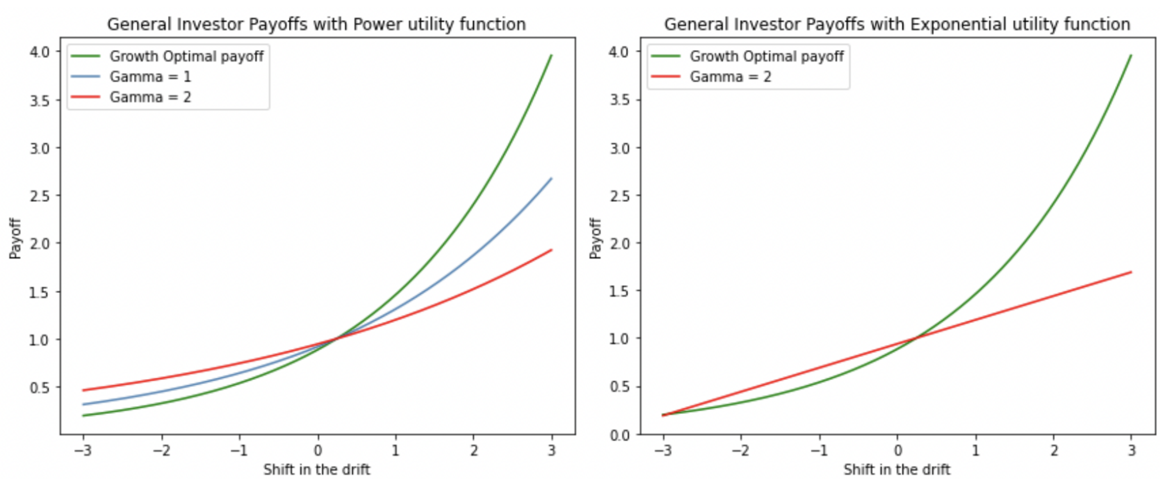
and for the case of the exponential utility function:

$$\gamma F(x) = \frac{f'(x)}{f(x)} \frac{F(x)}{F'(x)}$$
$$\iff \gamma F'(x) = \frac{f'(x)}{f(x)}$$
$$\iff \gamma F(x) = ln(f(x))$$
$$\iff F(x) = \frac{ln(f(x)}{\gamma}$$

3.2.1 Expected value shift

As we saw before, an investor who has a view that the expected value of the index should be higher than suggested by the market corresponds to a payoff similar to a call option.

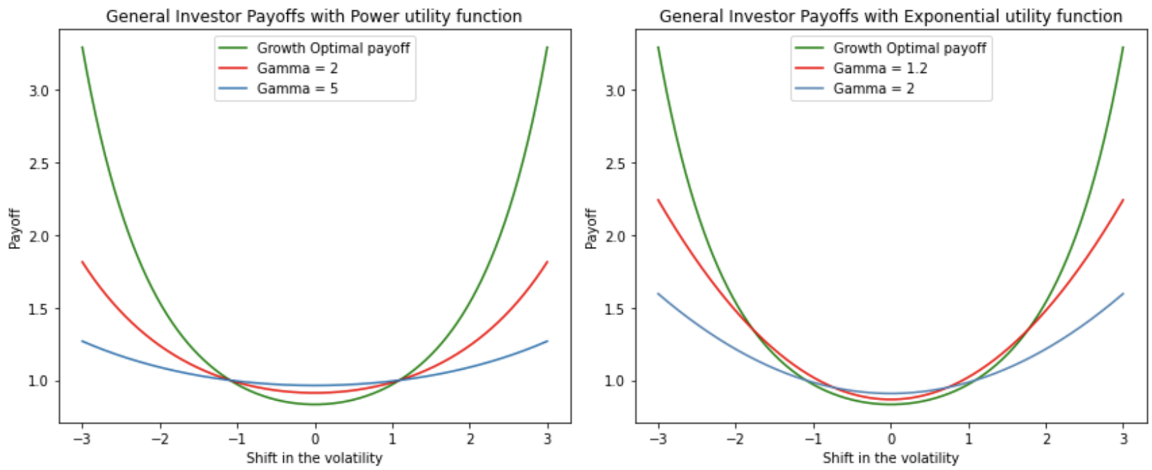
Now we assume some different scenarios for the risk aversion level of the investor. We will use the utilities functions we described before for different values of γ. The parameter γ describes the level of risk aversion, the higher the γ the higher the risk aversion.



For the case of the power utility when the risk aversion increases (parameter gamma increases) and so our investor does not like risk, we observe that his derived payoff is similar to the one of the growth optimizing payoff, as well as the call option payoff, and he is also willing to sacrifice his potential gain for some better downside protection.

In the case of the exponential utility function as the risk aversion increases the resulting payoff is similar to a forward contract which is less risky than the growth optimizer payoff which is similar to a call option. Again as gamma increases the investor is willing to sacrifice his potential gain for some better downside protection.

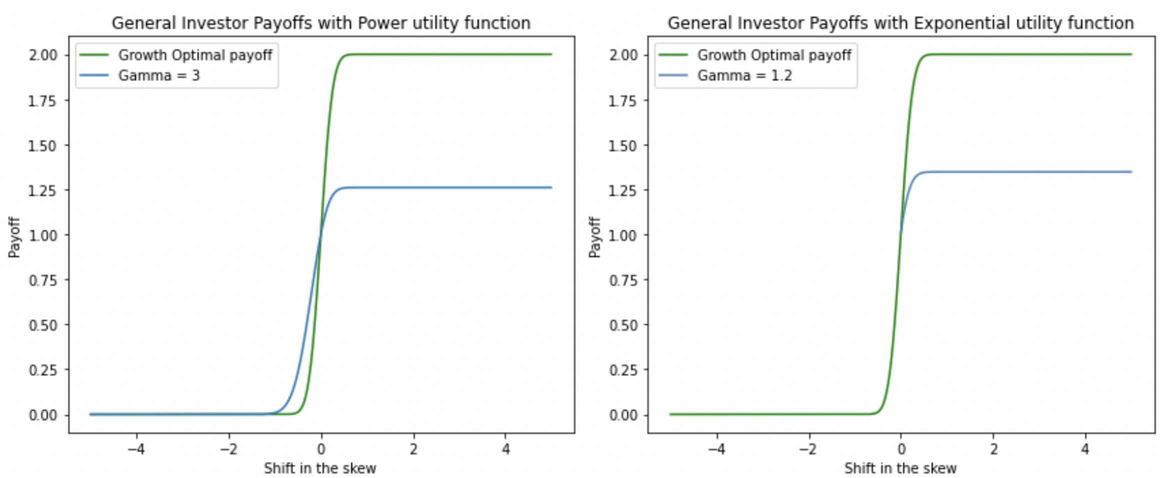
3.2.2 Volatility shift



For the case of the power utility when the risk aversion increases (parameter gamma increases) and so our investor does not like risk, we observe that his derived payoff is similar to the one of the growth optimizing payoff. He is also willing to sacrifice his potential gain in the sense that as we proceed to the tails the payoff becomes smaller but near ATM (where he loses) the payoff becomes higher, aiming for some better downside protection.

In the case of the exponential utility function, we take similar results with the power utility function. However, we observe that in this case the corresponding payoffs tend to be more "aggressive", in the sense that the investor demand for better returns e.g at the point -2 the investor with power utility (gamma=2) demands a return of 1.3 and in the same case the investor with exponential utility demands a return of 1.5.

3.2.3 Skew shift



For the Power utility function same as before, we observe that his derived payoff is similar to the one of the growth optimizing payoff, as well as the digital option payoff, he makes money for a larger area near ATM in comparison with the growth optimizing payoff but he sacrifices the potential gain. We have similar results for the case of the exponential utility function.

In general we observe that both utility functions provide satisfactory and similar results, where in some cases the power utility function results tend to be a little more conservative.

4 Applications with real data

Now that we have established that our formulas seem to provide a realistic representation of an investor's beliefs and risk preferences with artificial examples, it's time to test them with real-world data. The first step and the base on which we will work from now on is to extract a market-implied distribution for the underlying on which we will structure our products. This will be the risk-neutral distribution. Despite being reasonable to argue about which distribution will be used as prior, we have to keep in mind that this is just a starting point for our calculations. To extract the risk-neutral distribution we will use the Breeden - Litzenberger formula on call option prices.

4.1 Market-implied distribution

The SPDR S&P 500 ETF Trust (SPY) is the world's largest ETF and tracks the performance of the S&P 500 stock market index. Initially known only as the Standard & Poor's Depositary Receipts, it was launched in 1993 by State Street Global Advisors, an asset management company based in Boston, Massachusetts.

We download data for call options written on SPDR S&P 500 ETF Trust (SPY) on 29/09/2020 with maturity after 20 days on 19/10/20, with strike prices ranging from \$170 to \$425 and spot price \$332.

	strike	lastPrice	change	bid	ask	BxA	volume	openInterest	midprice
0	170	0.00	0.0	163.70	164.06	146 x 146	0	0	163.880
1	175	0.00	0.0	158.70	159.06	146 x 146	0	0	158.880
2	180	0.00	0.0	153.69	154.05	146 x 102	0	0	153.870
3	185	0.00	0.0	148.70	149.04	146 x 102	0	0	148.870
4	190	0.00	0.0	143.70	144.06	146 x 146	0	0	143.880
...	...	...	...	...	...	...	...	...	...
110	405	0.01	0.0	0.00	0.01	0 x 11065	0	28	0.005
111	410	0.02	0.0	0.00	0.01	0 x 8620	0	200	0.005
112	415	0.01	0.0	0.00	0.01	0 x 8630	0	5	0.005
113	420	0.02	0.0	0.00	0.01	0 x 8641	0	55	0.005
114	425	0.05	0.0	0.00	0.01	0 x 9471	0	20	0.005

Figure 3: SPDR S&P 500 ETF Trust (SPY) Call options

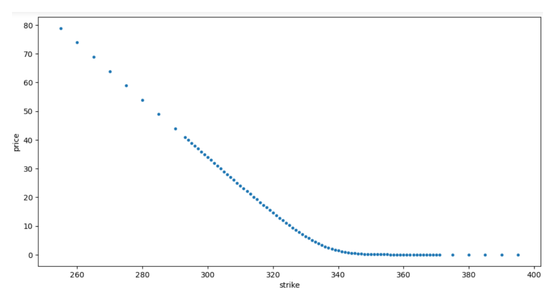


Figure 4: Call options prices

First, we drop the far ATM values since the strikes in these regions are too sparse to get a trustworthy result. Then we get the price of the options as the mid-price of the bid/ask prices and we derive the implied volatilities on the given strikes using the Black & Scholes formula.

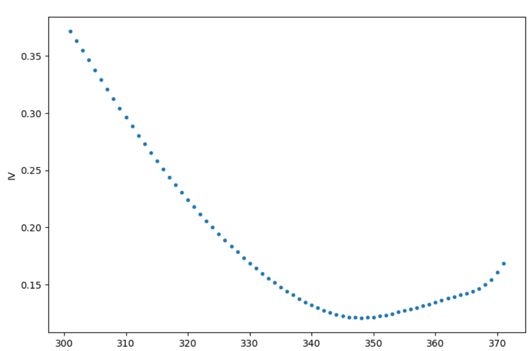


Figure 5: Implied volatilities

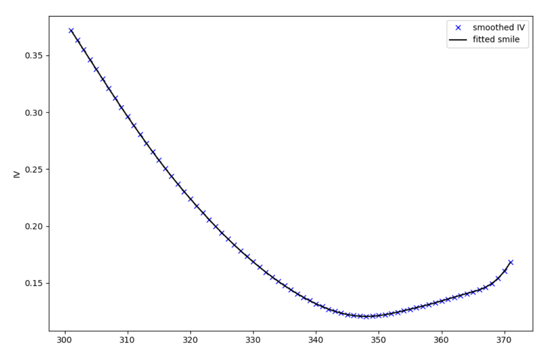


Figure 6: Smooth implied volatilities

Then by using the interpolated implied volatilities, we calculate the corresponding call options prices. Next, from these prices we use the Breeden-Litzenberger formula in order to derive the risk-neutral density.

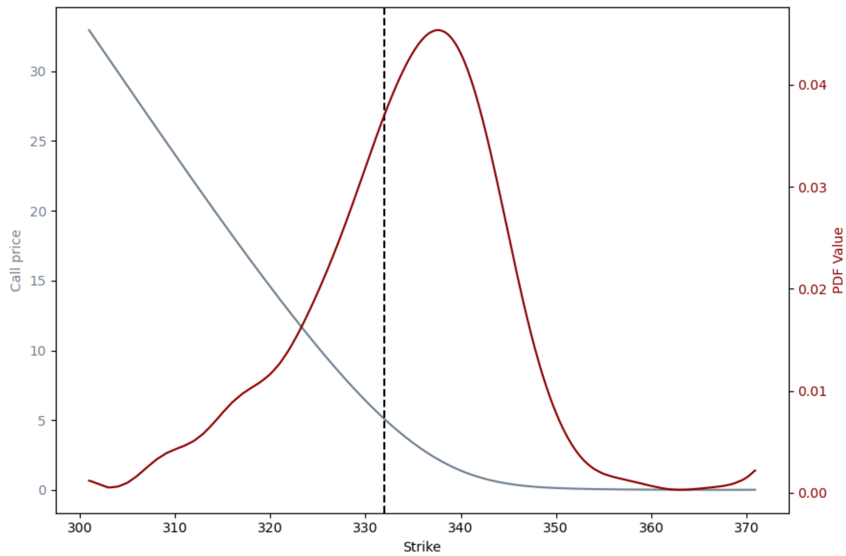


Figure 7: Learning process and investment structuring

We notice that the extreme values give irregular results, which cannot be trusted due to the sparsity of call option prices with strike far away from the money. So we drop the values outside the interval $[320, 360]$ and what remains is $\sim 90\%$ of the distribution (area under the curve ~ 0.9).

Now, in order to get a valid distribution we extrapolate on the left using exponential decay and derive a smoother representation of the risk neutral pdf.

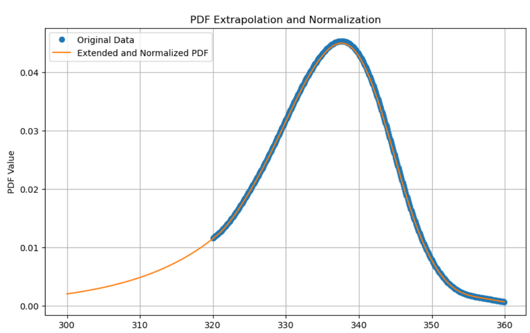


Figure 8: Extrapolated Risk Neutral pdf

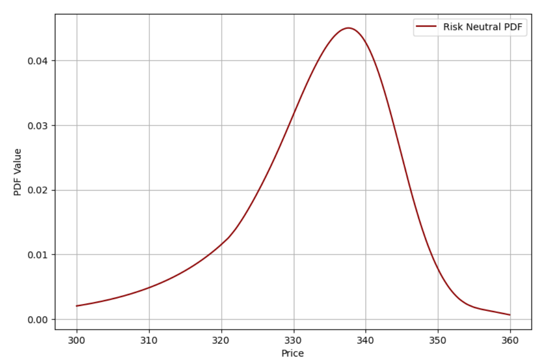


Figure 9: Risk Neutral pdf

4.2 Different investor beliefs

At the core of the structuring algorithm, lies the idea of custom beliefs an investor might have for the distribution of the underlying after some analysis he has conducted. More specifically, what matters most is how this belief compares to the market-implied distribution (risk-neutral density). Now we will consider some cases that represent possible beliefs of the investor.

4.2.1 Shift the expected value (drift)

First, consider that an investor who has the view that the expected value of the underlying at maturity should be higher than the one suggested by the market. In a simple case, that could be expressed by shifting the risk-neutral pdf by some units to the right. That corresponds to a believed distribution which is biased towards higher prices for the underlying.

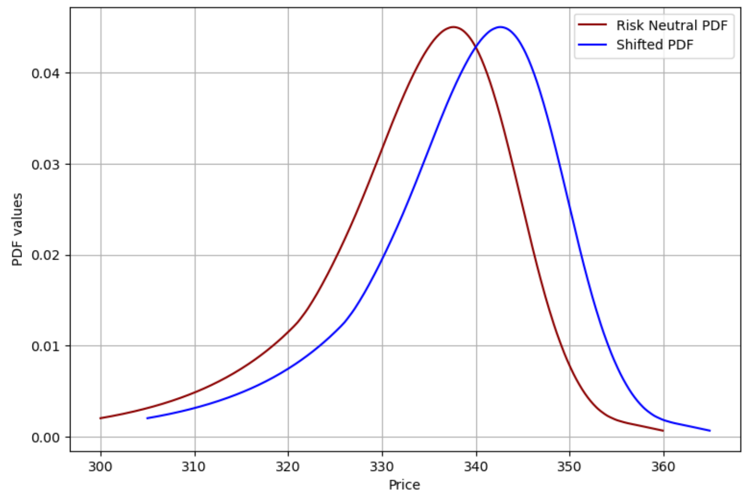


Figure 10: Shift of the expected value of the risk neutral pdf

Now we compute the optimal payoff for a growth-optimizing investor as suggested by the theory: $f(x) = \dfrac{b(x)}{m(x)}$, where $m(x)$ is the market-implied density and $b(x)$ is the pdf believed by the investor.

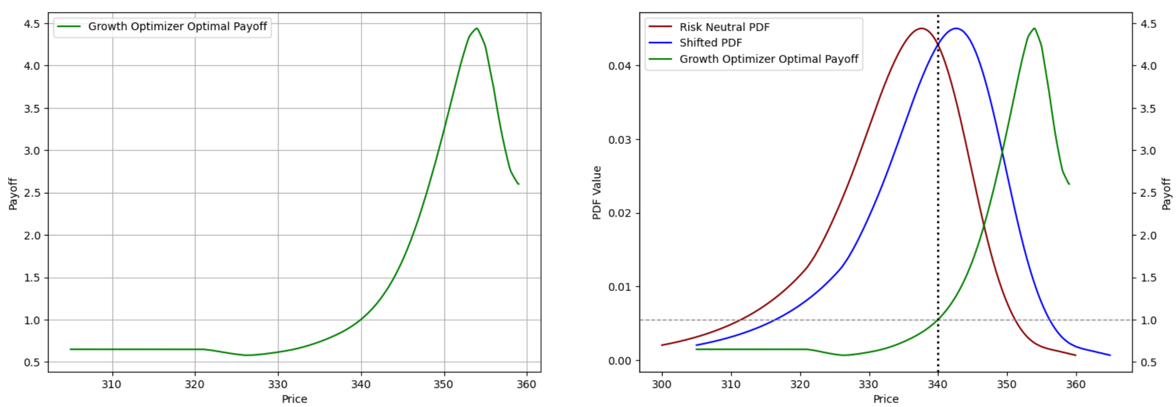


Figure 11: Derived Growth Optimal Payoff from the real and the shifted (drift) RNDs

We notice that the resulting payoff shares some characteristics with a call option. This aligns with the intuition that a call option expresses the expectation of a rise in the price of the underlying. The structured payoff however seems to capture more accurately the specific relation of the two densities by dropping as we get closer to the tails, where the difference in the believed and the implied probabilities becomes smaller. The exact form of the payoff on the far ITM values would depend on the exact way we model the tail probabilities.

4.2.2 Change the volatility

In the same manner as before, we now consider an investor that has a view on the volatility of the underlying. We choose a scenario where the investor expects that the volatility is larger than the one the market infers. This can be expressed by choosing a distribution with fatter tails.

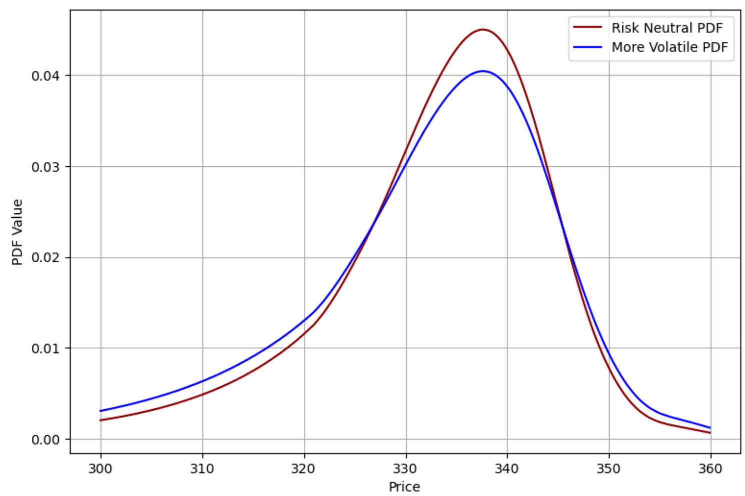


Figure 12: Change of the volatility of the risk-neutral pdf

Again we derive the growth optimal payoff for this scenario and we observe that this payoff resembles the market standards (when dealing with modified views of the volatility) straddles and strangles.

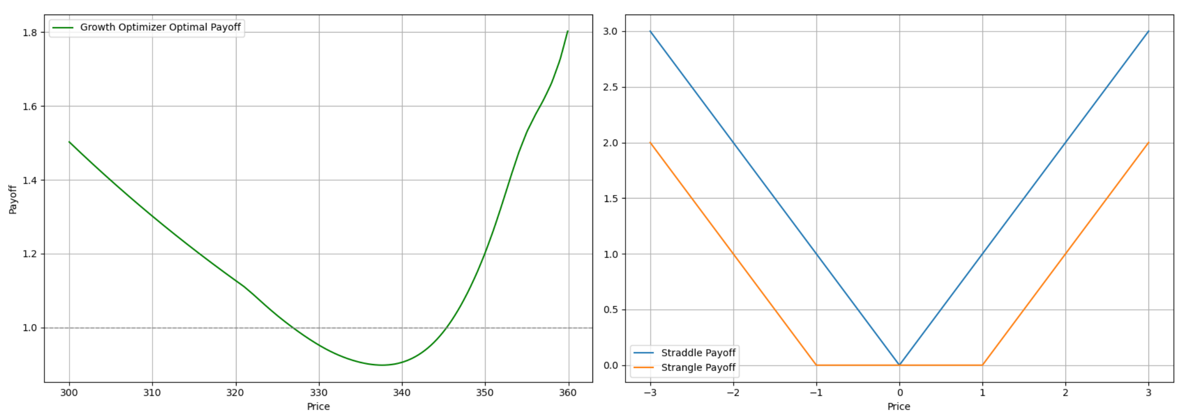


Figure 13: Derived Growth Optimal Payoff for the more volatile density (**left**) and payoff of a straddle and a strangle (**right**)

However, these existing products are symmetric around the strike. So if we talk about straddles and strangles with ATM strike, then they address the rise and fall of the price in a symmetric way. This doesn't seem ideal since the probabilities are skewed. On the other hand, the structured payoff seems to capture more efficiently the skewness in the fraction of believed and implied densities.

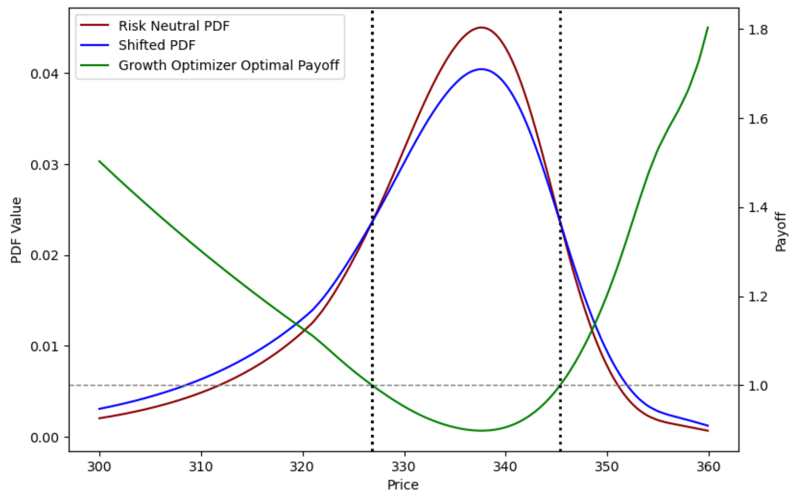


Figure 14: Derived Growth Optimal Payoff and the risk-neutral and adjusted RNDs

4.2.3 Change in drift and volatility

Now let's focus on a more specific case. Assume that we have an investor who buys a reverse convertible note with ATM strike. More details about this product will be given in the next section; for now this is just the motivation for looking at a specific belief. What such an investor usually expects is that the volatility will be low and that the price of the underlying might make a slight rise (not larger than the coupon value). In our case, the market (risk-neutral density) expects that the price of the underline will slightly rise from \$332 to \$333.5. Now let's assume that the investor believes that the price will rise to \$336.2 (as mean) and the volatility will be smaller than the market-implied. Then his belief might be expressed as depicted in the following figure:

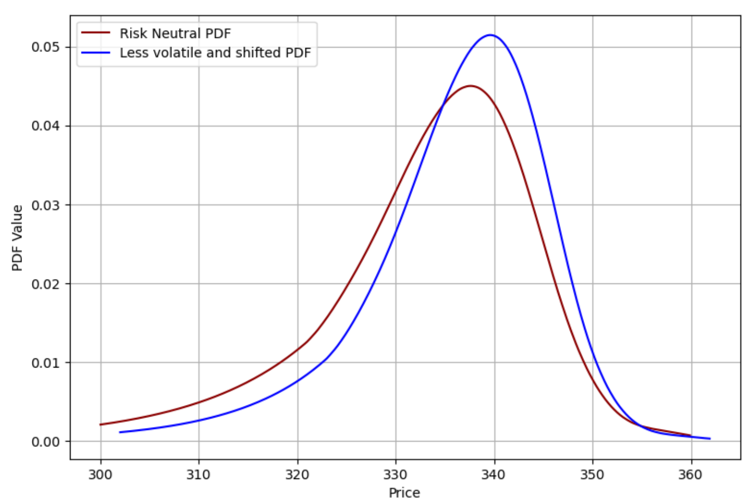


Figure 15: Change in the expected value and the volatility of the risk neutral pdf

The investor expects in this case that the price of the underlying will rise to \$336.2. We compute the optimal payoff in this case (according to our theory) for a growth-optimizing investor.

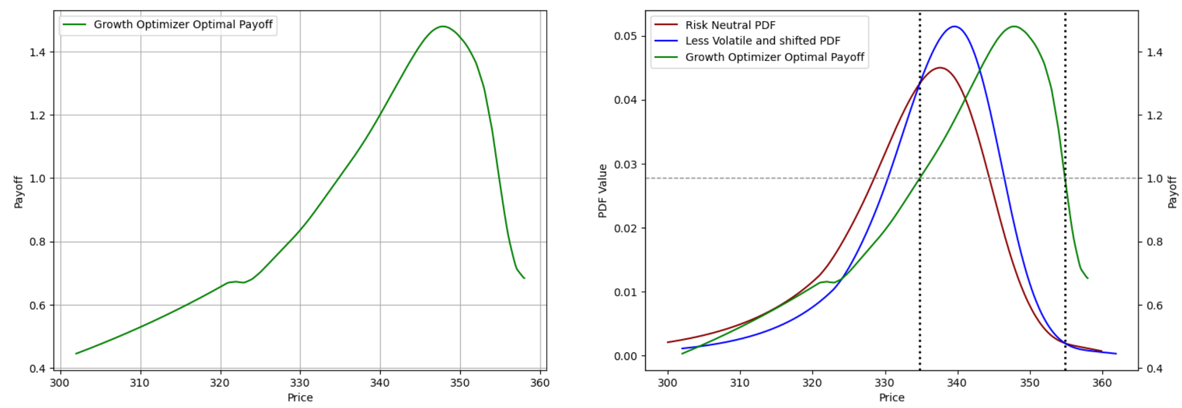


Figure 16: Derived Growth Optimal Payoff (scaled on the right figure)

We observe that this payoff has a shape similar to a short straddle, but assigning higher values to larger underlying prices. So it seems to capture both the beliefs for the density mean and the thinner tails.

4.2.4 Adjust the implied volatility smile

Up to this point, we only looked at examples where the investor's beliefs were directly depicted on the distribution of the underlying. However this is not necessary. We will now look at a case were the investor has a specific view on the implied volatility curve.

In this scenario, we suppose that the investor believes that the volatility smile is less skewed than the market-implied one. We adjust the volatility curve according to this expectation and using again the Breeden-Litzenberger formula we derive the corresponding density. Similarly as before, due to lack of option strikes away from the money, we are missing 10% of the distribution on the left tail. In contrary to what we did before though, we now won't extrapolate to complete the density, since this would require to make additional assumptions for the relationship of the two distributions (risk neutral and adjusted) that we preferred to avoid here.

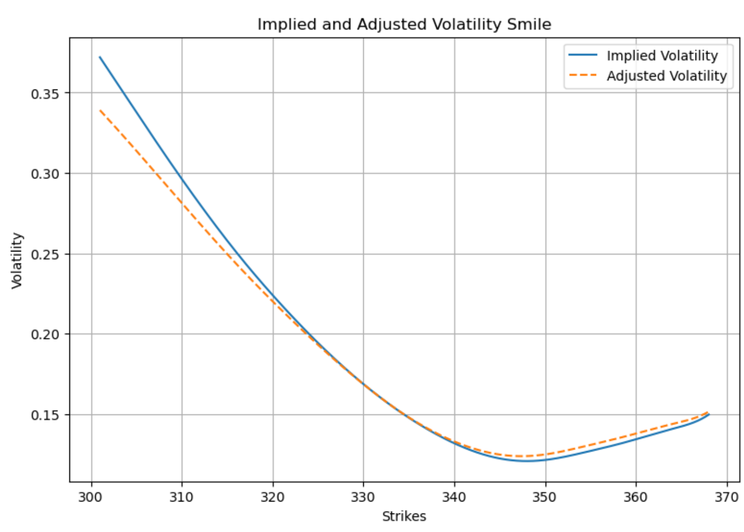


Figure 17: Implied volatilities curves: real and (skew) adjusted

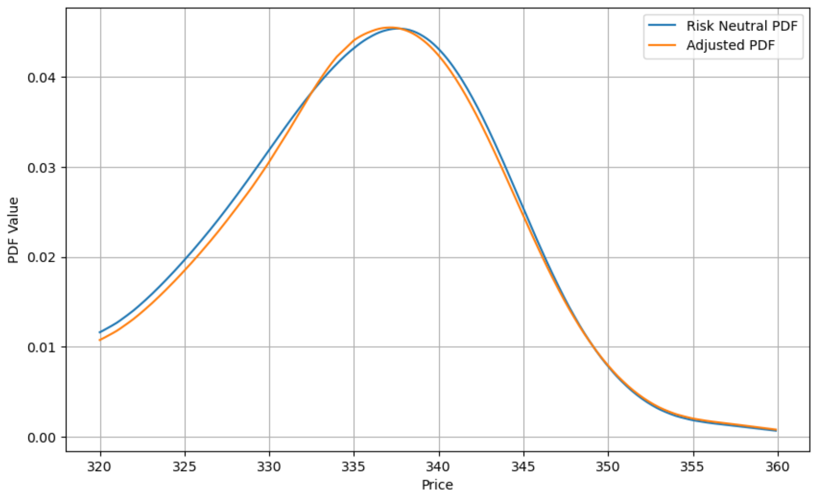


Figure 18: Risk neutral densities: real and adjusted

As before, we compute the optimal payoff for a growth-optimizing investor with the belief described above. The resulting function is shown in the following figure:

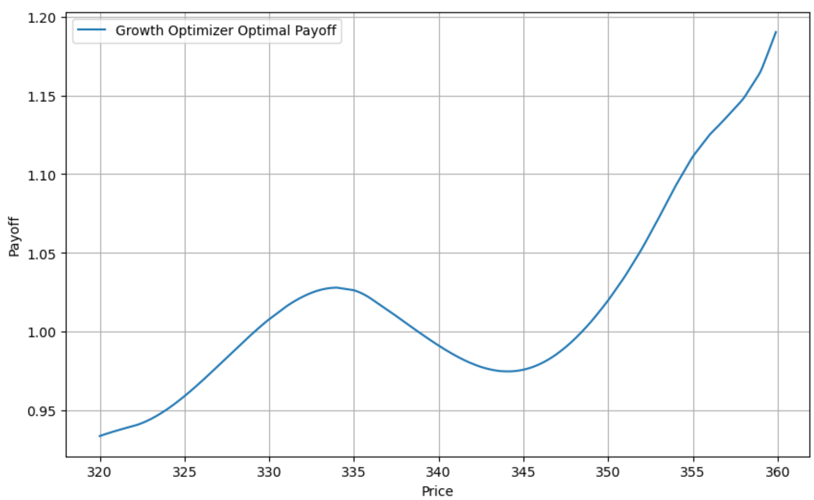


Figure 19: Growth Optimizer Optimal payoff with adjusted skew

We notice that the resulting payoff looks somewhat similar to that of a risk-reversal type (with adjusted slope) or digital payoff.

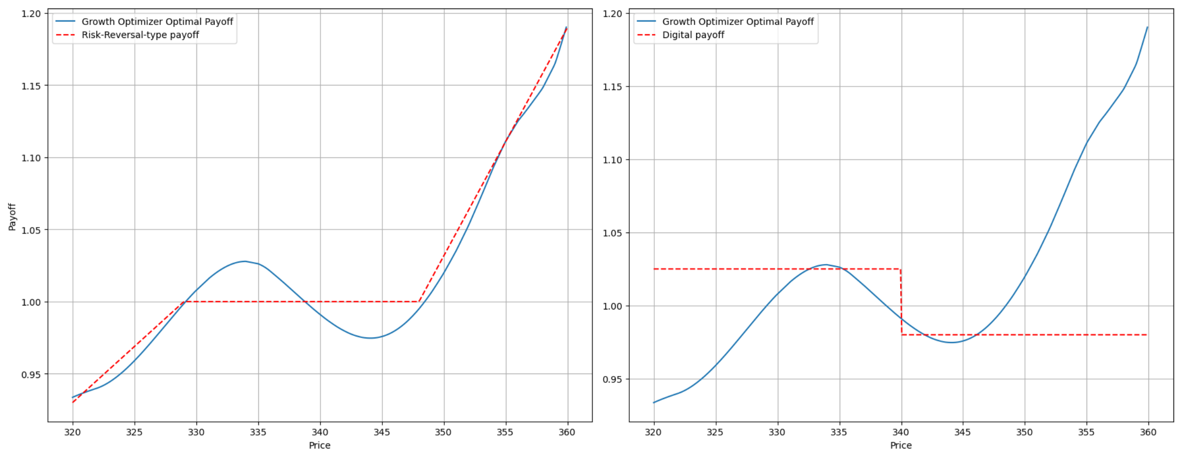


Figure 20: Growth Optimizer Optimal payoff with adjusted skew in the implied vol curve compared risk reversal and digital option payoffs

The payoff of the structured product though seems to be more expressive of the specific beliefs of the investor both around the money and the tails, instead of assuming piece-wise indifference as is implied by the constant parts of these existing products.

4.3 Risk aversion extraction

Now that we have established the growth optimal products for the scenarios we presented, we are ready to use our formulas to find the optimal payoffs for a client that doesn't only have a specific belief for the market, but also a given risk aversion, given as a Relative Risk Aversion (RRA) across moneyness which is implied by a utility function U with the formula $R(x) = \frac{-xU''(x)}{U'(x)}$. This can be done using our equation $\dfrac{dlnF}{dlnf} = \dfrac{1}{R}$ that connects the optimal payoff of the growth optimizing investor f, the risk appetite of the client R and the optimal payoff F of this client. Thus, given f, R we can extract the structured product F.

Before doing that though, we would like to approach the problem in the opposite direction. What if we use the previous formula, but instead of f, R we take as inputs f, F, where F is the payoff of an existing product? Then we could extract a risk aversion function R. And what does this risk aversion represent? If we look back at how f is constructed, we would realise that this R would represent the risk appetite of an investor that has the market belief b that leads to the construction of f through the formula $f(x) = \frac{b(x)}{m(x)}$ (m is the risk-neutral density) and who chooses to buy the product F. To state it more simply, the input is the belief and the preferred product and the output is the implied risk aversion. As we will see though, it is hard to get consistent sensible risk aversion results by working with real-world data.

The first step is to establish an investor belief. To do that, let's look back at some of the cases we encountered before. The scenarios we examined included:

1. Shifting the risk-neutral pdf
2. Increasing the volatility
3. Shifting to the right and reducing the volatility
4. Reducing the skewness of the implied volatility

4.3.1 Reverse Convertibles

Now we look on a specific product in order to apply the procedure. Reverse Convertibles are among the most popular yield enhancement products in Switzerland and are suited for investors who are anticipating a sideways or slightly upward trending market.

The holder of a reverse convertible gives up the potential upside exposure to the underlying asset in exchange for an enhanced coupon. The holder of the product remains exposed to the downside exposure.

The enhanced coupon of the reverse convertible is paid in any case. Because of this, the product will always outperform its underlying asset to the downside. The product will also outperform if the asset does not rise by more than the coupon. Hence, the ideal market scenario for reverse convertibles is the prospect of a sideways trending market.

In its most basic form, the reverse convertible is constructed by means of a short put and a money-market placement. In most cases, the strike is placed ATM. Obviously, the lower the strike, the lower the value of the put option and the lower the guaranteed coupon.

The below graph shows the payoff of a 1y reverse convertible with a ATM Put and a guaranteed coupon of 10.5%.

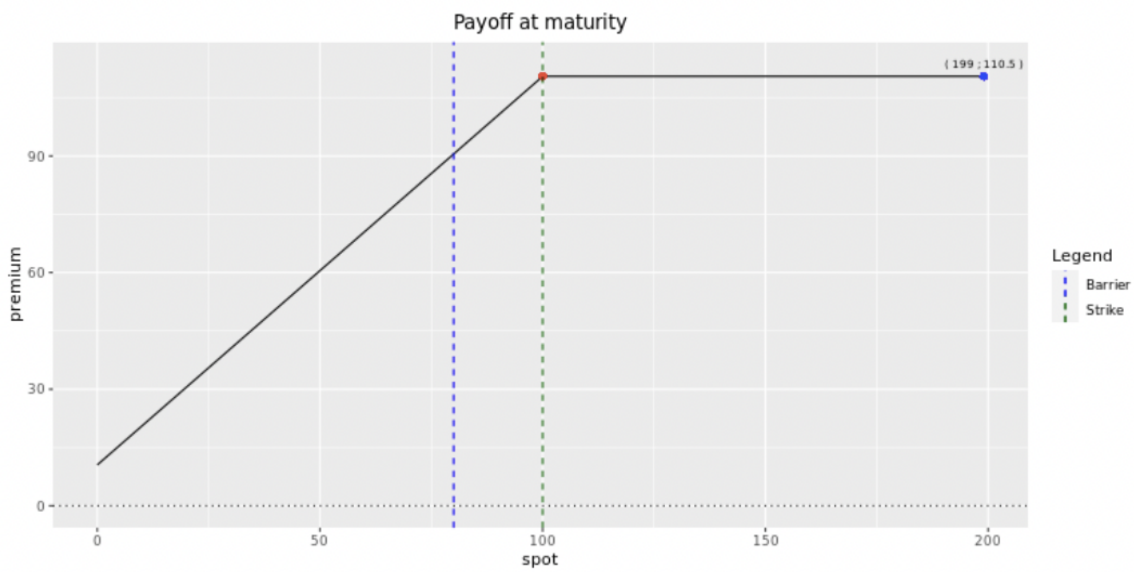


Figure 21: Reverse convertible payoff

The price of this reverse convertible is 99.81%, which is composed of:

- Long 1y Zero-Coupon Bond: 98.02% (assuming 2% interest rate for the sake of the example)
- Short 1y ATM put option: 8.50% (assuming 20% implied volatility and 3.5% of dividends)
- Guaranteed coupon: 10.5% paid at maturity → PV(10.5%) = 10.29%

Let us assume that the 19 bps left (100% - 99.81%) are earned by the bank as a commission.

4.3.2 Implementation of the procedure

As we showed before, each one of these cases gives rise to an optimal product for a growth-optimizing investor with payoff f. Now for each one of these cases, we will try to extract the implied risk aversion across moneyness of an investor that has bought a reverse convertible note (RC) with ATM strike and 10% coupon. So we use the formula $\dfrac{dlnF}{dlnf} = \dfrac{1}{R}$ and we get the following risk aversion functions:

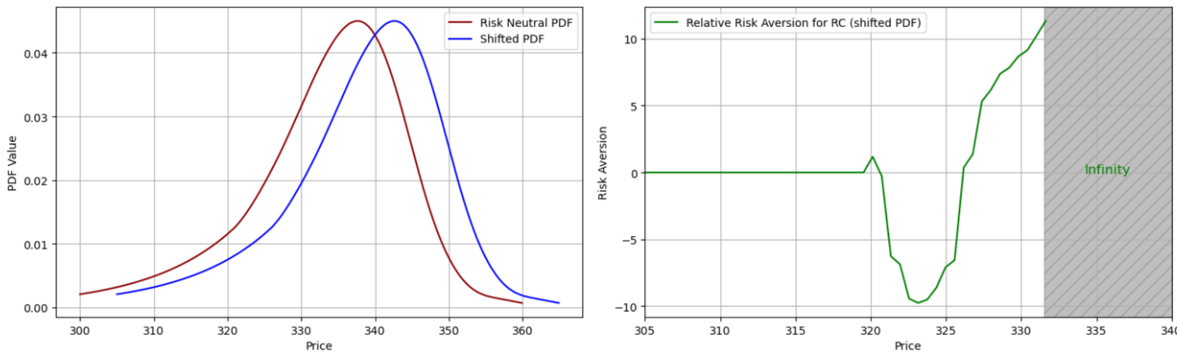


Figure 22: RND and shifted (drift) RND **(left)** and the derived Relative Risk Aversion for the Reverse Convertible **(right)**

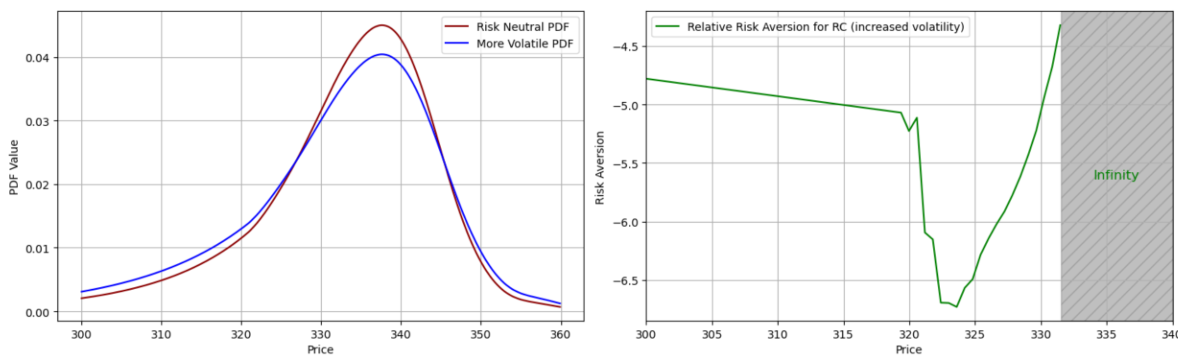


Figure 23: RND and vol adjusted PDF **(left)** and the derived Relative Risk Aversion for the Reverse Convertible **(right)**

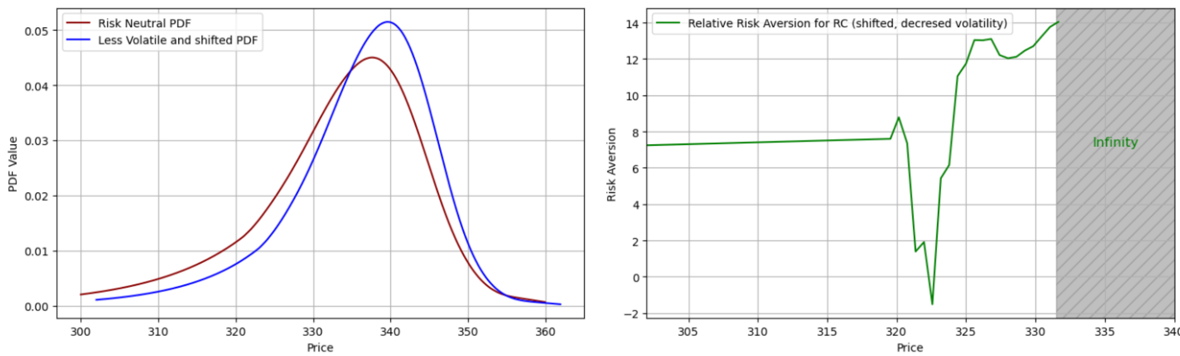


Figure 24: RND and shifted (drift + vol) RND **(left)** and the derived Relative Risk Aversion for the Reverse Convertible **(right)**

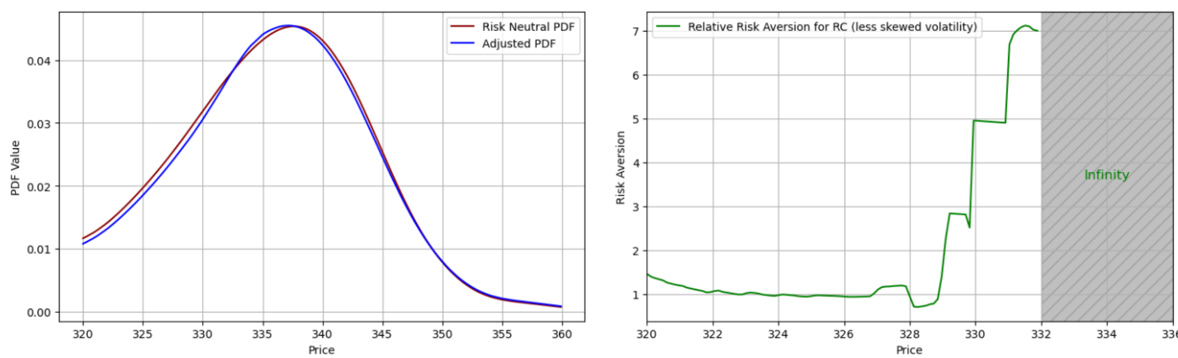


Figure 25: RND and skew-adjusted PDF **(left)** and the derived Relative Risk Aversion for the Reverse Convertible

A first observation is that for underlying prices larger than \$332, the risk aversion becomes infinite. Mathematically, this happens because the formula for RRA gives $R(F(x)) = \dfrac{f'(x)}{f(x)}\dfrac{F(x)}{F'(x)}$ and since F is the payoff of an RC, $F'(x)$ becomes 0 for $x \geq 332$ since the payoff is constant after the strike. Intuitively, this could be interpreted as inability to extract any risk preferences for values larger than the strike, since in this region small price movements don't change the payoff of the investor, so we can't get any information about how his utility is affected. In essence, what the RC payoff implies is that the investor is indifferent to small price movements in this area. This already looks like a limitation in the expressiveness of some existing products.

For the prices before the strike though, even though it also seems hard to extract consistent results, we can get some idea about the risk appetite of the client and try to explain it. In some cases, we observe that the line becomes negative entering the risk-loving territory of gambling. This especially makes sense in the case of the increased volatility (second set of graphs). Because if we think about it, what we expect for this extracted risk aversion to express, is the risk profile of an investor who believes that the real volatility is larger than the market-implied one, but chooses to buy a reverse convertible (which is suitable for investors who believe that the underlying price won't deviate much, so that the volatility will be low). So it is reasonable to assume that such an investor would have a gambling-like risk appetite. And this is what our analysis shows. On the other hand, the results for less volatily and shifted density and the density extracted from the change in the local volatility skew (third and fourth lines respectively), which would probably match the beliefs of an investor that buys an RC a bit better, seem to be more reasonable for the bigger part even though not smooth and easily interpretable. This could be assigned to the artificial-smooth form of the RC payoff, compared to the "rough" nature of the real data.

4.4 Derivative structuring for general investors

After examining how to extract the relative risk aversion from a given derivative, its time to look into the more straight-forward use of the proposed structuring algorithm. That is, given a belief b and a risk aversion R by the client, we construct an appropriate payoff that directly models these views. Assuming that this unknown payoff is F, it can be found as the solution to the ODE $\frac{dlnF}{dlnf} = \frac{1}{R(F)}$.

For the optimal payoff f of the growth-optimizing investor, we will look again at the four cases we had before. To recap, we made different assumptions for the believed pdf b by the investor and given the market-implied pdf m, we got f as $f = \frac{b}{m}$. For the RRA R we will look at three different cases:

1. If we consider the power utility function $U(x) = \frac{x^{1-\gamma}-1}{1-\gamma}$, we derive the constant RRA function $R(x) = \gamma$
2. If we consider the exponential utility function $U(x) = -\frac{e^{-\gamma x}}{\gamma}$, we derive the linear RRA function $R(x) = \gamma x$
3. As a third case we consider a custom RRA that is piecewise linear and expresses an increased risk aversion in the tails. The exact formula is shown in the following plot.

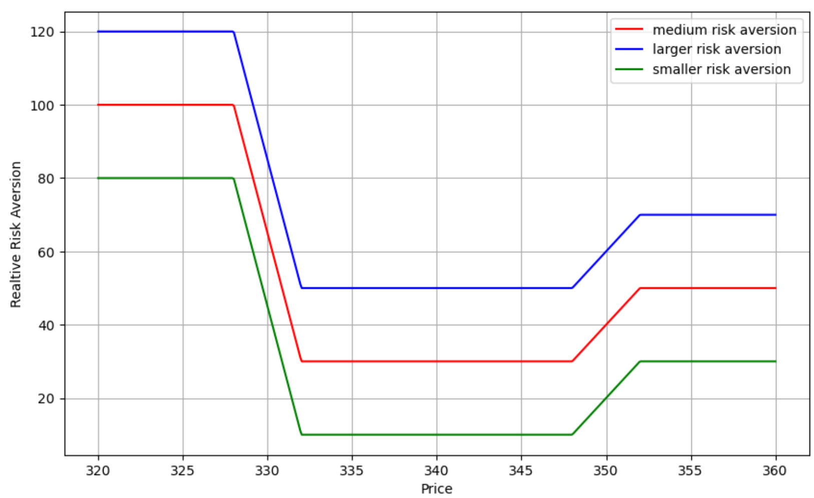


Figure 26: Different levels of custom risk aversion profiles

For the first two cases (power, exponential) the corresponding RRA has a simple expression so the ODEs have a closed-form solution:

1. For the power utility function with risk aversion parameter γ, the ODE gives $F(x) = f^{\frac{1}{\gamma}}(x)$.
2. For the exponential utility with risk aversion parameter γ, we get $F(x) = \frac{ln(f(x))}{\gamma}$.

For the custom RRA, since it is piecewise linear, it would again be possible to get a closed form solution for the ODE. However, we chose to solve it numerically in order to provide a general framework that would apply even for cases where an anlytical exact solution is not possible. So for each one of the four different beliefs, we define three ODEs (one for each custom RRA) and solve them using python's "solve_ivp" function.

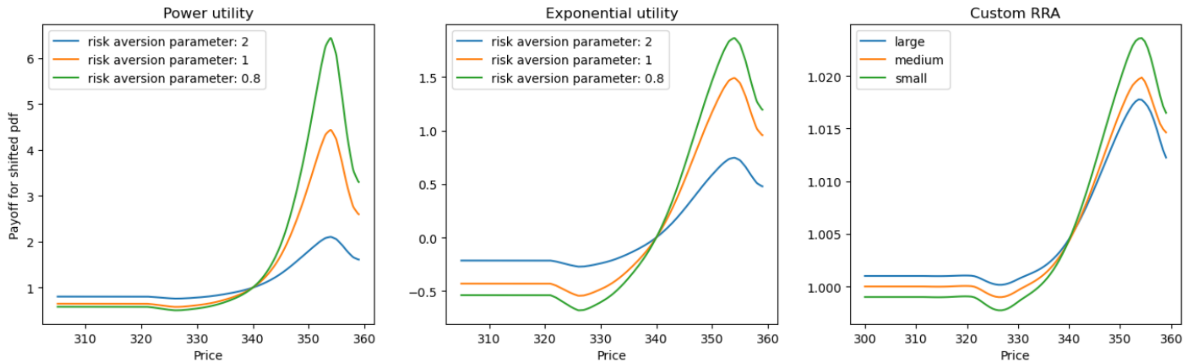


Figure 27: Different levels of custom risk aversion profiles for shifted PDF

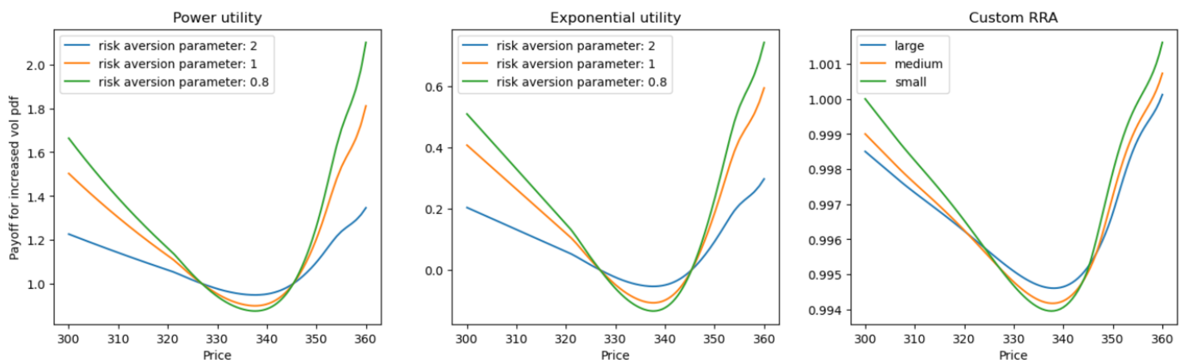


Figure 28: Different levels of custom risk aversion profiles for increased vol

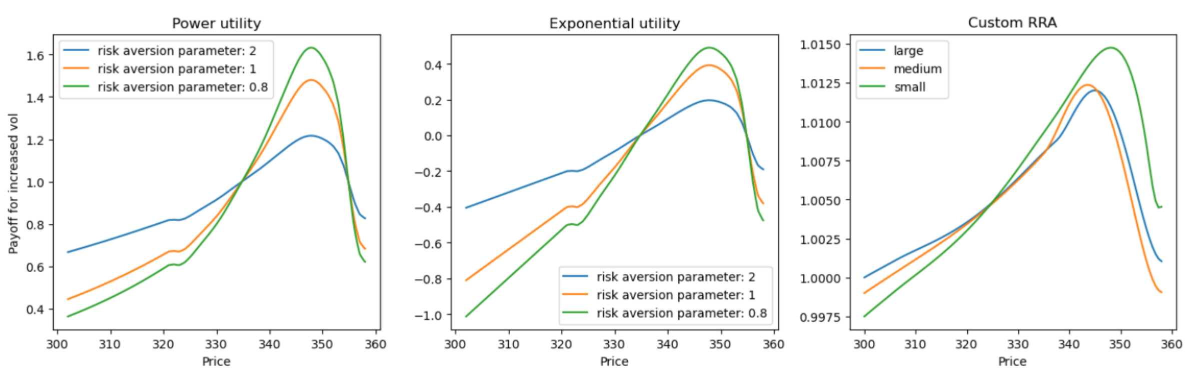


Figure 29: Different levels of custom risk aversion profiles for shifted PDF with reduced vol

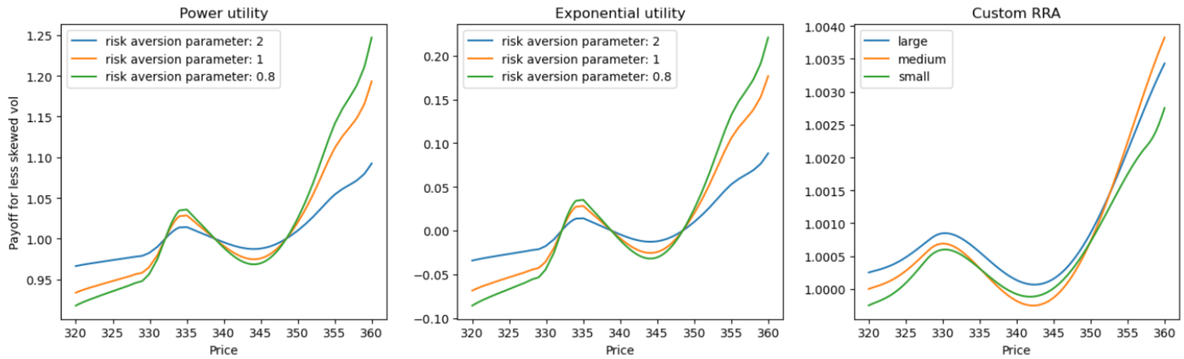


Figure 30: Different levels of custom risk aversion profiles for skew-adjusted PDF

These adjusted payoffs for the cases of risk aversion implied by the power and exponential utilities are just exponential and logarithmic transformations and scalings of the growth-optimizing payoffs. What they do is adjust the initial payoff to the specific risk appetite across moneyness of the investor. What is more obvious and interesting to observe is how the custom RRA's affect the optimal payoff. Since these functions are larger in the tails, thus making the investor more risk averse in these regions. This is depicted in the resulting payoffs, since if we notice the scale on the y-axis, we observe that they are much "flatter" than the previous cases (since the payoff in the tails is "cut" in exchange of more safety).

5 Conclusion

5.1 Summary

At the heart of structuring we always have optimization which reflects the goals of the client. This is preceded by preparatory steps defining the problem and followed by packaging of the solution into a tradeable product. Together these steps form the backbone of a manufacturing process which we summarized in Figure 2 The preparatory steps include deciding on the market variable, capturing the relevant market and client views and identifying the goals of the client. This leads us to the key solution stage where our equations come in: the growth-optimal $b = fm$ and the payoff elasticity equation.

In our study, we theoretically established this framework and then applied it in artificial cases to establish that the results that it produces are at least at an initial level reasonable. This stage already indicated that the proposed structuring method seems to provide a more accurate adjustment to the client's beliefs and risk preferences than traditional products. After that, we attempted to apply the same pipeline to using real market data. The results again indicate that the structured products in the cases we examined seem to more reasonably and accurately correspond to the specific investor profiles while raising some challenges that will shortly be discussed below. We also attempted to follow a different route and use the elasticity equation we introduced to derive relative risk aversion implied by a specific product.

5.2 Suggestion for further research

Throughout this project, we came across several issues and details that could constitute subject for further study.

One such issue is the choice of prior. The theoretical framework suggests that as prior distribution for our analysis, we should consider the market-implied density. But what choice should we make to represent the market view? The most straightforward answer is the risk-neutral density. However given that the risk-neutral density has been shown to be a poor predictor of future prices, the question is raised about how appropriate this choice is. Instead of that, we could use some empirical or risk-adjusted distributions. But then this raises the question of what we should use as a posterior belief. The effect of the specific choice of these functions on the resulting structured products would be interesting an interesting topic to study.

Another issue is how realistic it is to attempt to construct products that are so precisely designed to fit a client's preferences. The challenge in this area would be to align the roughness and irregularity that the market data imply with a certain level of "neatness" that would reasonably be demanded from a traded product. This could be seen as "smoothing" the payoffs that arise from the theoretical framework. In any case, the algorithm presented in this project, even if not used for precisely structuring products, could provide a very useful insight into the optimal payoffs, leading to much more efficiently structured derivatives than many of the intuitively constructed ones that are used today.

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