Quantitative Structuring & Implied Risk Aversion

A rigorous framework for the design of financial products

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Outline

Introduction

Structure products popularity

Structure products are popular for retail investors seeking

- higher yields
- leverage investments
- exotic payoffs
- specific risk exposure
- hedging

Reality check

However, on average they result in negative risk-adjusted returns ($\approx$ 7% annual loss) for investors due to:

- High fees ($\approx$ 7% annual fees)
- Overvaluation
- Misleading representation of risk profile
- Underlying securities are often too volatile
- Not enough downside protection
- Overpricing ($\approx$ 4.5%)

Structure products consistently under-perform alternative allocation to stocks and bonds.
They are often more advantageous to the sellers rather than the buyers.

Growth-optimizer Investor

The investor act as a specialist towards a specific market variable

- Expertise (confidence)
- Research: $P(x \mid research) = \frac{P(research \mid x)}{P(research)} P(x)$

Hence, the growth-optimizer forms a belief $b(x)$ and has for goal to

- Maximize expected rate of return,
- By leveraging expert knowledge on market variable

Market variable examples:

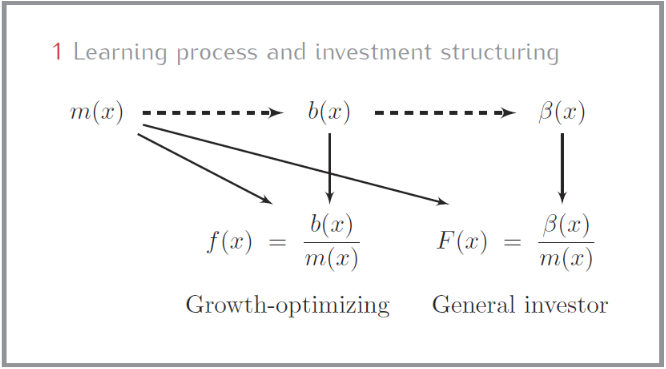
- Stock price
- Interest rate
- Weather forecast

Optimal investment strategy subject to belief reward non-zero payoff

Derivation

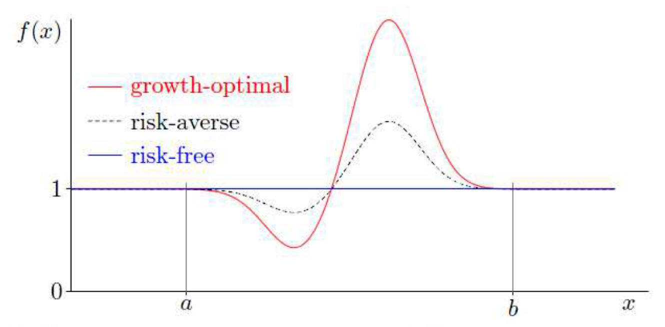
From expected utility maximization to the fundamental relationship between payoff elasticity and risk aversion

- $b(x) = f(x) \cdot m(x)$
- $\frac{dlnF}{dlnf} = \frac{u'(lnF)}{u'(lnF) - u''(lnF)}$
- Utility functions are a highly theoretical economic concept, for concrete application we get rid of it.
- Arrow-Pratt measure of relative risk aversion: $R(F) = -F\frac{u''(F)}{u'(F)}$
  - utility functions ignore addition and multiplication, mathematically not practical
  - remedy is a risk measure such as RAA
- main result: $\frac{dlnF}{dlnf} = \frac{1}{R} \implies R(F(x)) = \frac{f'(x)}{f(x)}\frac{F(x)}{F'(x)}$



- Market-implied distribution: $m(x)$
- Investor believed distribution: $b(x)$
- General investor capital splits: $\beta(x)$
- Growth-optimizing and general payoff function: $f(x)$ and $F(x)$

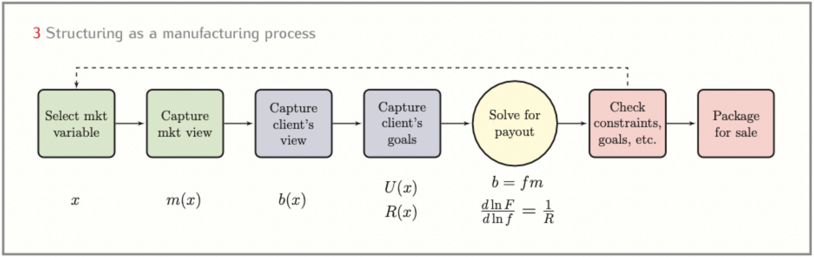
Payoff profiles



- Rational buy-and-hold investment.
- If no research, then no other view than $m(x)$ is rational so that $f(X) = 1$ (equivalent to a risk-free investment).
- When research is conducted, the investor has more knowledge of the market variable in an interval $a \leq x \leq b$ so that the believed and market-implied distribution will differ in $[a, b]$.

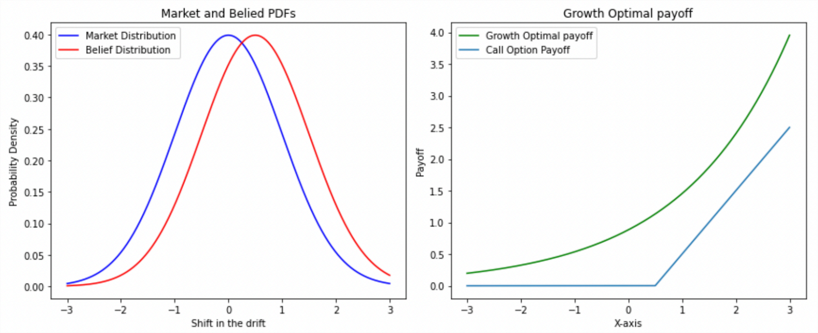
Method and artificial Examples

Structuring as a manufacturing process



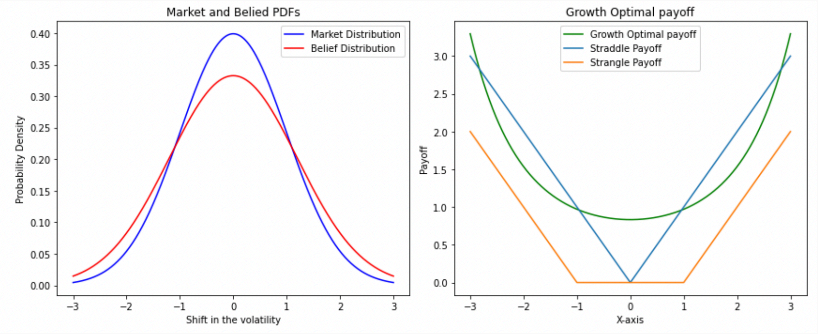
- **Correspondence principle:** verify intuitions behind product design.
- **Growth-Optimizing investor / Shifts in the belief distribution:** based on different beliefs in the drift, volatility and skew of the market distribution.
- **General investor / Incorporate different risk aversion levels:** we choose two different utility functions for the preferences of the investor (i) *the power utility* and (ii) *the exponential utility*.

Growth-Optimizing investor - Shift in the Expected Value



- We consider here simplistic scenarios i.e we assume a **Normal distribution** for the market-implied distribution for the index.
- Consider an investor who has a view that the **expected value** of the index should be **higher** than suggested by the market.
- This corresponds to a believed distribution which is **biased towards higher values of the index.**

Growth-Optimizing investor - Shift in the Volatility



- Again we assume a **Normal distribution** for the market-implied distribution for the index.
- Consider an investor who has a belief that the market **underestimates volatility.**
- This corresponds to a believed distribution which is **wider but not as tall as the market-implied distribution.**

Growth-Optimizing investor - Shift in the Skew

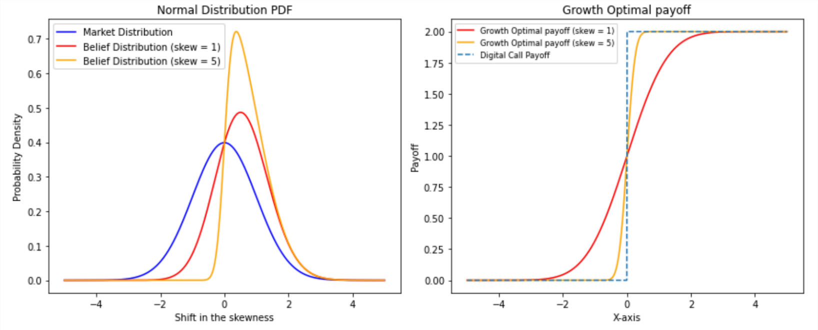
- Moving on to more complex examples, let us see how we could help investors with views on the **skew**.
- Here we assume that the probability density follows a skew-normal random variable which is defined as:

$$SN(x, \xi) = 2\phi(x)\Phi(\xi x)$$

  where ξ is a parameter which controls the skew.
- Imagine a market which implies a normal distribution, $m(x) = SN(x, 0)$, for some variable x.
- The investor believes that, in reality, there is a skew and uses a **skew-normal distribution**, $b(x) = SN(x, \xi)$, to describe this belief. For the growth-optimal payoff we compute:

$$f(x) = \frac{b(x)}{m(x)} = \frac{SN(x, \xi)}{SN(x, 0)} = 2\Phi(\xi x)$$

Growth-Optimizing investor - Shift in the Skew (2)



- We immediately recognize the profile of a cumulative distribution function which contains a **classic skew product – the digital – as a limiting case.**

General investor and utility functions

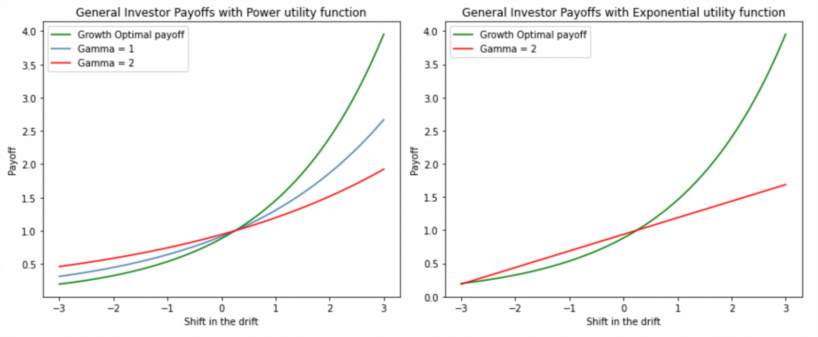
- Our main result is :

$$\frac{dlnF}{dlnf} = \frac{1}{R}$$

- We choose two different utility functions for the preferences of the investor :
- the **power utility function** $\frac{x^{1-\gamma}-1}{1-\gamma}$ with relative risk aversion (RRA) γ and
- the **exponential utility function** $-\frac{e^{-\gamma x}}{\gamma}$ with RRA $\gamma \cdot x$.
- The derived payoffs for the general investor are:

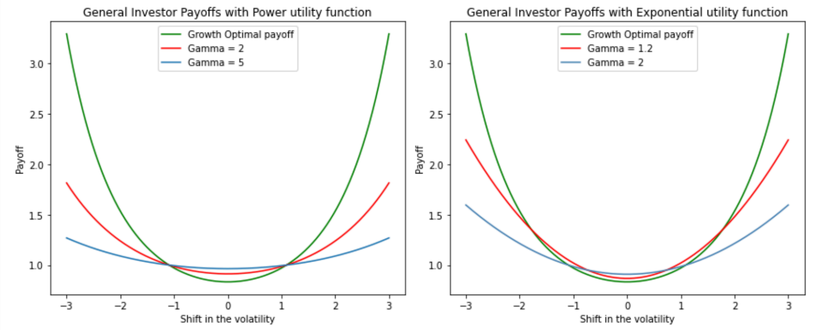
$$F^P(x) = f^{\frac{1}{\gamma}}(x), \qquad F^E(x) = \frac{ln(f(x))}{\gamma}$$

General investor - Shift in the Expected Value



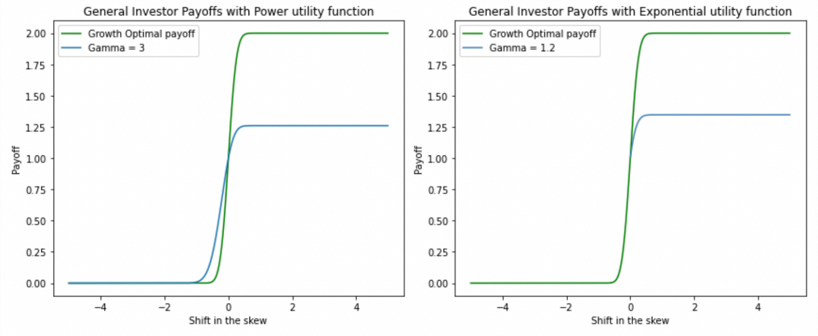
- Now we assume some different scenarios for the **risk aversion level** of the investor. The parameter γ describes the level of risk aversion, the higher the γ the higher the risk aversion.
- For the case of of the **power utility function** we obtain something similar to call option and for the **exponential utility function** something similar to a forward contract.

General investor - Shift in the Volatility



- For both utility functions we have similar results with the growth-optimizing payoff.
- When the **risk aversion increases**, he is willing to **sacrifice his potential gain**, aiming for some **better downside protection.**

General investor - Shift in the Skew



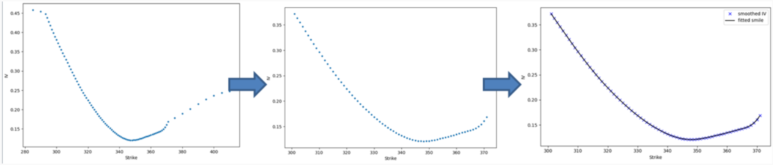
- For both utility function same as before, we observe that his derived payoff is similar to the one of the growth optimizing payoff, as well as the digital option payoff.
- He makes money for a larger area near ATM in comparison with the growth optimizing payoff but he sacrifices the potential gain.

Application on real data

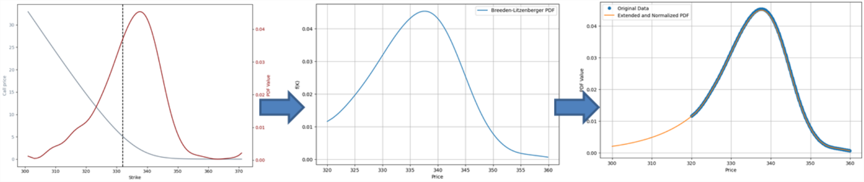
Risk neutral distribution

In order to try the algorithm on real data, the first step is to define a prior. In our case, this is the risk-neutral density. For that, we use price data of call options on SPDR S&P 500 ETF Trust (SPY) with 20-day maturity.

- From B&S formula we compute the implied volatilities.
- We drop the extreme values where the strikes are sparse.
- We use cubic interpolation and derive the implied volatility smile.



- Now that we have a coarse range of implied volatilites, we transform them back to call option prices for a wide range of strikes.
- We us the Breeden-Litzenbegrer formula $f_{\mathbb{Q}}(K, T) = e^{rT} \frac{\partial^2 C(K,\tau)}{\partial K^2}$ to derive the risk-neutral density
- Again drop the tail irregular values and end up with 90% of the distribution
- We extrapolate to the left using exponential decay and get the final RN distribution we will use.



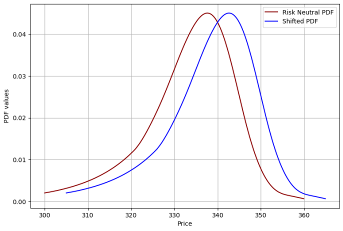
Optimal payoffs

Now that we have the market-implied density ($m(x)$) we just need the density believed by the investor ($b(x)$) and we are ready to construct the optimal payoff for a growth-optimizing investor, using the formula $f(x) = b(x)/m(x)$.

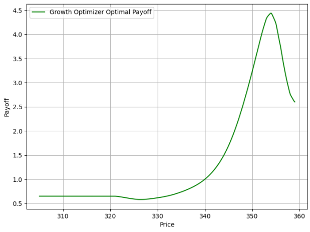
To get this we will consider four different cases.

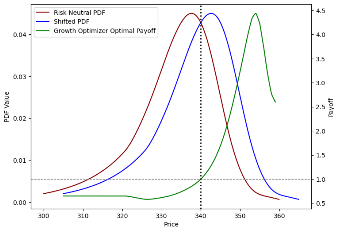
Density shift

First, we consider an investor who expects the underlying value at maturity to be higher than the market-implied one. A simple way to express this view is by shifting the pdf.



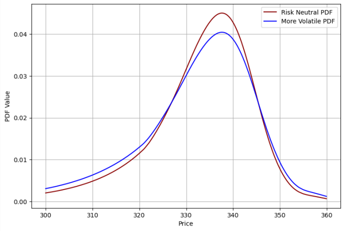
Now we construct the optimal payoff for a growth-optimizing investor and get the following function:



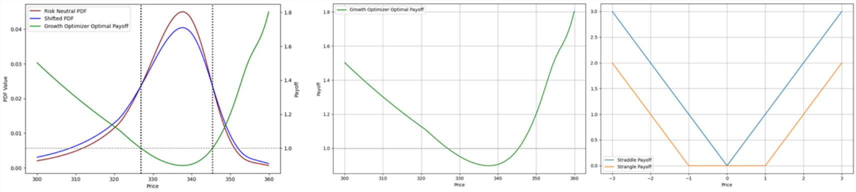


Volatility increase

Now we consider the case where an investor expects higher volatility (larger variance) than the market-implied. Thus he uses a distribution with fatter tails.

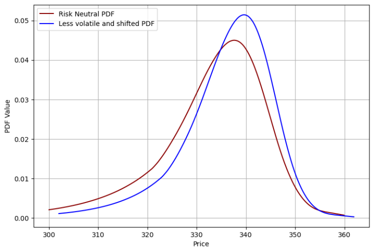


Now we construct the optimal payoff for a growth-optimizing investor and get the following function:

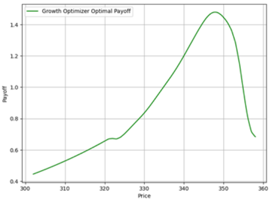


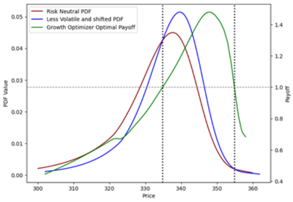
Shift and volatility change

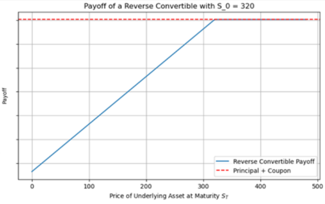
Now, we assume an investor who buys a reverse convertible note (short put + coupon) with strike ATM. What such an investor usually expects is low volatility and possibly a slight rise in price (not larger than the coupon value).



The optimal payoff for a growth-optimizing is:



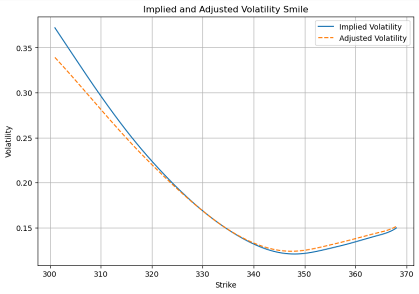


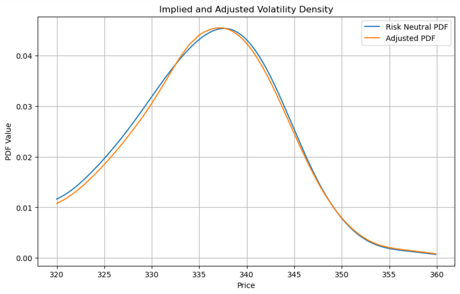


Adjust volatility smile

Instead of the underlying density immediately, the investor might have a different view on the implied volatility curve. So we consider the scenario where the investor believes that the volatility smile is less skewed than the market-implied one.

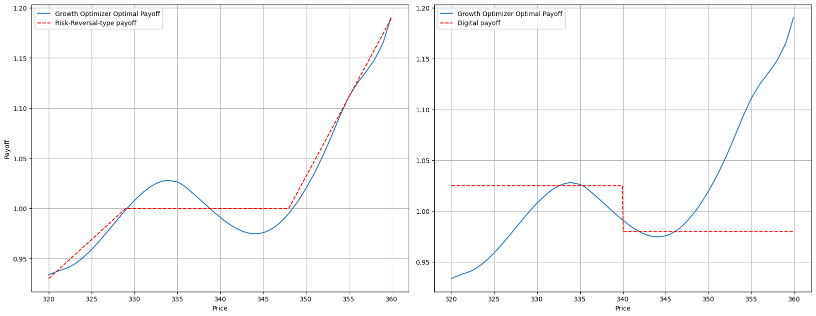
Then, we apply again the Breeden-Litzenberger formula to derive the corresponding density.





Adjust volatility smile (continued)

We compute the optimal payoff for the growth-optimizing investor and we notice some similarities with two existing derivatives: the **risk reversal** (short OTM put, long OTM call) and the **digital option**.



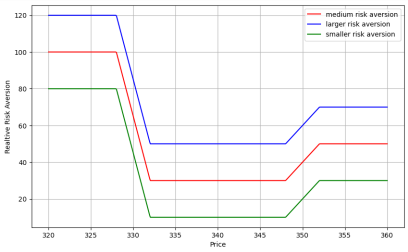
The payoff of the structured product though seems to be more expressive of the specific beliefs of the investor both around the money and the tails.

Derivative structuring for general investor (RRA)

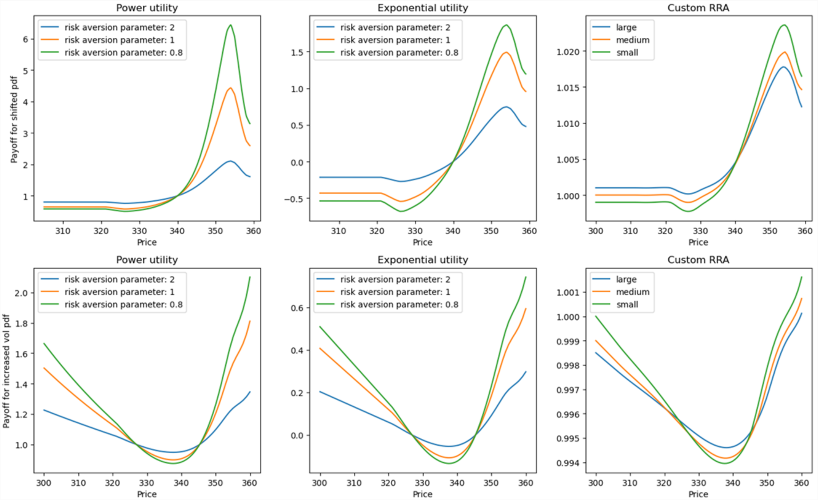
- We have the growth-optimizing payoff f.
- Given the relative risk aversion R of the client, we can get the optimal payoff for him by solving the ODE $\frac{dlnF}{dlnf} = \frac{1}{R(F)}$.

So for the **Relative Risk Aversion** we make three choices:

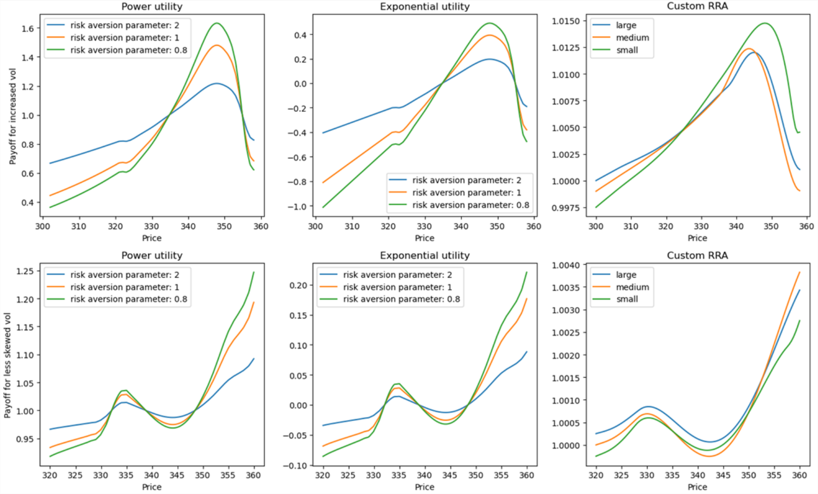
- The constant $R = \gamma$ (derived by the power utility $u(x) = \frac{x^{1-\gamma}-1}{1-\gamma}$).
- The linear $R = \gamma x$ (derived by the exponential utility $u(x) = -\frac{e^{-\gamma x}}{\gamma}$).
- Custom relative risk profiles an investor might have:



Derivative structuring for general investor (payoffs)



Derivative structuring for general investor (payoffs)

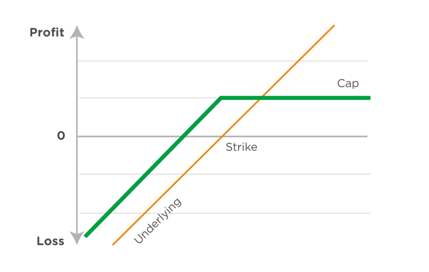


Implied risk aversion extraction

Lastly, we experimented with a different approach (not properly addressed in the paper):

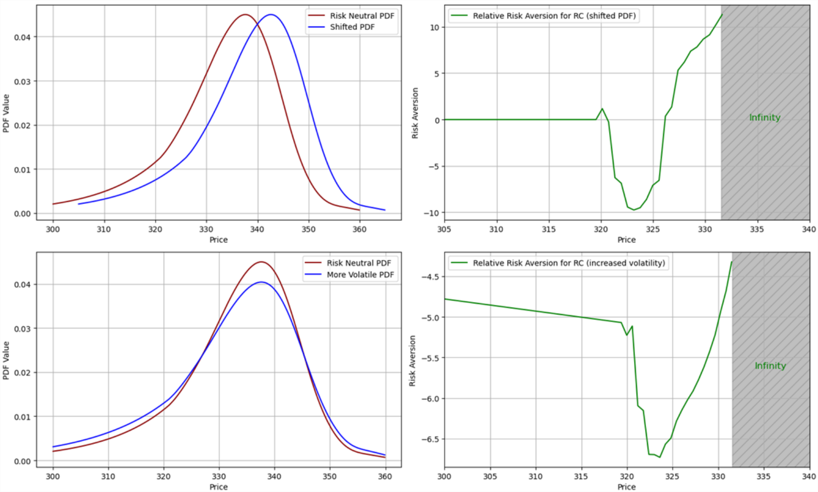
- What if we use the equation $\frac{dlnF}{dlnf} = \frac{1}{R(F)}$, but instead of the risk aversion R we input a payoff F and solve for R?
- Then the resulting function, would express the implied risk appetite of an investor with market belief $b(x)$ (which gave us f), who buys the derivative F.

We will experiment with a **reverse convertible** (short put + coupon):



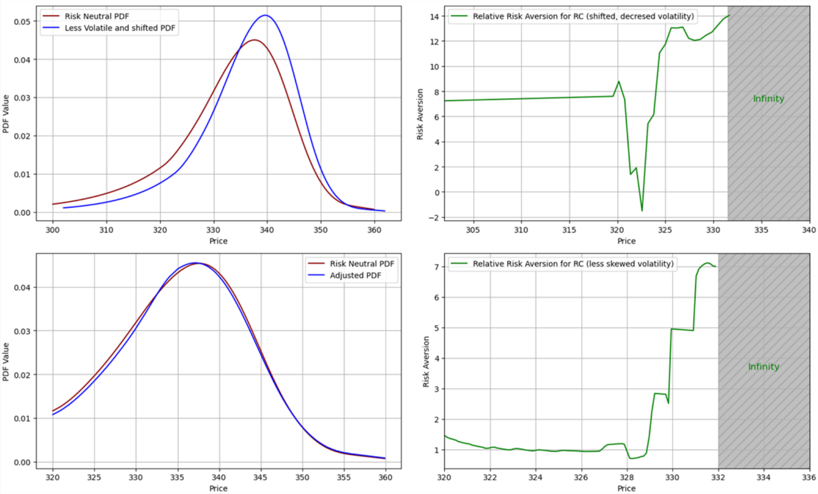
- Holder gives up upside exposure to underlying in exchange for enhanced coupon.
- Remains exposed to the downside movement.

Implied risk aversion extraction (results)



Implied risk aversion for investors who expect higher mean (**top row**) and lower volatility **(bottom row)**

Implied risk aversion extraction (results 2)



Implied risk aversion for investors who expect higher mean - lower vol (**top row**) and less skewed vol smile **(bottom row)**

Conclusion

Summary

- We presented the theoretical framework of efficient product structuring.
- We tested how this algorithm behaves in artificial market scenarios.
- We examined the behaviour of the framework on real market data and tried to extract some useful insights.
- We did an initial approach to using the structuring algorithm to extract the implied risk aversion from existing products.

Further steps

- Experiment with other priors than the risk-neutral density.
- Deal with lack of smoothness of the constructed derivatives.
- How to price the structured payoffs?

Thank you for your attention!