Applications of Erosion to Debris and Mudflows

Björn Birnir, David Cattan and Alejandro Stawsky

Department of Mathematics and the Center for Complex and Nonlinear Science, University of California, Santa Barbara

The Birnir-Bretherton-Smith Equations

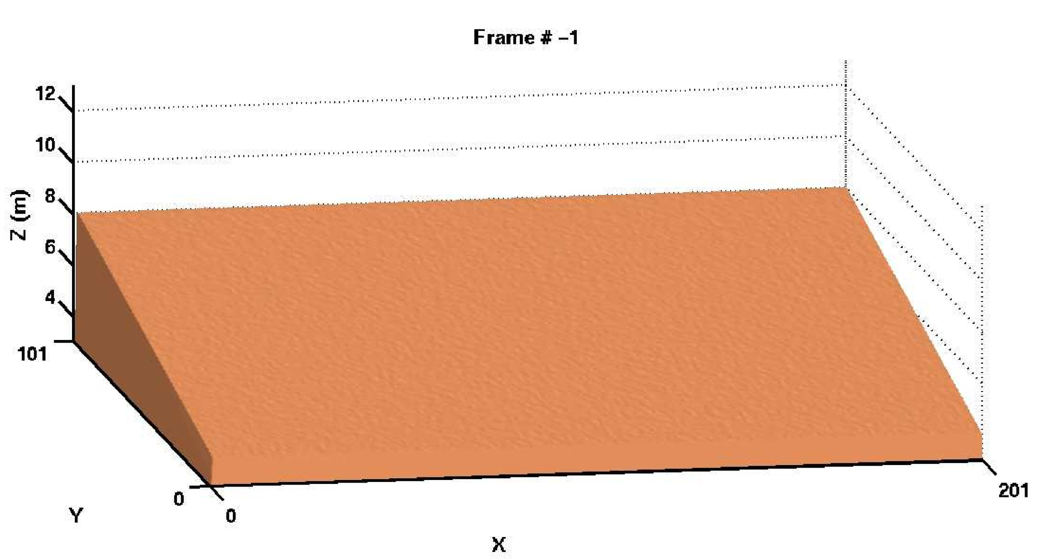
- Let $H = z + h$ be the *height of the free water surface*, where z is the height of the land surface and h is the water depth.

$$\eta^2 \frac{\partial h}{\partial t} = \nabla \cdot \left[h^{3/2} \, |\nabla H|^{1/2} \mathbf{u} \right] + R, \tag{1}$$

$$\frac{\partial H}{\partial t} = \nabla \cdot \left[h^{10/3} |\nabla H|^3 \mathbf{u} \right] - \delta h^{3/2} |\nabla H|. \tag{2}$$

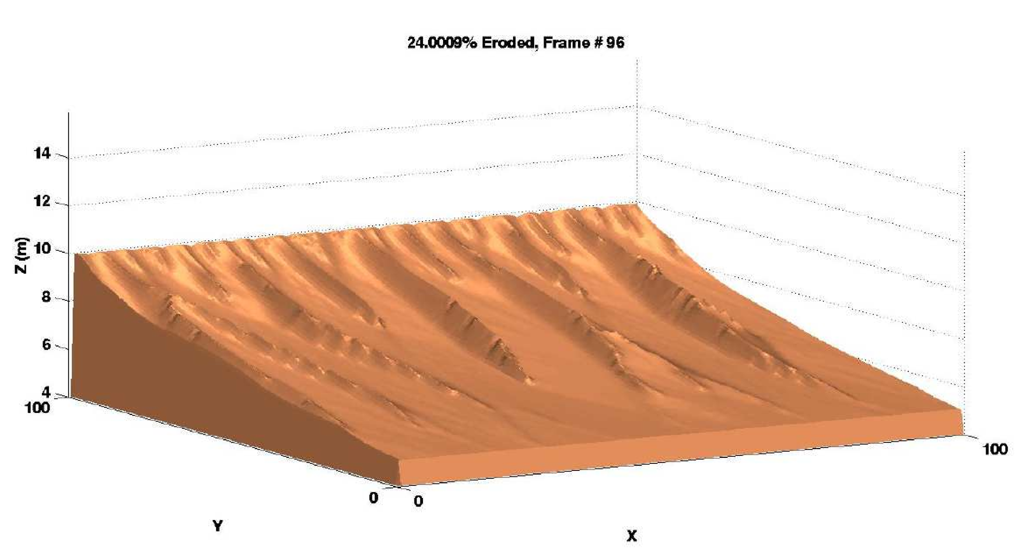
- $\mathbf{u} = \frac{\nabla H}{|\nabla H|}$ is the unit normal down the gradient of the water surface, R is the rainfall rate and η is small.
- The second term in Equation (2) models erosion and is inspired by Kramer and Marder 1992.

Initial Surface



The initial water surface, a flat ridge

A Typical Surface Simulated by David



A Pattern of Ridges and Valleys, at 60% eroded

Instabilities

Linearize the PDEs around the initial surface we get two instabilities:

- If the PDE (2) has no erosion term, the dispersion relation becomes

$$\omega = \frac{5}{3} d^{\frac{2}{3}} c^{\frac{1}{2}} [(2-d)k_1^2 + (\frac{1}{2} - 3d)k_2^2],$$

where d is small. It shows that all the spatial frequencies are unstable and that the highest frequencies grow the fastest.

- If the erosion term is included we get an additional instability

$$\omega = \frac{3}{2}\delta - k_1^2.$$

This instability gives rise to river channels.

Numerical Methods

- If the smallest frequencies grow the fastest, we have a real problem numerically.
- In nature there is a natural (lower) cutoff, when the scale of the grain size is reached.
- Nonlinearities also saturate the exponential growth of the instabilities.
- How does one capture this numerically? Cattan and Birnir (2017)
- Answer: Implicit methods work, explicit methods do not capture the small scales.
- Small viscosity is build into the Crank-Nickolson/Upwind scheme in a very controlled way. It is small and decreases with the discretization size.
- Both Predictor-Corrector and Crank-Nickolson/Upwind schemes capture the large scale features of the landscape. The number of ridges and the number of valleys are the same and the half-width of the valleys.

Scaling of the Variogram

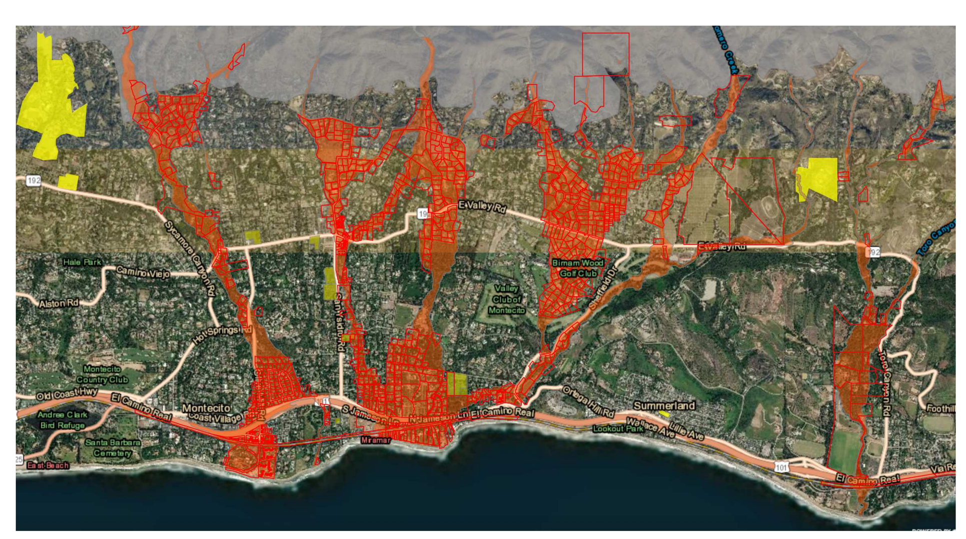
- The variogram

$$V_f(\mathbf{x}, t) = < |f(\mathbf{x}+\mathbf{y}, t) - f(\mathbf{y}, t)|^2 >^{\frac{1}{2}} \tag{3}$$

tion differences as a function of distances of separation (or lag) $|\ \mathbf{x}\ |$.

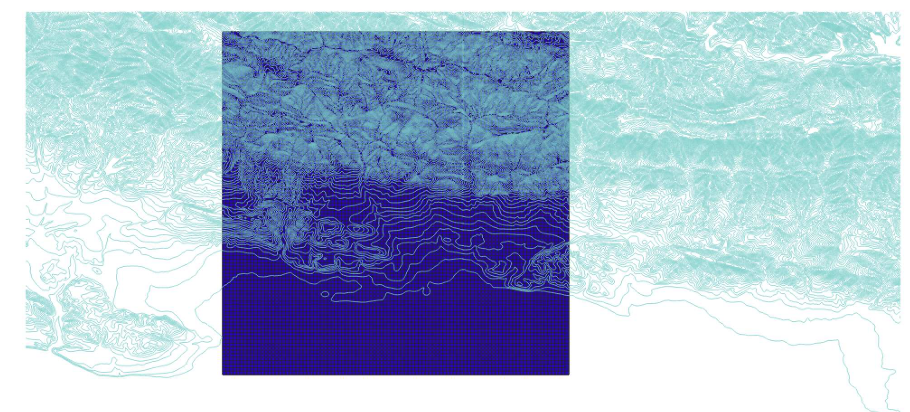
- This function, known as the *variogram, height-height correlation function*, *roughness function*, or *width function*, characterizes the roughness of the surface.
- The variogram is just the second structure function from turbulence.
- Crank-Nickolson/Upwind produces the scaling exponents $1/2$ for h and $3/4$ for H, see B., Smith and Merchant (2001).
- Predictor-Corrector produces the same (viscosity-dependent) scaling exponent for h and H.

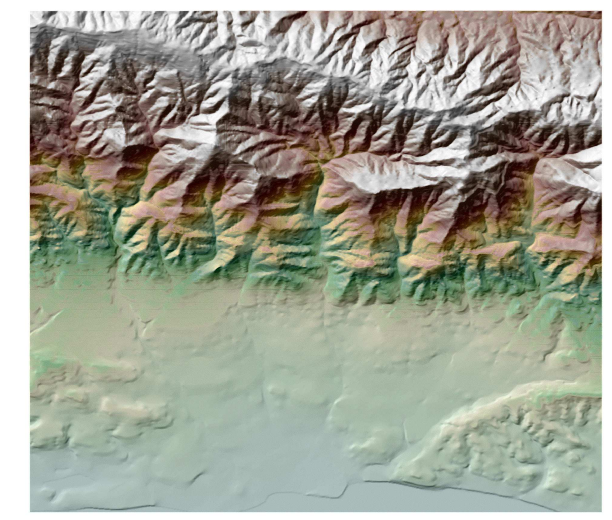
Debris Flows in Montecito, January 2017



The risk area in Montecito

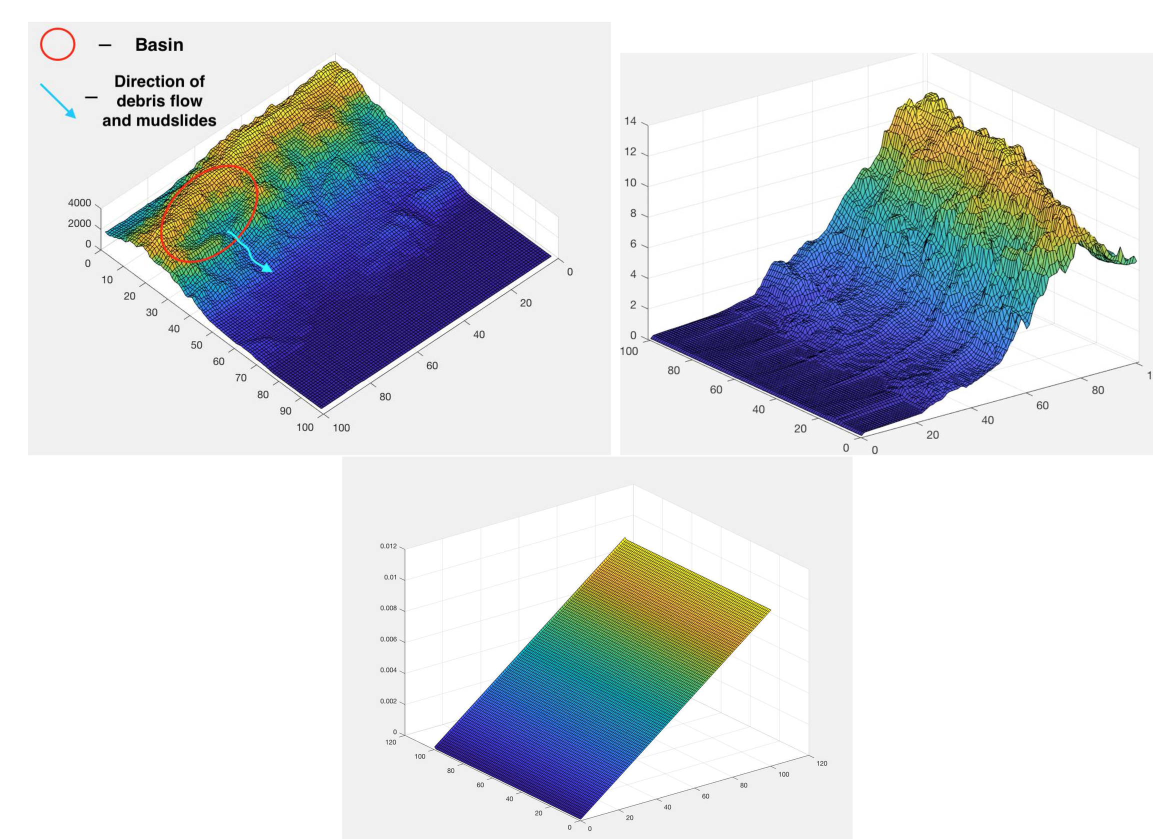
The Digital Elevation Model (DEM)





Area of Montecito/Santa Barbara mountains used for the simulations. The top figure is from the national map, the bottom figure is a DEM gif file.

The DEM of the watershed



(a) Aerial view of a basis and canyon on the Montecito/Santa Barbara mountains. The initial surface (b) and rainfall (c) used for the simulation

The Results: A Debris Flow and a Mudflow

- The model replicated two different types of flows: a debris flow and a mud flow.
- The simulations produce a debris flow on the scale of hours and a mudflow that last for two to three days.
- The debris flow run all the way to the ocean, but much of the mudflow stops below the foothills.
- The contours of the flow follow the risk area on the map to the left.