Math 104A

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HW 3

1. Let F be the set of best approximations to f by elements in W.

F is convex $\iff \forall p^*, q^* \in F \quad (1-\theta)p^* + \theta q^* \in F$
where
$F = \{p^* \in W \mid \|f-p^*\| \le \|f-p\| \ \forall p \in W\}$

let $p^*, q^* \in F$.

$\Rightarrow \|f-p^*\| = \|f-q^*\| = \min_{y\in F} \|f-y\| \qquad y^* = (1-\theta)p^* + \theta q^*$

$= \|(f-p^*)(1-\theta) + (f-q^*)\theta\| \le (1-\theta)\|f-p^*\| + \theta\|f-q^*\|$

$= \min_{y\in F} \|f-y\| \quad \Rightarrow \forall \theta \in (0,1) \ y^* \in F \ \square$

2. best unif. approx of degree at most n

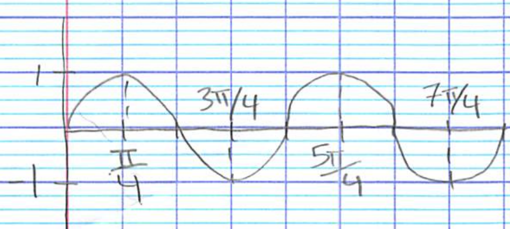
$\Updownarrow$

Chebyshev theorem: $e_n = f - p^*$ equioscillates between $\pm\|e_n\|_\infty$ $n+2$ times

$\Rightarrow e_n = \sin 2x - p^*$

$$\|e_n\|_\infty = \begin{cases} \max_{[a,b]} \sin 2x = 1 \\ \min_{[a,b]} \sin 2x = -1 \end{cases} \qquad n = 2 \Rightarrow 4 \text{ times } \pm\|e_n\|$$

By inspection choose $p^* = 0$.



Alternated 4 times

by Theorem 5, Chebyshev's Theorem, p^* is a best approximation to $f(x) = \sin 2x$ by elements in $W = \{$polys. of degree at most $2\}$ $\square$

3. Since f is continuous on a closed interval & p^* is a constant
let $p^* = \frac{1}{2}(\max_{x\in[a,b]} f(x) + \min_{x\in[a,b]} f(x))$

$f(x_1) = \max_x f(x)$ & $f(x_2) = \min_x f(x)$

then $e(x) = f(x) - p^*$ oscillates between $f(x_1)$ & $f(x_2)$ $0+2$ times (x_1, x_2)

$\|e(x)\|_\infty = |\frac{1}{2}(f(x_1) - f(x_2))| = |f(x) - p^*|$ for $x_1 = x$ or $x_2 = x$

$\Rightarrow p^*$ is best uniform approx. to f of degree 0. ▨

4. $V = \mathbb{R}^3$ w/ $\|f-p\|_\infty = \max\{|f_1 - p_1|, |f_2 - p_2|, |f_3 - p_3|\}$
$f = (3,6,4) = (f_1, f_2, f_3)$ & $p \in W = \text{span}\{(0,1,0), (0,0,1)\}$

$W \subset V \Rightarrow \forall p \in W$ $|f_1 - p_1| = 3$
$\Rightarrow 3 \leq \|f-p\|_\infty \forall p \in W$
$\Rightarrow \|f-p\| \forall p \in W$ s.t. $|f_i - p_i| \leq 3$ for $i = 2$ or 3

For example, $p^*, q^* \in W$ that are best approximations to f are $(0,7,3)$ & $(0,6,4)$

$\|f-p^*\|_\infty = \max\{3, |6-7=-1| = 1, |4-3| = 1\} = 3$

$\|f-q^*\|_\infty = \max\{3, 0, 0\} = 3$

$\therefore$ best approximation is not unique ▨

5. $p(x) = a_0 + a_1 T_1(x) + \cdots + a_n T_n(x)$
where T_j for $j = 1, \dots, n$ are chebyshev polynomials of degree j.

Prove unique representation of $p(x)$ through induction.

let $n = 0$ be the base step:
$p(x) = a_0 = b_0$ $\quad a_0, b_0$ coefficients
$p(x)$ is unique rep.

Assume from all $j = 1, \dots, n$. Show true for $j = n+1$.

$$\underline{a_{n+1} T_{n+1}(x)} + \cdots + a_0 = b_0 + \cdots + \underline{b_{n+1} T_{n+1}(x)}$$

$$a_{n+1} 2^n x^{n+1} + \cdots \qquad = \qquad \cdots + b_{n+1} 2^n x^{n+1}$$

$\Rightarrow a_{n+1} = b_{n+1} \Rightarrow a_{n+1} T_{n+1}(x) = b_{n+1} T_{n+1}(x)$
$\Rightarrow \cancel{a_{n+1} T_{n+1}(x)} + \cdots = \cancel{b_{n+1} T_{n+1}(x)} + \cdots$
True due to assumption

$\Rightarrow p(x)$ is unique by induction on n. $\blacksquare$

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T1 = @(x) x;
T2 = @(x) 2*x^2-1;
T3 = @(x) 4*x^3-3*x;
T6 = @(x) 32*x^6-48*x^4+2*x^2+1;
fplot(T1, [-1 1])
fplot(T2, [-1 1])
fplot(T3, [-1 1])
fplot(T6, [-1 1])
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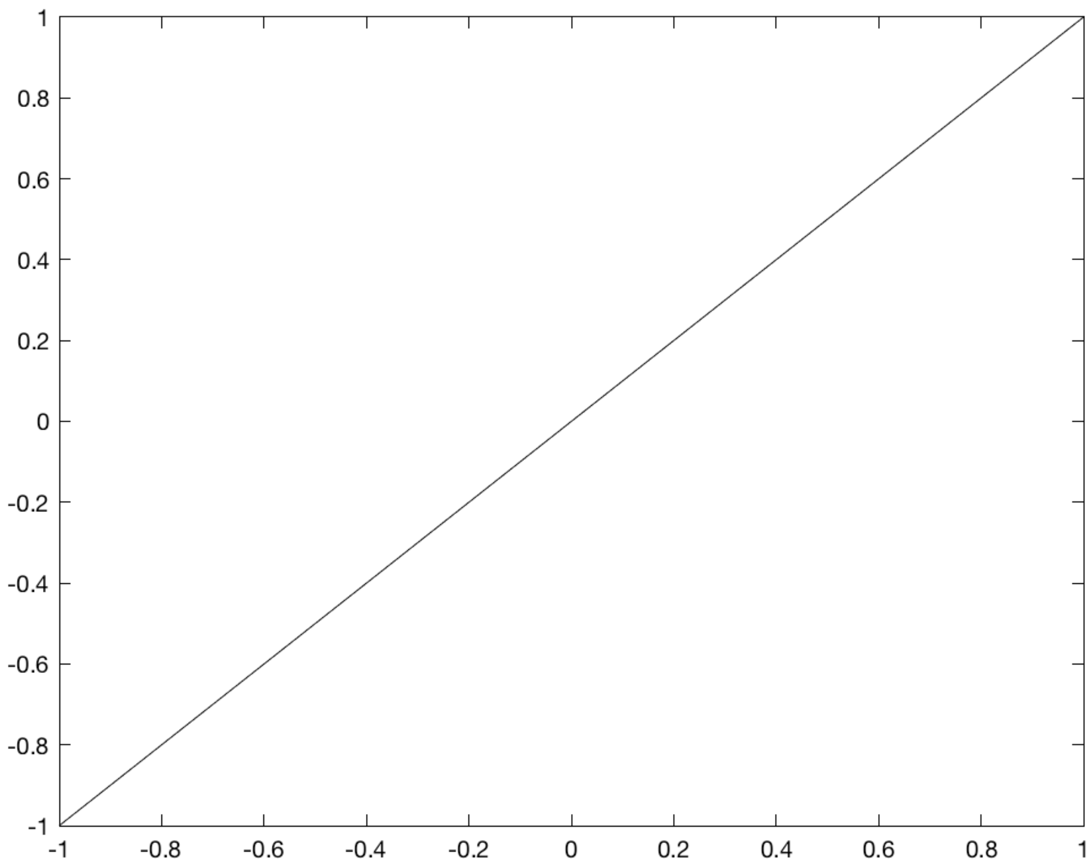
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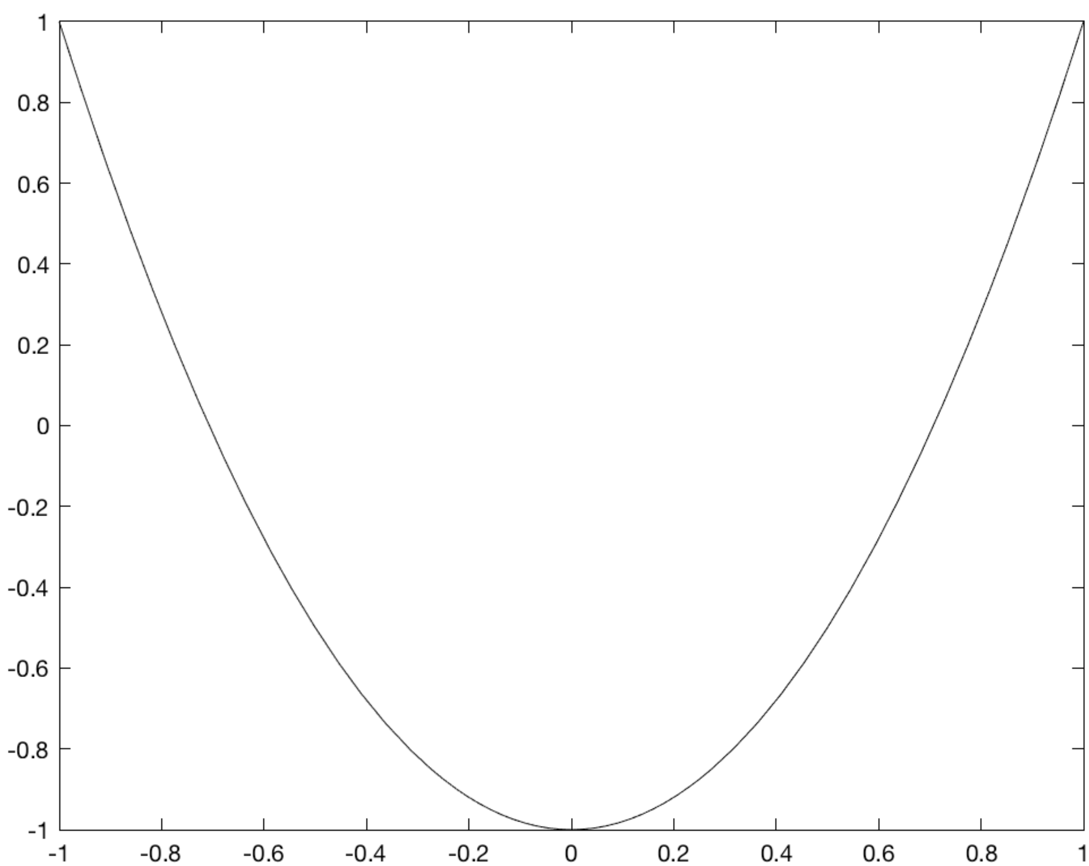
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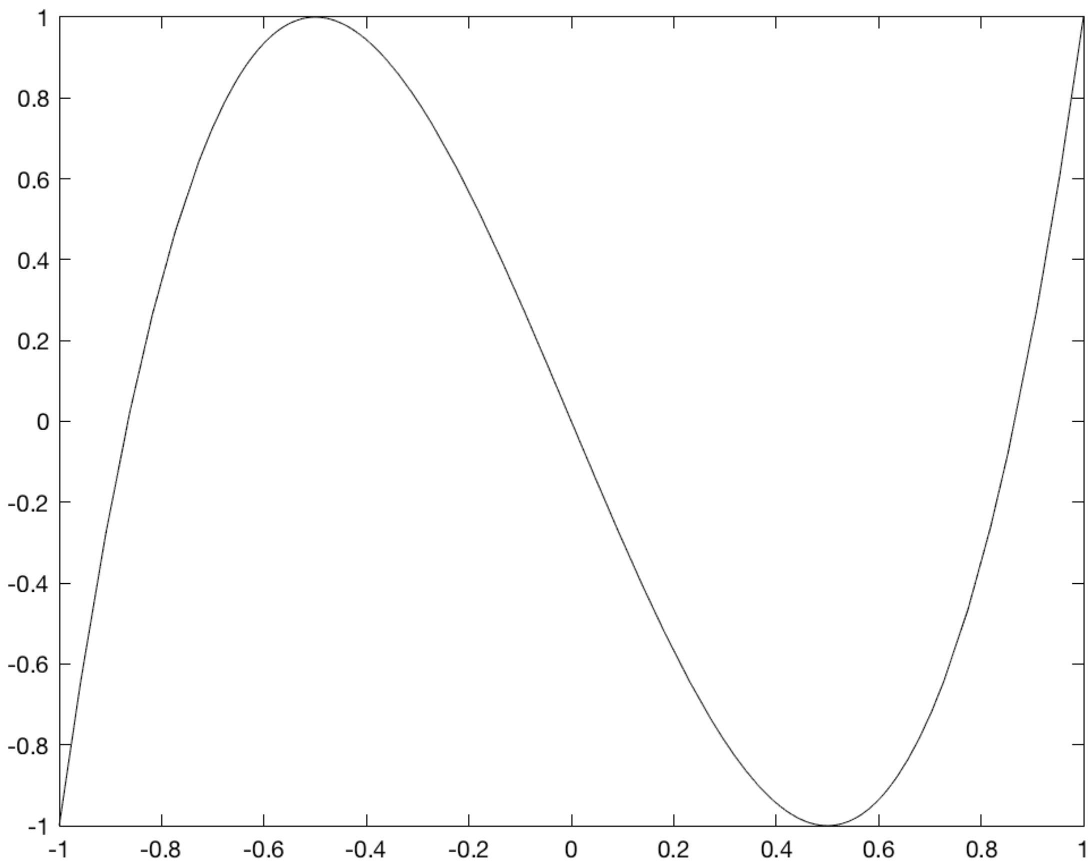
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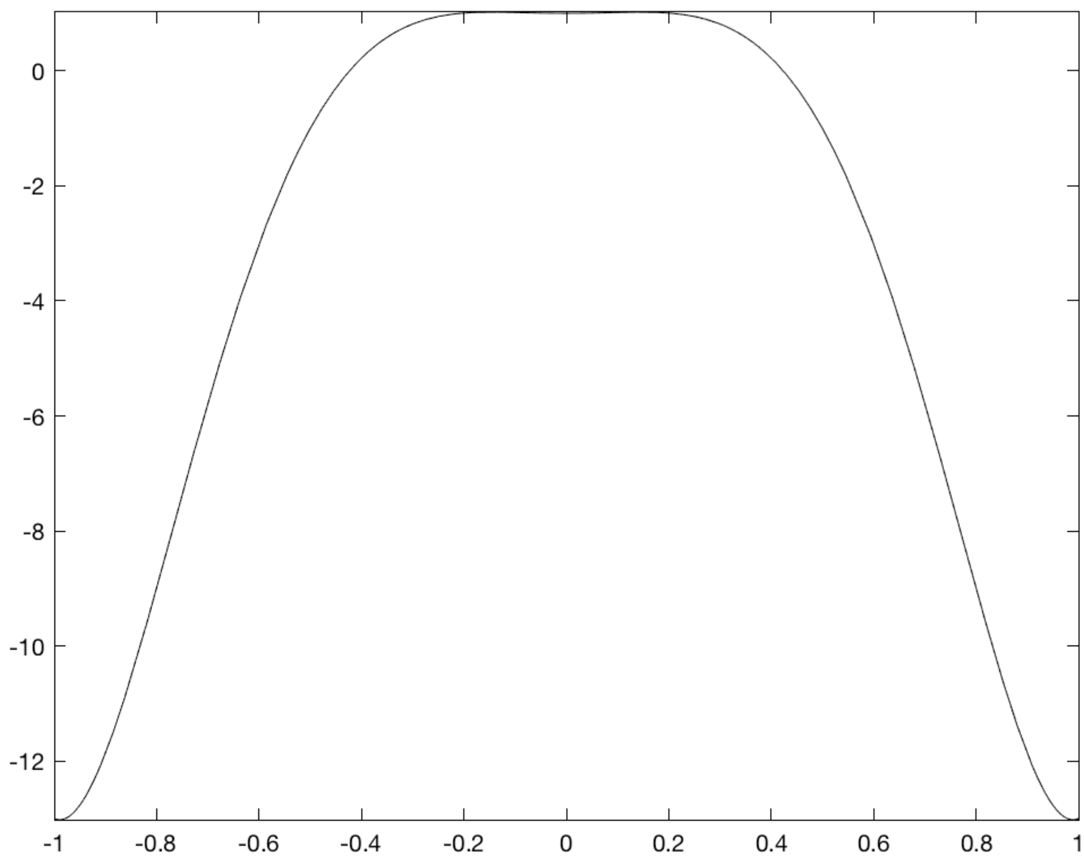
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