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Math 104A HW2

1. Let V be a vector space.
Prove that a norm $\|\cdot\|$ on V defines a continuous function $\|\cdot\|: V \to [0,\infty)$.

Pf:
Want to show that:
$\forall x \in V \ \forall \varepsilon > 0 \ \exists \delta > 0$ s.t.
$\|x-y\| < \delta \Rightarrow \big|\|x\| - \|y\|\big| < \varepsilon$ for some $y \in V$

Assume $x, y \in V \wedge \varepsilon > 0$. Let $\delta = \varepsilon$.

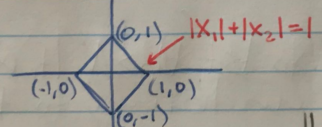
Suppose $\|x-y\| < \delta$, ie. $\|x-y\| < \varepsilon$.

Using the reverse triangle inequality, $(\|x\| - \|y\| \leq \|x-y\|)$
& the fact that the norm is always ≥ 0,
we deduce $\big|\|x\| - \|y\|\big| \leq \|x-y\| < \delta = \varepsilon$.

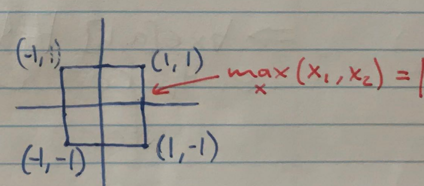
$\therefore \big|\|x\| - \|y\|\big| < \varepsilon$ and the norm $\|\cdot\|: V \to \mathbb{R}_+$ is indeed a continuous function $\square$

2. Let $V = \mathbb{R}^2$. Sketch the unit balls for $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$.

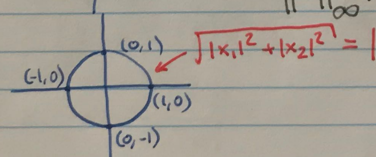
$\|\cdot\|_1$:



$\|\cdot\|_\infty$:



$\|\cdot\|_2$:



3. Let $M_n = \|f_n - f\|_\infty$. Prove that $\{f_n\}$ converges uniformly to f iff $M_n \to 0$ as $n \to \infty$.
Define the sequence of numbers M_n.

Pf:

"$\Longrightarrow$" Assume $\{f_n\}$ converges uniformly to f as $n \to \infty$.
By definition of uniform convergenge $|f_n(x) - f(x)|$ gets arbitrarily small for large enough $\forall x \in [a,b]$.
Since $M_n = \|f_n - f\|_\infty = \max\limits_{x\in[a,b]} \{|f_n(x) - f(x)|\}$,

as $n \to \infty$ M_n decreases & converges to zero as $f_n(x)$ converges to $f(x)$. $\square$

"$\Longleftarrow$" Assume $M_n = \|f_n - f\|_\infty$ converges to zero as $n \to \infty$.

$\Rightarrow M_n = \|f_n(x) - f\|_\infty = \max\limits_{x\in[a,b]} \{|f_n(x) - f(x)|\} \to 0$ as $n \to \infty$

$\Rightarrow \forall x \in [a,b]$ $\forall \varepsilon > 0$ $\exists N > 0$ s.t. $\forall n \geq N$ $|f_n(x) - f(x)| < \varepsilon$
i.e. f_n converges uniformly to f as $n \to \infty$ $\square$

4. ⓐ Prove that $f_n(x) = \left(\frac{n-1}{n}\right)x^2 + \frac{1}{n}x$, $0 \le x \le 1$ converges uniformly to $f(x) = x^2$ in $[0,1]$
i.e. $\forall x \in [0,1]$ $\forall \varepsilon > 0$ $\exists N > 0$ s.t. $\forall n \ge N$ $|f_n(x) - f(x)| < \varepsilon$.

Pf:

Since $f(x) = x^2$ is continuous on $[0,1]$ & $f_n(x) = B_n f(x)$ (the Bernstein polynomial) we will prove this via The Weierstrass Approximation Theorem.

Note: $f_n(x) = \left(\frac{n-1}{n}\right)x^2 + \frac{x}{n} = \sum_{k=0}^{n} \left(\frac{k}{n}\right)^2 \binom{n}{k} x^k (1-x)^{n-k} = B_n f(x)$

We know that, because f is continuous, $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $|f(x) - f(\frac{k}{n})| < \frac{\varepsilon}{2}$ if $|x - \frac{k}{n}| < \delta$. (1)

Also, $|f(x) - f(\frac{k}{n})| \le 2\|f\|_\infty$ $\forall x \in [0,1]$, $k = 0, \ldots, n$.

$$f(x) - B_n f(x) = \sum_{k=0}^{n} \left[f(x) - f(\tfrac{k}{n})\right] \binom{n}{k} x^k (1-x)^{n-k}$$

$$= \sum_{|k/n - x| < \delta} \left[f(x) - f(\tfrac{k}{n})\right] \binom{n}{k} x^k (1-x)^{n-k} + \sum_{|k/n - x| \ge \delta} \left[f(x) - f(\tfrac{k}{n})\right] \binom{n}{k} x^k (1-x)^{n-k}$$

(2) $$\le \frac{\varepsilon}{2} + \sum_{|k/n - x| \ge \delta} \left[f(x) - f(\tfrac{k}{n})\right] \binom{n}{k} x^k (1-x)^{n-k} \quad \text{(b/c of (1))}$$

Consider: $$\sum_{|k/n - x| \ge \delta} \left|f(x) - f(\tfrac{k}{n})\right| \binom{n}{k} x^k (1-x)^{n-k} \le 2\|f\|_\infty \sum \binom{n}{k} x^k (1-x)^{n-k}$$

$$\le \frac{2\|f\|_\infty}{\delta^2} \sum_{|k/n - x| \ge \delta} \left(\frac{k}{n} - x\right)^2 \binom{n}{k} x^k (1-x)^{n-k} \le \frac{2\|f\|_\infty}{\delta^2} \sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 \binom{n}{k} x^k (1-x)^{n-k}$$

$$= \frac{2\|f\|_\infty}{n\delta^2} x(1-x) \le \frac{\|f\|_\infty}{2n\delta^2}.$$

$\Rightarrow \exists N > 0$ s.t. $\forall n \ge N$ $\frac{\|f\|_\infty}{2n\delta^2} \le \frac{\varepsilon}{2}$.

(2) $\Rightarrow f(x) - B_n f(x) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\therefore$ $\forall x \in [0,1]$ $\forall \varepsilon > 0$ $\exists N > 0$ s.t. $\forall n \ge N$ $|f_n(x) - f(x)| = |f(x) - B_n f(x)| \le \varepsilon$ $\square$

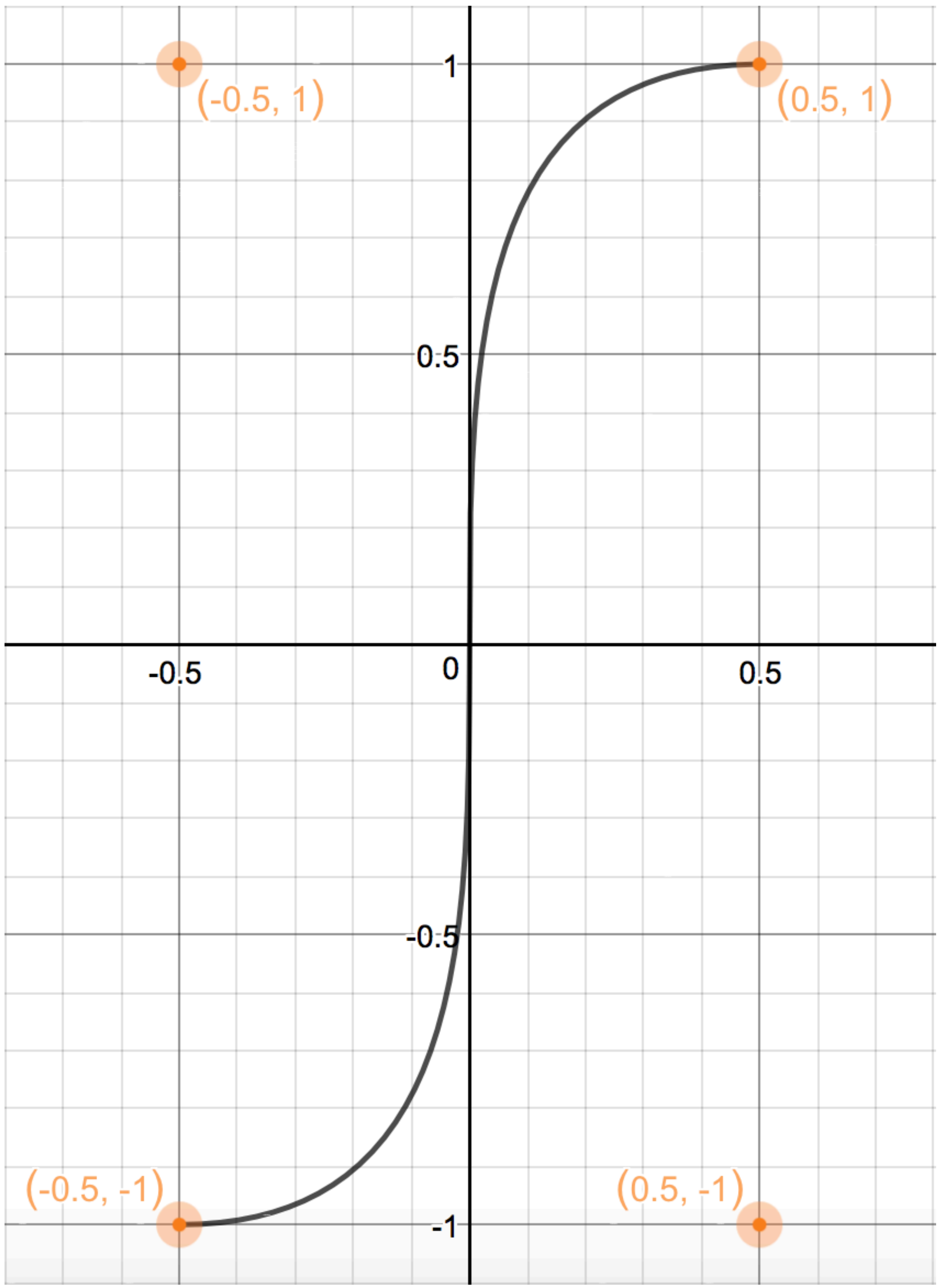
4. ⓑ The sequence $f_n(x) = x^n$ converge uniformly?

No because, depending on the value of n, $f_n(x) \to +\infty$ or $-\infty$ as $x \to -\infty$.

```
import math
import random
import numpy as np

# n choose m
def nCk(n,k):
f = math.factorial
return f(n) / f(k) / f(n-k)

#different functions for [0, 1/2)U[1/2, 1] (problem 5)
def f1(x):
return x
def f2(x):
return (1-x)

#Bernstein Polynomials and rate of convergence (problem 5)
def B(f,x,n):
l = 0.
for k in range(1,n) :
l += f(k/float(n))*nCk(n,k)*(x**k)*((1-x)**(n-k)) #Equation and Reimann sum
if k==4: #using 4 and 5 as arbitrary indeces to check rate of convergence h/g, but can use others
g = abs(f(x)-l)
if k==5: #
h = abs(f(x)-l)
return l, h/g #the value of the Bernstein approx. and the rate of convergence

#testing
for i in range(5):
x = random.uniform(0.,1.)
n = 60
if x>.5:
print 'x is', x
print 'f(x)=1-x :', f2(x)
print 'B poly is', B(f2,x,n)
print ('---------')
else:
print 'x is', x
print 'f(x)=x :', f1(x)
print 'B poly is', B(f1,x,n)
print ('---------')

#Bezier Points for Integral Sign (problem #6)
P0 = [-.5,-1]
P1 = [.5,-1]
P2 = [-.5,1]
P3 = [.5,1]

# creating the Cubic Bezier Curve (problem #6)
def BezierCurve(P0,P1,P2,P3):
t = np.linspace(0,1) #t between 0 and 1
x = (1-t)**3*P0[0]+3*(1-t)**2*t*P1[0]+3*(1-t)*t**2*P2[0]+t**3*P3[0] #x-coordinate
```

```python
y = (1-t)**3*P0[1]+3*(1-t)**2*t*P1[1]+3*(1-t)*t**2*P2[1]+t**3*P3[1] #y-coordinate
return x, y
```

```
Python 2.7.10 (default, Jul 14 2015, 19:46:27)
[GCC 4.8.2] on linux
>
x is 0.363316064404
f(x)=x : 0.363316064404
B poly is (0.3633160644039279, 0.999999869658076)
---------
x is 0.653530570456
f(x)=1-x : 0.346469429544
B poly is (0.3464694295444005, 1.0)
---------
x is 0.976761060822
f(x)=1-x : 0.0232389391784
B poly is (0.023238939178427027, 1.0)
---------
x is 0.776484077727
f(x)=1-x : 0.223515922273
B poly is (0.2235159222730429, 1.0)
---------
x is 0.20808671269
f(x)=x : 0.20808671269
B poly is (0.20808671268971568, 0.9977159281089483)
---------
>
```



1
0.5
-0.5
-1
-0.5
0
0.5
(-0.5, 1)
(0.5, 1)
(-0.5, -1)
(0.5, -1)