QMA AND THE POWER OF "POSITIVITY"

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joint with Roozbeh Bassirian, Bill Fefferman (arXiv)

see also Jeronimo and Wu (STOC '23)

OUTLINE

1. An introduction to QMA^+
2. Proof outline

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1. **An introduction to QMA^+**
2. Proof outline

CONTEXT: QMA AND $\mathrm{QMA}(2)$

What is the power of *unentangled* proofs?

REVIEW: $\mathrm{QMA}_{c,s}$

Set of decision problems such that:

- If YES (completeness): $\exists\, \mathrm{poly}(n)$-qubit $|\psi\rangle$ input to BQP machine, accepts w.p. $\geq c(n)$.
- If NO (soundness): $\forall\, \mathrm{poly}(n)$-qubit $|\psi\rangle$ input to BQP machine, accepts w.p. $\leq s(n)$.

REVIEW: $\mathrm{QMA}(2)_{c,s}$

Set of decision problems such that:

- If YES (completeness): $\exists\ \mathrm{poly}(n)$-qubit $|\psi_1\rangle \otimes |\psi_2\rangle$ input to BQP machine, accepts w.p. $\geq c(n)$.
- If NO (soundness): $\forall\ \mathrm{poly}(n)$-qubit $|\psi_1\rangle \otimes |\psi_2\rangle$ input to BQP machine, accepts w.p. $\leq s(n)$.

FACTS ABOUT QMA, $\mathrm{QMA}(2)$

Gap amplification (as long as $c(n) - s(n) \geq \frac{1}{\mathrm{poly}(n)}$):

- QMA: via parallel repetition
- $\mathrm{QMA}(2)$: using the *product test* [HM10]

Upper bounds (better than NEXP):

- $\mathrm{QMA} \subseteq \mathrm{PSPACE}$ using semidefinite programming (in fact, $\subseteq \mathrm{PP}$ by Kitaev and Watrous)
- Only $\mathrm{QMA}(2) \subseteq \mathrm{NEXP}$. *Why can't we do better?*

[JW23]: $\mathrm{QMA}^+_{c,s}$

Set of decision problems such that:

- If YES (completeness): $\exists \, \mathrm{poly}(n)$-qubit $|\psi\rangle$ **with non-negative amplitudes** input to BQP machine, accepts w.p. $\geq c(n)$.
- If NO (soundness): $\forall \, \mathrm{poly}(n)$-qubit $|\psi\rangle$ **with non-negative amplitudes** input to BQP machine, accepts w.p. $\leq s(n)$.

[JW23]: $\mathrm{QMA}^+(2)_{c,s}$

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HAVEN'T I SEEN THIS BEFORE?

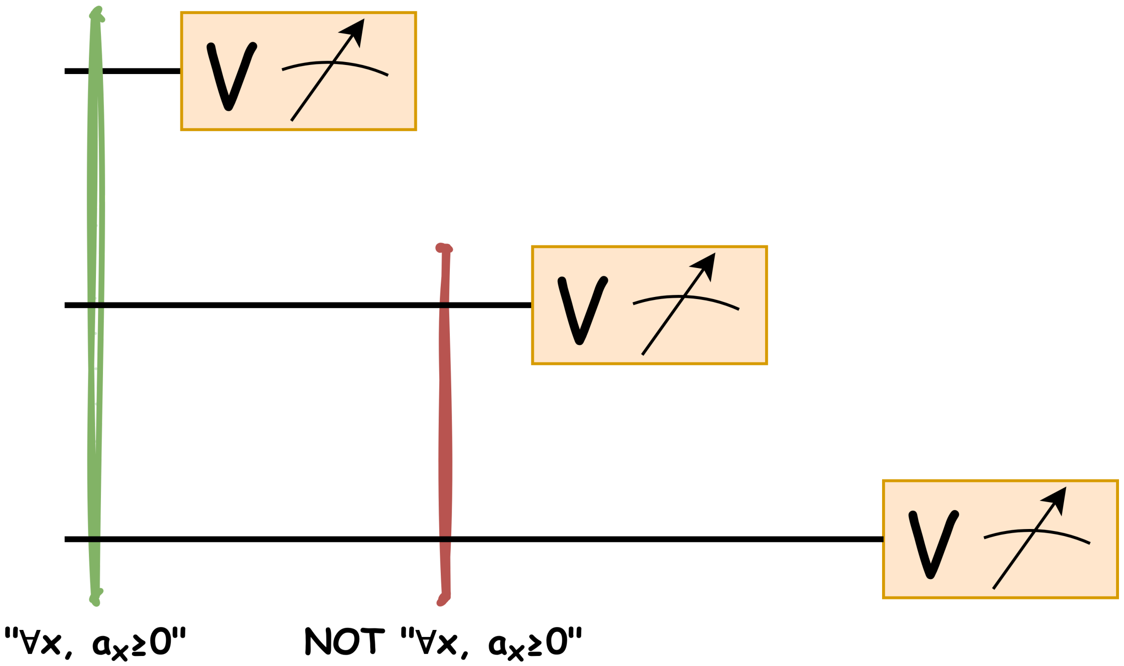
[GKS14] show that $\mathrm{SQMA} = \mathrm{QMA}$,
where we restrict to *subset states* **only in completeness**.

Lesson: Promise-symmetric restrictions can increase power, since they *restrict* Merlin's cheating in soundness.

"FACTS" ABOUT $\mathrm{QMA}^+, \mathrm{QMA}^+(2)$

Gap amplification:
Parallel repetition **fails**.

Reason: Partial measurements can reintroduce complex phases into remaining state.



“FACTS” ABOUT $\mathrm{QMA}^+, \mathrm{QMA}^+(2)$

Upper bounds (better than NEXP):

Using a semidefinite program **fails**.

Reason: Copositive programming is hard!

Optimizing $\max_{x \geq 0} x^\dagger A x$ can compute independence numbers of graphs, etc.

HOW POWERFUL IS THE $(^+)$?

Every state $|\psi\rangle$ has $\frac{1}{4}$ overlap with some state with non-negative amplitudes.

$\implies \exists$ constants $1 > c' > s' > 0$ s.t. $\mathrm{QMA}^+_{c',s'} = \mathrm{QMA}$ and $\mathrm{QMA}^+(2)_{c',s'} = \mathrm{QMA}(2)$.

[JW23]: ON $\mathrm{QMA}^+(2)$

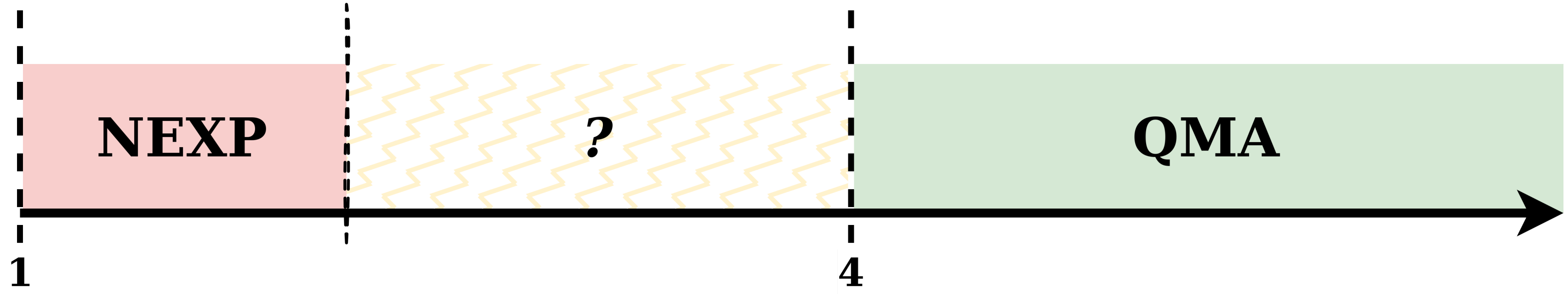
$\exists$ *other* constants $1 > c > s > 0$, s.t.
$\mathrm{QMA}^+(2)_{c,s} = \mathrm{NEXP}$.

New way to understand $\mathrm{QMA}(2)$:
$\mathrm{QMA}^+(2)$ gap amplification $\implies$ $\mathrm{QMA}(2) = \mathrm{NEXP}$!

OUR WORK: ON QMA^{+}

$\exists$ *other* constants $1 > c > s > 0$, s.t.

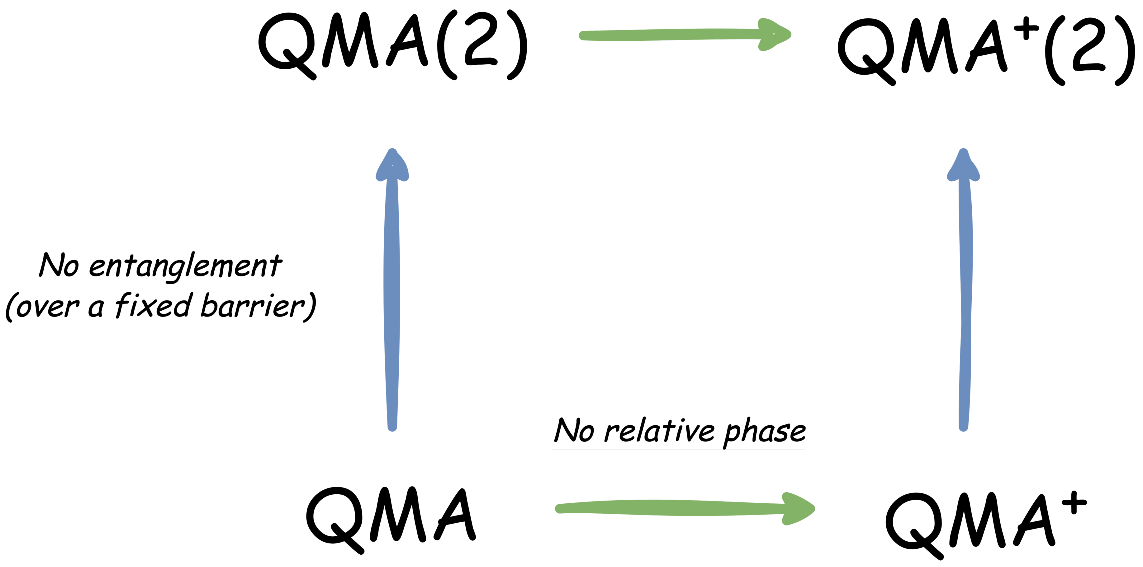
$$\mathrm{QMA}^{+}_{c,s} = \mathrm{NEXP}!$$



($\frac{c}{s}$ interpolates from QMA to NEXP)

NO gap amplification of QMA^{+}!

INTERPRETING THESE RESULTS



"Perhaps the power lies in the $^+$*, not the* (2)*..."*

Any technique to amplify the promise gap in $\mathrm{QMA}^+(2)$ must **fail** for QMA^+.

OUTLINE

1. An introduction to QMA^+
2. **Proof outline**

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1. An introduction to QMA^+
2. Proof outline:
    - $\mathrm{NP} \subseteq \mathrm{QMA}^+$ **with $O(\log n)$-qubit proof**
    - Scaling up to $\mathrm{NEXP} \subseteq \mathrm{QMA}^+$

CHOOSING A NP-HARD PROBLEM

Input: CSP instance (n variables, bounded alphabet Σ, q-uniform constraints $\{\mathcal{C}_1, \ldots, \mathcal{C}_R\}$)

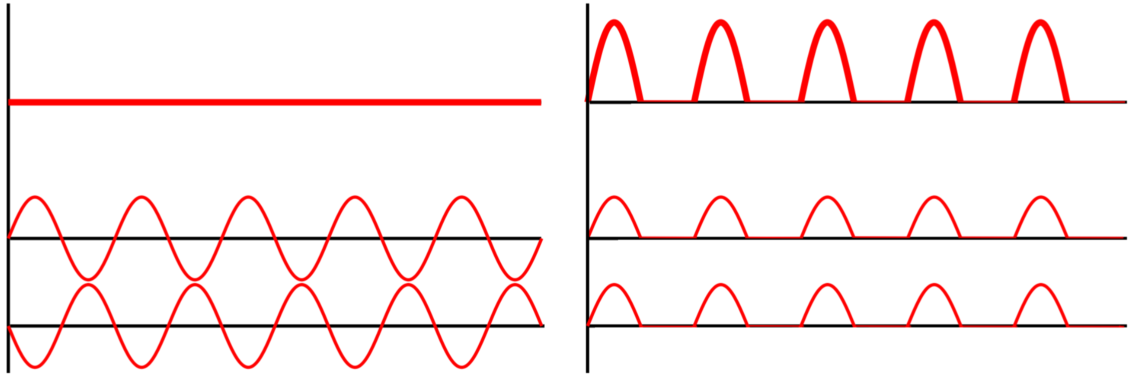
Output: Is instance fully satisfiable (for some $x \in \Sigma^n$)?

PCP Theorem: NP-hard for $q = O(1)$, with completeness $c = 1$ and soundness $s = \frac{1}{2}$ (i.e. *constant* gap).

Plan: Solve this in QMA^+ with $O(\log n)$-sized proof.

AMPLITUDE AND INTERFERENCE

edited from Wikipedia



Intuition: **non-negative amplitudes** “take the interference” out of $|\psi\rangle$ (they don’t “cancel out”!)

THE POWER OF PLUS ($^+$)

Goal: *require* Merlin to send a certain type of state.

Big idea of [JW23]: $\langle\psi|+\rangle \propto \||\psi\rangle\|_1$.

($\Pi_+ := |+\rangle\langle+|$ accepts $|\psi\rangle$ according to its ℓ_1 norm)

This is the **only use** of the ($^+$) assumption in both papers.

[JW23]'S USE OF $(^+)$

1. Notice that $\mathrm{QMA}^+(2) = \mathrm{QMA}^+(k)$ by [HM10] (so we can assume many copies of $|\psi\rangle$).
2. Project $\Pi_+|\psi\rangle$ on each copy; count the fraction that accept. (This *estimates* the ℓ_1 norm of $|\psi\rangle$)

$\longrightarrow$ check closeness to states of a *target* ℓ_1 norm.

OUR USE OF $(^{+})$

Consider two registers:
question ($\log n$ qubits) and *answer* ($O(1)$ qubits)

We choose one of two tests:

1. $\Pi_+ \otimes \Pi_+$: (dense as possible)
2. $\mathbb{I} \otimes (\mathbb{I} - \Pi_+)$

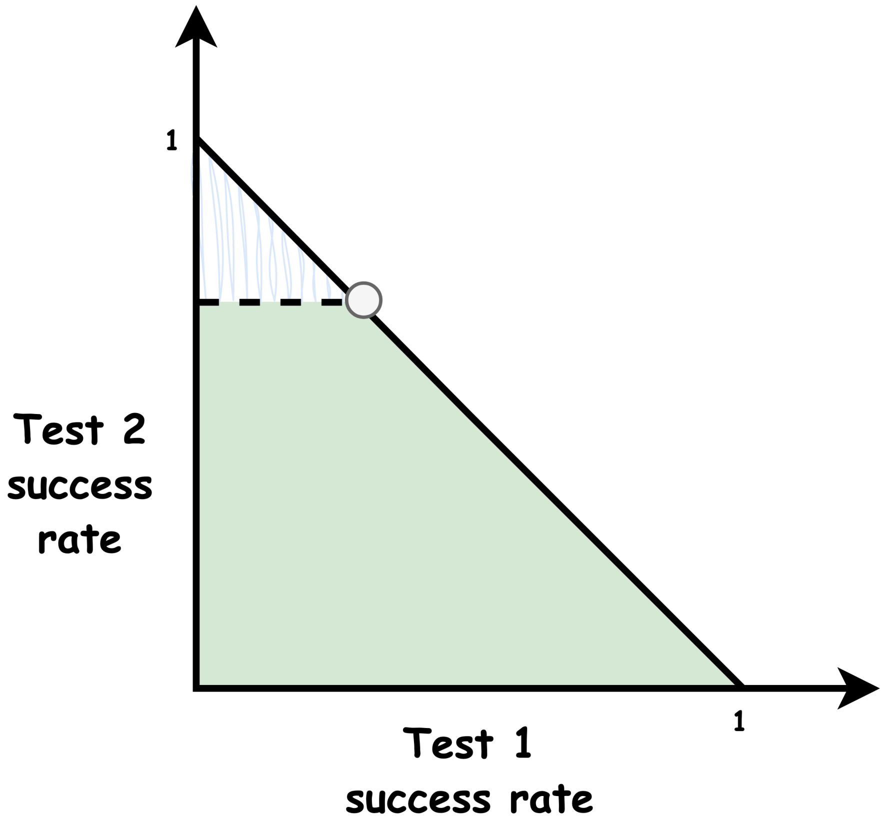
States with **non-negative amplitudes** can't perfectly pass Test 2. The best have *one answer per question*.

$\longrightarrow$ check closeness to states of a *rigid* form:

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} |j\rangle |f(j)\rangle$$

OUR USE OF ($^+$)

concept: Bryan O'Gorman



- Tests are orthogonal, so sum of success rates ≤ 1.
- Non-negative amplitude states live in green region.
- *Rigid* states at the circled point.

HOW CAN WE USE THIS POWER?

questions $\to$ constraints

answers $\to$ assignments of associated variables

In completeness, $\exists$ satisfying assignment $f : [R] \to \Sigma^q$ and proof $|\psi\rangle := \frac{1}{\sqrt{R}} \sum_{j=1}^{R} |j\rangle |f(j)\rangle$.

With some probability, we test for this *rigid* form. Otherwise, we check the *constraints* $\{\mathcal{C}_j\}$.

TESTING THE CONSTRAINTS

For *rigid* states, checking the constraints is easy: *measure in computational basis, and test* $\mathcal{C}_j(f(j))$.

But Merlin can cheat: sending different values for the same variable depending on the constraint.

So (with some probability), need to check for *consistency*.

CHECKING "CONSISTENCY"

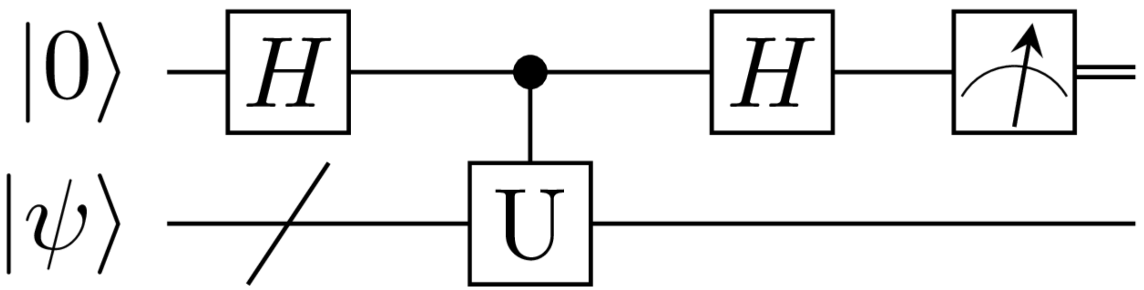
Goal: Construct unitaries $\{U_1, \ldots, U_d\}$ such that:

- Honest: $|\psi\rangle = U_k|\psi\rangle$ for all k
- Cheating: $|\psi\rangle$ "far" from $U_k|\psi\rangle$ for some k

Then, the consistency protocol would:

1. Choose uniform $k \in \{1, \ldots, d\}$
2. Run "Hadamard test" on $(|\psi\rangle, U_k|\psi\rangle)$

source: Victory Omole

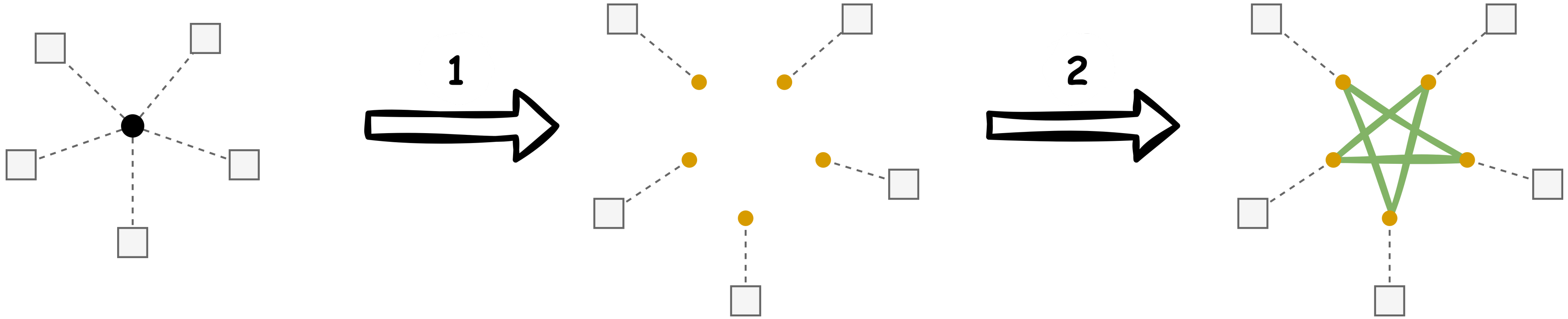


THE "CONSISTENCY" UNITARIES

We build the graph G using a step from [Dinur07]:

1. Turn each variable $i \in [n]$ into a *cluster* of vertices.
   (One vertex ● per constraint ■ involving i.)
2. Add a d-regular *expander* graph ✫ within each cluster.

Illustration for one variable ● and $d = 2$:



G is the *union* of all expander graphs ($R \cdot q$ vertices).

THE “CONSISTENCY” UNITARIES

Recall that a *rigid* state is of the form

$$|\psi\rangle := \frac{1}{\sqrt{R}} \sum_{j=1}^{R} |j\rangle|f(j)\rangle.$$

It is convenient to separate the variable assignments:

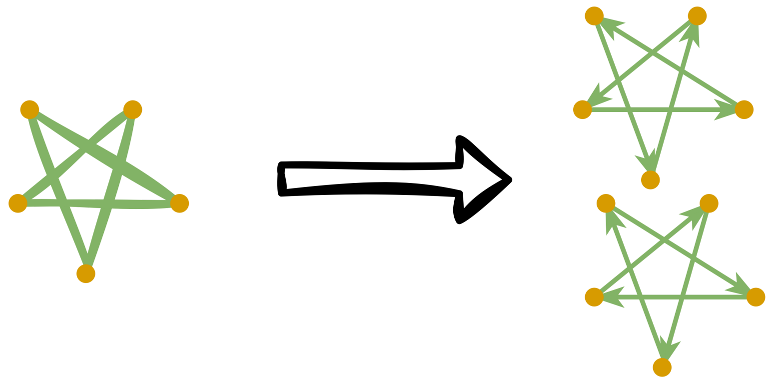
$$|\phi\rangle := \frac{1}{\sqrt{R \cdot q}} \sum_{j=1}^{R} \sum_{\iota=1}^{q} |j, \iota\rangle|f(j)[\iota]\rangle.$$

(Then all vertices of G are in superposition!)

The map $|\psi\rangle \mapsto |\phi\rangle$ is efficient; see paper for details.

THE “CONSISTENCY” UNITARIES

We can always decompose a d-regular graph into d permutations $\{\pi_1, \ldots, \pi_d\}$.



For $k \in [d]$, let $U_k : |j, \iota\rangle|value\rangle \mapsto |\pi_k(j, \iota)\rangle|value\rangle$ for each constraint $j \in [R]$ and variable index $\iota \in [q]$.

Honest assignment $\implies$ $|\phi\rangle = U_k|\phi\rangle$ for all k!

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A NEXP-HARD PROBLEM

Input: *Succinct* CSP instance ($N := 2^n$ variables, bounded alphabet Σ, q-uniform constraints $\{\mathcal{C}_1, \ldots, \mathcal{C}_R\}$)

Output: Is instance fully satisfiable (for some $x \in \Sigma^{2^n}$)?

PCP Theorem: NEXP-hard for $q = O(1)$, completeness $c = 1$ and soundness $s = \frac{1}{2}$ (i.e. *constant* gap).

Plan: Solve this in QMA^+ with $O(\mathrm{poly}(n))$-sized proof.

THE NP PROTOCOL *ALMOST* WORKS

Arthur’s protocol needs to be *efficient*.

Only issue: how to do “consistency” test?

- expanders are exponentially large
- could be exponentially many expanders
- adjacency lists could be exponentially large

SOLUTIONS ([JW23])

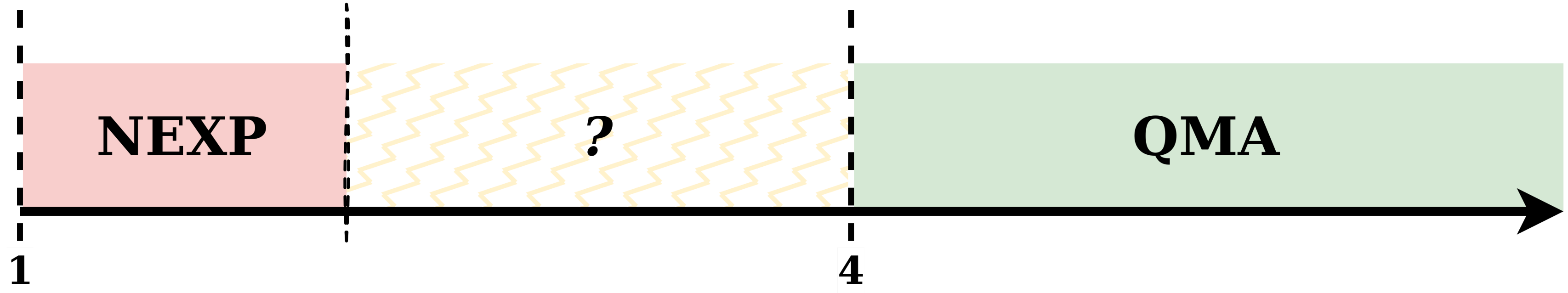
- expanders are exponentially large
  Doubly explicit expander constructions, to build the expanders and decompose them in $\mathrm{polylog}(N)$ time.
- could be exponentially many expanders
  Use a $O(1)$-*strongly uniform* PCP.
- adjacency lists could be exponentially large
  Use a $\mathrm{polylog}(N)$-*doubly explicit* PCP.

Everything else works after making these adjustments! 😎

CONCLUSIONS

Promise-symmetric classes can be very powerful.

But the promise gap *really matters*: $\mathrm{QMA}^{+}_{c,s}$ interpolates from QMA to NEXP for **constant** c, s.



(x-axis: $\frac{c}{s}$)

Is there a phase transition in constants?

CONCLUSIONS

Suppose you use $\mathrm{QMA}^+(2)$ to study $\mathrm{QMA}(2)$…

Any technique to amplify the promise gap in $\mathrm{QMA}^+(2)$ must **fail** for QMA^+.

What does $\mathrm{QMA}^+(2)$ have that QMA^+ doesn't have?

THANK YOU

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