Ch.4 Electric fields in Matters

4.1 Polarization
4.2 The Field of a Polarized Object
4.3 The Electric Displacement
4.4 Linear Dielectrics

4.1 Polarization

4.1.1 Dielectrics
4.1.2 Induced Dipoles
4.1.3 Alignment of Polar Molecules
4.1.4 Polarization

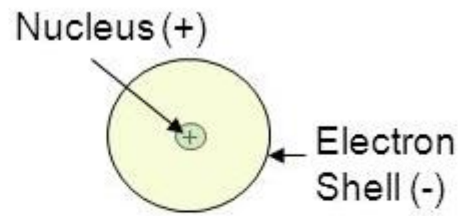
4.1.1 Dielectrics

Conductors: unlimited supply of charges free to move

Insulator (Dielectrics): Charges are attached to specific atoms or molecules
moves only a bit within atom or molecules

4.1.2 Induced Dipoles

No electric field



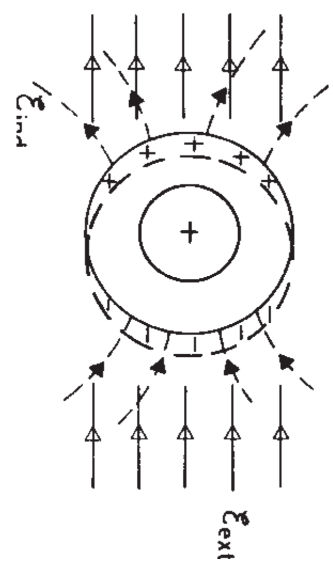
A neutral atom in a external electric field

Atom is polarized with dipole moment $P = \alpha E_{ext}$

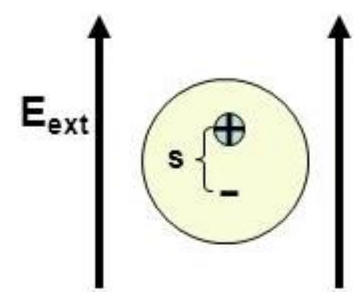
α is the atomic polarizability

H	He	Li	Be	C	Ne	Na	Ar	K	Cs
0.667	0.205	24.3	5.60	1.67	0.396	24.1	1.64	43.4	59.4

TABLE 4.1 Atomic Polarizabilities ($\alpha/4\pi\epsilon_0$, in units of 10^{-30} m^3).

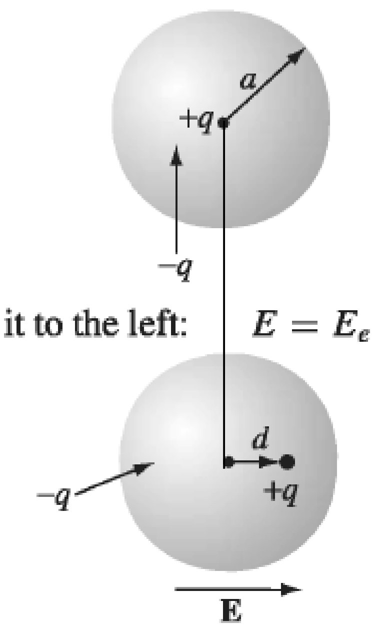


Induced Dipole



$$V_d = \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2} = \frac{\alpha\mathbf{E}_{ext}\cdot\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2}$$

Example 4.1. A primitive model for an atom consists of a point nucleus ($+q$) surrounded by a uniformly charged spherical cloud ($-q$) of radius a (Fig. 4.1). Calculate the atomic polarizability of such an atom.



In the presence of an external field $\mathbf{E}$, the nucleus will be shifted slightly to the right, the electron cloud to the left

the external field pushing the nucleus to the right exactly balances the internal field pulling it to the left: $E = E_e$

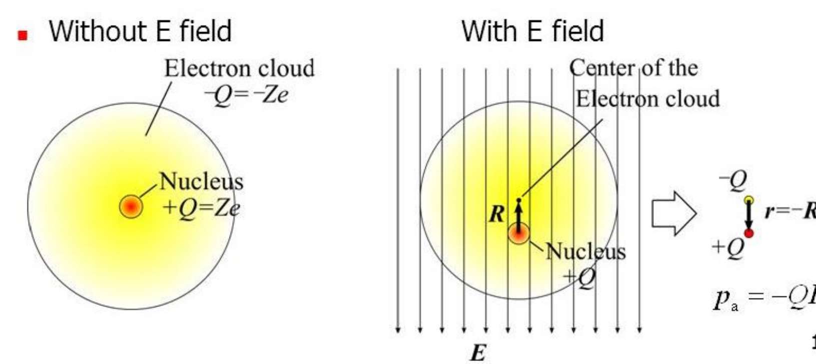
E_e is the field produced by the electron cloud $$E_e = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3}$$

At equilibrium, $$E = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3},$$

$$p = qd = (4\pi\epsilon_0 a^3)E$$

atomic polarizability $\alpha = 4\pi\epsilon_0 a^3 = 3\epsilon_0 v,$

v is the volume of the atom.

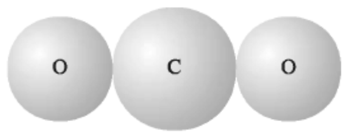


For molecules Carbon dioxide

$$\mathbf{p} = \alpha_{\perp}\mathbf{E}_{\perp} + \alpha_{\parallel}\mathbf{E}_{\parallel}$$

$\alpha_{\parallel}$ $4.5 \times 10^{-40}\ \mathrm{C^2{\cdot}m/N}$

$\alpha_{\perp}$ 2×10^{-40}



In general

$$\begin{aligned} p_x &= \alpha_{xx}E_x + \alpha_{xy}E_y + \alpha_{xz}E_z \\ p_y &= \alpha_{yx}E_x + \alpha_{yy}E_y + \alpha_{yz}E_z \\ p_z &= \alpha_{zx}E_x + \alpha_{zy}E_y + \alpha_{zz}E_z \end{aligned}$$

polarizability tensor α_{ij}

Problem 4.1 A hydrogen atom (with the Bohr radius of half an angstrom) is situated between two metal plates 1 mm apart, which are connected to opposite terminals of a 500 V battery. What fraction of the atomic radius does the separation distance d amount to, roughly? Estimate the voltage you would need with this apparatus to ionize the atom. [Use the value of α in Table 4.1. *Moral:* The displacements we're talking about are *minute*, even on an atomic scale.]

$E = V/x = 500/10^{-3} = 5\times10^{5}$. Table 4.1: $\alpha/4\pi\epsilon_0 = 0.66\times10^{-30}$, so $\alpha = 4\pi(8.85\times10^{-12})(0.66\times10^{-30}) = 7.34\times10^{-41}$. $p = \alpha E = ed \Rightarrow d = \alpha E/e = (7.34\times10^{-41})(5\times10^{5})/(1.6\times10^{-19}) = 2.29\times10^{-16}$ m. $d/R = (2.29\times10^{-16})/(0.5\times10^{-10}) = \boxed{4.6\times10^{-6}.}$ To ionize, say $d = R$. Then $R = \alpha E/e = \alpha V/ex \Rightarrow V = Rex/\alpha = (0.5\times10^{-10})(1.6\times10^{-19})(10^{-3})/(7.34\times10^{-41}) = \boxed{10^{8}\ \mathrm{V}.}$

Problem 4.2 According to quantum mechanics, the electron cloud for a hydrogen atom in the ground state has a charge density $\rho(r) = \dfrac{q}{\pi a^3} e^{-2r/a}$

where q is the charge of the electron and a is the Bohr radius. Find the atomic polarizability of such an atom. [*Hint:* First calculate the electric field of the electron cloud, $E_e(r)$; then expand the exponential, assuming $r \ll a$.[1]

First find the field, at radius r, using Gauss' law: $\int \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}$, or $E = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} Q_{\text{enc}}$.

$$Q_{\text{enc}} = \int_0^r \rho \, d\tau = \frac{4\pi q}{\pi a^3} \int_0^r e^{-2\bar{r}/a} \bar{r}^2 \, d\bar{r} = \frac{4q}{a^3} \left[-\frac{a}{2} e^{-2\bar{r}/a} \left(\bar{r}^2 + a\bar{r} + \frac{a^2}{2} \right) \right] \Bigg|_0^r$$

$$= -\frac{2q}{a^2} \left[e^{-2r/a} \left(r^2 + ar + \frac{a^2}{2} \right) - \frac{a^2}{2} \right] = q \left[1 - e^{-2r/a} \left(1 + 2\frac{r}{a} + 2\frac{r^2}{a^2} \right) \right] \qquad [\textit{Note: } Q_{\text{enc}}(r \to \infty) = q.]$$

the field of the electron cloud is $\quad E_e = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \left[1 - e^{-2r/a} \left(1 + 2\frac{r}{a} + 2\frac{r^2}{a^2} \right) \right]$

proton will be shifted from $r = 0$ to the point d where $E_e = E$ (the external field):

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{d^2} \left[1 - e^{-2d/a} \left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2} \right) \right]$$

Expanding in powers of (d/a): $\quad e^{-2d/a} = 1 - \left(\frac{2d}{a} \right) + \frac{1}{2} \left(\frac{2d}{a} \right)^2 - \frac{1}{3!} \left(\frac{2d}{a} \right)^3 + \cdots = 1 - 2\frac{d}{a} + 2\left(\frac{d}{a} \right)^2 - \frac{4}{3} \left(\frac{d}{a} \right)^3 + \cdots$

$$1 - e^{-2d/a} \left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2} \right) = 1 - \left(1 - 2\frac{d}{a} + 2\left(\frac{d}{a} \right)^2 - \frac{4}{3} \left(\frac{d}{a} \right)^3 + \cdots \right) \left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2} \right)$$

$$= \cancel{1} - \cancel{1} - 2\cancel{\frac{d}{a}} - 2\cancel{\frac{d^2}{a^2}} + 2\cancel{\frac{d}{a}} + 4\cancel{\frac{d^2}{a^2}} + 4\cancel{\frac{d^3}{a^3}} - 2\cancel{\frac{d^2}{a^2}} - 4\cancel{\frac{d^3}{a^3}} + \frac{4}{3}\frac{d^3}{a^3} + \cdots = \frac{4}{3} \left(\frac{d}{a} \right)^3 + \text{ higher order terms.}$$

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{d^2} \left(\frac{4}{3} \frac{d^3}{a^3} \right) = \frac{1}{4\pi\epsilon_0} \frac{4}{3a^3} (qd) = \frac{1}{3\pi\epsilon_0 a^3} p. \qquad \boxed{\alpha = 3\pi\epsilon_0 a^3.}$$

[Not so different from the *uniform* sphere model of Ex. 4.1 (see Eq. 4.2). Note that this result predicts $\frac{1}{4\pi\epsilon_0}\alpha = \frac{3}{4}a^3 = \frac{3}{4}\left(0.5 \times 10^{-10}\right)^3 = 0.09 \times 10^{-30}\,\text{m}^3$, compared with an experimental value (Table 4.1) of $0.66 \times 10^{-30}\,\text{m}^3$. Ironically the "classical" formula (Eq. 4.2) is slightly *closer* to the empirical value.]

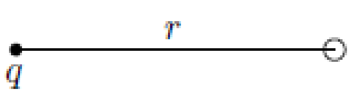
Problem 4.3 According to Eq. 4.1, the induced dipole moment of an atom is proportional to the external field. This is a "rule of thumb," not a fundamental law, and it is easy to concoct exceptions—in theory. Suppose, for example, the charge density of the electron cloud were proportional to the distance from the center, out to a radius R. To what power of E would p be proportional in that case? Find the condition on $\rho(r)$ such that Eq. 4.1 will hold in the weak-field limit.

$\rho(r) = Ar$. Electric field (by Gauss's Law): $\oint \mathbf{E}\cdot d\mathbf{a} = E\left(4\pi r^2\right) = \frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0}\int_0^r A\bar{r}\, 4\pi\bar{r}^2\, d\bar{r}$, or $E = \dfrac{1}{4\pi r^2}\dfrac{4\pi A\, r^4}{\epsilon_0\quad 4} = \dfrac{Ar^2}{4\epsilon_0}$. This "internal" field balances the external field $\mathbf{E}$ when nucleus is "off-center" an amount d: $ad^2/4\epsilon_0 = E \;\Rightarrow\; d = \sqrt{4\epsilon_0 E/A}$. So the induced dipole moment is $p = ed = 2e\sqrt{\epsilon_0/A}\sqrt{E}$. Evidently $\boxed{p \text{ is proportional to } E^{1/2}.}$

For Eq. 4.1 to hold in the weak-field limit, *E must be proportional to r*, for small r, which means that ρ must go to a constant (not zero) at the origin: $\boxed{\rho(0) \neq 0}$ (nor infinite).

Problem 4.4 A point charge q is situated a large distance r from a neutral atom of polarizability α. Find the force of attraction between them.

Problem 4.4



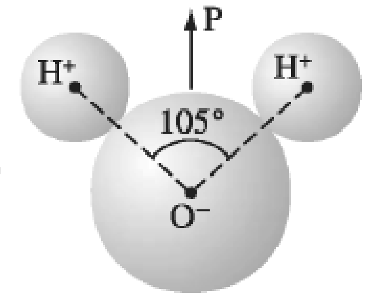
Field of q: $\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\mathbf{r}}$. Induced dipole moment of atom: $\mathbf{p} = \alpha\,\mathbf{E} = \frac{\alpha\, q}{4\pi\epsilon_0 r^2}\,\hat{\mathbf{r}}$.

Field of this dipole, at location of q ($\theta = \pi$, in Eq. 3.103): $E = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r^3}\left(\dfrac{2\alpha\, q}{4\pi\epsilon_0 r^2}\right)$ (to the right).

Force on q due to this field: $\boxed{F = 2\alpha\left(\dfrac{q}{4\pi\epsilon_0}\right)^2 \dfrac{1}{r^5}}$ (attractive).

$$\boxed{\mathbf{E}_{\text{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}).}$$

4.1.3 Alignment of Polar Molecules

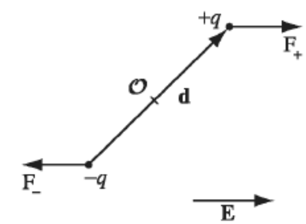


polar molecules

water molecule, built-in, permanent dipole electrons tend to cluster around the oxygen atom

dipole moment of water is unusually large: 6.1×10^{-30} C·m;

in an electric field If the field is uniform,



on the positive end, $\mathbf{F}_+ = q\mathbf{E}$,

on the negative end, $\mathbf{F}_- = -q\mathbf{E}$

torque: $\mathbf{N} = (\mathbf{r}_+ \times \mathbf{F}_+) + (\mathbf{r}_- \times \mathbf{F}_-) = \left[(\mathbf{d}/2) \times (q\mathbf{E})\right] + \left[(-\mathbf{d}/2) \times (-q\mathbf{E})\right] = q\mathbf{d} \times \mathbf{E}.$

a dipole $\mathbf{p} = q\mathbf{d}$ in a uniform field $\mathbf{E}$ experiences a torque $\boxed{\mathbf{N} = \mathbf{p} \times \mathbf{E}.}$

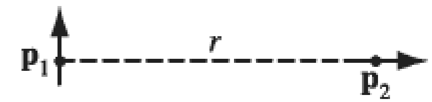
$\mathbf{N}$ is to line $\mathbf{p}$ up *parallel* to $\mathbf{E}$;

for the force on a dipole in a nonuniform field $\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_- = q(\mathbf{E}_+ - \mathbf{E}_-) = q(\Delta\mathbf{E})$,

Assuming the dipole is very short, $\Delta E_x \equiv (\nabla E_x) \cdot \mathbf{d}$, $\Delta\mathbf{E} = (\mathbf{d} \cdot \nabla)\mathbf{E}$, $\boxed{\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E}.}$

For a "perfect" dipole of infinitesimal length, Eq. 4.4 gives the torque *about the center of the dipole* even in a *non*uniform field; about any *other* point $\mathbf{N} = (\mathbf{p} \times \mathbf{E}) + (\mathbf{r} \times \mathbf{F})$

Problem 4.5 In Fig. 4.6, $\mathbf{p}_1$ and $\mathbf{p}_2$ are (perfect) dipoles a distance r apart. What is the torque on $\mathbf{p}_1$ due to $\mathbf{p}_2$? What is the torque on $\mathbf{p}_2$ due to $\mathbf{p}_1$? [In each case, I want the torque on the dipole *about its own center*. If it bothers you that the answers are not equal and opposite, see Prob. 4.29.]



Field of $\mathbf{p}_1$ at $\mathbf{p}_2$ ($\theta = \pi/2$ in Eq. 3.103): $\mathbf{E}_1 = \dfrac{p_1}{4\pi\epsilon_0 r^3}\,\hat{\boldsymbol{\theta}}$ (points *down*).

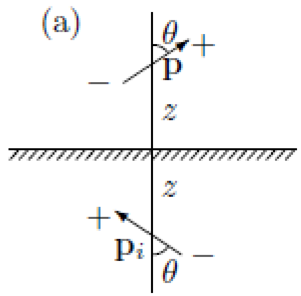
Torque on $\mathbf{p}_2$: $\mathbf{N}_2 = \mathbf{p}_2 \times \mathbf{E}_1 = p_2 E_1 \sin 90^\circ = p_2 E_1 = \boxed{\dfrac{p_1 p_2}{4\pi\epsilon_0 r^3}}$ (points *into* the page).

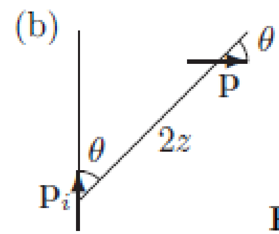
Field of $\mathbf{p}_2$ at $\mathbf{p}_1$ ($\theta = \pi$ in Eq. 3.103): $\mathbf{E}_2 = \dfrac{p_2}{4\pi\epsilon_0 r^3}\,(-2\,\hat{\mathbf{r}})$ (points to the *right*).

Torque on $\mathbf{p}_1$: $\mathbf{N}_1 = \mathbf{p}_1 \times \mathbf{E}_2 = \boxed{\dfrac{2 p_1 p_2}{4\pi\epsilon_0 r^3}}$ (points *into* the page).

$$\boxed{\mathbf{E}_{\text{dip}}(r, \theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}).}$$

Problem 4.6 A (perfect) dipole **p** is situated a distance z above an infinite grounded conducting plane (Fig. 4.7). The dipole makes an angle θ with the perpendicular to the plane. Find the torque on **p**. If the dipole is free to rotate, in what orientation will it come to rest?





Use image dipole as shown in Fig. (a). Redraw, placing $\mathbf{p}_i$ at the origin, Fig. (b).

$$\mathbf{E}_i = \frac{p}{4\pi\epsilon_0(2z)^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}); \quad \mathbf{p} = p\cos\theta\,\hat{\mathbf{r}} + p\sin\theta\,\hat{\boldsymbol{\theta}}.$$

$$\mathbf{N} = \mathbf{p}\times\mathbf{E}_i = \frac{p^2}{4\pi\epsilon_0(2z)^3}\left[(\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})\times(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})\right] = \frac{p^2}{4\pi\epsilon_0(2z)^3}\left[\cos\theta\sin\theta\,\hat{\boldsymbol{\phi}} + 2\sin\theta\cos\theta(-\hat{\boldsymbol{\phi}})\right]$$

$$= \frac{p^2\sin\theta\cos\theta}{4\pi\epsilon_0(2z)^3}(-\hat{\boldsymbol{\phi}}) \quad \text{(out of the page).}$$

$\sin\theta\cos\theta = (1/2)\sin 2\theta$, $\boxed{N = \dfrac{p^2\sin 2\theta}{4\pi\epsilon_0(16z^3)}}$

For $0 < \theta < \pi/2$, **N** tends to rotate **p** counterclockwise; for $\pi/2 < \theta < \pi$, **N** rotates **p** clockwise.

$\boxed{\text{stable orientation is perpendicular to the surface—either } \uparrow \text{ or } \downarrow.}$

Problem 4.7 Show that the energy of an ideal dipole **p** in an electric field **E** is given by

$$\boxed{U = -\mathbf{p}\cdot\mathbf{E}.}$$

If the potential is zero at infinity, the energy of a point charge Q is (Eq. 2.39) $W = QV(\mathbf{r})$. For a physical dipole, with $-q$ at $\mathbf{r}$ and $+q$ at $\mathbf{r}+\mathbf{d}$,

$$U = qV(\mathbf{r}+\mathbf{d}) - qV(\mathbf{r}) = q[V(\mathbf{r}+\mathbf{d}) - V(\mathbf{r})] = q\left[-\int_{\mathbf{r}}^{\mathbf{r}+\mathbf{d}} \mathbf{E}\cdot d\mathbf{l}\right].$$

For an ideal dipole the integral reduces to $\mathbf{E}\cdot\mathbf{d}$, and

$$U = -q\mathbf{E}\cdot\mathbf{d} = -\mathbf{p}\cdot\mathbf{E},$$

since $\mathbf{p} = q\mathbf{d}$. If you do not (or cannot) use infinity as the reference point, the result still holds, as long as you bring the two charges in from the *same point*, $\mathbf{r}_0$ (or two points at the same potential). In that case $W = Q[V(\mathbf{r}) - V(\mathbf{r}_0)]$, and

$$U = q[V(\mathbf{r}+\mathbf{d}) - V(\mathbf{r}_0)] - q[V(\mathbf{r}) - V(\mathbf{r}_0)] = q[V(\mathbf{r}+\mathbf{d}) - V(\mathbf{r})],$$

Problem 4.8 Show that the interaction energy of two dipoles separated by a displacement **r** is

$$U = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[\mathbf{p}_1\cdot\mathbf{p}_2 - 3(\mathbf{p}_1\cdot\hat{\mathbf{r}})(\mathbf{p}_2\cdot\hat{\mathbf{r}})].$$

$U = -\mathbf{p}_1\cdot\mathbf{E}_2$, but $\mathbf{E}_2 = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\mathbf{p}_2\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}_2\right]$. So $U = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[\mathbf{p}_1\cdot\mathbf{p}_2 - 3(\mathbf{p}_1\cdot\hat{\mathbf{r}})(\mathbf{p}_2\cdot\hat{\mathbf{r}})\right]$.

Problem 4.9 A dipole $\mathbf{p}$ is a distance r from a point charge q, and oriented so that $\mathbf{p}$ makes an angle θ with the vector $\mathbf{r}$ from q to $\mathbf{p}$.

(a) What is the force on $\mathbf{p}$? (b) What is the force on q?

(a) $\mathbf{F} = (\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{E}$ (Eq. 4.5); $\mathbf{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r^2}\hat{\mathbf{r}} = \dfrac{q}{4\pi\epsilon_0}\dfrac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{(x^2+y^2+z^2)^{3/2}}$.

$$\begin{aligned}
F_x &= \left(p_x\frac{\partial}{\partial x} + p_y\frac{\partial}{\partial y} + p_z\frac{\partial}{\partial z}\right)\frac{q}{4\pi\epsilon_0}\frac{x}{(x^2+y^2+z^2)^{3/2}} \\
&= \frac{q}{4\pi\epsilon_0}\left\{p_x\left[\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{2}x\frac{2x}{(x^2+y^2+z^2)^{5/2}}\right] + p_y\left[-\frac{3}{2}x\frac{2y}{(x^2+y^2+z^2)^{5/2}}\right]\right. \\
&\quad \left. + p_z\left[-\frac{3}{2}x\frac{2z}{(x^2+y^2+z^2)^{5/2}}\right]\right\} = \frac{q}{4\pi\epsilon_0}\left[\frac{p_x}{r^3} - \frac{3x}{r^5}(p_x x + p_y y + p_z z)\right] = \frac{q}{4\pi\epsilon_0}\left[\frac{\mathbf{p}}{r^3} - \frac{3\mathbf{r}(\mathbf{p}\cdot\mathbf{r})}{r^5}\right]_x
\end{aligned}$$

$$\mathbf{F} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\left[\mathbf{p} - 3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}\right].}$$

(b) $\mathbf{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r^3}\{3\,[\mathbf{p}\cdot(-\hat{\mathbf{r}})]\,(-\hat{\mathbf{r}}) - \mathbf{p}\} = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p}\right]$. (This is from Eq. 3.104; the minus signs are because $\mathbf{r}$ points *toward* $\mathbf{p}$, in this problem.)

$$\mathbf{F} = q\mathbf{E} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p}\right].}$$

[Note that the forces are equal and opposite, as you would expect from Newton's third law.]

4.1.4 Polarization

What happens to a piece of dielectric material when it is placed in an electric field?

If the substance consists of neutral atoms nonpolar molecules), the field will induce in each a tiny dipole moment, in the same direction as the field.

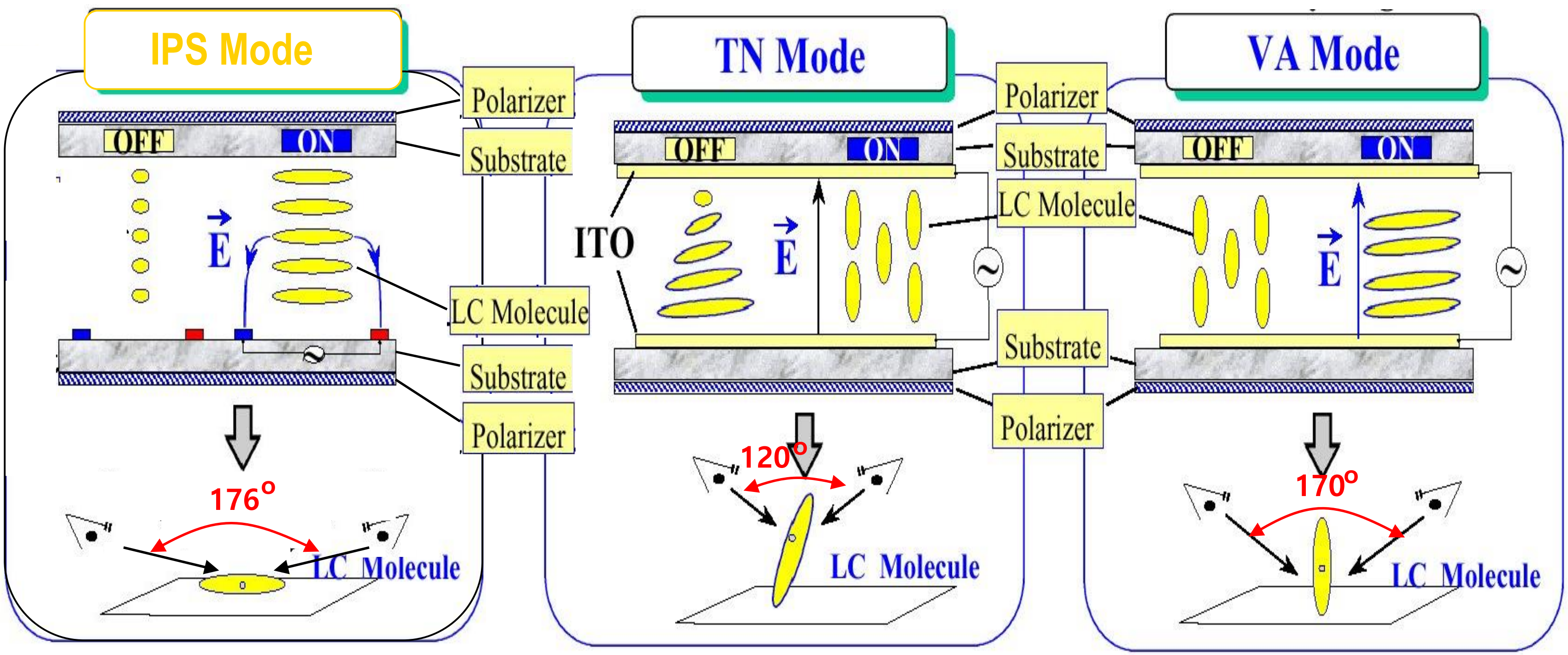
If the material is made up of polar molecules, each permanent dipole will experience a torque, tending to line it up along the field direction.

(Random thermal motions compete with this process,

result: *dipoles pointing along the direction of the field*— the material becomes **polarized**.

Polarization density $\mathbf{P} \equiv$ *dipole moment per unit volume,*

液晶顯示器模式的比較



IPS Mode
TN Mode
VA Mode
OFF
ON
Polarizer
Substrate
LC Molecule
ITO
E
176°
120°
170°
LC Molecule

4.2 The Field of a Polarized Object

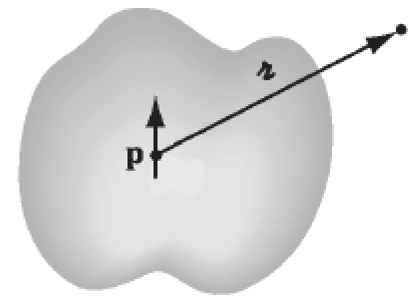
4.2.1 Bound Charges
4.2.2 Physical Interpretation of Bound Charges
4.2.3 The Field Inside a Dielectric

The Field of a Polarized Object

4.2.1 Bound Charges

What is the field produced by an object containing dipole moment per unit volume P

For a single dipole $\mathbf{p}$ $\quad V(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{\mathbf{p}\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}, \qquad \mathbf{p} = \mathbf{P}\,d\tau'$



total potential is $\quad V(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int_{\mathcal{V}} \dfrac{\mathbf{P}(\mathbf{r}')\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$

$\nabla'\left(\dfrac{1}{\mathscr{r}}\right) = \dfrac{\hat{\mathscr{r}}}{\mathscr{r}^2},$ the differentiation is with respect to the *source* coordi nates ($\mathbf{r}'$),

$$V = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \mathbf{P}\cdot\nabla'\left(\frac{1}{\mathscr{r}}\right)d\tau'.$$

Integrating by parts, $\quad V = \dfrac{1}{4\pi\epsilon_0}\left[\displaystyle\int_{\mathcal{V}} \nabla'\cdot\left(\dfrac{\mathbf{P}}{\mathscr{r}}\right)d\tau' - \int_{\mathcal{V}} \dfrac{1}{\mathscr{r}}(\nabla'\cdot\mathbf{P})\,d\tau'\right]$

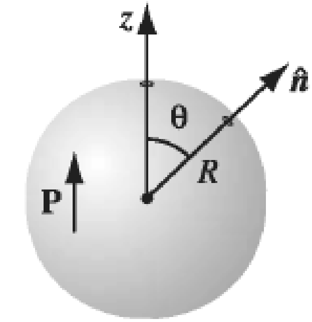
divergence theorem, $\quad V = \dfrac{1}{4\pi\epsilon_0}\displaystyle\oint_{\mathcal{S}} \dfrac{1}{\mathscr{r}}\mathbf{P}\cdot d\mathbf{a}' - \dfrac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \dfrac{1}{\mathscr{r}}(\nabla'\cdot\mathbf{P})\,d\tau'. = \dfrac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \dfrac{\sigma_b}{\mathscr{r}}\,da' + \dfrac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \dfrac{\rho_b}{\mathscr{r}}\,d\tau'.$

surface charge density $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$ $\quad \boxed{\sigma_b \equiv \mathbf{P}\cdot\hat{\mathbf{n}}} \quad \boxed{\rho_b \equiv -\nabla\cdot\mathbf{P}.}$ $\quad$ volume charge density $\rho_b = -\nabla\cdot\mathbf{P}$

bound charges

potential of a surface charge $\qquad$ potential of a volume charge

Example 4.2. Find the electric field produced by a uniformly polarized sphere of radius R.



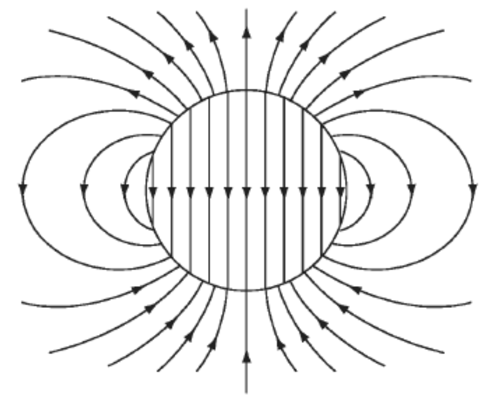
choose the z axis to coincide with the direction of polarization

The volume bound charge density ρ_b is zero, since $\mathbf{P}$ is uniform,

surface charge density $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = P\cos\theta,$

Ex. 3.9:

$$V(r,\theta) = \begin{cases} \dfrac{P}{3\epsilon_0} r\cos\theta, & \text{for} \quad r \le R, \\[2ex] \dfrac{P}{3\epsilon_0}\dfrac{R^3}{r^2}\cos\theta, & \text{for} \quad r \ge R. \end{cases}$$



$r\cos\theta = z$, the *field* inside the sphere is *uniform*:

$$\mathbf{E} = -\nabla V = -\frac{P}{3\epsilon_0}\hat{\mathbf{z}} = -\frac{1}{3\epsilon_0}\mathbf{P}, \quad \text{for} \quad r < R.$$

Outside potential is identical to that of a perfect dipole at the origin, $V = \dfrac{1}{4\pi\epsilon_0}\dfrac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}, \quad \text{for} \quad r \ge R,$

$$\mathbf{p} = \frac{4}{3}\pi R^3 \mathbf{P}.$$

Example 3.9. A specified charge density $\sigma_0(\theta)$ is glued over the surface of a spherical shell of radius R. Find the resulting potential inside and outside the sphere.

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma_0}{\mathscr{r}}\,da$$

Or separation of variable

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \qquad (r \le R)$$

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \qquad (r \ge R)$$

V continuous at $r = R$

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) \longrightarrow B_l = A_l R^{2l+1}$$

V derivative discontinuous at $r = R$

$$\left(\frac{\partial V_{\text{out}}}{\partial r} - \frac{\partial V_{\text{in}}}{\partial r}\right)\bigg|_{r=R} = -\frac{1}{\epsilon_0}\sigma_0(\theta)$$

$$-\sum_{l=0}^{\infty}(l+1)\frac{B_l}{R^{l+2}}P_l(\cos\theta) - \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = -\frac{1}{\epsilon_0}\sigma_0(\theta)$$

$$\sum_{l=0}^{\infty}(2l+1)A_l R^{l-1} P_l(\cos\theta) = \frac{1}{\epsilon_0}\sigma_0(\theta)$$

$$\int_{-1}^{1} P_l(x)P_{l'}(x)\,dx = \int_0^{\pi} P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\,d\theta = \begin{cases} 0, & \text{if } l' \ne l, \\ \frac{2}{2l+1}, & \text{if } l' = l. \end{cases}$$

$$\longrightarrow \quad A_l = \frac{1}{2\epsilon_0 R^{l-1}}\int_0^{\pi} \sigma_0(\theta) P_l(\cos\theta)\sin\theta\,d\theta$$

If $\sigma_0(\theta) = k\cos\theta = kP_1(\cos\theta)$

all the A_l's are zero except for $l = 1$

$$A_1 = \frac{k}{2\epsilon_0}\int_0^{\pi}[P_1(\cos\theta)]^2\sin\theta\,d\theta = \frac{k}{3\epsilon_0}$$

$$V(r,\theta) = \frac{k}{3\epsilon_0} r\cos\theta \quad (r \le R) \qquad V(r,\theta) = \frac{kR^3}{3\epsilon_0}\frac{1}{r^2}\cos\theta \quad (r \ge R)$$

Same as Ex 3.8

if $\sigma_0(\theta)$ is the induced charge on a metal sphere in an external field $E_0\hat{\mathbf{z}}$, $k = 3\epsilon_0 E_0$

the potential inside is $E_0 r\cos\theta = E_0 z$, the field is $-E_0\hat{\mathbf{z}}$ cancel off the external field,

r>R V= $E_0\frac{R^3}{r^2}\cos\theta$ Due to the induced charge

Problem 4.10 A sphere of radius R carries a polarization

$$\mathbf{P}(\mathbf{r}) = k\mathbf{r},$$

where k is a constant and $\mathbf{r}$ is the vector from the center.

(a) Calculate the bound charges σ_b and ρ_b.

(b) Find the field inside and outside the sphere.

(a) $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \boxed{kR;}$ $\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P} = -\dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 kr) = -\dfrac{1}{r^2}3kr^2 = \boxed{-3k.}$

(b) For $r < R$, $\mathbf{E} = \frac{1}{3\epsilon_0}\rho r\,\hat{\mathbf{r}}$ (Prob. 2.12), so $\mathbf{E} = \boxed{-(k/\epsilon_0)\,\mathbf{r}.}$

For $r > R$, same as if all charge at center; but $Q_{\text{tot}} = (kR)(4\pi R^2) + (-3k)(\frac{4}{3}\pi R^3) = 0$, so $\boxed{\mathbf{E} = \mathbf{0}.}$

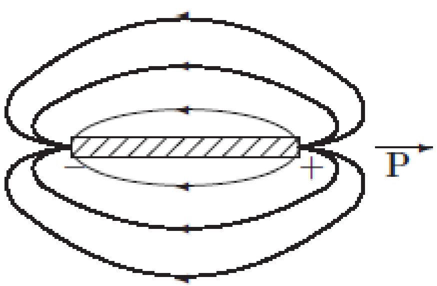
Problem 4.11 A short cylinder, of radius a and length L, carries a "frozen-in" uniform polarization $\mathbf{P}$, parallel to its axis. Find the bound charge, and sketch the electric field (i) for $L \gg a$, (ii) for $L \ll a$, and (iii) for $L \approx a$. [This is known as a **bar electret**; it is the electrical analog to a bar magnet. In practice, only very special materials—barium titanate is the most "familiar" example—will hold a permanent electric polarization. That's why you can't buy electrets at the toy store.]

$\rho_b = 0$; $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \pm P$ (plus sign at one end—the one $\mathbf{P}$ points *toward*; minus sign at the other—the one $\mathbf{P}$ points *away* from).

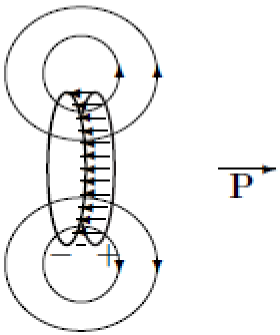
(i) $L \gg a$. Then the ends look like point charges, and the whole thing is like a physical dipole, of length L and charge $P\pi a^2$. See Fig. (a).

(ii) $L \ll a$. Then it's like a circular parallel-plate capacitor. Field is nearly uniform inside; nonuniform "fringing field" at the edges. See Fig. (b).

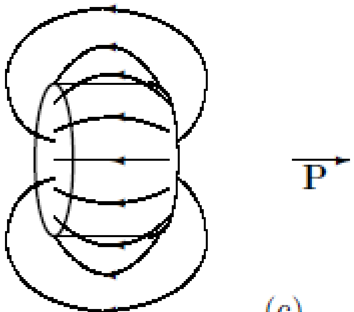
(iii) $L \approx a$. See Fig. (c).



(a) Like a dipole



(b) Like a parallel-plate capacitor



(c)

Problem 4.12 Calculate the potential of a uniformly polarized sphere (Ex. 4.2) directly from Eq. 4.9.

$V = \frac{1}{4\pi\epsilon_0}\int \frac{\mathbf{P}\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau = \mathbf{P}\cdot\left\{\frac{1}{4\pi\epsilon_0}\int\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau\right\}$. But the term in curly brackets is precisely the *field* of a uniformly *charged* sphere, divided by ρ. The integral was done explicitly in Probs. 2.7 and 2.8:

$$\frac{1}{4\pi\epsilon_0}\int\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau = \frac{1}{\rho}\left\{\begin{array}{l} \frac{1}{4\pi\epsilon_0}\frac{(4/3)\pi R^3\rho}{r^2}\,\hat{\mathbf{r}},\ (r > R), \\ \frac{1}{4\pi\epsilon_0}\frac{(4/3)\pi R^3\rho}{R^3}\,\mathbf{r},\ (r < R). \end{array}\right\} \quad \text{So } V(r,\theta) = \left\{\begin{array}{ll} \frac{R^3}{3\epsilon_0 r^2}\mathbf{P}\cdot\hat{\mathbf{r}} = \boxed{\frac{R^3 P\cos\theta}{3\epsilon_0 r^2}}, & (r > R), \\ \frac{1}{3\epsilon_0}\mathbf{P}\cdot\mathbf{r} = \boxed{\frac{Pr\cos\theta}{3\epsilon_0}}, & (r < R). \end{array}\right\}$$

Problem 2.7 Find the electric field a distance z from the center of a spherical surface of radius R (Fig. 2.11) that carries a uniform charge density σ. Treat the case $z < R$ (inside) as well as $z > R$ (outside). Express your answers in terms of the total charge q on the sphere. [*Hint:* Use the law of cosines to write $\mathscr{r}$ in terms of R and θ. Be sure to take the *positive* square root: $\sqrt{R^2 + z^2 - 2Rz} = (R - z)$ if $R > z$, but it's $(z - R)$ if $R < z$.]

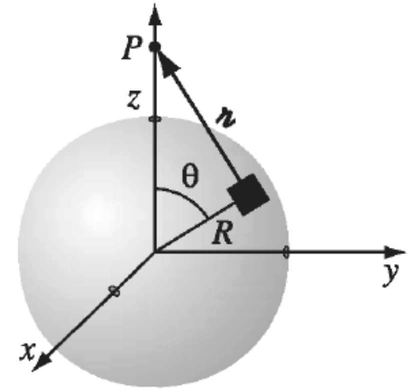


FIGURE 2.11

$\mathbf{E}$ is clearly in the z direction. From the diagram,

$dq = \sigma da = \sigma R^2 \sin\theta \, d\theta \, d\phi$,

$\mathscr{r}^2 = R^2 + z^2 - 2Rz\cos\theta$,

$\cos\psi = \frac{z - R\cos\theta}{\mathscr{r}}$.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(\vec{r}\,')}{R^2}\hat{R}\, da'$$

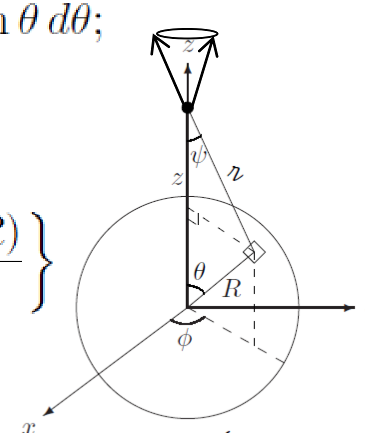
$$E_z = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma R^2 \sin\theta \, d\theta \, d\phi (z - R\cos\theta)}{(R^2 + z^2 - 2Rz\cos\theta)^{3/2}}. \qquad \int d\phi = 2\pi.$$

$$= \frac{1}{4\pi\epsilon_0}(2\pi R^2\sigma)\int_0^\pi \frac{(z - R\cos\theta)\sin\theta}{(R^2 + z^2 - 2Rz\cos\theta)^{3/2}}d\theta.$$

Let $u = \cos\theta$; $du = -\sin\theta \, d\theta$; $\left\{ \begin{array}{l} \theta = 0 \Rightarrow u = +1 \\ \theta = \pi \Rightarrow u = -1 \end{array} \right\}$

$$= \frac{1}{4\pi\epsilon_0}(2\pi R^2\sigma)\int_{-1}^{1} \frac{z - Ru}{(R^2 + z^2 - 2Rzu)^{3/2}}du.$$

$$= \frac{1}{4\pi\epsilon_0}(2\pi R^2\sigma)\left[\frac{1}{z^2}\frac{zu - R}{\sqrt{R^2 + z^2 - 2Rzu}}\right]_{-1}^{1} = \frac{1}{4\pi\epsilon_0}\frac{2\pi R^2\sigma}{z^2}\left\{\frac{(z - R)}{|z - R|} - \frac{(-z - R)}{|z + R|}\right\}$$



For $z > R$ (outside the sphere), $E_z = \frac{1}{4\pi\epsilon_0}\frac{4\pi R^2\sigma}{z^2} = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}$, so $\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\hat{\mathbf{z}}.}$

For $z < R$ (inside), $E_z = 0$, so $\boxed{\mathbf{E} = \mathbf{0}.}$

Problem 2.8 Use your result in Prob. 2.7 to find the field inside and outside a solid sphere of radius R that carries a uniform volume charge density ρ. Express your answers in terms of the total charge of the sphere, q. Draw a graph of $|\mathbf{E}|$ as a function of the distance from the center.

According to Prob. 2.7, all shells *interior* to the point (i.e. at smaller r) contribute as though their charge were concentrated at the center, while all exterior shells contribute nothing. Therefore:

$$\mathbf{E}(r) = \frac{1}{4\pi\epsilon_0}\frac{Q_{\text{int}}}{r^2}\,\hat{\mathbf{r}},$$

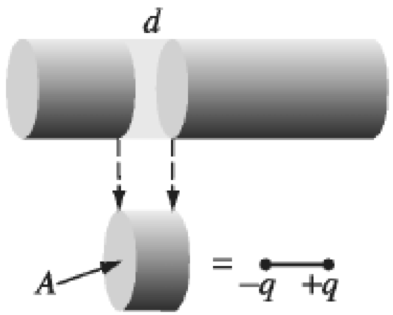
where Q_{int} is the total charge interior to the point. *Outside* the sphere, *all* the charge is interior, so

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\,\hat{\mathbf{r}}.}$$

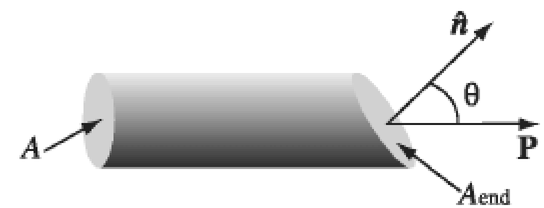
Inside the sphere, only that fraction of the total which is interior to the point counts:

$$Q_{\text{int}} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}Q = \frac{r^3}{R^3}Q, \quad \text{so} \quad \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{r^3}{R^3}Q\frac{1}{r^2}\,\hat{\mathbf{r}} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\mathbf{r}.}$$

4.2.2 Physical Interpretation of Bound Charges



Polarization density P



Dipole moment =P(Ad)=qd

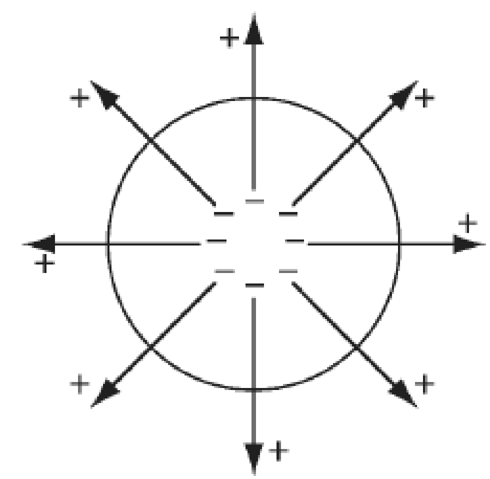
→ q=PA

→ Surface charge density $\sigma_b = \dfrac{q}{A} = P$

$$\sigma_b = \frac{q}{A_{end}} = \frac{q}{A/\cos\theta} = P\cos\theta = \mathbf{P}\cdot\hat{n}$$

Non-uniform polarization → bound charge density inside

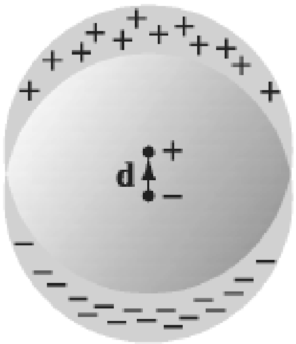
Diverging P results in a pile up of surface charge



$$\int_{\mathcal{V}} \rho_b \, d\tau = -\oint_{\mathcal{S}} \mathbf{P}\cdot d\mathbf{a} = -\int_{\mathcal{V}} (\nabla\cdot\mathbf{P})\, d\tau$$

→ $$\rho_b = -\nabla\cdot\mathbf{P}$$

Example 4.3. There is another way of analyzing the uniformly polarized sphere (Ex. 4.2), which nicely illustrates the idea of a bound charge. What we have, really, is *two* spheres of charge: a positive sphere and a negative sphere. Without polarization the two are superimposed and cancel completely. But when the material is uniformly polarized, all the plus charges move slightly *upward* (the z direction), and all the minus charges move slightly *downward* (Fig. 4.15). The two spheres no longer overlap perfectly: at the top there's a "cap" of leftover positive charge and at the bottom a cap of negative charge. This "leftover" charge is precisely the bound surface charge σ_b.



In Prob. 2.18, the field in the region of overlap between two uniformly charged spheres;

$$\mathbf{E} = -\frac{1}{4\pi\epsilon_0}\frac{q\mathbf{d}}{R^3}$$

in terms of the polarization of the sphere, $\mathbf{p} = q\mathbf{d} = (\frac{4}{3}\pi R^3)\mathbf{P}$, $\mathbf{E} = -\frac{1}{3\epsilon_0}\mathbf{P}$

outside, all the charge on each sphere were concentrated at the respective center. $\longrightarrow$ a dipole,

$$V = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

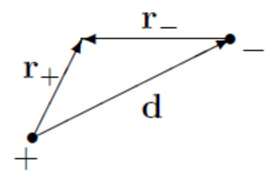
These answers agree, of course, with the results of Ex. 4.2.

Problem 2.18 Two spheres, each of radius R and carrying uniform volume charge densities $+\rho$ and $-\rho$, respectively, are placed so that they partially overlap (Fig. 2.28). Call the vector from the positive center to the negative center $\mathbf{d}$. Show that the field in the region of overlap is constant, and find its value. [*Hint:* Use the answer to Prob. 2.12.]

From Prob. 2.12, the field inside the positive sphere is $\mathbf{E}_+ = \frac{\rho}{3\epsilon_0}\mathbf{r}_+$, where $\mathbf{r}_+$ is the vector from the positive center to the point in question. Likewise, the field of the negative sphere is $-\frac{\rho}{3\epsilon_0}\mathbf{r}_-$. So the *total* field is

$$\mathbf{E} = \frac{\rho}{3\epsilon_0}(\mathbf{r}_+ - \mathbf{r}_-)$$

But (see diagram) $\mathbf{r}_+ - \mathbf{r}_- = \mathbf{d}$. So $\boxed{\mathbf{E} = \frac{\rho}{3\epsilon_0}\mathbf{d}.}$



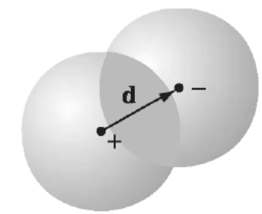


FIGURE 2.28

Problem 4.14 When you polarize a neutral dielectric, the charge moves a bit, but the *total* remains zero. This fact should be reflected in the bound charges σ_b and ρ_b. Prove from Eqs. 4.11 and 4.12 that the total bound charge vanishes.

Total charge on the dielectric is $Q_{\text{tot}} = \oint_S \sigma_b\, da + \int_V \rho_b\, d\tau = \oint_S \mathbf{P}\cdot d\mathbf{a} - \int_V \boldsymbol{\nabla}\cdot\mathbf{P}\, d\tau$. But the divergence theorem says $\oint_S \mathbf{P}\cdot d\mathbf{a} = \int_V \boldsymbol{\nabla}\cdot\mathbf{P}\, d\tau$, so $Q_{\text{enc}} = 0$. qed

Problem 4.13 A very long cylinder, of radius a, carries a uniform polarization $\mathbf{P}$ perpendicular to its axis. Find the electric field inside the cylinder. Show that the field *outside* the cylinder can be expressed in the form

$$\mathbf{E}(\mathbf{r}) = \frac{a^2}{2\epsilon_0 s^2}[2(\mathbf{P}\cdot\hat{\mathbf{s}})\hat{\mathbf{s}} - \mathbf{P}]$$

Think of it as two cylinders of opposite uniform charge density $\pm\rho$. *Inside*, the field at a distance s from the axis of a uniformly charge cylinder is given by Gauss's law: $E2\pi s\ell = \frac{1}{\epsilon_0}\rho\pi s^2\ell \Rightarrow \mathbf{E} = (\rho/2\epsilon_0)\mathbf{s}$. For *two* such cylinders, one plus and one minus, the net field (inside) is $\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = (\rho/2\epsilon_0)(\mathbf{s}_+ - \mathbf{s}_-)$. But $\mathbf{s}_+ - \mathbf{s}_- = -\mathbf{d}$, so $\mathbf{E} = \boxed{-\rho\mathbf{d}/(2\epsilon_0),}$ where $\mathbf{d}$ is the vector from the negative axis to positive axis. In this case the total dipole moment of a chunk of length ℓ is $\mathbf{P}(\pi a^2\ell) = (\rho\pi a^2\ell)\,\mathbf{d}$. So $\rho\mathbf{d} = \mathbf{P}$, and $\boxed{\mathbf{E} = -\mathbf{P}/(2\epsilon_0),}$ for $s < a$.

Outside, Gauss's law gives $E2\pi s\ell = \frac{1}{\epsilon_0}\rho\pi a^2\ell \Rightarrow \mathbf{E} = \frac{\rho a^2}{2\epsilon_0}\frac{\hat{\mathbf{s}}}{s}$, for *one* cylinder. For the combination, $\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \frac{\rho a^2}{2\epsilon_0}\left(\frac{\hat{\mathbf{s}}_+}{s_+} - \frac{\hat{\mathbf{s}}_-}{s_-}\right)$, where

$$\mathbf{s}_\pm = \mathbf{s} \mp \frac{\mathbf{d}}{2};$$

$$\frac{\mathbf{s}_\pm}{s_\pm^2} = \left(\mathbf{s} \mp \frac{\mathbf{d}}{2}\right)\left(s^2 + \frac{d^2}{4} \mp \mathbf{s}\cdot\mathbf{d}\right)^{-1} \cong \frac{1}{s^2}\left(\mathbf{s} \mp \frac{\mathbf{d}}{2}\right)\left(1 \mp \frac{\mathbf{s}\cdot\mathbf{d}}{s^2}\right)^{-1} \cong \frac{1}{s^2}\left(\mathbf{s} \mp \frac{\mathbf{d}}{2}\right)\left(1 \pm \frac{\mathbf{s}\cdot\mathbf{d}}{s^2}\right)$$

$$= \frac{1}{s^2}\left(\mathbf{s} \pm \mathbf{s}\frac{(\mathbf{s}\cdot\mathbf{d})}{s^2} \mp \frac{\mathbf{d}}{2}\right) \quad \text{(keeping only 1st order terms in } \mathbf{d}\text{)}.$$

$$\left(\frac{\hat{\mathbf{s}}_+}{s_+} - \frac{\hat{\mathbf{s}}_-}{s_-}\right) = \frac{1}{s^2}\left[\left(\mathbf{s} + \mathbf{s}\frac{(\mathbf{s}\cdot\mathbf{d})}{s^2} - \frac{\mathbf{d}}{2}\right) - \left(\mathbf{s} - \mathbf{s}\frac{(\mathbf{s}\cdot\mathbf{d})}{s^2} + \frac{\mathbf{s}}{2}\right)\right] = \frac{1}{s^2}\left(2\frac{\mathbf{s}(\mathbf{s}\cdot\mathbf{d})}{s^2} - \mathbf{d}\right).$$

$$\boxed{\mathbf{E}(\mathbf{s}) = \frac{a^2}{2\epsilon_0}\frac{1}{s^2}\left[2(\mathbf{P}\cdot\hat{\mathbf{s}})\,\hat{\mathbf{s}} - \mathbf{P}\right],} \quad \text{for } s > a.$$

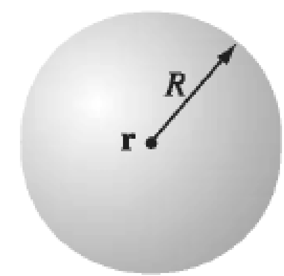
4.2.3 The Field Inside a Dielectric

Microscopic view of dipole ⟶ Macroscopic view

Average field
Mean field

to calculate the macroscopic field at some point $\mathbf{r}$ within a dielectric

average the true (microscopic) field over an appropriate volume



The macroscopic field at $\mathbf{r}$, consists of two parts:

average field over the sphere due to all charges *outside*,

average due to all charges *inside:*

$$\mathbf{E} = \mathbf{E}_{\text{out}} + \mathbf{E}_{\text{in}}$$

in Prob. 3.47(d)

the average field (over a sphere), produced by charges *outside*, is equal to the field they produce at the center,

$\mathbf{E}_{\text{out}}$ is the field at $\mathbf{r}$ due to the dipoles exterior to the sphere. These are far enough away

$$V_{\text{out}} = \frac{1}{4\pi\epsilon_0} \int_{\text{outside}} \frac{\mathbf{P}(\mathbf{r}') \cdot \hat{\boldsymbol{\imath}}}{\imath^2}\, d\tau'$$

The dipoles *inside* the sphere their *average* field, according to Eq. 3.105, $\mathbf{E}_{\text{in}} = -\dfrac{1}{4\pi\epsilon_0}\dfrac{\mathbf{p}}{R^3}$

regardless of the details of the charge distribution within the sphere.

$$\mathbf{p} = (\tfrac{4}{3}\pi R^3)\,\mathbf{P} \qquad \mathbf{E}_{\text{in}} = -\frac{1}{3\epsilon_0}\mathbf{P}.$$

by assumption, the sphere is small enough that **P** does not vary signif icantly over its volume,

$$V_{\text{out}} = \frac{1}{4\pi\epsilon_0}\int_{\text{outside}} \frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'.$$

the term *left out* of the integral

the field at the center of a *uniformly* polarized sphere, $-(1/3\epsilon_0)\mathbf{P}$

this is precisely what $\mathbf{E}_{\text{in}}$ (Eq. 4.18) puts back in!

The macroscopic field, is given by the potential $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'$

where the integral runs over the *entire* volume of the dielectric.

the average field over *any* sphere (due to the charge inside) is the same as

= the field at the center of a *uniformly polarized* sphere with the same total dipole moment.

Extra

Problem 3.47 Show that the average field inside a sphere of radius R, due to all the charge within the sphere, is

$$\mathbf{E}_{\text{ave}} = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3}, \qquad \text{Eq. 3.105}$$

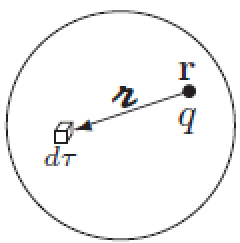
where $\mathbf{p}$ is the total dipole moment. There are several ways to prove this delightfully simple result. Here's one method:[22]

(a) Show that the average field due to a single charge q at point $\mathbf{r}$ inside the sphere is the same as the field at $\mathbf{r}$ due to a uniformly charged sphere with $\rho = -q/(\frac{4}{3}\pi R^3)$, namely

$$\frac{1}{4\pi\epsilon_0}\frac{1}{(\frac{4}{3}\pi R^3)}\int \frac{q}{\boldsymbol{\mathscr{r}}^2}\hat{\boldsymbol{\mathscr{r}}}\,d\tau',$$

where $\boldsymbol{\mathscr{r}}$ is the vector from $\mathbf{r}$ to $d\tau'$.

(a) The average field due to a point charge q at $\mathbf{r}$ is



$$\mathbf{E}_{\text{ave}} = \frac{1}{(\frac{4}{3}\pi R^3)}\int \mathbf{E}\,d\tau, \quad \text{where } \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{\boldsymbol{\mathscr{r}}^2}\hat{\boldsymbol{\mathscr{r}}},$$

$$\text{so} \quad \mathbf{E}_{\text{ave}} = \frac{1}{(\frac{4}{3}\pi R^3)}\frac{1}{4\pi\epsilon_0}\int q\frac{\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2}\,d\tau.$$

(Here $\mathbf{r}$ is the source point, $d\tau$ is the field point, so $\boldsymbol{\mathscr{r}}$ goes from $\mathbf{r}$ to $d\tau$.) The field at $\mathbf{r}$ due to uniform charge ρ over the sphere is $\mathbf{E}_\rho = \frac{1}{4\pi\epsilon_0}\int \rho\frac{\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2}\,d\tau$. This time $d\tau$ is the source point and $\mathbf{r}$ is the field point, so $\boldsymbol{\mathscr{r}}$ goes from $d\tau$ to $\mathbf{r}$, and hence carries the opposite sign. So with $\rho = -q/\left(\frac{4}{3}\pi R^3\right)$, the two expressions agree: $\mathbf{E}_{\text{ave}} = \mathbf{E}_\rho$.

(b) The latter can be found from Gauss's law (see Prob. 2.12). Express the answer in terms of the dipole moment of q.

$$\mathbf{E}_\rho = \frac{1}{3\epsilon_0}\rho\mathbf{r} = -\frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{R^3} = -\frac{\mathbf{p}}{4\pi\epsilon_0 R^3}.$$

(c) Use the superposition principle to generalize to an arbitrary charge distribution.

(c) If there are many charges inside the sphere, $\mathbf{E}_{\text{ave}}$ is the sum of the individual averages, and $\mathbf{p}_{\text{tot}}$ is the sum of the individual dipole moments. So $\mathbf{E}_{\text{ave}} = -\frac{\mathbf{p}}{4\pi\epsilon_0 R^3}$. qed

(d) While you're at it, show that the average field over the volume of a sphere, due to all the charges *outside*, is the same as the field they produce at the center.

(d) The same argument, only with q placed at $\mathbf{r}$ *outside* the sphere, gives

$$\mathbf{E}_{\text{ave}} = \mathbf{E}_{\rho} = \frac{1}{4\pi\epsilon_0}\frac{\left(\frac{4}{3}\pi R^3\rho\right)}{r^2}\,\hat{\mathbf{r}} \quad \text{(field at } \mathbf{r} \text{ due to uniformly charged sphere)} = \frac{1}{4\pi\epsilon_0}\frac{-q}{r^2}\,\hat{\mathbf{r}}.$$

But this is precisely the field produced by q (at $\mathbf{r}$) at the *center* of the sphere. So the average field (over the sphere) due to a point charge *outside* the sphere is the same as the field that same charge produces at the center. And by superposition, this holds for any *collection* of exterior charges.

4.3 The Electric Displacement

4.3.1 Gauss’s Law in the Presence of Dielectrics
4.3.2 A Deceptive Parallel
4.3.3 Boundary Conditions

4.3.1 Gauss's Law in the Presence of Dielectrics

the effect of polarization is to produce accumulations of (bound) charge, $\rho_b = -\nabla \cdot \mathbf{P}$ within the dielectric

$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$ on the surface

The field due to polarization of the medium is just the field of this bound charge.

Within the dielectric, the total charge density $\rho = \rho_b + \rho_f$

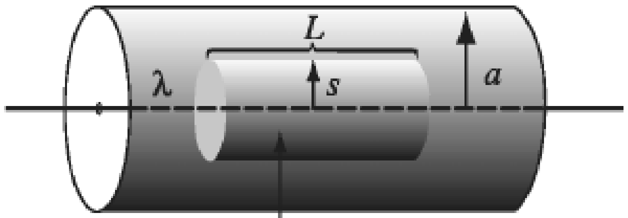
Gauss's law $\epsilon_0 \nabla \cdot \mathbf{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \mathbf{P} + \rho_f$ $\quad$ $\mathbf{E}$ is now the *total* field

$\longrightarrow \nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$ $\quad$ **electric displacement** $\boxed{\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P},}$

$\longrightarrow \boxed{\nabla \cdot \mathbf{D} = \rho_f,}$ or $\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$

$Q_{f_{\text{enc}}}$ total free charge enclosed in the volume

Example 4.4. A long straight wire, carrying uniform line charge λ, is surrounded by rubber insulation out to a radius a (Fig. 4.17). Find the electric displacement.



Drawing a cylindrical Gaussian surface, of radius s and length L,

$$D(2\pi s L) = \lambda L \longrightarrow \mathbf{D} = \frac{\lambda}{2\pi s}\hat{\mathbf{s}}$$

this formula holds both within the insulation and outside

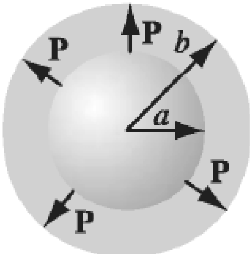
for $s > a$ $\quad \mathbf{P} = 0 \quad \mathbf{E} = \frac{1}{\epsilon_0}\mathbf{D} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}},$

Inside the rubber, the electric field cannot be determined, since we do not know **P**.

Problem 4.15 A thick spherical shell (inner radius a, outer radius b) is made of dielectric material with a "frozen-in" polarization

$$\mathbf{P}(\mathbf{r}) = \frac{k}{r}\hat{\mathbf{r}},$$

where k is a constant and r is the distance from the center (Fig. 4.18). (There is no *free* charge in the problem.) Find the electric field in all three regions by two different methods:



(a) Locate all the bound charge, and use Gauss's law (Eq. 2.13) to calculate the field it produces.

(a) $\rho_b = -\nabla\cdot\mathbf{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{k}{r}\right) = -\frac{k}{r^2}$; $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \begin{cases} +\mathbf{P}\cdot\hat{\mathbf{r}} = k/b & (\text{at } r = b), \\ -\mathbf{P}\cdot\hat{\mathbf{r}} = -k/a & (\text{at } r = a). \end{cases}$

Gauss's law $\Rightarrow \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q_{\text{enc}}}{r^2}\hat{\mathbf{r}}$. For $r < a$, $Q_{\text{enc}} = 0$, so $\boxed{\mathbf{E} = \mathbf{0}.}$ For $r > b$, $Q_{\text{enc}} = 0$ (Prob. 4.14), so $\boxed{\mathbf{E} = \mathbf{0}.}$

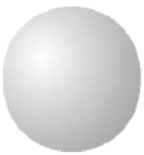
For $a < r < b$, $Q_{\text{enc}} = \left(\frac{-k}{a}\right)(4\pi a^2) + \int_a^r \left(\frac{-k}{\bar{r}^2}\right) 4\pi\bar{r}^2 d\bar{r} = -4\pi ka - 4\pi k(r-a) = -4\pi kr$; so $\boxed{\mathbf{E} = -(k/\epsilon_0 r)\,\hat{\mathbf{r}}.}$

(b) Use Eq. 4.23 to find **D**, and then get **E** from Eq. 4.21. [Notice that the second method is much faster, and it avoids any explicit reference to the bound charges.]

(b) $\oint \mathbf{D}\cdot d\mathbf{a} = Q_{f_{\text{enc}}} = 0 \Rightarrow \mathbf{D} = \mathbf{0}$ everywhere. $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \mathbf{0} \Rightarrow \mathbf{E} = (-1/\epsilon_0)\mathbf{P}$, so

$\boxed{\mathbf{E} = \mathbf{0} \text{ (for } r < a \text{ and } r > b);}$ $\boxed{\mathbf{E} = -(k/\epsilon_0 r)\,\hat{\mathbf{r}} \text{ (for } a < r < b).}$

Problem 4.16 Suppose the field inside a large piece of dielectric is $\mathbf{E}_0$, so that the electric displacement is $\mathbf{D}_0 = \epsilon_0 \mathbf{E}_0 + \mathbf{P}$.



(a) Sphere

(a) Now a small spherical cavity (Fig. 4.19a) is hollowed out of the material. Find the field at the center of the cavity in terms of $\mathbf{E}_0$ and $\mathbf{P}$. Also find the displacement at the center of the cavity in terms of $\mathbf{D}_0$ and $\mathbf{P}$. Assume the polarization is "frozen in," so it doesn't change when the cavity is excavated.

(b) Needle

(b) Do the same for a long needle-shaped cavity running parallel to $\mathbf{P}$ (Fig. 4.19b).

(c) Do the same for a thin wafer-shaped cavity perpendicular to $\mathbf{P}$ (Fig. 4.19c).

Assume the cavities are small enough that $\mathbf{P}$, $\mathbf{E}_0$, and $\mathbf{D}_0$ are essentially uniform. [*Hint:* Carving out a cavity is the same as superimposing an object of the same shape but opposite polarization.]

(c) Wafer

(a) Same as $\mathbf{E}_0$ minus the field at the center of a sphere with uniform polarization $\mathbf{P}$. The latter (Eq. 4.14) is $-\mathbf{P}/3\epsilon_0$. So $\boxed{\mathbf{E} = \mathbf{E}_0 + \frac{1}{3\epsilon_0}\mathbf{P}.}$ $\mathbf{D} = \epsilon_0\mathbf{E} = \epsilon_0\mathbf{E}_0 + \frac{1}{3}\mathbf{P} = \mathbf{D}_0 - \mathbf{P} + \frac{1}{3}\mathbf{P}$, so $\boxed{\mathbf{D} = \mathbf{D}_0 - \frac{2}{3}\mathbf{P}.}$

(b) Same as $\mathbf{E}_0$ minus the field of $\pm$ charges at the two ends of the "needle"—but these are small, and far away, so $\boxed{\mathbf{E} = \mathbf{E}_0.}$ $\mathbf{D} = \epsilon_0\mathbf{E} = \epsilon_0\mathbf{E}_0 = \mathbf{D}_0 - \mathbf{P}$, so $\boxed{\mathbf{D} = \mathbf{D}_0 - \mathbf{P}.}$

(c) Same as $\mathbf{E}_0$ minus the field of a parallel-plate capacitor with upper plate at $\sigma = P$. The latter is $-(1/\epsilon_0)P$, so $\boxed{\mathbf{E} = \mathbf{E}_0 + \frac{1}{\epsilon_0}\mathbf{P}.}$ $\mathbf{D} = \epsilon_0\mathbf{E} = \epsilon_0\mathbf{E}_0 + \mathbf{P}$, so $\boxed{\mathbf{D} = \mathbf{D}_0.}$

4.3.2 A Deceptive Parallel

there is no "Coulomb's law" for **D**:

$$\boxed{\nabla \cdot \mathbf{D} = \rho_f,} \longrightarrow \mathbf{D}(\mathbf{r}) \neq \frac{1}{4\pi}\int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\rho_f(\mathbf{r}')\,d\tau'$$

curl of **D** is *not* always zero $\quad \nabla \times \mathbf{D} = \epsilon_0(\nabla \times \mathbf{E}) + (\nabla \times \mathbf{P}) = \nabla \times \mathbf{P}$

$\nabla \times \mathbf{D} \neq 0,$ **D** cannot be expressed as the gradient of a scalar

there is no "potential" for **D**.

Advice: When you are asked to compute the electric displacement, first look for symmetry. If the problem exhibits spherical, cylindrical, or plane symmetry, then you can get **D** directly from Eq. 4.23 by the usual Gauss's law methods. (Evidently in such cases $\nabla \times \mathbf{P}$ is automatically zero, but since symmetry alone dictates the answer, you're not really obliged to worry about the curl.) If the requisite symmetry is absent, you'll have to think of another approach, and, in particular, you must *not* assume that **D** is determined exclusively by the free charge.

4.3.3 Boundary Conditions

$$\boxed{\nabla \cdot \mathbf{D} = \rho_f,}$$

$$D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f$$

$$\nabla \times \mathbf{D} = \epsilon_0(\nabla \times \mathbf{E}) + (\nabla \times \mathbf{P}) = \nabla \times \mathbf{P}$$

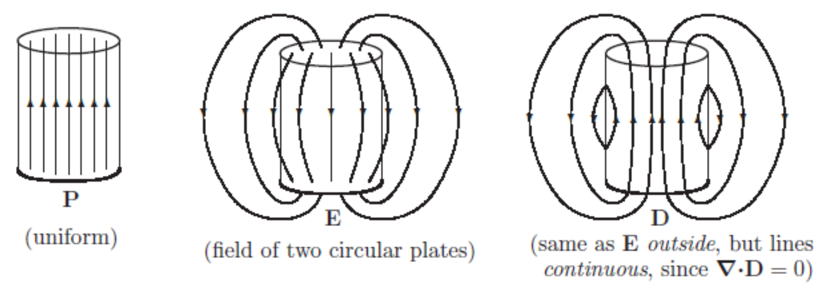
$$\mathbf{D}^{\parallel}_{\text{above}} - \mathbf{D}^{\parallel}_{\text{below}} = \mathbf{P}^{\parallel}_{\text{above}} - \mathbf{P}^{\parallel}_{\text{below}}$$

In the presence of dielectrics, also

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{1}{\epsilon_0}\sigma$$

$$\mathbf{E}^{\parallel}_{\text{above}} - \mathbf{E}^{\parallel}_{\text{below}} = \mathbf{0}$$

Problem 4.17 For the bar electret of Prob. 4.11, make three careful sketches: one of $\mathbf{P}$, one of $\mathbf{E}$, and one of $\mathbf{D}$. Assume L is about $2a$. [*Hint:* $\mathbf{E}$ lines terminate on charges; $\mathbf{D}$ lines terminate on *free* charges.]



4.4 Linear Dielectrics

4.4.1 Susceptibility, Permittivity, Dielectric Constant
4.4.2 Boundary Value Problems with Linear Dielectrics
4.4.3 Energy in Dielectric Systems
4.4.4 Forces on Dielectrics

4.4.1 Susceptibility, Permittivity, Dielectric Constant

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

χ_e Electric susceptibility (dimensionless)

χ_e depends on the microscopic structure of the substance

linear dielectrics

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} = \epsilon_0 (1 + \chi_e) \mathbf{E},$$

$$\mathbf{D} = \epsilon \mathbf{E} \qquad \epsilon \equiv \epsilon_0 (1 + \chi_e)$$

permittivity

$$\varepsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{N \cdot m^2}$$

$$\varepsilon_r = 1 + \chi_e = \frac{\varepsilon}{\varepsilon_0}$$

Relative permittivity

Material	Dielectric Constant	Material	Dielectric Constant
Vacuum	1	Benzene	2.28
Helium	1.000065	Diamond	5.7-5.9
Neon	1.00013	Salt	5.9
Hydrogen (H_2)	1.000254	Silicon	11.7
Argon	1.000517	Methanol	33.0
Air (dry)	1.000536	Water	80.1
Nitrogen (N_2)	1.000548	Ice (-30° C)	104
Water vapor (100° C)	1.00589	$KTaNbO_3$ (0° C)	34,000

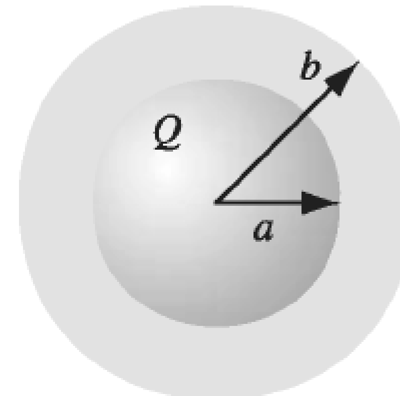
TABLE 4.2 Dielectric Constants (unless otherwise specified, values given are for 1 atm, 20° C). *Data from Handbook of Chemistry and Physics*, 91st ed. (Boca Raton: CRC Press, 2010).

Note that **E** in Eq. 4.30 is the *total* field; it may be due in part to free charges and in part to the polarization itself. If, for instance, we put a piece of dielectric into an external field $\mathbf{E}_0$, we cannot compute **P** directly from Eq. 4.30; the external field will polarize the material, and this polarization will produce its own field, which then contributes to the total field, and this in turn modifies the polarization, which ... Breaking out of this infinite regress is not always easy. You'll see some examples in a moment. The simplest approach is to begin with the *displacement*, at least in those cases where **D** can be deduced directly from the free charge distribution.

[7] In modern optical applications, especially, *non*linear materials have become increasingly important. For these there is a second term in the formula for **P** as a function of **E**—typically a *cubic* term. In general, Eq. 4.30 can be regarded as the first (nonzero) term in the Taylor expansion of **P** in powers of **E**.

[8] As long as we are engaged in this orgy of unnecessary terminology and notation, I might as well mention that formulas for **D** in terms of **E** (Eq. 4.32, in the case of linear dielectrics) are called **constitutive relations**.

Example 4.5. A metal sphere of radius a carries a charge Q (Fig. 4.20). It is surrounded, out to radius b, by linear dielectric material of permittivity ϵ. Find the potential at the center (relative to infinity).



To compute V, we need to know $\mathbf{E}$; to find $\mathbf{E}$, we might first try to locate the bound charge; we could get the bound charge from $\mathbf{P}$, but we can't calculate $\mathbf{P}$ unless we already know $\mathbf{E}$ (Eq. 4.30). We seem to be in a bind. What we *do* know is the *free* charge Q, and fortunately the arrangement is spherically symmetric, so let's begin by calculating $\mathbf{D}$, using Eq. 4.23:

$$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}}, \quad \text{for all points } r > a.$$

(Inside the metal sphere, of course, $\mathbf{E} = \mathbf{P} = \mathbf{D} = \mathbf{0}$.)

$$\mathbf{D} = \epsilon\mathbf{E}, \quad \longrightarrow \quad \mathbf{E} = \begin{cases} \dfrac{Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}, & \text{for } a < r < b, \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}, & \text{for } r > b. \end{cases}$$

$$V = -\int_\infty^0 \mathbf{E}\cdot d\mathbf{l} = -\int_\infty^b \left(\frac{Q}{4\pi\epsilon_0 r^2}\right) dr - \int_b^a \left(\frac{Q}{4\pi\epsilon r^2}\right) dr - \int_a^0 (0)\, dr \quad = \frac{Q}{4\pi}\left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b}\right)$$

As it turns out, it was not necessary for us to compute the polarization or the bound charge explicitly, though this can easily be done:

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} = \frac{\epsilon_0 \chi_e Q}{4\pi \epsilon r^2} \hat{\mathbf{r}},$$ in the dielectric

$$\rho_b = -\nabla \cdot \mathbf{P} = 0$$

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = \begin{cases} \dfrac{\epsilon_0 \chi_e Q}{4\pi \epsilon b^2}, & \text{at the outer surface,} \\ \dfrac{-\epsilon_0 \chi_e Q}{4\pi \epsilon a^2}, & \text{at the inner surface.} \end{cases}$$

surface bound charge at a is *negative*

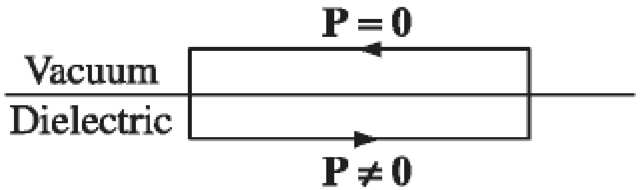
($\hat{\mathbf{n}}$ points outward *with respect to the dielectric,* which is $+\hat{\mathbf{r}}$ at b but $-\hat{\mathbf{r}}$ at a)

the charge on the metal sphere attracts its opposite in all the dielectric molecules.

It is this layer of negative charge that reduces the field, within the dielectric, $1/4\pi\epsilon_0(Q/r^2)\hat{\mathbf{r}}$ to $1/4\pi\epsilon(Q/r^2)\hat{\mathbf{r}}$.

a dielectric is rather like an imperfect conductor: the dielectric cancellation is only partial.

on a *conducting* shell the induced surface charge would cancel the field of Q *completely* in the region $a < r < b$



$$\oint \mathbf{P} \cdot d\mathbf{l} \neq 0, \qquad \nabla \times P \neq 0$$

In homogeneous linear dielectric $\quad \nabla \cdot \mathbf{D} = \rho_f \quad \text{and} \quad \nabla \times \mathbf{D} = \mathbf{0},$

D can be found from the free charge just as though the dielectric were not there:

$$\mathbf{D} = \epsilon_0 \mathbf{E}_{\text{vac}}$$

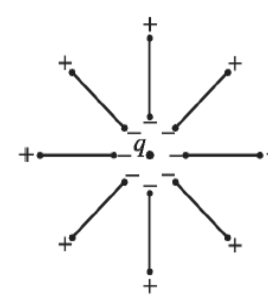
where $\mathbf{E}_{\text{vac}}$ is the field the same free charge distribution would produce in the absence of any dielectric.

$$\mathbf{E} = \frac{1}{\epsilon}\mathbf{D} = \frac{1}{\epsilon_r}\mathbf{E}_{\text{vac}}$$

field everywhere is simply reduced by a factor of one over the dielectric constant.

if a free charge q is embedded in a large dielectric, $\quad \mathbf{E} = \frac{1}{4\pi\epsilon}\frac{q}{r^2}\hat{\mathbf{r}}$

(that's ϵ, not ϵ_0), and the force it exerts on nearby charges is reduced accordingly. But it's not that there is anything wrong with Coulomb's law; rather, the polarization of the medium partially "shields" the charge, by surrounding it with bound charge of the opposite sign (Fig. 4.22).[11]

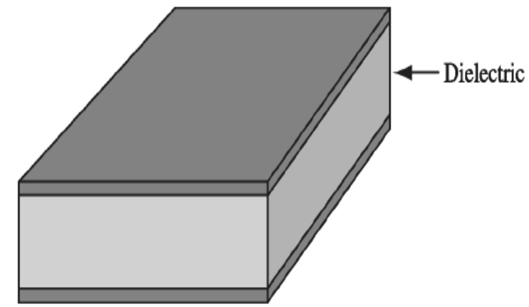


Example 4.6. A parallel-plate capacitor (Fig. 4.23) is filled with insulating material of dielectric constant ϵ_r. What effect does this have on its capacitance?

Since the field is confined to the space between the plates, the dielectric will reduce $\mathbf{E}$, and hence also the potential difference V, by a factor $1/\epsilon_r$. Accordingly, the capacitance $C = Q/V$ is *increased by a factor of the dielectric constant*,

$$C = \epsilon_r C_{\text{vac}}$$

This is, in fact, a common way to beef up a capacitor.

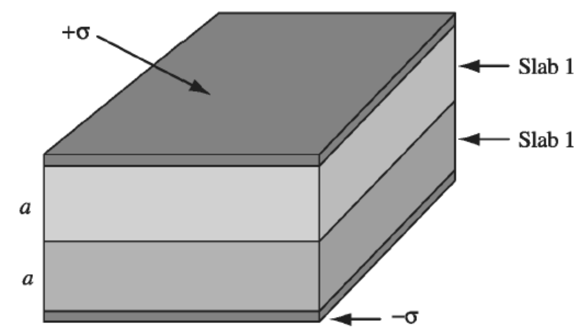


A *crystal* is generally easier to polarize in some directions than in others,[12] and in this case Eq. 4.30 is replaced by the general linear relation

$$\left.\begin{aligned}
P_x &= \epsilon_0(\chi_{e_{xx}} E_x + \chi_{e_{xy}} E_y + \chi_{e_{xz}} E_z)\\
P_y &= \epsilon_0(\chi_{e_{yx}} E_x + \chi_{e_{yy}} E_y + \chi_{e_{yz}} E_z)\\
P_z &= \epsilon_0(\chi_{e_{zx}} E_x + \chi_{e_{zy}} E_y + \chi_{e_{zz}} E_z)
\end{aligned}\right\}, \tag{4.38}$$

just as Eq. 4.1 was superseded by Eq. 4.3 for asymmetrical molecules. The nine coefficients, $\chi_{e_{xx}}, \chi_{e_{xy}}, \ldots,$ constitute the **susceptibility tensor.**

Problem 4.18 The space between the plates of a parallel-plate capacitor (Fig. 4.24) is filled with two slabs of linear dielectric material. Each slab has thickness a, so the total distance between the plates is $2a$. Slab 1 has a dielectric constant of 2, and slab 2 has a dielectric constant of 1.5. The free charge density on the top plate is σ and on the bottom plate $-\sigma$.



(a) Find the electric displacement **D** in each slab.

(b) Find the electric field **E** in each slab.

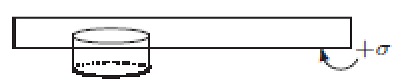
(c) Find the polarization **P** in each slab.

(d) Find the potential difference between the plates.

(e) Find the location and amount of all bound charge.

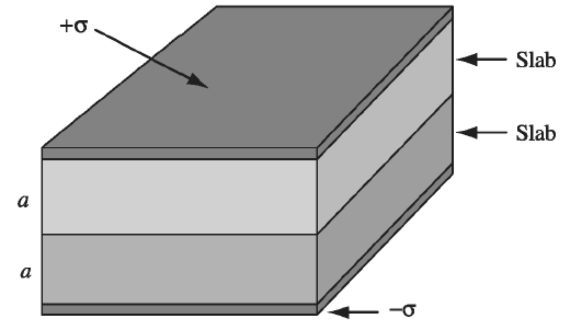
(f) Now that you know all the charge (free and bound), recalculate the field in each slab, and confirm your answer to (b).

(a) Apply $\int \mathbf{D} \cdot d\mathbf{a} = Q_{f_{enc}}$ to the gaussian surface shown. $DA = \sigma A \Rightarrow \boxed{D = \sigma.}$ (*Note:* $\mathbf{D} = \mathbf{0}$ inside the metal plate.) This is true in both slabs; **D** points *down*.



(b) $\mathbf{D} = \epsilon \mathbf{E} \Rightarrow E = \sigma/\epsilon_1$ in slab 1, $E = \sigma/\epsilon_2$ in slab 2. But $\epsilon = \epsilon_0 \epsilon_r$, so $\epsilon_1 = 2\epsilon_0$; $\epsilon_2 = \frac{3}{2}\epsilon_0$. $\boxed{E_1 = \sigma/2\epsilon_0,}$ $\boxed{E_2 = 2\sigma/3\epsilon_0.}$

(c) $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$, so $P = \epsilon_0 \chi_e d/(\epsilon_0 \epsilon_r) = (\chi_e/\epsilon_r)\sigma$; $\chi_e = \epsilon_r - 1 \Rightarrow P = (1 - \epsilon_r^{-1})\sigma$. $\boxed{P_1 = \sigma/2,}$ $\boxed{P_2 = \sigma/3.}$



(d) Find the potential difference between the plates.

(e) Find the location and amount of all bound charge.

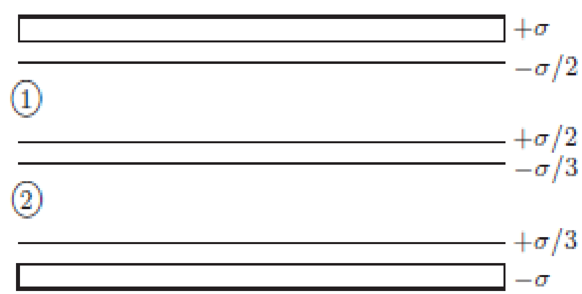
(f) Now that you know all the charge (free and bound), recalculate the field in each slab, and confirm your answer to (b).

(d) $V = E_1 a + E_2 a = (\sigma a/6\epsilon_0)(3+4) = \boxed{7\sigma a/6\epsilon_0.}$

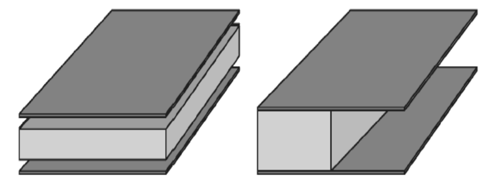
(e) $\rho_b = 0;$ $\boxed{\begin{array}{l}\sigma_b = +P_1 \text{ at bottom of slab (1)} = \sigma/2, \\ \sigma_b = -P_1 \text{ at top of slab (1)} = -\sigma/2;\end{array}}$ $\boxed{\begin{array}{l}\sigma_b = +P_2 \text{ at bottom of slab (2)} = \sigma/3, \\ \sigma_b = -P_2 \text{ at top of slab (2)} = -\sigma/3.\end{array}}$

(f) In slab 1: $\left\{\begin{array}{l}\text{total surface charge } \textit{above}\text{: } \sigma - (\sigma/2) = \sigma/2, \\ \text{total surface charge } \textit{below}\text{: } (\sigma/2) - (\sigma/3) + (\sigma/3) - \sigma = -\sigma/2,\end{array}\right\} \Longrightarrow E_1 = \dfrac{\sigma}{2\epsilon_0}.\ \checkmark$

In slab 2: $\left\{\begin{array}{l}\text{total surface charge } \textit{above}\text{: } \sigma - (\sigma/2) + (\sigma/2) - (\sigma/3) = 2\sigma/3, \\ \text{total surface charge } \textit{below}\text{: } (\sigma/3) - \sigma = -2\sigma/3,\end{array}\right\} \Longrightarrow E_2 = \dfrac{2\sigma}{3\epsilon_0}.\ \checkmark$



Problem 4.19 Suppose you have enough linear dielectric material, of dielectric constant ϵ_r, to *half*-fill a parallel-plate capacitor (Fig. 4.25). By what fraction is the capacitance increased when you distribute the material as in Fig. 4.25(a)? How about Fig. 4.25(b)? For a given potential difference V between the plates, find $\mathbf{E}$, $\mathbf{D}$, and $\mathbf{P}$, in each region, and the free and bound charge on all surfaces, for both cases.



(a) (b)

With no dielectric, $C_0 = A\epsilon_0/d$ (Eq. 2.54).

In configuration (a), with $+\sigma$ on upper plate, $-\sigma$ on lower, $D = \sigma$ between the plates. $E = \sigma/\epsilon_0$ (in air) and $E = \sigma/\epsilon$ (in dielectric). So $V = \frac{\sigma}{\epsilon_0}\frac{d}{2} + \frac{\sigma}{\epsilon}\frac{d}{2} = \frac{Qd}{2\epsilon_0 A}\left(1 + \frac{\epsilon_0}{\epsilon}\right)$.

$$C_a = \frac{Q}{V} = \frac{\epsilon_0 A}{d}\left(\frac{2}{1+1/\epsilon_r}\right) \Longrightarrow \boxed{\frac{C_a}{C_0} = \frac{2\epsilon_r}{1+\epsilon_r}.}$$

In configuration (b), with potential difference V: $E = V/d$, so $\sigma = \epsilon_0 E = \epsilon_0 V/d$ (in air). $P = \epsilon_0\chi_e E = \epsilon_0\chi_e V/d$ (in dielectric), so $\sigma_b = -\epsilon_0\chi_e V/d$ (at top surface of dielectric). $\sigma_{\text{tot}} = \epsilon_0 V/d = \sigma_f + \sigma_b = \sigma_f - \epsilon_0\chi_e V/d$, so $\sigma_f = \epsilon_0 V(1+\chi_e)/d = \epsilon_0\epsilon_r V/d$ (on top plate above dielectric).

$$\Longrightarrow C_b = \frac{Q}{V} = \frac{1}{V}\left(\sigma\frac{A}{2} + \sigma_f\frac{A}{2}\right) = \frac{A}{2V}\left(\epsilon_0\frac{V}{d} + \epsilon_0\frac{V}{d}\epsilon_r\right) = \frac{A\epsilon_0}{d}\left(\frac{1+\epsilon_r}{2}\right). \quad \boxed{\frac{C_b}{C_0} = \frac{1+\epsilon_r}{2}.}$$

[Which is greater? $\frac{C_b}{C_0} - \frac{C_a}{C_0} = \frac{1+\epsilon_r}{2} - \frac{2\epsilon_r}{1+\epsilon_r} = \frac{(1+\epsilon_r)^2 - 4\epsilon_r}{2(1+\epsilon_r)} = \frac{1+2\epsilon_r+4\epsilon_r^2-4\epsilon_r}{2(1+\epsilon_r)} = \frac{(1-\epsilon_r)^2}{2(1+\epsilon_r)} > 0$. So $C_b > C_a$.]

If the x axis points *down*:

	σ_b (top surface)	σ_f (top plate)
(a)	$-\frac{2(\epsilon_r-1)}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}$	$\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}$
(b)	$-(\epsilon_r - 1)\frac{\epsilon_0 V}{d}$	$\epsilon_r\frac{\epsilon_0 V}{d}$ (left); $\frac{\epsilon_0 V}{d}$ (right)

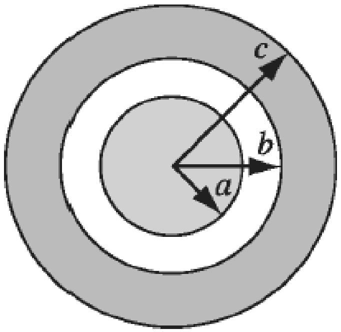
	E	**D**	**P**
(a) air	$\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{V}{d}\,\hat{\mathbf{x}}$	$\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	0
(a) dielectric	$\frac{2}{(\epsilon_r+1)}\frac{V}{d}\,\hat{\mathbf{x}}$	$\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	$\frac{2(\epsilon_r-1)}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$
(b) air	$\frac{V}{d}\,\hat{\mathbf{x}}$	$\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	0
(b) dielectric	$\frac{V}{d}\,\hat{\mathbf{x}}$	$\epsilon_r\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	$(\epsilon_r - 1)\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$

Problem 4.20 A sphere of linear dielectric material has embedded in it a uniform free charge density ρ. Find the potential at the center of the sphere (relative to infinity), if its radius is R and the dielectric constant is ϵ_r.

$\int \mathbf{D}\cdot d\mathbf{a} = Q_{f_{\text{enc}}} \Rightarrow D4\pi r^2 = \rho\frac{4}{3}\pi r^3 \Rightarrow D = \frac{1}{3}\rho r \Rightarrow \mathbf{E} = (\rho r/3\epsilon)\,\hat{\mathbf{r}}$, for $r < R$; $D4\pi r^2 = \rho\frac{4}{3}\pi R^3 \Rightarrow D = \rho R^3/3r^2 \Rightarrow \mathbf{E} = (\rho R^3/3\epsilon_0 r^2)\,\hat{\mathbf{r}}$, for $r > R$.

$$V = -\int_\infty^0 \mathbf{E}\cdot d\mathbf{l} = \frac{\rho R^3}{3\epsilon_0}\left.\frac{1}{r}\right|_\infty^R - \frac{\rho}{3\epsilon}\int_R^0 r\,dr = \frac{\rho R^2}{3\epsilon_0} + \frac{\rho}{3\epsilon}\frac{R^2}{2} = \boxed{\frac{\rho R^2}{3\epsilon_0}\left(1 + \frac{1}{2\epsilon_r}\right)}.$$

Problem 4.21 A certain coaxial cable consists of a copper wire, radius a, surrounded by a concentric copper tube of inner radius c (Fig. 4.26). The space between is partially filled (from b out to c) with material of dielectric constant ϵ_r, as shown. Find the capacitance per unit length of this cable.



Let Q be the charge on a length ℓ of the inner conductor.

$$\oint \mathbf{D}\cdot d\mathbf{a} = D2\pi s\ell = Q \Rightarrow D = \frac{Q}{2\pi s\ell};$$

$$E = \frac{Q}{2\pi\epsilon_0 s\ell}\ (a < s < b),$$

$$E = \frac{Q}{2\pi\epsilon s\ell}\ (b < r < c)$$

$$V = -\int_c^a \mathbf{E}\cdot d\mathbf{l} = \int_a^b \left(\frac{Q}{2\pi\epsilon_0\ell}\right)\frac{ds}{s} + \int_b^c \left(\frac{Q}{2\pi\epsilon\ell}\right)\frac{ds}{s} = \frac{Q}{2\pi\epsilon_0\ell}\left[\ln\left(\frac{b}{a}\right) + \frac{\epsilon_0}{\epsilon}\ln\left(\frac{c}{b}\right)\right]$$

$$\frac{C}{\ell} = \frac{Q}{V\ell} = \boxed{\frac{2\pi\epsilon_0}{\ln(b/a) + (1/\epsilon_r)\ln(c/b)}}.$$

4.4.2 Boundary Value Problems with Linear Dielectrics

In a (homogeneous isotropic) linear dielectric,

bound charge density (ρ_b) ~ free charge density (ρ_f)

$$\rho_b = -\nabla \cdot \mathbf{P} = -\nabla \cdot \left(\epsilon_0 \frac{\chi_e}{\epsilon} \mathbf{D}\right) = -\left(\frac{\chi_e}{1+\chi_e}\right) \rho_f$$

any net charge must reside at the surface.

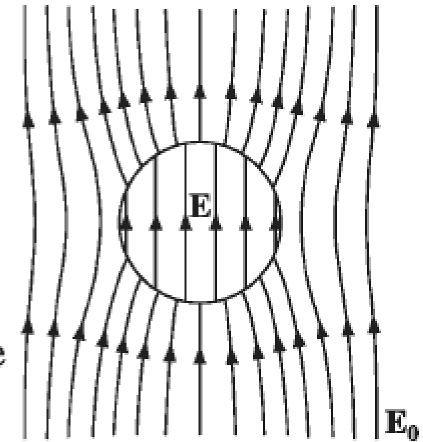
Within such a dielectric, potential obeys Laplace's equation

boundary conditions $\epsilon_{\text{above}} E_{\text{above}}^{\perp} - \epsilon_{\text{below}} E_{\text{below}}^{\perp} = \sigma_f$

$$\epsilon_{\text{above}} \frac{\partial V_{\text{above}}}{\partial n} - \epsilon_{\text{below}} \frac{\partial V_{\text{below}}}{\partial n} = -\sigma_f \qquad V_{\text{above}} = V_{\text{below}}$$

Example 4.7. A sphere of homogeneous linear dielectric material is placed in an otherwise uniform electric field $\mathbf{E}_0$ (Fig. 4.27). Find the electric field inside the sphere.

Similar to Ex. 3.8, Conductor sphere



the field of the induced charge canceled $\mathbf{E}_0$ within the sphere;

in a *dielectric*, the cancellation (from the bound charge) is incomplete.

solve Laplace's equation (i) $V_{\text{in}} = V_{\text{out}}$, at $r = R$,

B.C (ii) $\epsilon \dfrac{\partial V_{\text{in}}}{\partial r} = \epsilon_0 \dfrac{\partial V_{\text{out}}}{\partial r}$, at $r = R$, no free charge at the surface

(iii) $V_{\text{out}} \to -E_0 r \cos\theta$, for $r \gg R$.

$r \le R$ $\quad V_{\text{in}}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$

$r \ge R$ $\quad V_{\text{out}}(r,\theta) = -E_0 r\cos\theta + \sum_{l=0}^{\infty} \dfrac{B_l}{r^{l+1}} P_l(\cos\theta)$

(i) → $\displaystyle\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = -E_0 R\cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) \longrightarrow \begin{cases} A_l R^l = \dfrac{B_l}{R^{l+1}}, & \text{for } l \neq 1, \\ A_1 R = -E_0 R + \dfrac{B_1}{R^2}. \end{cases}$

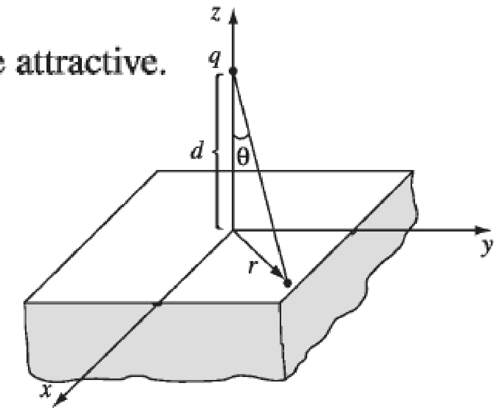
(ii) → $\displaystyle\epsilon_r \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = -E_0\cos\theta - \sum_{l=0}^{\infty} \frac{(l+1)B_l}{R^{l+2}} P_l(\cos\theta) \longrightarrow \begin{cases} \epsilon_r l A_l R^{l-1} = -\dfrac{(l+1)B_l}{R^{l+2}}, & \text{for } l \neq 1, \\ \epsilon_r A_1 = -E_0 - \dfrac{2B_1}{R^3}. \end{cases}$

$\longrightarrow \quad A_l = B_l = 0, \quad \text{for } l \neq 1, \quad A_1 = -\dfrac{3}{\epsilon_r + 2} E_0 \quad B_1 = \dfrac{\epsilon_r - 1}{\epsilon_r + 2} R^3 E_0$

$$V_{\text{in}}(r,\theta) = -\frac{3E_0}{\epsilon_r + 2} r\cos\theta = -\frac{3E_0}{\epsilon_r + 2} z \qquad \mathbf{E} = \frac{3}{\epsilon_r + 2}\mathbf{E}_0$$

uniform field inside the sphere!

Example 4.8. Suppose the entire region below the plane $z = 0$ in Fig. 4.28 is filled with uniform linear dielectric material of susceptibility χ_e. Calculate the force on a point charge q situated a distance d above the origin.



The surface bound charge on the xy plane is of opposite sign to q, so the force will be attractive.

$\rho_b = -\nabla \cdot \mathbf{P} = -\nabla \cdot \left(\epsilon_0 \frac{\chi_e}{\epsilon}\mathbf{D}\right) = -\left(\frac{\chi_e}{1+\chi_e}\right)\rho_f$ there is no volume bound charge

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = P_z = \epsilon_0 \chi_e E_z$$

$\uparrow z = 0$

due to q and the bound charge

E_q: $$-\frac{1}{4\pi\epsilon_0}\frac{q}{(r^2+d^2)}\cos\theta = -\frac{1}{4\pi\epsilon_0}\frac{qd}{(r^2+d^2)^{3/2}} \qquad r = \sqrt{x^2+y^2}$$

E_{σ_b}: The z component of the field of the bound charge is $-\sigma_b/2\epsilon_0$

$$\sigma_b = \epsilon_0\chi_e\left[-\frac{1}{4\pi\epsilon_0}\frac{qd}{(r^2+d^2)^{3/2}} - \frac{\sigma_b}{2\epsilon_0}\right]$$

solve for σ_b: $$\sigma_b = -\frac{1}{2\pi}\left(\frac{\chi_e}{\chi_e+2}\right)\frac{qd}{(r^2+d^2)^{3/2}}$$ Vs. grounded conductor $$\sigma_b = -\frac{1}{2\pi}\frac{qd}{(r^2+d^2)^{3/2}}$$

total bound charge is $$q_b = -\left(\frac{\chi_e}{\chi_e+2}\right)q$$

the field of σ_b $$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\sigma_b\,da$$

Or by method of image

by method of image

replace the dielectric by a single point charge q_b at the image position $(0, 0, -d)$

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{q_b}{\sqrt{x^2+y^2+(z+d)^2}}\right] \qquad z > 0 \qquad (4.52)$$

a charge $(q + q_b)$ at $(0, 0, d)$

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{q+q_b}{\sqrt{x^2+y^2+(z-d)^2}}\right] \qquad z < 0 \qquad (4.53)$$

Eqs. 4.52 and 4.53 constitute a function that satisfies Poisson's equation with a point charge q at $(0, 0, d)$

zero at infinity, continuous at the boundary $z = 0$, normal derivative discontinuity σ_b at $z = 0$

$$-\epsilon_0\left(\left.\frac{\partial V}{\partial z}\right|_{z=0^+} - \left.\frac{\partial V}{\partial z}\right|_{z=0^-}\right) = -\frac{1}{2\pi}\left(\frac{\chi_e}{\chi_e+2}\right)\frac{qd}{(x^2+y^2+d^2)^{3/2}}$$

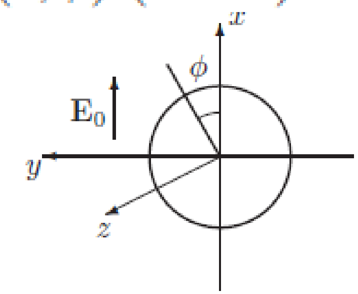
$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qq_b}{(2d)^2}\hat{\mathbf{z}} = -\frac{1}{4\pi\epsilon_0}\left(\frac{\chi_e}{\chi_e+2}\right)\frac{q^2}{4d^2}\hat{\mathbf{z}}$$

Problem 4.22 A very long cylinder of linear dielectric material is placed in an otherwise uniform electric field $\mathbf{E}_0$. Find the resulting field within the cylinder. (The radius is a, the susceptibility χ_e, and the axis is perpendicular to $\mathbf{E}_0$.)

as Ex. 4.7: solve Laplace's equation for $V_{\text{in}}(s,\phi)$ $(s<a)$ and $V_{\text{out}}(s,\phi)$ $(s>a)$

$$\begin{cases} \text{(i)} & V_{\text{in}} = V_{\text{out}} & \text{at } s=a, \\ \text{(ii)} & \epsilon\frac{\partial V_{\text{in}}}{\partial s} = \epsilon_0\frac{\partial V_{\text{out}}}{\partial s} & \text{at } s=a, \\ \text{(iii)} & V_{\text{out}} \to -E_0 s\cos\phi & \text{for } s \gg a. \end{cases}$$

From Prob. 3.24



$$\boxed{V(s,\phi) = a_0 + b_0\ln s + \sum_{k=1}^{\infty}\left[s^k(a_k\cos k\phi + b_k\sin k\phi) + s^{-k}(c_k\cos k\phi + d_k\sin k\phi)\right].}$$

eliminated the constant terms by setting $V=0$ on the yz plane

$$V_{\text{in}}(s,\phi) = \sum_{k=1}^{\infty} s^k(a_k\cos k\phi + b_k\sin k\phi),$$

(iii) $\longrightarrow$

$$V_{\text{out}}(s,\phi) = -E_0 s\cos\phi + \sum_{k=1}^{\infty} s^{-k}(c_k\cos k\phi + d_k\sin k\phi)$$

(i) $\longrightarrow$ $$\sum a^k(a_k\cos k\phi + b_k\sin k\phi) = -E_0 a\cos\phi + \sum a^{-k}(c_k\cos k\phi + d_k\sin k\phi)$$

(ii) $\longrightarrow$ $$\epsilon_r\sum k a^{k-1}(a_k\cos k\phi + b_k\sin k\phi) = -E_0\cos\phi - \sum k a^{-k-1}(c_k\cos k\phi + d_k\sin k\phi)$$

$b_k = d_k = 0$ for all k,

$a_k = c_k = 0$ unless $k=1$, for $k=1$,

$$aa_1 = -E_0 a + a^{-1}c_1,$$
$$\epsilon_r a_1 = -E_0 - a^{-2}c_1$$

$\longrightarrow$ $$a_1 = -\frac{E_0}{(1+\chi_e/2)},$$

$$V_{\text{in}}(s,\phi) = -\frac{E_0}{(1+\chi_e/2)}s\cos\phi = -\frac{E_0}{(1+\chi_e/2)}x, \qquad \mathbf{E}_{\text{in}}(s,\phi) = -\frac{\partial V_{\text{in}}}{\partial x}\hat{\mathbf{x}} = \boxed{\frac{\mathbf{E}_0}{(1+\chi_e/2)}}.$$

As in the spherical case (Ex. 4.7) the field inside is *uniform*.

Problem 4.23 Find the field inside a sphere of linear dielectric material in an otherwise uniform electric field $\mathbf{E}_0$ (Ex. 4.7) by the following method of successive approximations: First pretend the field inside is just $\mathbf{E}_0$, and use Eq. 4.30 to write down the resulting polarization $\mathbf{P}_0$. This polarization generates a field of its own, $\mathbf{E}_1$ (Ex. 4.2), which in turn modifies the polarization by an amount $\mathbf{P}_1$, which further changes the field by an amount $\mathbf{E}_2$, and so on. The resulting field is $\mathbf{E}_0 + \mathbf{E}_1 + \mathbf{E}_2 + \cdots$. Sum the series, and compare your answer with Eq. 4.49.

$\mathbf{P}_0 = \epsilon_0\chi_e\mathbf{E}_0$; $\mathbf{E}_1 = -\frac{1}{3\epsilon_0}\mathbf{P}_0 = -\frac{\chi_e}{3}\mathbf{E}_0$; $\mathbf{P}_1 = \epsilon_0\chi_e\mathbf{E}_1 = -\frac{\epsilon_0\chi_e^2}{3}\mathbf{E}_0$; $\mathbf{E}_2 = -\frac{1}{3\epsilon_0}\mathbf{P}_1 = \frac{\chi_e^2}{9}\mathbf{E}_0$; .... Evidently $\mathbf{E}_n = \left(-\frac{\chi_e}{3}\right)^n \mathbf{E}_0$, so

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1 + \mathbf{E}_2 + \cdots = \left[\sum_{n=0}^{\infty}\left(-\frac{\chi_e}{3}\right)^n\right]\mathbf{E}_0.$$

The geometric series can be summed explicitly:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \text{so} \quad \boxed{\mathbf{E} = \frac{1}{(1+\chi_e/3)}\mathbf{E}_0,}$$

which agrees with Eq. 4.49. [Curiously, this method formally requires that $\chi_e < 3$ (else the infinite series diverges), yet the *result* is subject to no such restriction, since we can also get it by the method of Ex. 4.7.]

Problem 4.24 An uncharged conducting sphere of radius a is coated with a thick insulating shell (dielectric constant ϵ_r) out to radius b. This object is now placed in an otherwise uniform electric field $\mathbf{E}_0$. Find the electric field in the insulator.

Potentials:

$$\begin{cases} V_{\text{out}}(r,\theta) &= -E_0 r\cos\theta + \sum \frac{B_l}{r^{l+1}} P_l(\cos\theta), & (r > b); \\ V_{\text{med}}(r,\theta) &= \sum \left(A_l r^l + \frac{\bar{B}_l}{r^{l+1}} \right) P_l(\cos\theta), & (a < r < b); \\ V_{\text{in}}(r,\theta) &= 0, & (r < a). \end{cases}$$

Boundary Conditions:

$$\begin{cases} \text{(i)} & V_{\text{out}} = V_{\text{med}}, & (r = b); \\ \text{(ii)} & \epsilon \frac{\partial V_{\text{med}}}{\partial r} = \epsilon_0 \frac{\partial V_{\text{out}}}{\partial r}, & (r = b); \\ \text{(iii)} & V_{\text{med}} = 0, & (r = a). \end{cases}$$

$$\text{(i)} \Rightarrow -E_0 b\cos\theta + \sum \frac{B_l}{b^{l+1}} P_l(\cos\theta) = \sum \left(A_l b^l + \frac{\bar{B}_l}{b^{l+1}} \right) P_l(\cos\theta);$$

$$\text{(ii)} \Rightarrow \epsilon_r \sum \left[l A_l b^{l-1} - (l+1)\frac{\bar{B}_l}{b^{l+2}} \right] P_l(\cos\theta) = -E_0\cos\theta - \sum (l+1)\frac{B_l}{b^{l+2}} P_l(\cos\theta);$$

$$\text{(iii)} \Rightarrow A_l a^l + \frac{\bar{B}_l}{a^{l+1}} = 0 \Rightarrow \bar{B}_l = -a^{2l+1} A_l.$$

For $l \neq 1$:

$$\text{(i)} \quad \frac{B_l}{b^{l+1}} = \left(A_l b^l - \frac{a^{2l+1} A_l}{b^{l+1}} \right) \Rightarrow B_l = A_l \left(b^{2l+1} - a^{2l+1} \right);$$

$$\text{(ii)} \quad \epsilon_r \left[l A_l b^{l-1} + (l+1)\frac{a^{2l+1} A_l}{b^{l+2}} \right] = -(l+1)\frac{B_l}{b^{l+2}} \Rightarrow B_l = -\epsilon_r A_l \left[\left(\frac{l}{l+1} \right) b^{2l+1} + a^{2l+1} \right] \Rightarrow A_l = B_l = 0.$$

For $l = 1$:

$$\text{(i)} \quad -E_0 b + \frac{B_1}{b^2} = A_1 b - \frac{a^3 A_1}{b^2} \Rightarrow B_1 - E_0 b^3 = A_1 \left(b^3 - a^3 \right);$$

$$\text{(ii)} \quad \epsilon_r \left(A_1 + 2\frac{a^3 A_1}{b^3} \right) = -E_0 - 2\frac{B_1}{b^3} \Rightarrow -2B_1 - E_0 b^3 = \epsilon_r A_1 \left(b^3 + 2a^3 \right)$$

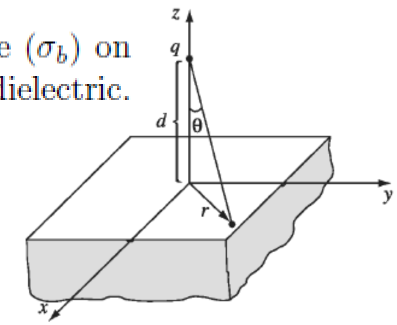
$$-3E_0 b^3 = A_1 \left[2\left(b^3 - a^3\right) + \epsilon_r \left(b^3 + 2a^3\right) \right]$$

$$A_1 = \frac{-3E_0}{2[1 - (a/b)^3] + \epsilon_r [1 + 2(a/b)^3]}.$$

$$V_{\text{med}}(r,\theta) = \frac{-3E_0}{2[1 - (a/b)^3] + \epsilon_r [1 + 2(a/b)^3]} \left(r - \frac{a^3}{r^2} \right) \cos\theta,$$

$$\mathbf{E}(r,\theta) = -\nabla V_{\text{med}} = \boxed{\frac{3E_0}{2[1 - (a/b)^3] + \epsilon_r [1 + 2(a/b)^3]} \left\{ \left(1 + \frac{2a^3}{r^3} \right) \cos\theta\,\hat{\mathbf{r}} - \left(1 - \frac{a^3}{r^3} \right) \sin\theta\,\hat{\boldsymbol{\theta}} \right\}}.$$

Problem 4.25 Suppose the region *above* the xy plane in Ex. 4.8 is *also* filled with linear dielectric but of a different susceptibility χ_e'. Find the potential everywhere.



There are four charges involved: (i) q, (ii) polarization charge surrounding q, (iii) surface charge (σ_b) on the top surface of the lower dielectric, (iv) surface charge (σ_b') on the lower surface of the upper dielectric.

In view of Eq. 4.39, the bound charge (ii) is $q_p = -q(\chi_e'/(1+\chi_e')$, so the *total* (point) charge at $(0,0,d)$ is $q_t = q + q_p = q/(1+\chi_e') = q/\epsilon_r'$. As in Ex. 4.8,

$$\text{(a)}\ \sigma_b = \epsilon_0\chi_e\left[\frac{-1}{4\pi\epsilon_0}\frac{qd/\epsilon_r'}{(r^2+d^2)^{\frac{3}{2}}} - \frac{\sigma_b}{2\epsilon_0} - \frac{\sigma_b'}{2\epsilon_0}\right] \quad (\text{here } \sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = +P_z = \epsilon_0\chi_e E_z);$$

$$\text{(b)}\ \sigma_b' = \epsilon_0\chi_e'\left[\frac{1}{4\pi\epsilon_0}\frac{qd/\epsilon_r'}{(r^2+d^2)^{\frac{3}{2}}} - \frac{\sigma_b}{2\epsilon_0} - \frac{\sigma_b'}{2\epsilon_0}\right] \quad (\text{here } \sigma_b = -P_z = -\epsilon_0\chi_e' E_z).$$

Solve for σ_b, σ_b': first divide by χ_e and χ_e' (respectively) and subtract:

$$\frac{\sigma_b'}{\chi_e'} - \frac{\sigma_b}{\chi_e} = \frac{1}{2\pi}\frac{qd/\epsilon_r'}{(r^2+d^2)^{\frac{3}{2}}} \Rightarrow \sigma_b' = \chi_e'\left[\frac{\sigma_b}{\chi_e} + \frac{1}{2\pi}\frac{qd/\epsilon_r'}{(r^2+d^2)^{\frac{3}{2}}}\right].$$

Plug this into (a) and solve for σ_b, using $\epsilon_r' = 1+\chi_e'$:

$$\sigma_b = \frac{-1}{4\pi}\frac{qd/\epsilon_r'}{(r^2+d^2)^{\frac{3}{2}}}\chi_e(1+\chi_e') - \frac{\sigma_b}{2}(\chi_e+\chi_e'), \text{ so } \boxed{\sigma_b = \frac{-1}{4\pi}\frac{qd}{(r^2+d^2)^{\frac{3}{2}}}\frac{\chi_e}{[1+(\chi_e+\chi_e')/2]}};$$

$$\sigma_b' = \chi_e'\left\{\frac{-1}{4\pi}\frac{qd}{(r^2+d^2)^{\frac{3}{2}}}\frac{1}{[1+(\chi_e+\chi_e')/2]} + \frac{1}{2\pi}\frac{qd/\epsilon_r'}{(r^2+d^2)^{\frac{3}{2}}}\right\}, \text{ so } \boxed{\sigma_b' = \frac{1}{4\pi}\frac{qd}{(r^2+d^2)^{\frac{3}{2}}}\frac{\epsilon_r\chi_e'/\epsilon_r'}{[1+(\chi_e+\chi_e')/2]}}.$$

The *total* bound surface charge is $\sigma_t = \sigma_b + \sigma_b' = \frac{1}{4\pi}\frac{qd}{(r^2+d^2)^{\frac{3}{2}}}\frac{(\chi_e'-\chi_e)}{\epsilon_r'[1+(\chi_e+\chi_e')/2]}$ (which vanishes, as it should, when $\chi_e' = \chi_e$). The total bound charge is (compare Eq. 4.51):

$$q_t = \frac{(\chi_e'-\chi_e)q}{2\epsilon_r'[1+(\chi_e+\chi_e')/2]} = \boxed{\left(\frac{\epsilon_r'-\epsilon_r}{\epsilon_r'+\epsilon_r}\right)\frac{q}{\epsilon_r'}}, \text{ and hence}$$

$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q/\epsilon_r'}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{q_t}{\sqrt{x^2+y^2+(z+d)^2}}\right\}} \quad (\text{for } z>0).$$

Meanwhile, since $\frac{q}{\epsilon_r'} + q_t = \frac{q}{\epsilon_r'}\left[1+\frac{\epsilon_r'-\epsilon_r}{\epsilon_r'+\epsilon_r}\right] = \frac{2q}{\epsilon_r'+\epsilon_r}$, $\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{[2q/(\epsilon_r'+\epsilon_r)]}{\sqrt{x^2+y^2+(z-d)^2}}}$ (for $z<0$).

4.4.3 Energy in Dielectric Systems

work to charge up a capacitor (Eq. 2.55): $W = \frac{1}{2}CV^2$

If the capacitor is filled with linear dielectric, $C = \epsilon_r C_{\text{vac}}$ in Ex. 4.6

more (free) charge, to achieve a given potential, because part of the field is canceled off by the bound charges.

Chapter 2, $$W = \frac{\epsilon_0}{2}\int E^2\, d\tau \quad (4.55)$$

The case of the dielectric-filled capacitor $$W = \frac{\epsilon_0}{2}\int \epsilon_r E^2\, d\tau = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\, d\tau$$

bring in the free charge, $$\Delta W = \int (\Delta\rho_f)V\, d\tau$$

$\nabla\cdot\mathbf{D} = \rho_f$ $\quad \Delta\rho_f = \nabla\cdot(\Delta\mathbf{D})$ $\rightarrow$ $$\Delta W = \int [\nabla\cdot(\Delta\mathbf{D})]V\, d\tau$$

$$\nabla\cdot[(\Delta\mathbf{D})V] = [\nabla\cdot(\Delta\mathbf{D})]V + \Delta\mathbf{D}\cdot(\nabla V)$$

$$\Delta W = \int \nabla\cdot[(\Delta\mathbf{D})V]\, d\tau + \int (\Delta\mathbf{D})\cdot\mathbf{E}\, d\tau$$

Surface integral=0

$$\Delta W = \int (\Delta\mathbf{D})\cdot\mathbf{E}\, d\tau$$

this applies to *any* material.

if the medium is a linear dielectric, $\mathbf{D} = \epsilon\mathbf{E}$

$$\frac{1}{2}\Delta(\mathbf{D}\cdot\mathbf{E}) = \frac{1}{2}\Delta(\epsilon E^2) = \epsilon(\Delta\mathbf{E})\cdot\mathbf{E} = (\Delta\mathbf{D})\cdot\mathbf{E}$$

$$\Delta W = \Delta\left(\frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\, d\tau\right) \longrightarrow W = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\, d\tau \quad (4.58)$$

$$W = \frac{\epsilon_0}{2}\int E^2\, d\tau \tag{4.55}$$

1. We bring in all the charges (free *and* bound), one by one, with tweezers, and glue each one down in its proper final location. If *this* is what you mean by "assemble the system," then Eq. 4.55 is your formula for the energy stored. Notice, however, that this will *not* include the work involved in stretching and twisting the dielectric molecules (if we picture the positive and negative charges as held together by tiny springs, it does not include the spring energy, $\frac{1}{2}kx^2$, associated with polarizing each molecule).[18]

$$W = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\, d\tau \tag{4.58}$$

2. With the unpolarized dielectric in place, we bring in the *free* charges, one by one, allowing the dielectric to respond as it sees fit. If *this* is what you mean by "assemble the system" (and ordinarily it *is,* since free charge is what we actually push around), then Eq. 4.58 is the formula you want. In this case the "spring" energy *is* included, albeit indirectly, because the force you must apply to the *free* charge depends on the disposition of the *bound* charge; as you move the free charge, you are automatically stretching those "springs."

Example 4.9. A sphere of radius R is filled with material of dielectric constant ϵ_r and uniform embedded free charge ρ_f. What is the energy of this configuration?

From Gauss's law

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}} \qquad (4.23)$$

$$\mathbf{D}(r) = \begin{cases} \dfrac{\rho_f}{3}\mathbf{r} & (r < R), \\[2ex] \dfrac{\rho_f}{3}\dfrac{R^3}{r^2}\hat{\mathbf{r}} & (r > R). \end{cases} \longrightarrow \qquad \mathbf{E}(r) = \begin{cases} \dfrac{\rho_f}{3\epsilon_0\epsilon_r}\mathbf{r} & (r < R), \\[2ex] \dfrac{\rho_f}{3\epsilon_0}\dfrac{R^3}{r^2}\hat{\mathbf{r}} & (r > R). \end{cases}$$

The purely *electrostatic* energy (Eq. 4.55) is

$$W_1 = \frac{\epsilon_0}{2}\left[\left(\frac{\rho_f}{3\epsilon_0\epsilon_r}\right)^2 \int_0^R r^2\, 4\pi r^2\, dr + \left(\frac{\rho_f}{3\epsilon_0}\right)^2 R^6 \int_R^\infty \frac{1}{r^4}\, 4\pi r^2\, dr\right] = \frac{2\pi}{9\epsilon_0}\rho_f^2 R^5\left(\frac{1}{5\epsilon_r^2} + 1\right)$$

the *total* energy (Eq. 4.58) is

$$W_2 = \frac{1}{2}\left[\left(\frac{\rho_f}{3}\right)\left(\frac{\rho_f}{3\epsilon_0\epsilon_r}\right) \int_0^R r^2\, 4\pi r^2\, dr + \left(\frac{\rho_f R^3}{3}\right)\left(\frac{\rho_f R^3}{3\epsilon_0}\right) \int_R^\infty \frac{1}{r^4}\, 4\pi r^2\, dr\right] = \frac{2\pi}{9\epsilon_0}\rho_f^2 R^5\left(\frac{1}{5\epsilon_r} + 1\right)$$

$W_1 < W_2$ because W_1 does not include the energy involved in stretching the molecules.

Let's check that W_2 is the work done on the *free* charge in assembling the system.

start with the (uncharged, unpolarized) dielectric sphere,
bring in the free charge in infinitesimal installments (dq),
filling out the sphere layer by layer.
reached radius r',

$$\mathbf{E}(r) = \begin{cases} \dfrac{\rho_f}{3\epsilon_0\epsilon_r}\mathbf{r} & (r < r'), \\[2ex] \dfrac{\rho_f}{3\epsilon_0\epsilon_r}\dfrac{r'^3}{r^2}\hat{\mathbf{r}} & (r' < r < R), \\[2ex] \dfrac{\rho_f}{3\epsilon_0}\dfrac{r'^3}{r^2}\hat{\mathbf{r}} & (r > R). \end{cases}$$

The work required to bring the next dq in from infinity to r' is

$$dW = -dq\left[\int_\infty^R \mathbf{E}\cdot d\mathbf{l} + \int_R^{r'} \mathbf{E}\cdot d\mathbf{l}\right] = -dq\left[\frac{\rho_f r'^3}{3\epsilon_0}\int_\infty^R \frac{1}{r^2}\,dr + \frac{\rho_f r'^3}{3\epsilon_0\epsilon_r}\int_R^{r'} \frac{1}{r^2}\,dr\right] = \frac{\rho_f r'^3}{3\epsilon_0}\left[\frac{1}{R} + \frac{1}{\epsilon_r}\left(\frac{1}{r'} - \frac{1}{R}\right)\right]dq$$

This increases the radius (r'): $dq = \rho_f 4\pi r'^2\,dr'$

the *total* work done, in going from $r' = 0$ to $r' = R$,

$$W = \frac{4\pi\rho_f^2}{3\epsilon_0}\left[\frac{1}{R}\left(1 - \frac{1}{\epsilon_r}\right)\int_0^R r'^5\,dr' + \frac{1}{\epsilon_r}\int_0^R r'^4\,dr'\right] = \frac{2\pi}{9\epsilon_0}\rho_f^2 R^5\left(\frac{1}{5\epsilon_r} + 1\right) = W_2.\ \checkmark$$

the energy "stored in the springs" is $$W_{\text{spring}} = W_2 - W_1 = \frac{2\pi}{45\epsilon_0\epsilon_r^2}\rho_f^2 R^5(\epsilon_r - 1)$$

to confirm this in an explicit model.

the dielectric a collection of tiny proto-dipoles, each consisting of $+q$ and $-q$ attached to a spring of constant k and equilibrium length 0,

in the absence of any field the positive and negative ends coincide.

One end of each dipole is nailed in position (like the nuclei in a solid),

the other end is free to move in response to any imposed field.

Let $d\tau$ be the volume assigned to each proto-dipole occupy only a small portion of this space

With the field turned on, the electric force on the free end is balanced by the spring force;

the charges separate by a distance d: $qE = kd$ $\quad \mathbf{E} = \dfrac{\rho_f}{3\epsilon_0\epsilon_r}\mathbf{r}$

The resulting dipole moment is $p = qd$, the polarization is $P = p/d\tau$,

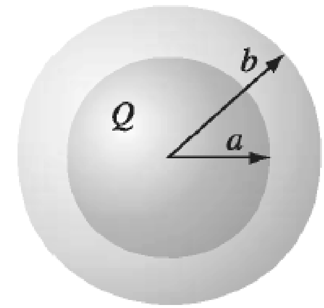
$$k = \frac{\rho_f}{3\epsilon_0\epsilon_r d^2} P r\, d\tau$$

The energy of this particular spring is $\quad dW_{\text{spring}} = \dfrac{1}{2}kd^2 = \dfrac{\rho_f}{6\epsilon_0\epsilon_r} P r\, d\tau$

$$W_{\text{spring}} = \frac{\rho_f}{6\epsilon_0\epsilon_r}\int P r\, d\tau \qquad \mathbf{P} = \epsilon_0\chi_e\mathbf{E} = \epsilon_0\chi_e\frac{\rho_f}{3\epsilon_0\epsilon_r}\mathbf{r} = \frac{(\epsilon_r - 1)\rho_f}{3\epsilon_r}\mathbf{r}$$

$$W_{\text{spring}} = \frac{\rho_f}{6\epsilon_0\epsilon_r}\frac{(\epsilon_r - 1)\rho_f}{3\epsilon_r}4\pi\int_0^R r^4\, dr = \frac{2\pi}{45\epsilon_0\epsilon_r^2}\rho_f^2 R^5(\epsilon_r - 1) \quad !$$

Problem 4.26 A spherical conductor, of radius a, carries a charge Q (Fig. 4.29). It is surrounded by linear dielectric material of susceptibility χ_e, out to radius b. Find the energy of this configuration (Eq. 4.58).



From Ex. 4.5:

$$\mathbf{D} = \begin{Bmatrix} 0, & (r<a) \\ \dfrac{Q}{4\pi r^2}\hat{\mathbf{r}}, & (r>a) \end{Bmatrix}, \quad \mathbf{E} = \begin{Bmatrix} 0, & (r<a) \\ \dfrac{Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}, & (a<r<b) \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}, & (r>b) \end{Bmatrix}$$

$$W = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,d\tau = \frac{1}{2}\frac{Q^2}{(4\pi)^2}4\pi\left\{\frac{1}{\epsilon}\int_a^b \frac{1}{r^2}\frac{1}{r^2}r^2 dr + \frac{1}{\epsilon_0}\int_b^\infty \frac{1}{r^2}dr\right\} = \frac{Q^2}{8\pi}\left\{\frac{1}{\epsilon}\left(\frac{-1}{r}\right)\Big|_a^b + \frac{1}{\epsilon_0}\left(\frac{-1}{r}\right)\Big|_b^\infty\right\}$$

$$= \frac{Q^2}{8\pi\epsilon_0}\left\{\frac{1}{(1+\chi_e)}\left(\frac{1}{a}-\frac{1}{b}\right)+\frac{1}{b}\right\} = \boxed{\frac{Q^2}{8\pi\epsilon_0(1+\chi_e)}\left(\frac{1}{a}+\frac{\chi_e}{b}\right).}$$

Problem 4.27 Calculate W, using both Eq. 4.55 and Eq. 4.58, for a sphere of radius R with frozen-in uniform polarization $\mathbf{P}$ (Ex. 4.2). Comment on the discrepancy. Which (if either) is the "true" energy of the system?

Using Eq. 4.55: $W = \frac{\epsilon_0}{2}\int E^2\,d\tau$. From Ex. 4.2 and Eq. 3.103,

$$\mathbf{E} = \left\{ \begin{array}{ll} \dfrac{-1}{3\epsilon_0}P\,\hat{\mathbf{z}}, & (r < R) \\ \dfrac{R^3 P}{3\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}), & (r > R) \end{array} \right\}, \quad \text{so}$$

$$W_{r<R} = \frac{\epsilon_0}{2}\left(\frac{P}{3\epsilon_0}\right)^2 \frac{4}{3}\pi R^3 = \frac{2\pi}{27}\frac{P^2R^3}{\epsilon_0}.$$

$$W_{r>R} = \frac{\epsilon_0}{2}\left(\frac{R^3P}{3\epsilon_0}\right)^2 \int \frac{1}{r^6}\left(4\cos^2\theta + \sin^2\theta\right) r^2 \sin\theta\,dr\,d\theta\,d\phi$$

$$= \frac{(R^3P)^2}{18\epsilon_0}2\pi \int_0^{\pi}(1+3\cos^2\theta)\sin\theta\,d\theta \int_R^{\infty}\frac{1}{r^4}\,dr = \frac{\pi(R^3P)^2}{9\epsilon_0}\left(-\cos\theta - \cos^3\theta\right)\Big|_0^{\pi}\left(-\frac{1}{3r^3}\right)\Big|_R^{\infty}$$

$$= \frac{\pi(R^3P)^2}{9\epsilon_0}\left(\frac{4}{3R^3}\right) = \frac{4\pi R^3P^2}{27\epsilon_0}.$$

$$W_{\text{tot}} = \boxed{\frac{2\pi R^3P^2}{9\epsilon_0}.}$$

This is the correct electrostatic energy of the configuration, but it is not the "total work necessary to assemble the system," because it leaves out the mechanical energy involved in polarizing the molecules.

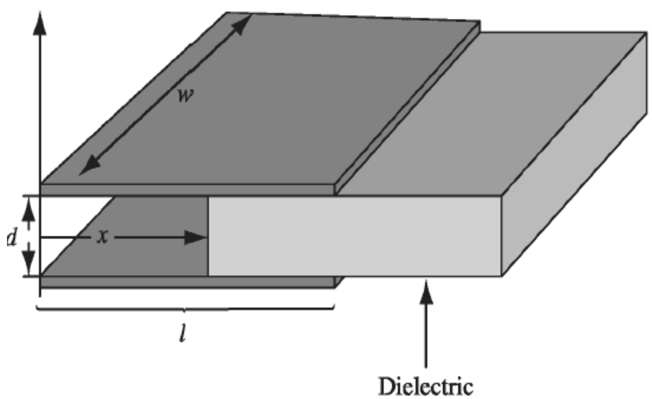
Using Eq. 4.58: $W = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,d\tau$. For $r > R$, $\mathbf{D} = \epsilon_0\mathbf{E}$, so this contribution is the same as before. For $r < R$, $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = -\frac{1}{3}\mathbf{P} + \mathbf{P} = \frac{2}{3}\mathbf{P} = -2\epsilon_0\mathbf{E}$, so $\frac{1}{2}\mathbf{D}\cdot\mathbf{E} = -2\frac{\epsilon_0}{2}E^2$, and this contribution is now $(-2)\left(\frac{2\pi}{27}\frac{P^2R^3}{\epsilon_0}\right) = -\frac{4\pi}{27}\frac{R^3P^2}{\epsilon_0}$, exactly cancelling the exterior term. *Conclusion:* $\boxed{W_{\text{tot}} = 0.}$ This is not surprising, since the derivation in Sect. 4.4.3 calculates the work done on the *free* charge, and in this problem there is no free charge in sight. Since this is a nonlinear dielectric, however, the result cannot be interpreted as the "work necessary to assemble the configuration"—the latter would depend entirely on *how* you assemble it.

4.4.4 Forces on Dielectrics

Just as a conductor is attracted into an electric field

dielectric is attracted into an electric field

the bound charge tends to accumulate near the free charge of the opposite sign.

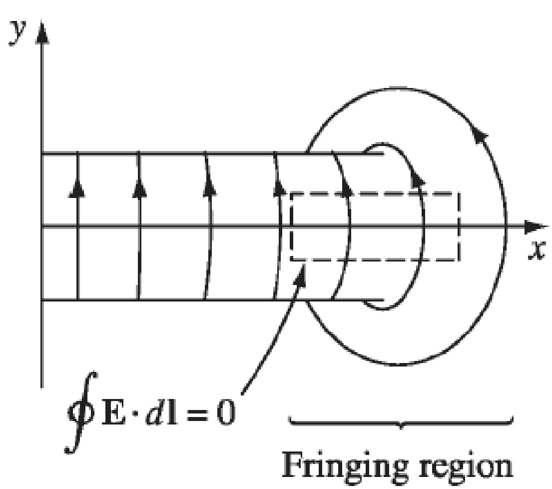


It is this nonuniform fringing field around the edges,

that pulls the dielectric into the capacitor.

$F_{me} = -F$

Fringing fields are notoriously difficult to calculate;



W be the energy of the system

depends on the amount of overlap.

out an infinitesimal distance dx,

the energy is changed by an amount equal to the work done:

$$dW = F_{me}\, dx$$

F_{me} is the force I must exert, to counteract the electrical force F on the dielectric:

$$F_{me} = -F$$

the electrical force on the slab $F = -\frac{dW}{dx}$

the energy stored in the capacitor $W = \frac{1}{2}CV^2$

the capacitance $C = \frac{\epsilon_0 w}{d}(\epsilon_r l - \chi_e x)$

assume that the total charge on the plates ($Q = CV$) is held constant,

$$W = \frac{1}{2}\frac{Q^2}{C}$$

$$F = -\frac{dW}{dx} = \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx} = \frac{1}{2}V^2\frac{dC}{dx}$$

$$\frac{dC}{dx} = -\frac{\epsilon_0 \chi_e w}{d}$$

$$F = -\frac{\epsilon_0 \chi_e w}{2d}V^2$$

(The minus sign indicates that the force is in the negative x direction; the dielectric is pulled *into* the capacitor.)

(with V constant) $F = -\frac{dW}{dx}$ $\longrightarrow$ $F = -\frac{1}{2}V^2\frac{dC}{dx}$ Wrong!

the *battery also does* work as the dielectric moves;

$$dW = F_{\text{me}}\,dx + V\,dQ$$

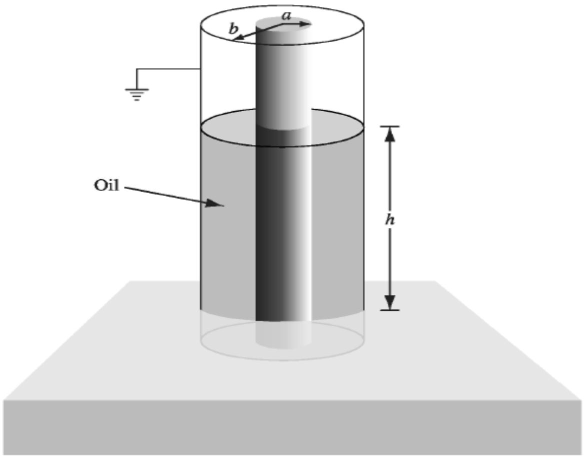
work done by the battery

$$F = -\frac{dW}{dx} + V\frac{dQ}{dx} = -\frac{1}{2}V^2\frac{dC}{dx} + V^2\frac{dC}{dx} = \frac{1}{2}V^2\frac{dC}{dx}$$

Please understand: The force on the dielectric cannot possibly depend on whether you plan to hold Q constant or V constant—it is determined entirely by the distribution of charge, free and bound. It's simpler to *calculate* the force assuming constant Q, because then you don't have to worry about work done by the battery; but if you insist, it can be done correctly either way.

Notice that we were able to determine the force *without knowing anything about the fringing fields that are ultimately responsible for it!* Of course, it's built into the whole structure of electrostatics that $\nabla \times \mathbf{E} = \mathbf{0}$, and hence that the fringing fields must be present; we're not really getting something for nothing here—just cleverly exploiting the internal consistency of the theory. The energy stored in the fringing fields themselves (which was not accounted for in this derivation) stays constant, as the slab moves; what *does* change is the energy well *inside* the capacitor, where the field is nice and uniform.

Problem 4.28 Two long coaxial cylindrical metal tubes (inner radius a, outer radius b) stand vertically in a tank of dielectric oil (susceptibility χ_e, mass density ρ). The inner one is maintained at potential V, and the outer one is grounded (Fig. 4.32). To what height (h) does the oil rise, in the space between the tubes?



First find the capacitance, as a function of h:

$$\left.\begin{array}{l} \textit{Air part: } E = \frac{2\lambda}{4\pi\epsilon_0 s} \Longrightarrow V = \frac{2\lambda}{4\pi\epsilon_0}\ln(b/a), \\[2ex] \textit{Oil part: } D = \frac{2\lambda'}{4\pi s} \Longrightarrow E = \frac{2\lambda'}{4\pi\epsilon s} \Longrightarrow V = \frac{2\lambda'}{4\pi\epsilon}\ln(b/a), \end{array}\right\} \Longrightarrow \frac{\lambda}{\epsilon_0} = \frac{\lambda'}{\epsilon};\ \lambda' = \frac{\epsilon}{\epsilon_0}\lambda = \epsilon_r\lambda.$$

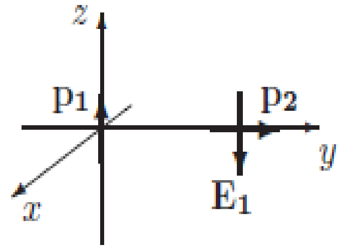
$Q = \lambda' h + \lambda(\ell - h) = \epsilon_r\lambda h - \lambda h + \lambda\ell = \lambda[(\epsilon_r - 1)h + \ell] = \lambda(\chi_e h + \ell)$, where ℓ is the total height

$$C = \frac{Q}{V} = \frac{\lambda(\chi_e h + \ell)}{2\lambda\ln(b/a)}4\pi\epsilon_0 = 2\pi\epsilon_0\frac{(\chi_e h + \ell)}{\ln(b/a)}.$$

$$\left.\begin{array}{l} \text{The net upward force is given by Eq. 4.64: } F = \frac{1}{2}V^2\frac{dC}{dh} = \frac{1}{2}V^2\frac{2\pi\epsilon_0\chi_e}{\ln(b/a)}. \\ \text{The gravitational force } \textit{down} \text{ is } F = mg = \rho\pi(b^2 - a^2)gh. \end{array}\right\} \boxed{h = \frac{\epsilon_0\chi_e V^2}{\rho(b^2 - a^2)g\ln(b/a)}}.$$

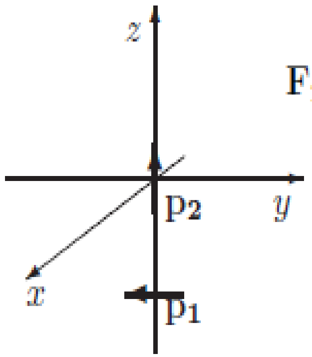
Extra 4.2.2 **Problem 4.29**

(a) For the configuration in Prob. 4.5, calculate the *force* on $\mathbf{p}_2$ due to $\mathbf{p}_1$, and the force on $\mathbf{p}_1$ due to $\mathbf{p}_2$. Are the answers consistent with Newton's third law?



(a) Eq. 4.5 $\Rightarrow \mathbf{F}_2 = (\mathbf{p}_2 \cdot \boldsymbol{\nabla})\,\mathbf{E}_1 = p_2 \dfrac{\partial}{\partial y}(\mathbf{E}_1)$;

Eq. 3.103 $\Rightarrow \mathbf{E}_1 = \dfrac{p_1}{4\pi\epsilon_0 r^3}\,\hat{\boldsymbol{\theta}} = -\dfrac{p_1}{4\pi\epsilon_0 y^3}\,\hat{\mathbf{z}}$. Therefore



$$\mathbf{F}_2 = -\frac{p_1 p_2}{4\pi\epsilon_0}\left[\frac{d}{dy}\left(\frac{1}{y^3}\right)\right]\hat{\mathbf{z}} = \frac{3p_1p_2}{4\pi\epsilon_0 y^4}\,\hat{\mathbf{z}}, \text{ or } \boxed{\mathbf{F}_2 = \frac{3p_1p_2}{4\pi\epsilon_0 r^4}\,\hat{\mathbf{z}}} \text{ (upward).}$$

To calculate $\mathbf{F}_1$, put $\mathbf{p}_2$ at the origin, pointing in the z direction; then $\mathbf{p}_1$ is at $-r\,\hat{\mathbf{z}}$, and it points in the $-\hat{\mathbf{y}}$ direction. So $\mathbf{F}_1 = (\mathbf{p}_1 \cdot \boldsymbol{\nabla})\,\mathbf{E}_2 = -p_1 \dfrac{\partial \mathbf{E}_2}{\partial y}\bigg|_{x=y=0,\,z=-r}$; we need $\mathbf{E}_2$ as a function of x, y, and z.

From Eq. 3.104: $\mathbf{E}_2 = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r^3}\left[\dfrac{3(\mathbf{p}_2\cdot\mathbf{r})\mathbf{r}}{r^2} - \mathbf{p}_2\right]$, where $\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$, $\mathbf{p}_2 = p_2\,\hat{\mathbf{z}}$, and hence $\mathbf{p}_2\cdot\mathbf{r} = p_2 z$.

$$\mathbf{E}_2 = \frac{p_2}{4\pi\epsilon_0}\left[\frac{3z(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) - (x^2+y^2+z^2)\,\hat{\mathbf{z}}}{(x^2+y^2+z^2)^{5/2}}\right] = \frac{p_2}{4\pi\epsilon_0}\left[\frac{3xz\,\hat{\mathbf{x}} + 3yz\,\hat{\mathbf{y}} - (x^2+y^2-2z^2)\,\hat{\mathbf{z}}}{(x^2+y^2+z^2)^{5/2}}\right]$$

$$\frac{\partial \mathbf{E}_2}{\partial y} = \frac{p_2}{4\pi\epsilon_0}\left\{-\frac{5}{2}\frac{2y}{r^7}[3xz\,\hat{\mathbf{x}} + 3yz\,\hat{\mathbf{y}} - (x^2+y^2-2z^2)\,\hat{\mathbf{z}}] + \frac{1}{r^5}(3z\,\hat{\mathbf{y}} - 2y\,\hat{\mathbf{z}})\right\};$$

$$\frac{\partial \mathbf{E}_2}{\partial y}\bigg|_{(0,0)} = \frac{p_2}{4\pi\epsilon_0}\frac{3z}{r^5}\,\hat{\mathbf{y}}; \quad \mathbf{F}_1 = -p_1\left(\frac{p_2}{4\pi\epsilon_0}\frac{-3r}{r^5}\,\hat{\mathbf{y}}\right) = \boxed{\frac{3p_1p_2}{4\pi\epsilon_0 r^4}\,\hat{\mathbf{y}}.}$$

But $\hat{\mathbf{y}}$ in these coordinates corresponds to $-\hat{\mathbf{z}}$ in the original system, so these results *are* consistent with Newton's third law: $\mathbf{F}_1 = -\mathbf{F}_2$.

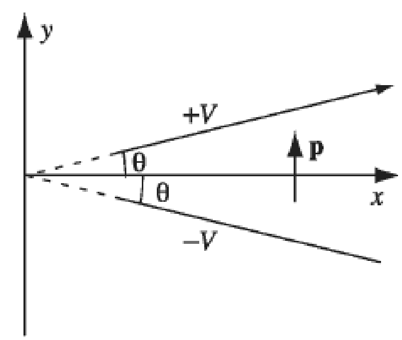
(b) Find the total torque on $\mathbf{p}_2$ *with respect to the center of* $\mathbf{p}_1$, and compare it with the torque on $\mathbf{p}_1$ *about that same point*. [*Hint:* combine your answer to (a) with the result of Prob. 4.5.]

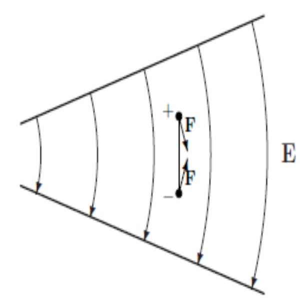
(b) From the remark following Eq. 4.5, $\mathbf{N}_2 = (\mathbf{p}_2 \times \mathbf{E}_1) + (\mathbf{r} \times \mathbf{F}_2)$. The first term was calculated in Prob. 4.5; the second we get from (a), using $\mathbf{r} = r\,\hat{\mathbf{y}}$:

$$\mathbf{p}_2 \times \mathbf{E}_1 = \frac{p_1 p_2}{4\pi\epsilon_0 r^3}(-\hat{\mathbf{x}}); \quad \mathbf{r} \times \mathbf{F}_2 = (r\,\hat{\mathbf{y}}) \times \left(\frac{3p_1p_2}{4\pi\epsilon_0 r^4}\,\hat{\mathbf{z}}\right) = \frac{3p_1p_2}{4\pi\epsilon_0 r^3}\,\hat{\mathbf{x}}; \text{ so } \boxed{\mathbf{N}_2 = \frac{2p_1p_2}{4\pi\epsilon_0 r^3}\,\hat{\mathbf{x}}.}$$

This is equal and opposite to the torque on $\mathbf{p}_1$ due to $\mathbf{p}_2$, with respect to the center of $\mathbf{p}_1$ (see Prob. 4.5).

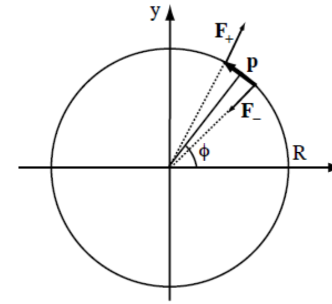
Problem 4.30 An electric dipole $\mathbf{p}$, pointing in the y direction, is placed midway between two large conducting plates, as shown in Fig. 4.33. Each plate makes a small angle θ with respect to the x axis, and they are maintained at potentials $\pm V$. What is the *direction* of the net force on $\mathbf{p}$? (There's nothing to *calculate*, here, but do explain your answer qualitatively.)





Net force is $\boxed{\text{to the right}}$ (see diagram). Note that the field lines must bulge to the right, as shown, because $\mathbf{E}$ is perpendicular to the surface of each conductor.

Problem 4.31 A point charge Q is "nailed down" on a table. Around it, at radius R, is a frictionless circular track on which a dipole $\mathbf{p}$ rides, constrained always to point tangent to the circle. Use Eq. 4.5 to show that the electric force on the dipole is



$$\mathbf{F} = \frac{Q}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3}.$$

Notice that this force is always in the "forward" direction (you can easily confirm this by drawing a diagram showing the forces on the two ends of the dipole). Why isn't this a perpetual motion machine?[21]

Problem 4.31 In cylindrical coordinates (in the $z = 0$ plane), $\mathbf{p} = p\,\hat{\boldsymbol{\phi}}$, $\mathbf{p}\cdot\nabla = p\frac{1}{s}\frac{\partial}{\partial\phi}$, and $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{s^2}\,\hat{\mathbf{s}}$, so

$$\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E} = \left(\frac{p}{s}\frac{\partial}{\partial\phi}\right)\frac{1}{4\pi\epsilon_0}\frac{Q}{s^2}\,\hat{\mathbf{s}} = \frac{pQ}{4\pi\epsilon_0 s^3}\frac{\partial\hat{\mathbf{s}}}{\partial\phi} = \frac{pQ}{4\pi\epsilon_0 s^3}\,\hat{\boldsymbol{\phi}} = \frac{Q}{4\pi\epsilon_0 R^3}\,\mathbf{p}. \quad \checkmark$$

Qualitatively, the forces on the negative and positive ends, though equal in magnitude, point in slightly different directions, and they combine to make a net force in the "forward" direction:

To keep the dipole going in a circle, there must be a centripetal force exerted by the track (we may as well take it to act at the center of the dipole, and it is irrelevant to the problem), and to keep it aiming in the tangential direction there must be a torque (which we could model by radial forces of equal magnitude acting at the two ends). Indeed, if the dipole has the orientation indicated in the figure, and is moving in the $\hat{\phi}$ direction, the torque exerted by Q is clockwise, whereas the rotation is counterclockwise, so these constraint forces must actually be *larger* than the forces exerted by Q, and the *net* force will be in the "backward" direction—tending to slow the dipole down. [If the motion is in the $-\hat{\phi}$ direction, then the *electrical* forces will dominate, and the net force will be in the direction of $\mathbf{p}$, but this again will tend to slow it down.]

Problem 4.32 Earnshaw's theorem (Prob. 3.2) says that you cannot trap a charged particle in an electrostatic field. *Question:* Could you trap a neutral (but polarizable) atom in an electrostatic field?

(a) Show that the force on the atom is $\mathbf{F} = \frac{1}{2}\alpha\nabla(E^2)$.

(b) The question becomes, therefore: Is it possible for E^2 to have a local maximum (in a charge-free region)? In that case the force would push the atom back to its equilibrium position. Show that the answer is *no*. [*Hint:* Use Prob. 3.4(a).][22]

(a) According to Eqs. 4.1 and 4.5, $\mathbf{F} = \alpha(\mathbf{E}\cdot\nabla)\mathbf{E}$. From the product rule,

$$\nabla E^2 = \nabla(\mathbf{E}\cdot\mathbf{E}) = 2\mathbf{E}\times(\nabla\times\mathbf{E}) + 2(\mathbf{E}\cdot\nabla)\mathbf{E}.$$

But in electrostatics $\nabla\times\mathbf{E} = \mathbf{0}$, so $(\mathbf{E}\cdot\nabla)\mathbf{E} = \frac{1}{2}\nabla(E^2)$, and hence

$$\mathbf{F} = \frac{1}{2}\alpha\nabla(E^2). \quad \checkmark$$

[It is tempting to start with Eq. 4.6, and write $\mathbf{F} = -\nabla U = \nabla(\mathbf{p}\cdot\mathbf{E}) = \alpha\nabla(\mathbf{E}\cdot\mathbf{E}) = \alpha\nabla(E^2)$. The error occurs in the third step: $\mathbf{p}$ should not have been differentiated, but after it is replaced by $\alpha\mathbf{E}$ we are differentiating *both* $\mathbf{E}$'s.]

(b) Suppose E^2 has a local maximum at point P. Then there is a sphere (of radius R) about P such that $E^2(P') < E^2(P)$, and hence $|\mathbf{E}(P')| < |\mathbf{E}(P)|$, for all points on the surface. But if there is no charge inside the sphere, then Problem 3.4a says the average field over the spherical surface is equal to the value at the center: $\frac{1}{4\pi R^2}\int \mathbf{E}\,da = \mathbf{E}(P)$, or, choosing the z axis to lie along $\mathbf{E}(P)$,

$$\frac{1}{4\pi R^2}\int E_z\, da = E(P).$$

But if E^2 has a maximum at P, then

$$\int E_z\, da \le \int |\mathbf{E}|\, da < \int |\mathbf{E}(P)|\, da = 4\pi R^2 E(P),$$

and it follows that $E(P) < E(P)$, a contradiction. Therefore, E^2 cannot have a maximum in a charge-free region. [It can have a minimum, however; at the midpoint between two equal charges the field is zero, and this is obviously a minimum.]

Problem 4.33 A dielectric cube of side a, centered at the origin, carries a "frozen-in" polarization $\mathbf{P} = k\mathbf{r}$, where k is a constant. Find all the bound charges, and check that they add up to zero.

$\mathbf{P} = k\mathbf{r} = k(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) \Longrightarrow \rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P} = -k(1+1+1) = \boxed{-3k.}$

Total volume bound charge: $\boxed{Q_{\text{vol}} = -3ka^3.}$

$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$. At top surface, $\hat{\mathbf{n}} = \hat{\mathbf{z}}$, $z = a/2$; so $\sigma_b = ka/2$. Clearly, $\boxed{\sigma_b = ka/2}$ on all six surfaces.

Total surface bound charge: $\boxed{Q_{\text{surf}} = 6(ka/2)a^2 = 3ka^3.}$ Total bound charge is zero. ✓

Problem 4.34 The space between the plates of a parallel-plate capacitor is filled with dielectric material whose dielectric constant varies linearly from 1 at the bottom plate ($x = 0$) to 2 at the top plate ($x = d$). The capacitor is connected to a battery of voltage V. Find all the bound charge, and check that the total is zero.

Say the high voltage is connected to the bottom plate, so the electric field points in the x direction, while the free charge density (σ_f) is positive on the lower plate and negative on the upper plate. (If you connect the battery the other way, all the signs will switch.) The susceptibility is $\chi_e = \frac{x}{d}$, and the permittivity is $\epsilon = \epsilon_0\left(1 + \frac{x}{d}\right)$. Between the plates

$$\mathbf{D} = \sigma_f\,\hat{\mathbf{x}},\ \mathbf{E} = \frac{1}{\epsilon}\mathbf{D} = \frac{\sigma_f}{\epsilon_0(1 + x/d)}\,\hat{\mathbf{x}};\ V = -\int_d^0 \mathbf{E}\cdot d\mathbf{l} = \frac{\sigma_f}{\epsilon_0}\int_0^d \frac{1}{(1 + x/d)}\,dx = \frac{\sigma_f}{\epsilon_0} d\ln\left(1 + \frac{x}{d}\right)\Big|_0^d = \frac{\sigma_f d}{\epsilon_0}\ln 2.$$

So

$$\sigma_f = \frac{\epsilon_0 V}{d\ln 2},\quad \mathbf{E} = \frac{V}{d\ln 2}\frac{1}{(1 + x/d)}\,\hat{\mathbf{x}},\quad \mathbf{P} = \epsilon_0\chi_e\mathbf{E} = \frac{\epsilon_0 V}{d^2\ln 2}\frac{x}{(1 + x/d)}\,\hat{\mathbf{x}}.$$

The bound charges are therefore

$$\rho_b = -\nabla\cdot\mathbf{P} = -\frac{\epsilon_0 V}{d^2\ln 2}\left[\frac{1}{(1 + x/d)} - \frac{x/d}{(1 + x/d)^2}\right] = \boxed{-\frac{\epsilon_0 V}{d^2\ln 2}\frac{1}{(1 + x/d)^2}};\ \sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \begin{cases}\boxed{0} & (x = 0),\\ \boxed{\dfrac{\epsilon_0 V}{2d\ln 2}} & (x = d).\end{cases}$$

The total bound charge is

$$Q_b = \int \rho_b\,d\tau + \int \sigma_b\,da = -\frac{\epsilon_0 V}{d^2\ln 2}\int_0^d \frac{1}{(1 + x/d)^2}A\,dx + \frac{\epsilon_0 V}{2d\ln 2}A = \frac{\epsilon_0 V A}{d\ln 2}\left[-\frac{1}{d}\frac{-d}{(1 + x/d)}\Big|_0^d + \frac{1}{2}\right]$$

$$= \frac{\epsilon_0 V A}{d\ln 2}\left(\frac{1}{2} - 1 + \frac{1}{2}\right) = 0,\quad \checkmark$$

Problem 4.35 A point charge q is imbedded at the center of a sphere of linear dielectric material (with susceptibility χ_e and radius R). Find the electric field, the polarization, and the bound charge densities, ρ_b and σ_b. What is the total bound charge on the surface? Where is the compensating negative bound charge located?

$$\oint \mathbf{D}\cdot d\mathbf{a} = Q_{f_{\text{enc}}} \Rightarrow \mathbf{D} = \frac{q}{4\pi r^2}\,\hat{\mathbf{r}};\ \mathbf{E} = \frac{1}{\epsilon}\mathbf{D} = \boxed{\frac{q}{4\pi\epsilon_0(1+\chi_e)}\frac{\hat{\mathbf{r}}}{r^2}};\ \mathbf{P} = \epsilon_0\chi_e\mathbf{E} = \boxed{\frac{q\chi_e}{4\pi(1+\chi_e)}\frac{\hat{\mathbf{r}}}{r^2}}.$$

$$\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P} = -\frac{q\chi_e}{4\pi(1+\chi_e)}\left(\boldsymbol{\nabla}\cdot\frac{\hat{\mathbf{r}}}{r^2}\right) = \boxed{-q\frac{\chi_e}{1+\chi_e}\delta^3(\mathbf{r})}\ \text{(Eq. 1.99)};\ \sigma_b = \mathbf{P}\cdot\hat{\mathbf{r}} = \boxed{\frac{q\chi_e}{4\pi(1+\chi_e)R^2}};$$

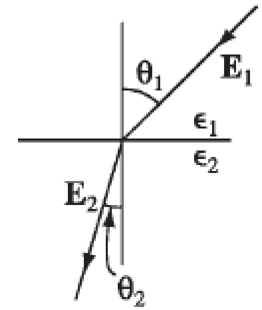
$$Q_{\text{surf}} = \sigma_b(4\pi R^2) = \boxed{q\frac{\chi_e}{1+\chi_e}}.$$ The compensating negative charge is at the center:

$$\int \rho_b\,d\tau = -\frac{q\chi_e}{1+\chi_e}\int \delta^3(\mathbf{r})\,d\tau = -q\frac{\chi_e}{1+\chi_e}.$$

Problem 4.36 At the interface between one linear dielectric and another, the electric field lines bend (see Fig. 4.34). Show that

$$\tan\theta_2/\tan\theta_1 = \epsilon_2/\epsilon_1, \tag{4.68}$$

assuming there is no *free* charge at the boundary. [*Comment:* Eq. 4.68 is reminiscent of Snell's law in optics. Would a convex "lens" of dielectric material tend to "focus," or "defocus," the electric field?]



$\mathbf{E}^{\parallel}$ is continuous (Eq. 4.29); $D_\perp$ is continuous (Eq. 4.26, with $\sigma_f = 0$). So $E_{x_1} = E_{x_2}$, $D_{y_1} = D_{y_2} \Rightarrow$ $\epsilon_1 E_{y_1} = \epsilon_2 E_{y_2}$, and hence

$$\frac{\tan\theta_2}{\tan\theta_1} = \frac{E_{x_2}/E_{y_2}}{E_{x_1}/E_{y_1}} = \frac{E_{y_1}}{E_{y_2}} = \frac{\epsilon_2}{\epsilon_1}. \quad \text{qed}$$

If 1 is air and 2 is dielectric, $\tan\theta_2/\tan\theta_1 = \epsilon_2/\epsilon_0 > 1$, and the field lines bend *away* from the normal. This is the opposite of light rays, so a convex "lens" would *defocus* the field lines.

Problem 4.37 A point dipole $\mathbf{p}$ is imbedded at the center of a sphere of linear dielectric material (with radius R and dielectric constant ϵ_r). Find the electric potential inside and outside the sphere.

$$\left[\mathit{Answer:}\ \frac{p\cos\theta}{4\pi\epsilon r^2}\left(1+2\frac{r^3}{R^3}\frac{(\epsilon_r-1)}{(\epsilon_r+2)}\right),\ (r\le R);\ \frac{p\cos\theta}{4\pi\epsilon_0 r^2}\left(\frac{3}{\epsilon_r+2}\right),\ (r\ge R)\right]$$

In view of Eq. 4.39, the *net* dipole moment at the center is $\mathbf{p}'=\mathbf{p}-\frac{\chi_e}{1+\chi_e}\mathbf{p}=\frac{1}{1+\chi_e}\mathbf{p}=\frac{1}{\epsilon_r}\mathbf{p}$. We want the potential produced by $\mathbf{p}'$ (at the center) and σ_b (at R). Use separation of variables:

$$\left\{\begin{array}{ll} \textit{Outside:}\ V(r,\theta)=\displaystyle\sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta) & \text{(Eq. 3.72)} \\ \textit{Inside:}\quad V(r,\theta)=\displaystyle\frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{\epsilon_r r^2}+\sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta) & \text{(Eqs. 3.66, 3.102)} \end{array}\right\}.$$

$$V\text{ continuous at }R\Rightarrow\left\{\begin{array}{ll} \displaystyle\frac{B_l}{R^{l+1}}=A_lR^l, & \text{or } B_l=R^{2l+1}A_l\ (l\ne 1) \\ \displaystyle\frac{B_1}{R^2}=\frac{1}{4\pi\epsilon_0}\frac{p}{\epsilon_r R^2}+A_1R,\ \text{or} & B_1=\frac{p}{4\pi\epsilon_0\epsilon_r}+A_1R^3 \end{array}\right\}.$$

$$\left.\frac{\partial V}{\partial r}\right|_{R+}-\left.\frac{\partial V}{\partial r}\right|_{R-}=-\sum(l+1)\frac{B_l}{R^{l+2}}P_l(\cos\theta)+\frac{1}{4\pi\epsilon_0}\frac{2p\cos\theta}{\epsilon_r R^3}-\sum lA_lR^{l-1}P_l(\cos\theta)=-\frac{1}{\epsilon_0}\sigma_b$$

$$=-\frac{1}{\epsilon_0}\mathbf{P}\cdot\hat{\mathbf{r}}=-\frac{1}{\epsilon_0}(\epsilon_0\chi_e\mathbf{E}\cdot\hat{\mathbf{r}})=\chi_e\left.\frac{\partial V}{\partial r}\right|_{R-}=\chi_e\left\{-\frac{1}{4\pi\epsilon_0}\frac{2p\cos\theta}{\epsilon_r R^3}+\sum lA_lR^{l-1}P_l(\cos\theta)\right\}.$$

$$-(l+1)\frac{B_l}{R^{l+2}}-lA_lR^{l-1}=\chi_e lA_lR^{l-1}\ (l\ne1);\ \text{or}\ -(2l+1)A_lR^{l-1}=\chi_e lA_lR^{l-1}\ \Rightarrow A_l=0\ (\ell\ne1).$$

For $l=1$: $$-2\frac{B_1}{R^3}+\frac{1}{4\pi\epsilon_0}\frac{2p}{\epsilon_r R^3}-A_1=\chi_e\left(-\frac{1}{4\pi\epsilon_0}\frac{2p}{\epsilon_r R^3}+A_1\right)-B_1+\frac{p}{4\pi\epsilon_0\epsilon_r}-\frac{A_1R^3}{2}=-\frac{1}{4\pi\epsilon_0}\frac{\chi_e p}{\epsilon_r}+\chi_e\frac{A_1R^3}{2};$$

$$-\frac{p}{4\pi\epsilon_0\epsilon_r} - A_1R^3 + \frac{p}{4\pi\epsilon_0\epsilon_r} - \frac{A_1R^3}{2} = -\frac{1}{4\pi\epsilon_0}\frac{\chi_e p}{\epsilon_r} + \chi_e\frac{A_1R^3}{2} \;\Rightarrow\; \frac{A_1R^3}{2}(3+\chi_e) = \frac{1}{4\pi\epsilon_0}\frac{\chi_e p}{\epsilon_r}.$$

$$\Rightarrow A_1 = \frac{1}{4\pi\epsilon_0}\frac{2\chi_e p}{R^3\epsilon_r(3+\chi_e)} = \frac{1}{4\pi\epsilon_0}\frac{2(\epsilon_r-1)p}{R^3\epsilon_r(\epsilon_r+2)}; \quad B_1 = \frac{p}{4\pi\epsilon_0\epsilon_r}\left[1+\frac{2(\epsilon_r-1)}{(\epsilon_r+2)}\right] = \frac{p}{4\pi\epsilon_0\epsilon_r}\frac{3\epsilon_r}{\epsilon_r+2}.$$

$$\boxed{V(r,\theta) = \left(\frac{p\cos\theta}{4\pi\epsilon_0 r^2}\right)\left(\frac{3}{\epsilon_r+2}\right) \quad (r \ge R).}$$

Meanwhile, for $r \le R$, $V(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{\epsilon_r r^2} + \frac{1}{4\pi\epsilon_0}\frac{pr\cos\theta}{R^3}\frac{2(\epsilon_r-1)}{\epsilon_r(\epsilon_r+2)}$

$$= \boxed{\frac{p\cos\theta}{4\pi\epsilon_0 r^2\epsilon_r}\left[1+2\left(\frac{\epsilon_r-1}{\epsilon_r+2}\right)\frac{r^3}{R^3}\right] \quad (r \le R).}$$

Problem 4.38 Prove the following uniqueness theorem: A volume $\mathcal{V}$ contains a specified free charge distribution, and various pieces of linear dielectric material, with the susceptibility of each one given. If the potential is specified on the boundaries $\mathcal{S}$ of $\mathcal{V}$ ($V = 0$ at infinity would be suitable) then the potential throughout $\mathcal{V}$ is uniquely determined. [*Hint:* Integrate $\nabla \cdot (V_3 \mathbf{D}_3)$ over $\mathcal{V}$.]

Given two solutions, V_1 (and $\mathbf{E}_1 = -\nabla V_1$, $\mathbf{D}_1 = \epsilon \mathbf{E}_1$) and V_2 ($\mathbf{E}_2 = -\nabla V_2$, $\mathbf{D}_2 = \epsilon \mathbf{E}_2$), define $V_3 \equiv V_2 - V_1$ ($\mathbf{E}_3 = \mathbf{E}_2 - \mathbf{E}_1$, $\mathbf{D}_3 = \mathbf{D}_2 - \mathbf{D}_1$).

$\int_{\mathcal{V}} \nabla \cdot (V_3 \mathbf{D}_3)\, d\tau = \int_{\mathcal{S}} V_3 \mathbf{D}_3 \cdot d\mathbf{a} = 0$, ($V_3 = 0$ on $\mathcal{S}$), so $\int (\nabla V_3) \cdot \mathbf{D}_3\, d\tau + \int V_3 (\nabla \cdot \mathbf{D}_3)\, d\tau = 0$.
But $\nabla \cdot \mathbf{D}_3 = \nabla \cdot \mathbf{D}_2 - \nabla \cdot \mathbf{D}_1 = \rho_f - \rho_f = 0$, and $\nabla V_3 = \nabla V_2 - \nabla V_1 = -\mathbf{E}_2 + \mathbf{E}_1 = -\mathbf{E}_3$, so $\int \mathbf{E}_3 \cdot \mathbf{D}_3\, d\tau = 0$. But $\mathbf{D}_3 = \mathbf{D}_2 - \mathbf{D}_1 = \epsilon \mathbf{E}_2 - \epsilon \mathbf{E}_1 = \epsilon \mathbf{E}_3$, so $\int \epsilon (E_3)^2\, d\tau = 0$. But $\epsilon > 0$, so $\mathbf{E}_3 = 0$, so $V_2 - V_1 = $ constant. But at surface, $V_2 = V_1$, so $V_2 = V_1$ everywhere. qed

Problem 4.39 A conducting sphere at potential V_0 is half embedded in linear dielectric material of susceptibility χ_e, which occupies the region $z < 0$ (Fig. 4.35). *Claim:* the potential everywhere is exactly the same as it would have been in the absence of the dielectric! Check this claim, as follows:

(a) Write down the formula for the proposed potential $V(r)$, in terms of V_0, R, and r. Use it to determine the field, the polarization, the bound charge, and the free charge distribution on the sphere.

(b) Show that the resulting charge configuration would indeed produce the potential $V(r)$.

(c) Appeal to the uniqueness theorem in Prob. 4.38 to complete the argument.

(d) Could you solve the configurations in Fig. 4.36 with the same potential? If not, explain *why*.

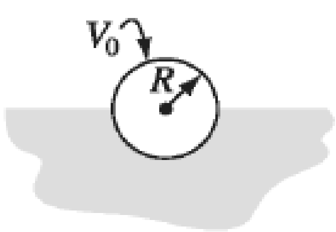


FIGURE 4.35

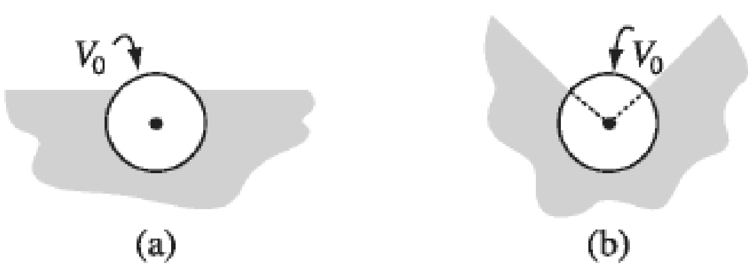


FIGURE 4.36

(a) Proposed potential: $\boxed{V(r) = V_0\frac{R}{r}.}$ If so, then $\boxed{\mathbf{E} = -\nabla V = V_0\frac{R}{r^2}\,\hat{\mathbf{r}},}$ in which case $\boxed{\mathbf{P} = \epsilon_0\chi_e V_0\frac{R}{r^2}\,\hat{\mathbf{r}},}$ in the region $z < 0$. ($\mathbf{P} = 0$ for $z > 0$, of course.) Then $\sigma_b = \epsilon_0\chi_e V_0\frac{R}{R^2}(\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}) = \boxed{-\frac{\epsilon_0\chi_e V_0}{R}.}$ (*Note:* $\hat{\mathbf{n}}$ points *out* of dielectric $\Rightarrow \hat{\mathbf{n}} = -\hat{\mathbf{r}}$.) This σ_b is on the surface at $r = R$. The flat surface $z = 0$ carries no bound charge, since $\hat{\mathbf{n}} = \hat{\mathbf{z}} \perp \hat{\mathbf{r}}$. Nor is there any volume bound charge (Eq. 4.39). If V is to have the required spherical symmetry, the *net* charge must be uniform:

$\sigma_{\text{tot}}4\pi R^2 = Q_{\text{tot}} = 4\pi\epsilon_0 R V_0$ (since $V_0 = Q_{\text{tot}}/4\pi\epsilon_0 R$), so $\sigma_{\text{tot}} = \epsilon_0 V_0/R$. Therefore

$$\boxed{\sigma_f = \left\{ \begin{array}{l} (\epsilon_0 V_0/R), \text{ on northern hemisphere} \\ (\epsilon_0 V_0/R)(1+\chi_e), \text{ on southern hemisphere} \end{array} \right\}.}$$

(b) By construction, $\sigma_{\text{tot}} = \sigma_b + \sigma_f = \epsilon_0 V_0/R$ is uniform (on the northern hemisphere $\sigma_b = 0$, $\sigma_f = \epsilon_0 V_0/R$; on the southern hemisphere $\sigma_b = -\epsilon_0\chi_e V_0/R$, so $\sigma_f = \epsilon V_0/R$). The potential of a uniformly charged sphere is

$$V_0 = \frac{Q_{\text{tot}}}{4\pi\epsilon_0 r} = \frac{\sigma_{\text{tot}}(4\pi R^2)}{4\pi\epsilon_0 r} = \frac{\epsilon_0 V_0}{R}\frac{R^2}{\epsilon_0 r} = V_0\frac{R}{r}. \quad \checkmark$$

(c) Since everything is consistent, and the boundary conditions ($V = V_0$ at $r = R$, $V \to 0$ at ∞) are met, Prob. 4.38 guarantees that this is *the* solution.

(d) Figure (b) works the same way, but Fig. (a) does *not:* on the flat surface, $\mathbf{P}$ is *not* perpendicular to $\hat{\mathbf{n}}$, so we'd get bound charge on this surface, spoiling the symmetry.

Problem 4.40 According to Eq. 4.5, the force on a single dipole is $(\mathbf{p} \cdot \nabla)\mathbf{E}$, so the *net* force on a dielectric object is

$$\mathbf{F} = \int (\mathbf{P} \cdot \nabla)\mathbf{E}_{\text{ext}}\, d\tau. \tag{4.69}$$

[Here $\mathbf{E}_{\text{ext}}$ is the field of everything *except* the dielectric. You might assume that it wouldn't matter if you used the *total* field; after all, the dielectric can't exert a force on *itself*. However, because the field of the dielectric is discontinuous at the location of any bound surface charge, the derivative introduces a spurious delta function, and it is safest to stick with $\mathbf{E}_{\text{ext}}$.] Use Eq. 4.69 to determine the force on a tiny sphere, of radius R, composed of linear dielectric material of susceptibility χ_e, which is situated a distance s from a fine wire carrying a uniform line charge λ.

$\mathbf{E}_{\text{ext}} = \dfrac{\lambda}{2\pi\epsilon_0 s}\,\hat{\mathbf{s}}$. Since the sphere is tiny, this is essentially constant, and hence $\mathbf{P} = \dfrac{\epsilon_0\chi_e}{1+\chi_e/3}\mathbf{E}_{\text{ext}}$ (Ex. 4.7).

$$\mathbf{F} = \int \left(\frac{\epsilon_0\chi_e}{1+\chi_e/3}\right)\left(\frac{\lambda}{2\pi\epsilon_0 s}\right)\frac{d}{ds}\left(\frac{\lambda}{2\pi\epsilon_0 s}\right)\hat{\mathbf{s}}\, d\tau = \left(\frac{\epsilon_0\chi_e}{1+\chi_e/3}\right)\left(\frac{\lambda}{2\pi\epsilon_0}\right)^2\left(\frac{1}{s}\right)\left(\frac{-1}{s^2}\right)\hat{\mathbf{s}}\int d\tau$$

$$= \frac{-\chi_e}{1+\chi_e/3}\left(\frac{\lambda^2}{4\pi^2\epsilon_0}\right)\frac{1}{s^3}\frac{4}{3}\pi R^3\,\hat{\mathbf{s}} = \boxed{-\left(\frac{\chi_e}{3+\chi_e}\right)\frac{\lambda^2 R^3}{\pi\epsilon_0 s^3}\,\hat{\mathbf{s}}.}$$

Problem 4.41 In a linear dielectric, the polarization is proportional to the field: $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$. If the material consists of atoms (or nonpolar molecules), the induced dipole moment of each one is likewise proportional to the field $\mathbf{p} = \alpha \mathbf{E}$. *Question:* What is the relation between the atomic polarizability α and the susceptibility χ_e?

Since $\mathbf{P}$ (the dipole moment per unit volume) is $\mathbf{p}$ (the dipole moment per atom) times N (the number of atoms per unit volume), $\mathbf{P} = N\mathbf{p} = N\alpha\mathbf{E}$, one's first inclination is to say that

$$\chi_e = \frac{N\alpha}{\epsilon_0}. \tag{4.70}$$

And in fact this is not far off, if the density is low. But closer inspection reveals a subtle problem, for the field $\mathbf{E}$ in Eq. 4.30 is the *total macroscopic* field in the medium, whereas the field in Eq. 4.1 is due to everything *except* the particular atom under consideration (polarizability was defined for an isolated atom subject to a specified external field); call this field $\mathbf{E}_{\text{else}}$. Imagine that the space allotted to each atom is a sphere of radius R, and show that

$$\mathbf{E} = \left(1 - \frac{N\alpha}{3\epsilon_0}\right) \mathbf{E}_{\text{else}}. \tag{4.71}$$

Use this to conclude that

$$\chi_e = \frac{N\alpha/\epsilon_0}{1 - N\alpha/3\epsilon_0}, \qquad \alpha = \frac{3\epsilon_0}{N}\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)$$

Equation 4.72 is known as the **Clausius-Mossotti** formula, or, in its application to optics, the **Lorentz-Lorenz** equation.

The density of atoms is $N = \frac{1}{(4/3)\pi R^3}$. The macroscopic field $\mathbf{E}$ is $\mathbf{E}_{\text{self}} + \mathbf{E}_{\text{else}}$, where $\mathbf{E}_{\text{self}}$ is the average field over the sphere due to the atom itself.

$$\mathbf{p} = \alpha \mathbf{E}_{\text{else}} \;\Rightarrow\; \mathbf{P} = N\alpha\mathbf{E}_{\text{else}}.$$

[Actually, it is the field at the *center*, not the average over the sphere, that belongs here, but the two are in fact equal, as we found in Prob. 3.47d.] Now

$$\mathbf{E}_{\text{self}} = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3}$$

(Eq. 3.105), so

$$\mathbf{E} = -\frac{1}{4\pi\epsilon_0}\frac{\alpha}{R^3}\mathbf{E}_{\text{else}} + \mathbf{E}_{\text{else}} = \left(1 - \frac{\alpha}{4\pi\epsilon_0 R^3}\right)\mathbf{E}_{\text{else}} = \left(1 - \frac{N\alpha}{3\epsilon_0}\right)\mathbf{E}_{\text{else}}.$$

So

$$\mathbf{P} = \frac{N\alpha}{(1 - N\alpha/3\epsilon_0)}\mathbf{E} = \epsilon_0\chi_e\mathbf{E},$$

and hence

$$\chi_e = \frac{N\alpha/\epsilon_0}{(1 - N\alpha/3\epsilon_0)}.$$

Solving for α:

$$\chi_e - \frac{N\alpha}{3\epsilon_0}\chi_e = \frac{N\alpha}{\epsilon_0} \;\Rightarrow\; \frac{N\alpha}{\epsilon_0}\left(1 + \frac{\chi_e}{3}\right) = \chi_e,$$

or

$$\alpha = \frac{\epsilon_0}{N}\frac{\chi_e}{(1+\chi_e/3)} = \frac{3\epsilon_0}{N}\frac{\chi_e}{(3+\chi_e}. \quad \text{But } \chi_e = \epsilon_r - 1, \text{ so } \alpha = \frac{3\epsilon_0}{N}\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right). \quad \text{qed}$$

Problem 4.42 Check the Clausius-Mossotti relation (Eq. 4.72) for the gases listed in Table 4.1. (Dielectric constants are given in Table 4.2.) (The densities here are so small that Eqs. 4.70 and 4.72 are indistinguishable. For experimental data that confirm the Clausius-Mossotti correction term see, for instance, the first edition of Purcell's *Electricity and Magnetism*, Problem 9.28.)[23]

For an ideal gas, N = Avagadro's number/22.4 liters $= (6.02 \times 10^{23})/(22.4 \times 10^{-3}) = 2.7 \times 10^{25}$. $N\alpha/\epsilon_0 = (2.7 \times 10^{25})(4\pi\epsilon_0 \times 10^{-30})\beta/\epsilon_0 = 3.4 \times 10^{-4}\beta$, where β is the number listed in Table 4.1.

$$\left.\begin{array}{ll} \text{H:} & \beta = 0.667,\ N\alpha/\epsilon_0 = (3.4 \times 10^{-4})(0.67) = 2.3 \times 10^{-4},\ \chi_e = 2.5 \times 10^{-4} \\ \text{He:} & \beta = 0.205,\ N\alpha/\epsilon_0 = (3.4 \times 10^{-4})(0.21) = 7.1 \times 10^{-5},\ \chi_e = 6.5 \times 10^{-5} \\ \text{Ne:} & \beta = 0.396,\ N\alpha/\epsilon_0 = (3.4 \times 10^{-4})(0.40) = 1.4 \times 10^{-4},\ \chi_e = 1.3 \times 10^{-4} \\ \text{Ar:} & \beta = 1.64,\ \ \ N\alpha/\epsilon_0 = (3.4 \times 10^{-4})(1.64) = 5.6 \times 10^{-4},\ \chi_e = 5.2 \times 10^{-4} \end{array}\right\} \text{agreement is quite good.}$$

Problem 4.43 The Clausius-Mossotti equation (Prob. 4.41) tells you how to calculate the susceptibility of a *nonpolar* substance, in terms of the atomic polarizability α. The **Langevin equation** tells you how to calculate the susceptibility of a *polar* substance, in terms of the permanent molecular dipole moment p. Here's how it goes:

(a) The energy of a dipole in an external field $\mathbf{E}$ is $u = -\mathbf{p} \cdot \mathbf{E} = -pE\cos\theta$ (Eq. 4.6), where θ is the usual polar angle, if we orient the z axis along $\mathbf{E}$. Statistical mechanics says that for a material in equilibrium at absolute temperature T, the probability of a given molecule having energy u is proportional to the Boltzmann factor,

$$\exp(-u/kT).$$

The average energy of the dipoles is therefore

$$<u> = \frac{\int u e^{-(u/kT)}\, d\Omega}{\int e^{-(u/kT)}\, d\Omega},$$

where $d\Omega = \sin\theta\, d\theta\, d\phi$, and the integration is over all orientations ($\theta : 0 \to \pi$; $\phi : 0 \to 2\pi$). Use this to show that the polarization of a substance containing N molecules per unit volume is

$$P = Np[\coth(pE/kT) - (kT/pE)]. \tag{4.73}$$

That's the Langevin formula. Sketch P/Np as a function of pE/kT.

(a) Doing the (trivial) ϕ integral, and changing the remaining integration variable from θ to u ($du = pE\sin\theta\, d\theta$),

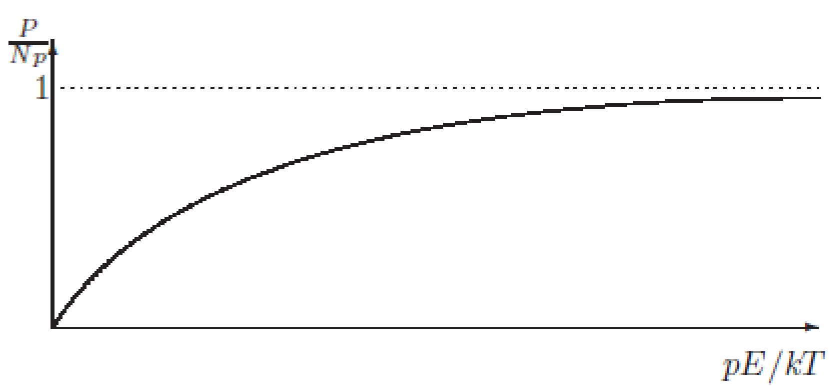
$$\langle u\rangle = \frac{\int_{-pE}^{pE} u e^{-u/kT}\,du}{\int_{-pE}^{pE} e^{-u/kT}\,du} = \frac{(kT)^2 e^{-u/kT}\left[-(u/kT)-1\right]\Big|_{-pE}^{pE}}{-kTe^{-u/kT}\Big|_{-pE}^{pE}}$$

$$= kT\left\{\frac{\left[e^{-pE/kT}-e^{pE/kT}\right]+\left[(pE/kT)e^{-pE/kT}+(pE/kT)e^{pE/kT}\right]}{e^{-pE/kT}-e^{pE/kT}}\right\}$$

$$= kT - pE\left[\frac{e^{pE/kT}+e^{-pE/kT}}{e^{pE/kT}-e^{-pE/kT}}\right] = kT - pE\coth\left(\frac{pE}{kT}\right).$$

$$\mathbf{P} = N\langle\mathbf{p}\rangle;\ \mathbf{p} = \langle p\cos\theta\rangle\hat{\mathbf{E}} = \langle\mathbf{p}\cdot\mathbf{E}\rangle(\hat{\mathbf{E}}/E) = -\langle u\rangle(\hat{\mathbf{E}}/E);\ P = Np\frac{-\langle u\rangle}{pE} = \boxed{Np\left\{\coth\left(\frac{pE}{kT}\right)-\frac{kT}{pE}\right\}}.$$

Let $y \equiv P/Np$, $x \equiv pE/kT$. Then $y = \coth x - 1/x$. As $x \to 0$, $y = \left(\frac{1}{x}+\frac{x}{3}-\frac{x^3}{45}+\cdots\right)-\frac{1}{x} = \frac{x}{3}-\frac{x^3}{45}+\cdots \to 0$, so the graph starts at the origin, with an initial slope of $1/3$. As $x\to\infty$, $y\to\coth(\infty)=1$, so the graph goes asymptotically to $y=1$ (see Figure).



(b) Notice that for large fields/low temperatures, virtually *all* the molecules are lined up, and the material is *non*linear. Ordinarily, however, kT is much greater than pE. Show that in this régime the material *is* linear, and calculate its susceptibility, in terms of N, p, T, and k. Compute the susceptibility of water at 20°C, and compare the experimental value in Table 4.2. (The dipole moment of water is 6.1×10^{-30} C·m.) This is rather far off, because we have again neglected the distinction between $\mathbf{E}$ and $\mathbf{E}_{\text{else}}$. The agreement is better in low-density gases, for which the difference between $\mathbf{E}$ and $\mathbf{E}_{\text{else}}$ is negligible. Try it for water vapor at 100°C and 1 atm.

(b) For small x, $y \approx \frac{1}{3}x$, so $\frac{P}{Np} \approx \frac{pE}{3kT}$, or $P \approx \frac{Np^2}{3kT}E = \epsilon_0\chi_e E \Rightarrow P$ is proportional to E, and $\boxed{\chi_e = \dfrac{Np^2}{3\epsilon_0 kT}}$.

For water at $20° = 293\,\mathrm{K}$, $p = 6.1 \times 10^{-30}\,\mathrm{C\,m}$; $N = \frac{\text{molecules}}{\text{volume}} = \frac{\text{molecules}}{\text{mole}} \times \frac{\text{moles}}{\text{gram}} \times \frac{\text{grams}}{\text{volume}}$.
$N = (6.0 \times 10^{23}) \times \left(\frac{1}{18}\right) \times (10^6) = 0.33 \times 10^{29}$; $\chi_e = \frac{(0.33\times10^{29})(6.1\times10^{-30})^2}{(3)(8.85\times10^{-12})(1.38\times10^{-23})(293)} = \boxed{12.}$ Table 4.2 gives an experimental value of 79, so it's pretty far off.

For water vapor at $100° = 373\,\mathrm{K}$, treated as an ideal gas, $\frac{\text{volume}}{\text{mole}} = (22.4 \times 10^{-3}) \times \left(\frac{373}{293}\right) = 2.85 \times 10^{-2}\ \mathrm{m}^3$.

$$N = \frac{6.0 \times 10^{23}}{2.85 \times 10^{-2}} = 2.11 \times 10^{25}; \quad \chi_e = \frac{(2.11 \times 10^{25})(6.1 \times 10^{-30})^2}{(3)(8.85 \times 10^{-12})(1.38 \times 10^{-23})(373)} = \boxed{5.7 \times 10^{-3}.}$$

Table 4.2 gives 5.9×10^{-3}, so this time the agreement is quite good.