Branch and Price

MIP problem

(P)

$$\begin{aligned} z = \min \quad & \sum_{i \in I} c_i x_i \\ \text{s.t.} \quad & \sum_{i \in I} a_i x_i \geq b \\ & x_i \in \mathbb{Z}_+, \quad i \in I \end{aligned} \tag{1}$$

a_i and b are column vectors.

Cutting Stock exemple:

There is m orders with widths w_j each one with demand b_j, and the standard roll with W width.

We want to cut the ordered rolls to fullfill theirs demand by using a minimum quantity of standard rolls.

Let I be the set of patterns of cut that do not violate the max roll width:

$I = \{i \in {\mathbb{Z}_+}^m : \sum_{j=1}^m w_j i_j \leq W\}.$

The set I_R will now on denote a subset of I, for exemplo, for the patterns presented in the image above it is

$$I_R = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right\}$$

The variable x_i is the amount of pattern i to processed. Therefore the problem can be formulated as following:

$$
\begin{aligned}
\min \quad & \sum_{i \in I} x_i \\
\text{s.t.} \quad & \sum_{i \in I} i_j x_i \geq b \\
& x_i \in \mathbb{Z}_+, \quad i \in I
\end{aligned} \tag{2}
$$

Back to the (P) problem

I_R is the restricted subset of I, hence it defines the Restricted Master Problem RMP as the linear relaxation of P over I_R.

(RMP)

$$
\begin{aligned}
z_{RMP} = \quad \min \quad & \sum_{i \in I_R} c_i x_i \\
\text{s.t.} \quad & \sum_{i \in I_R} a_i x_i \geq b \\
& x_i \geq 0
\end{aligned} \tag{3}
$$

The (RMP) can be represented in a compact matricial notation:

$$
\begin{aligned}
z_{RMP} = \quad \min \quad & c_R x_R \\
\text{s.t.} \quad & A_R x_R \geq b
\end{aligned} \tag{4}
$$

Let x^*_R and π^*_R the primal and dual optimum solutions of RMP.

The strong duality of linear optimization problems gives $z_{RMP} = \pi^*_R b$.

Besides, consider the z_{LP} the optimum value of the linear relaxation of P over I, which is a not known value at this moment.

Despite that, as $I_R \subseteq I$ than $z_{LP} \leq z_{RMP}$.

Therefore, $\pi^*_R b$ is an upper bound for the linear relaxation of P. Would the equality happens here? If that is true, as $c_R x^*_R = \pi^*_R b$ than x_R is an optimum solution for the LR of P.

In order to awnser that question, it is intended to check if π^*_R would be dual feasible for the LR of P as well. As for simplex method, the reduced cost should be greater or equal to zero.

It should be done implicitily over the set I, that is, we should solve the following pricing problem:

(PP)

$$
\begin{aligned}
\sigma = \quad \min \quad & c_i - \pi^*_R a_i \\
\text{s.t.} \quad & i \in I
\end{aligned} \tag{5}
$$

Therefore,

- if $\sigma \geq 0$ then π_R is an dual optimum of the LR of (P).
- otherwise, the optimum solution of (PP) is a candidate to be a new collumn for the (RMP).

Back to Cutting stock exemple:

The columns vectors a_i is not explicitily available, but it is defined through the rule that defines I
.

Accordingly, considering the m vector variable y, the PP problem becomes

$$
\begin{aligned}
\sigma = \quad & \min \quad 1 - \sum_{j=1}^{m} \pi_j^* y_j \\
& \text{s.t.} \quad \sum_{j=1}^{m} w_j y_j \leq W \\
& \qquad\quad y_j \geq 0.
\end{aligned}
\tag{6}
$$