Some Recent and Rediscovered Developments in Bayes Linear Statistics

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Statistics without probability

Defining Expectation Without Probability

We now define expectation (note, we still have not defined probability) of random quantity X, $\mathrm{E}[\mathrm{X}]$, as the value $\bar{x}$ you would choose if you must suffer penalty

$$L = \left(\frac{\mathrm{X} - \bar{x}}{k}\right)^2$$

once you observe X.

Assumption: Coherence. You do not have a preference for a given penalty if you have the option for one that is certainly smaller.

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- $\mathscr{B}$ requires the specification of $\mathrm{E}[\mathrm{X}]$, $\mathrm{E}[\mathrm{D}]$, $\mathrm{var}[\mathrm{X}]$, $\mathrm{var}[\mathrm{D}]$, and $\mathrm{cov}[\mathrm{X}, \mathrm{D}]$

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- **Expectation is the fundamental unit of belief and $\mathscr{B}$ is the analogy of the joint probability measure in a standard Bayesian analysis.**

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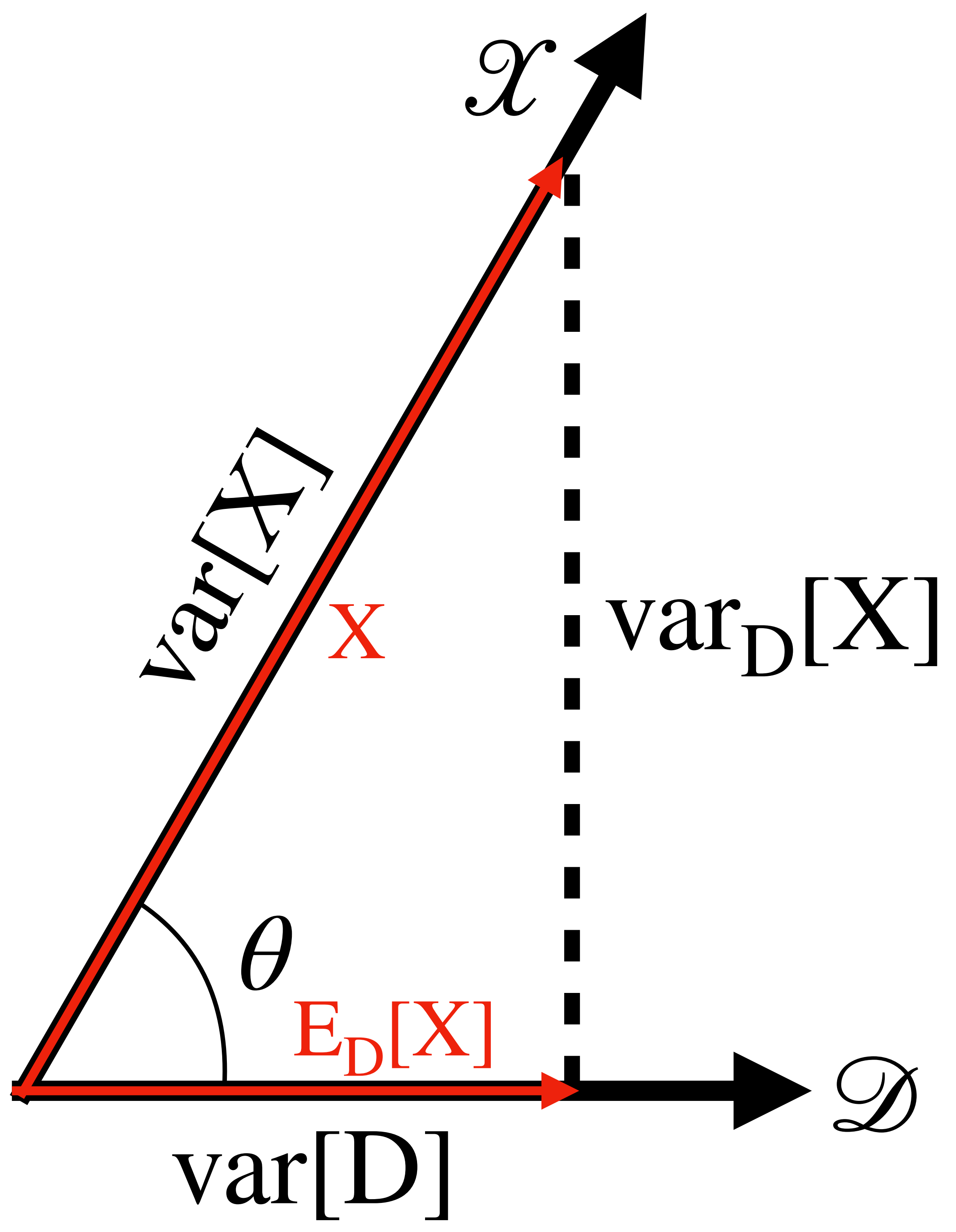
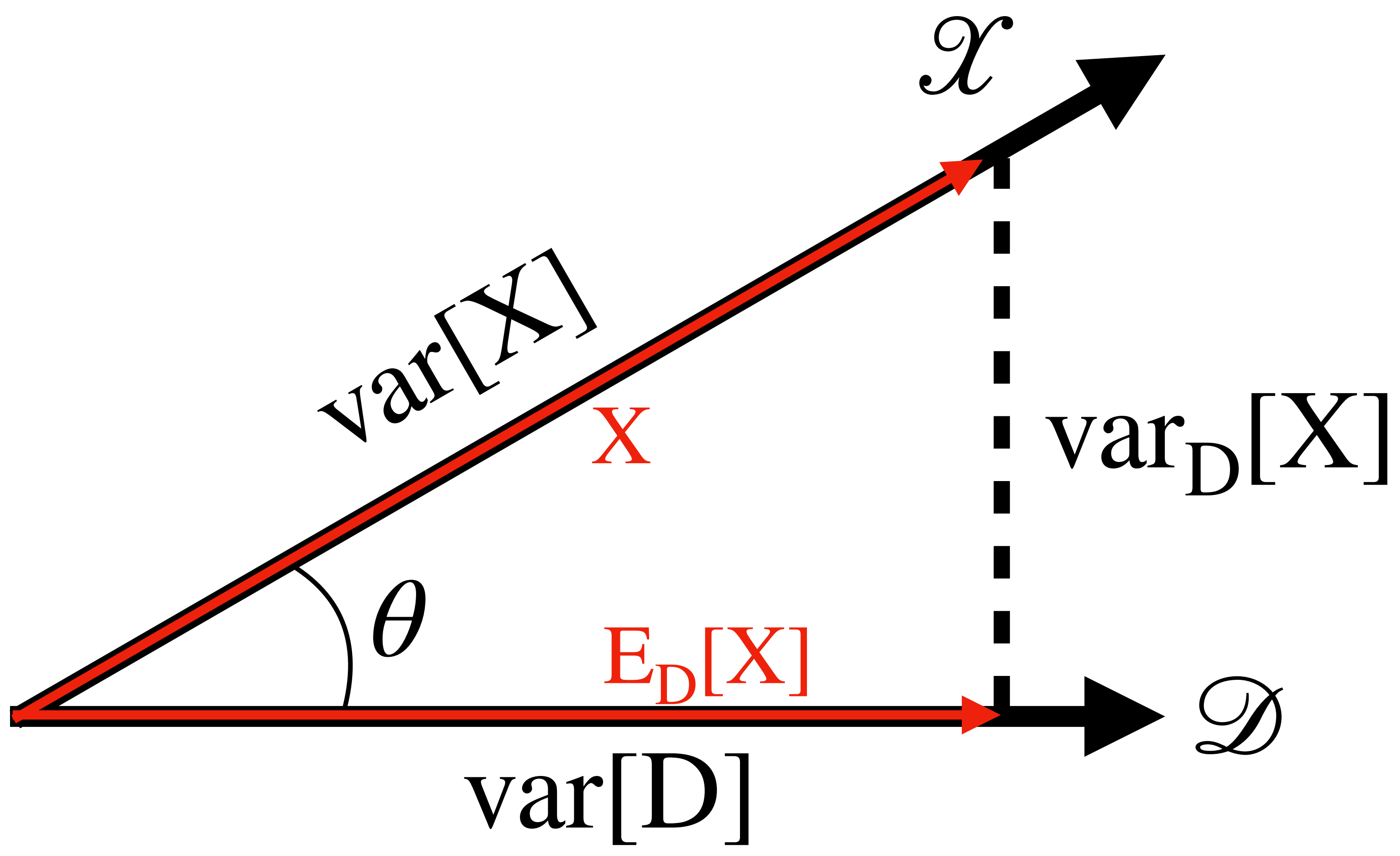
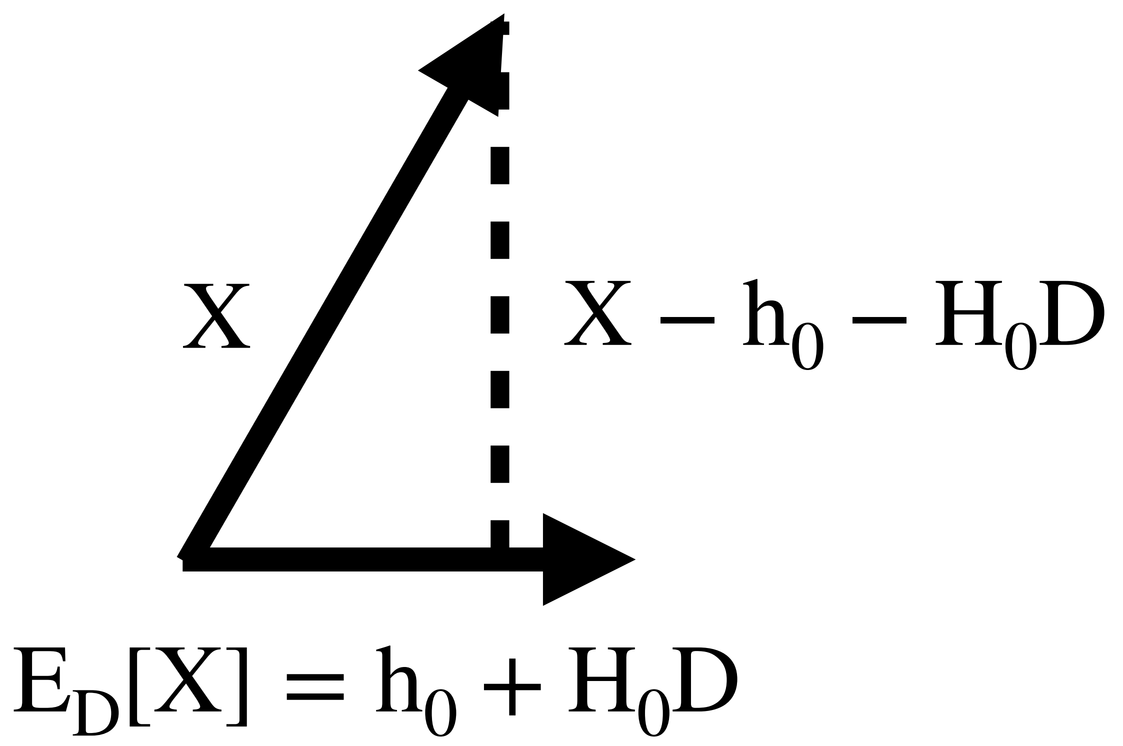




$$\theta = \cos^{-1}(\mathrm{cor[X, D]})$$

“the Bayes linear equations”



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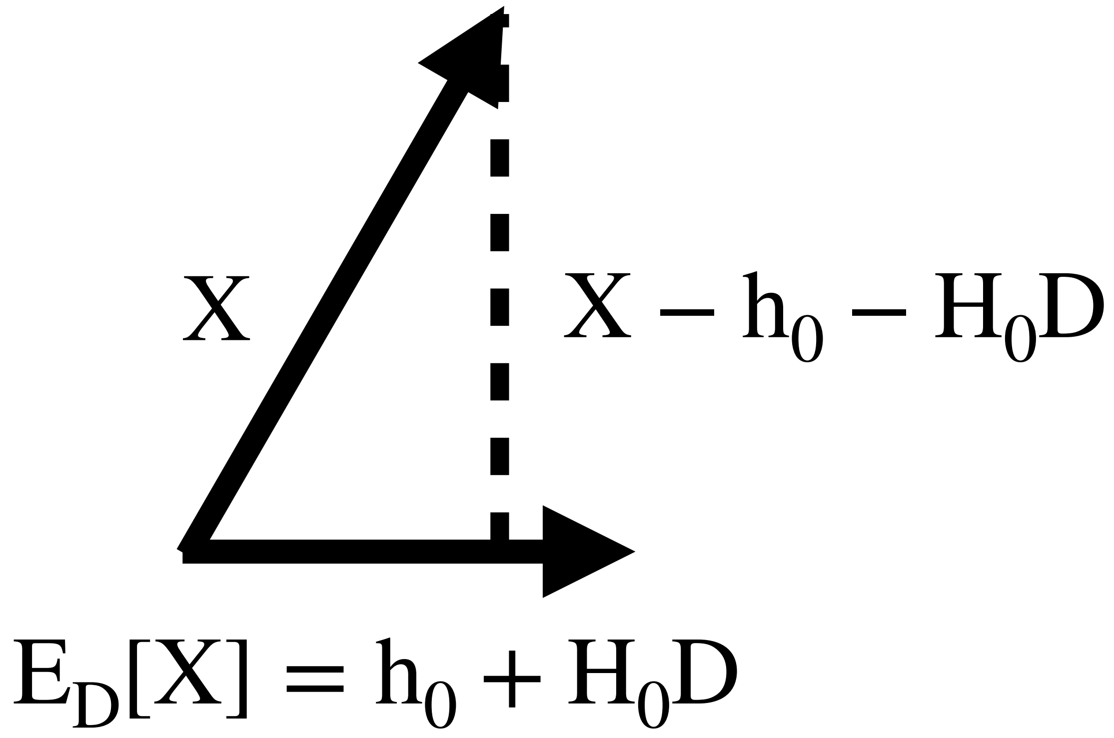

The orthogonal projection of X onto $h_0 + H_0 D$ solves:

$$\langle X - h_0 - H_0 D, h_0 + H_0 D \rangle = E[(X - h_0 - H_0 D)^\intercal (h_0 + H_0 D)] = 0,$$

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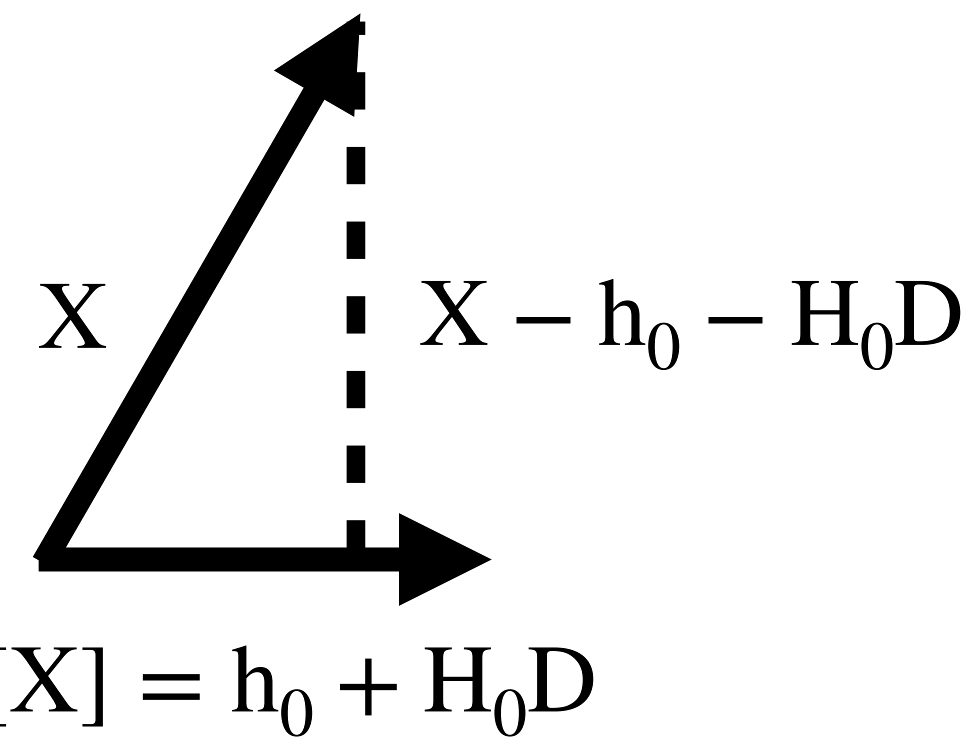

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so $h_0 = E[X] - H_0 E[D]$, $H_0 = \mathrm{cov}[X, D]\mathrm{var}[D]^{-1}$, and

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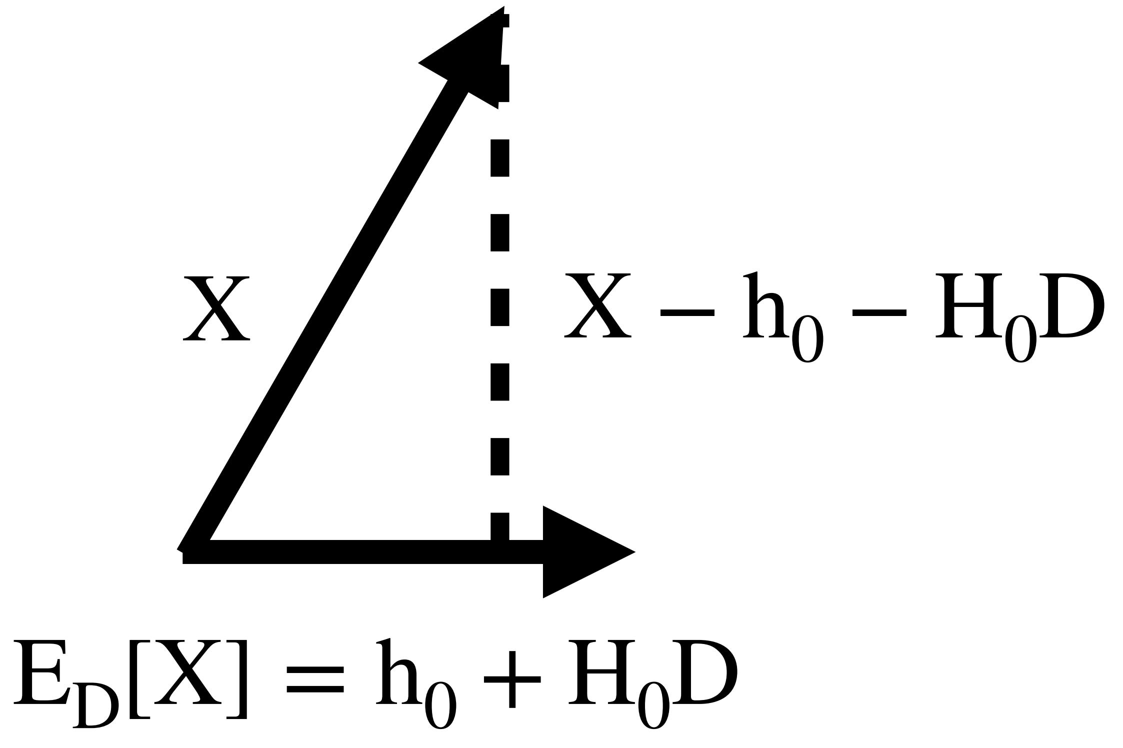

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The length $\mathrm{var}_D[X] = \|X - E_D[X]\|^2 = \langle X - E_D[X], X - E_D[X] \rangle$, so

$$\mathrm{var}_D[X] = \mathrm{var}[X] - \mathrm{cov}[X, D]\mathrm{var}[D]^{-1}\mathrm{cov}[D, X]$$

Bayes Linear

Belief space $\mathscr{B}$

E[X], E[D], var[X], var[D], cov[X, D]

Probabilistic Bayes

Probability measure P

p(X), p(D | X)

Bayes Linear		Probabilistic Bayes
Belief space $\mathscr{B}$	$\longleftrightarrow$	Probability measure P
$E[X], E[D], var[X], var[D], cov[X, D]$		$p(X), p(D \mid X)$
Adjusted Expectation/Variance	$\longleftrightarrow$	Posterior Distribution
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Bayes Linear		Probabilistic Bayes
Belief space $\mathscr{B}$	$\longleftrightarrow$	Probability measure P
$\mathrm{E}[\mathrm{X}], \mathrm{E}[\mathrm{D}], \mathrm{var}[\mathrm{X}], \mathrm{var}[\mathrm{D}], \mathrm{cov}[\mathrm{X}, \mathrm{D}]$		$\mathrm{p}(\mathrm{X}), \mathrm{p}(\mathrm{D} \mid \mathrm{X})$
Adjusted Expectation/Variance	$\longleftrightarrow$	Posterior Distribution
$\mathrm{E}_{\mathrm{D}}[\mathrm{X}], \mathrm{var}_{\mathrm{D}}[\mathrm{X}]$		$\mathrm{E}[\mathrm{X} \mid \mathrm{D}], p(\mathrm{X} \mid \mathrm{D})$
Probability from expectation	$\longleftrightarrow$	Expectation from probability
$\mathrm{P}(E) = \mathrm{E}[\mathbf{1}_E]$		$\mathrm{E}[\mathrm{X}] = \int x\mathrm{p}(x)\,\mathrm{d}x$

Normal without normality?

$$\mathrm{E_D[X] = E[X] + cov[X, D] var[D]^{-1}(E[D] - D)}$$

$$\mathrm{var_D[X] = var[X] - cov[X, D] var[D]^{-1} cov[D, X]}$$

$$E_D[X] = E[X] + \mathrm{cov}[X, D]\mathrm{var}[D]^{-1}(E[D] - D)$$

$$\mathrm{var}_D[X] = \mathrm{var}[X] - \mathrm{cov}[X, D]\mathrm{var}[D]^{-1}\mathrm{cov}[D, X]$$

Isn't this just the posterior equations for an update with normal prior and likelihood?

$$E[X \mid D] = E[X] + \mathrm{cov}[X, D]\mathrm{var}[D \mid X]^{-1}(E[D \mid X] - D)$$

$$\mathrm{var}[X \mid D] = \mathrm{var}[X] - \mathrm{cov}[X, D]\mathrm{var}[D \mid X]^{-1}\mathrm{cov}[D, X]$$

This isn’t the whole story…

(proof extended from the results of Hartigan, 1969)

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Assume in a probabilistic Bayesian analysis that the posterior expectation is linear in D, $\mathrm{E}[\mathrm{X} \mid \mathrm{D}] = A\mathrm{D} + B$

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And again: $\mathrm{E}[\mathrm{D}\mathrm{X}^\intercal] = \mathrm{E}_{\mathrm{D}}\left[\mathrm{D}\mathrm{E}_{\mathrm{X}}[\mathrm{X} \mid \mathrm{D}]\right] = \mathrm{E}_{\mathrm{D}}\left[\mathrm{D}(A\mathrm{D} + B)^\intercal\right]$

$$= \mathrm{var}[\mathrm{D}]A^\intercal + \mathrm{E}[\mathrm{D}]\mathrm{E}[\mathrm{X}]^\intercal$$

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$$= \mathrm{var}[\mathrm{D}]A^\intercal + \mathrm{E}[\mathrm{D}]\mathrm{E}[\mathrm{X}]^\intercal$$

Definition of covariance: $\mathrm{E}[\mathrm{D}\mathrm{X}^\intercal] = \mathrm{cov}[\mathrm{X}, \mathrm{D}] + \mathrm{E}[\mathrm{D}]\mathrm{E}[\mathrm{X}]^\intercal$

$$A = \mathrm{cov}[\mathrm{X}, \mathrm{D}]\mathrm{var}[\mathrm{D}]^{-1}, \qquad B = \mathrm{E}[\mathrm{X}] - A\mathrm{E}[\mathrm{D}]$$

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Substitute A and B into $\mathrm{E}[\mathrm{X} \mid \mathrm{D}] = A\mathrm{D} + B$

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$$\mathrm{var}[\mathrm{X} \mid \mathrm{D}] = \mathrm{var}[\mathrm{X}] - \mathrm{cov}[\mathrm{X}, \mathrm{D}]\mathrm{var}[\mathrm{D}]^{-1}\mathrm{cov}[\mathrm{D}, \mathrm{X}]$$

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We can recover the Bayes linear equations only with the assumption that the posterior expectation is linear in D

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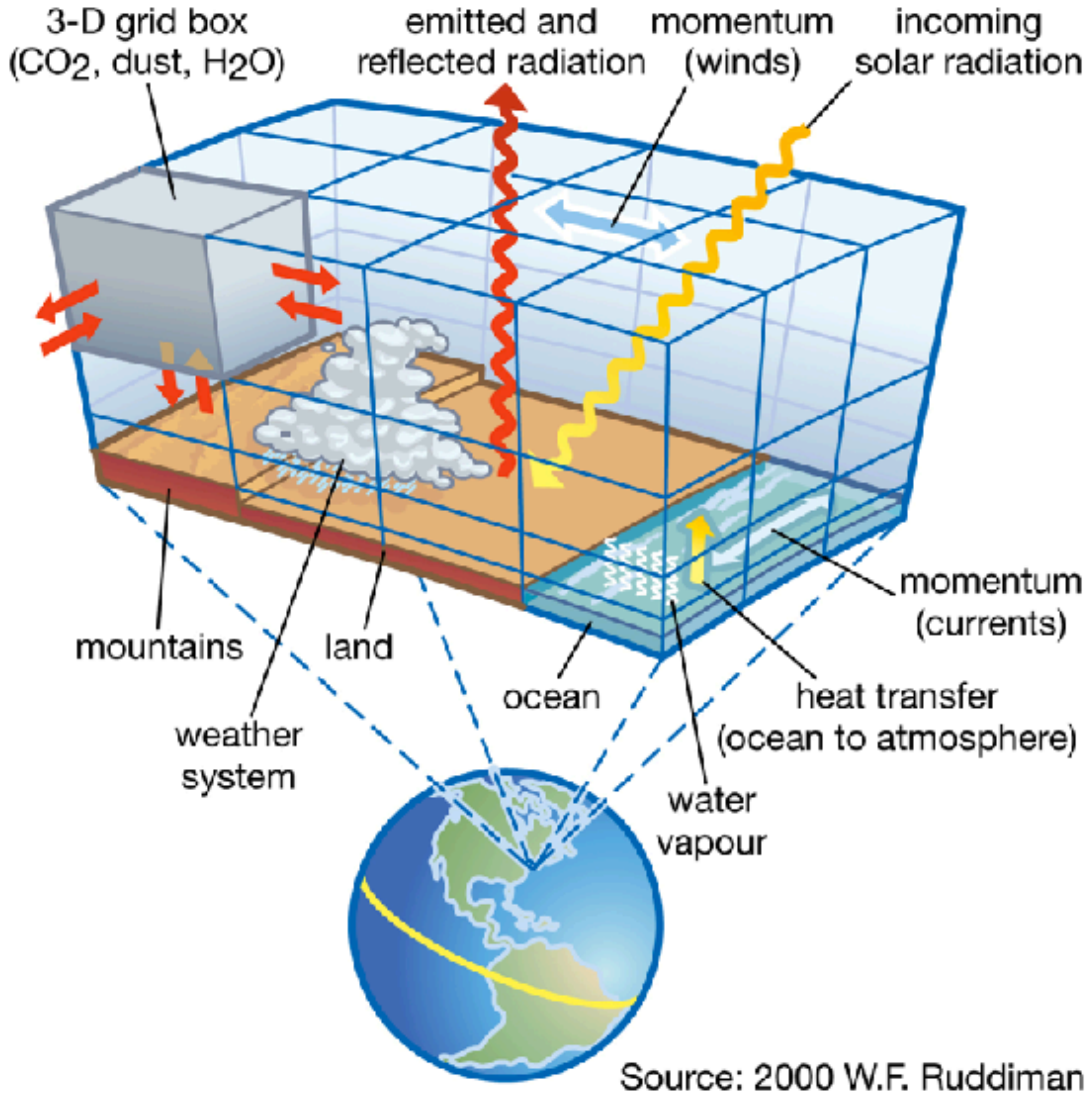
And some mixture models (Ericson, 1969)

Hierarchical Bayes Linear

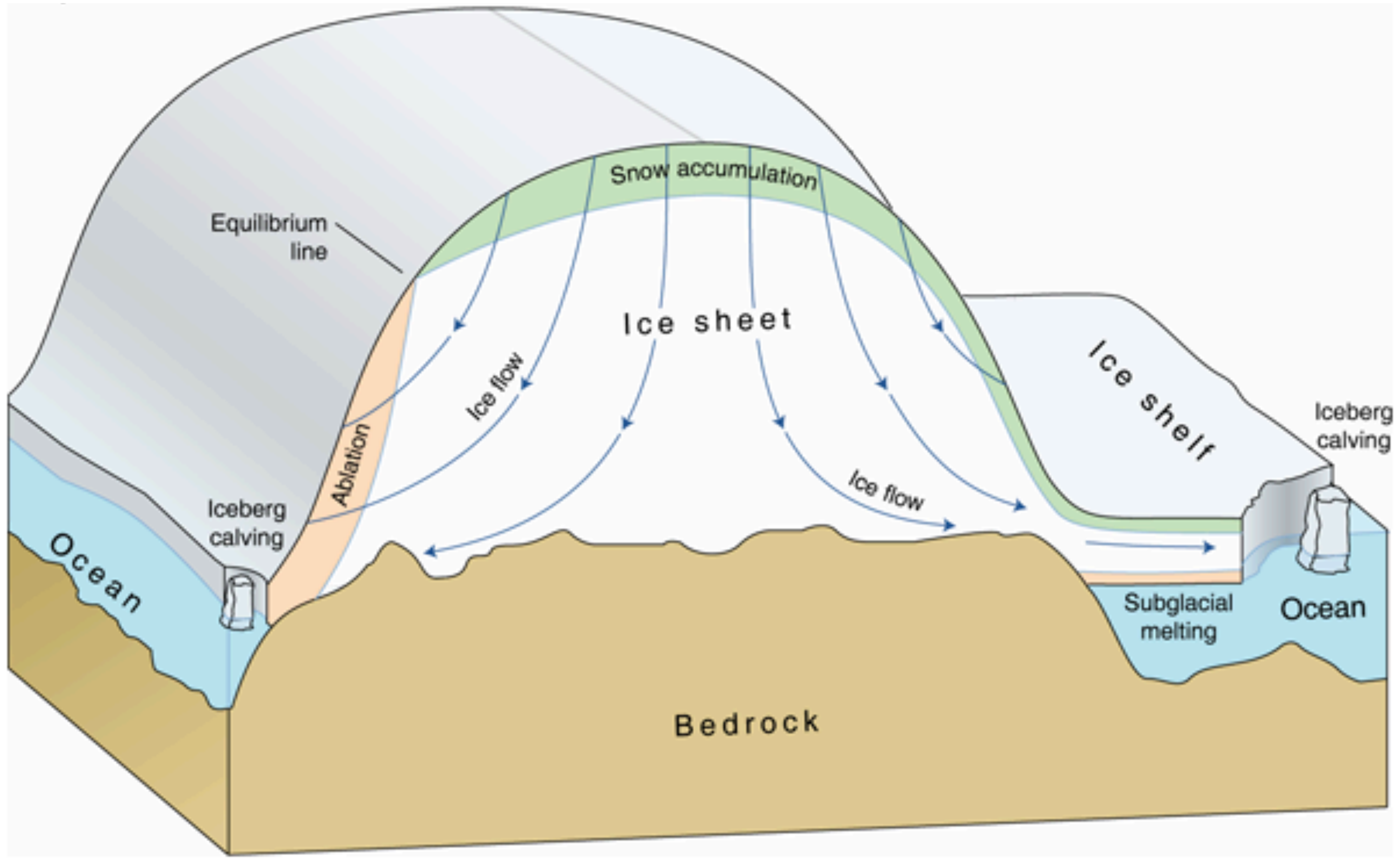
Modelling glacier dynamics is hard…



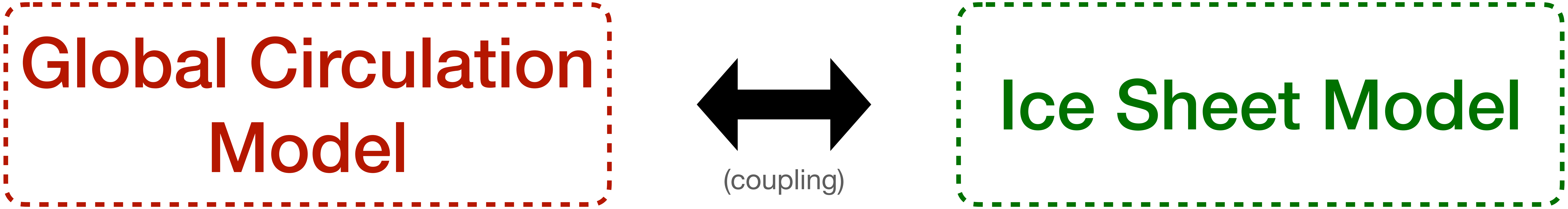
Global Circulation Models (GCM)

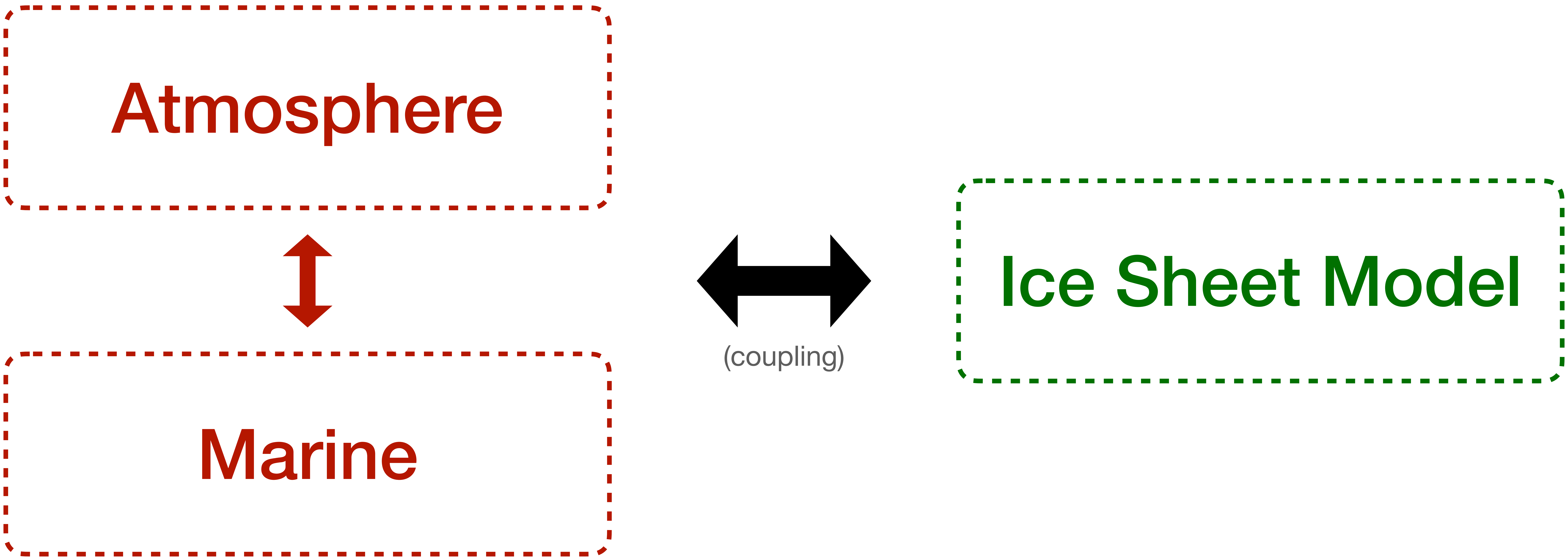
⟷

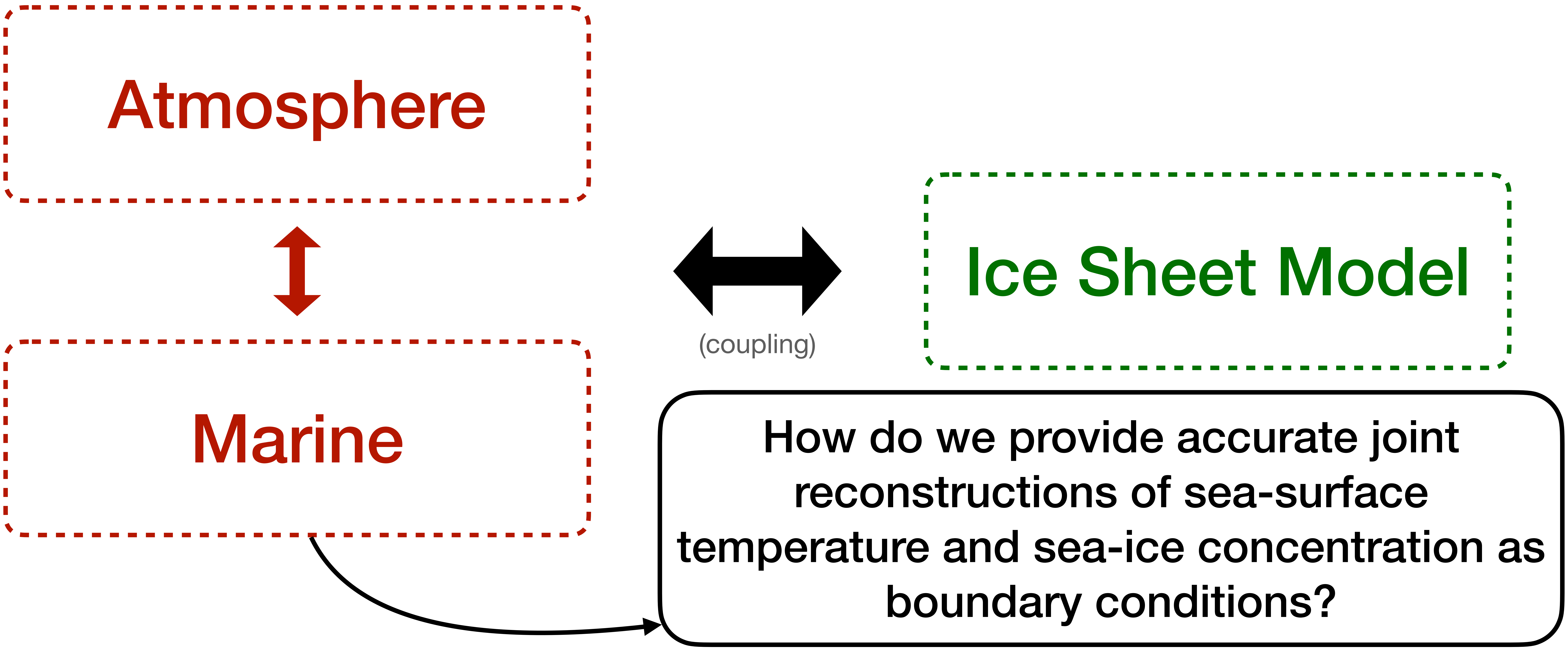
(coupling)



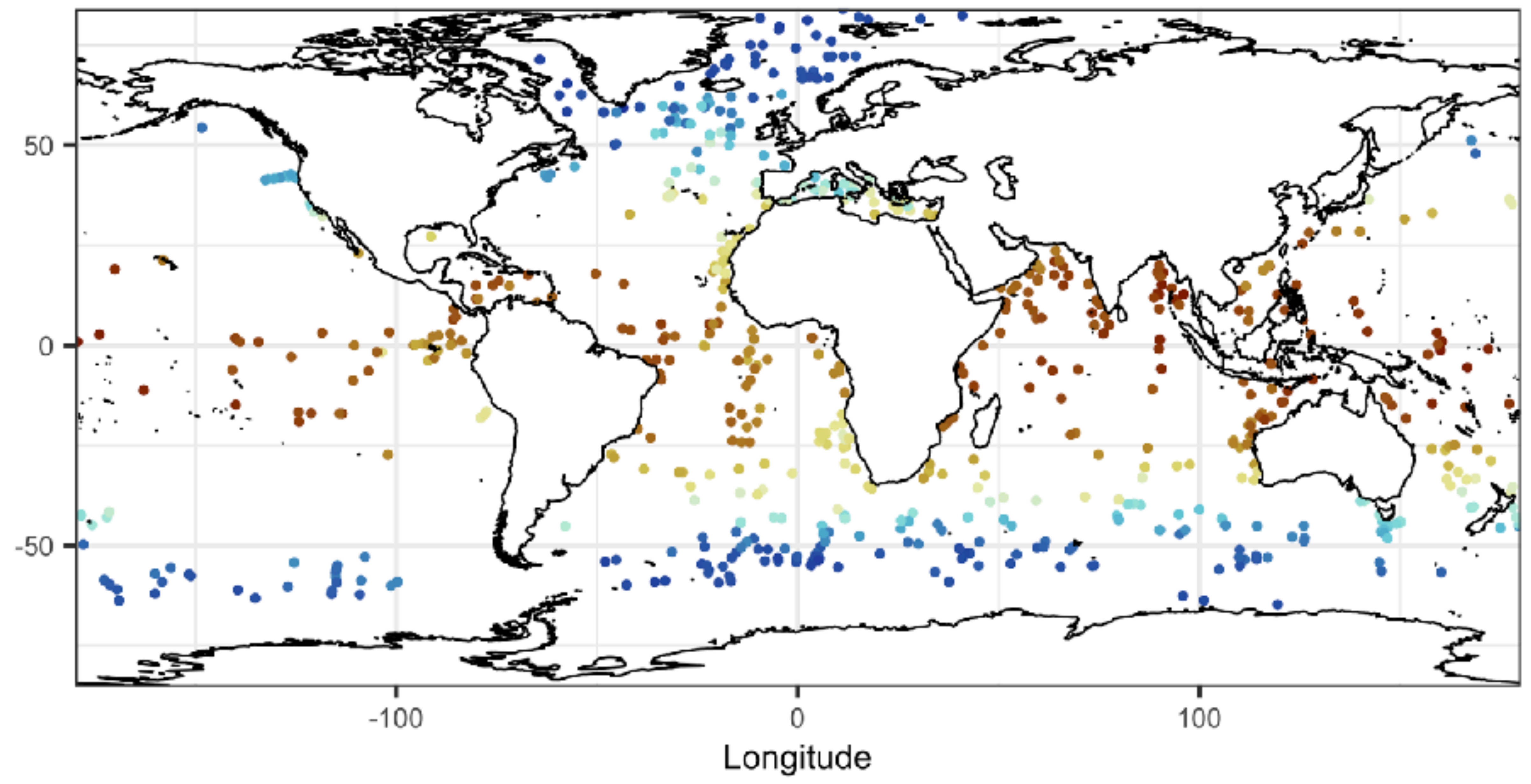
Regional Ice Sheet Models


Global Circulation Model
(coupling)
Ice Sheet Model


Atmosphere
Marine
(coupling)
Ice Sheet Model


Atmosphere
Marine
(coupling)
Ice Sheet Model
How do we provide accurate joint reconstructions of sea-surface temperature and sea-ice concentration as boundary conditions?

Data



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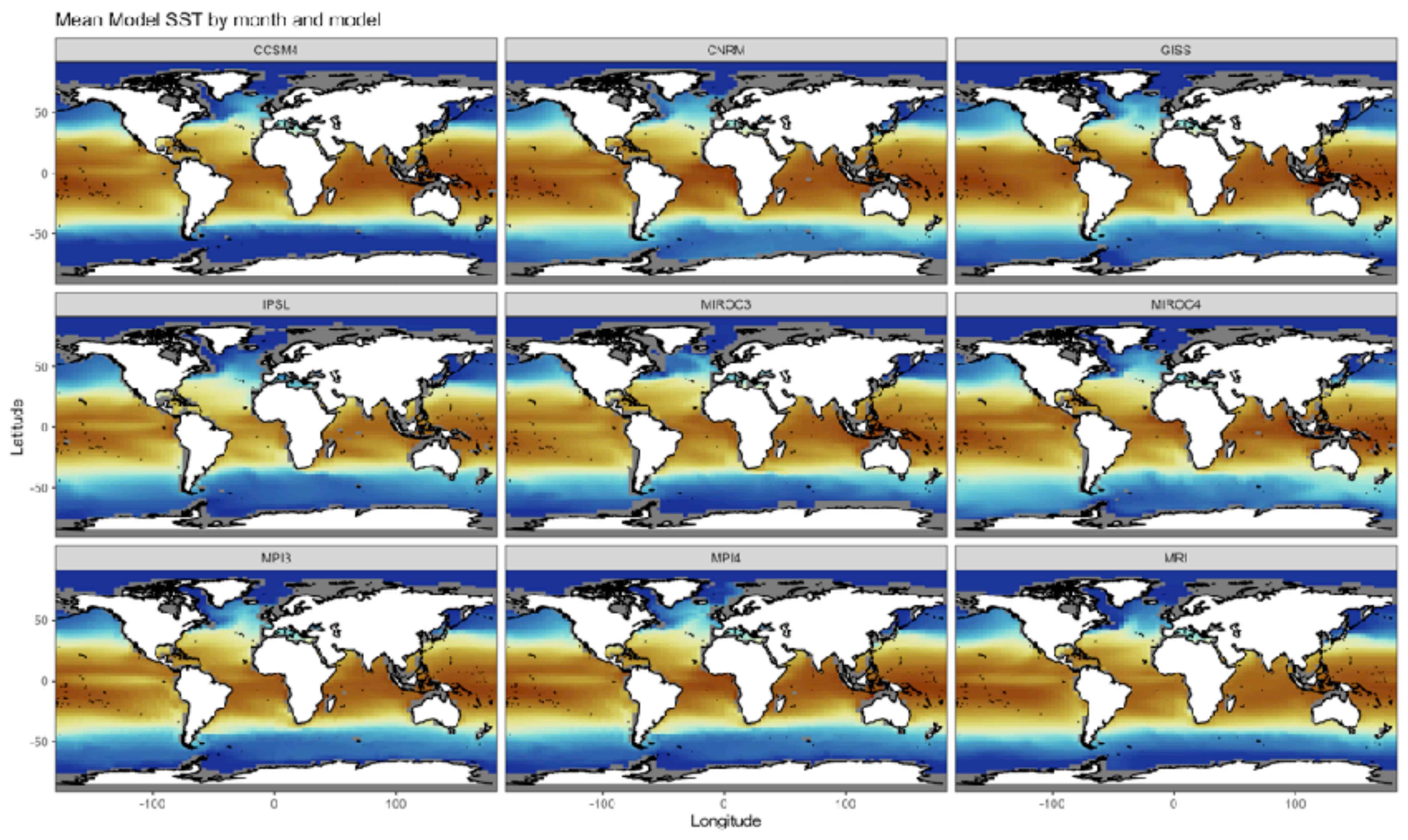
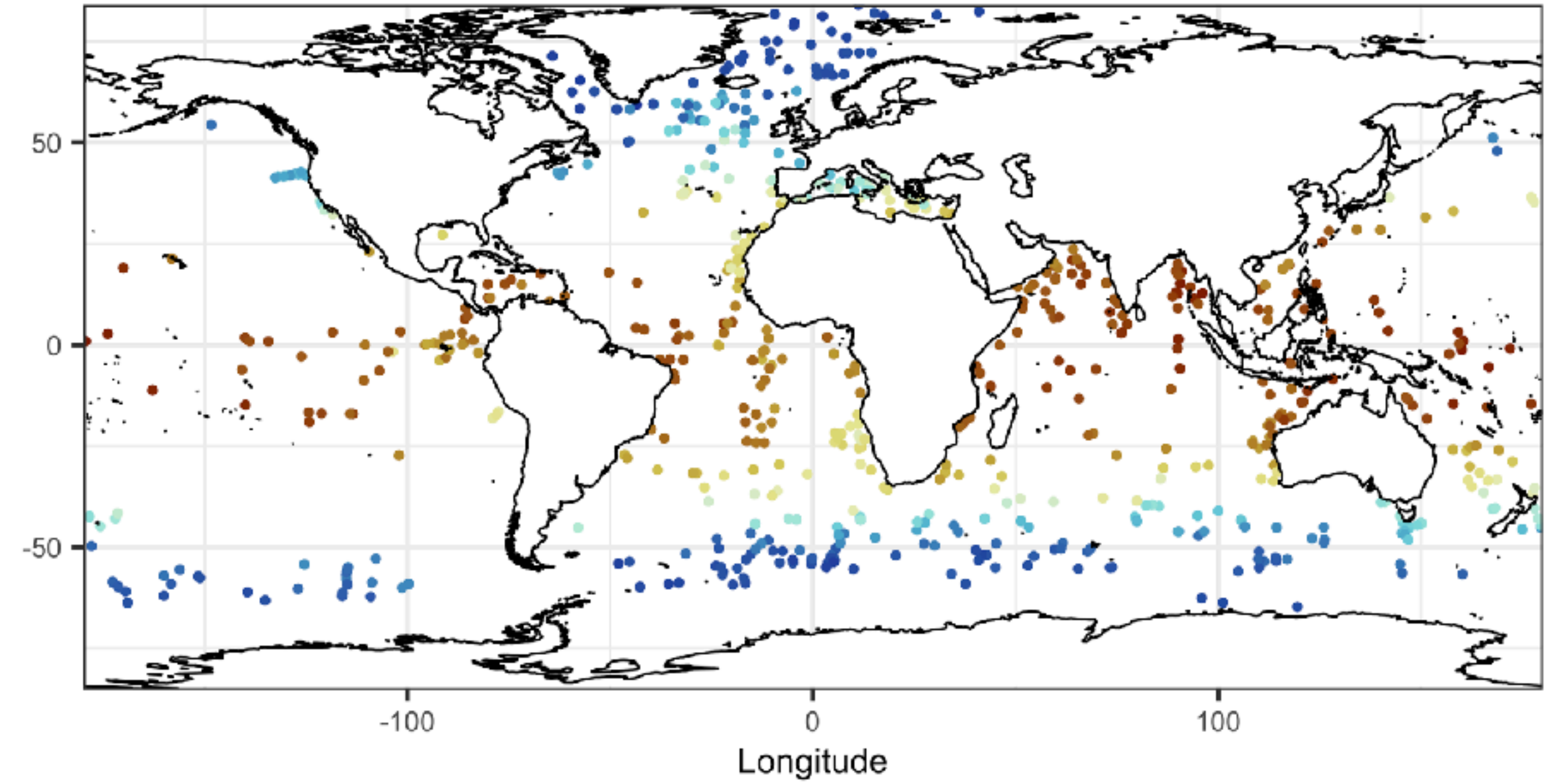
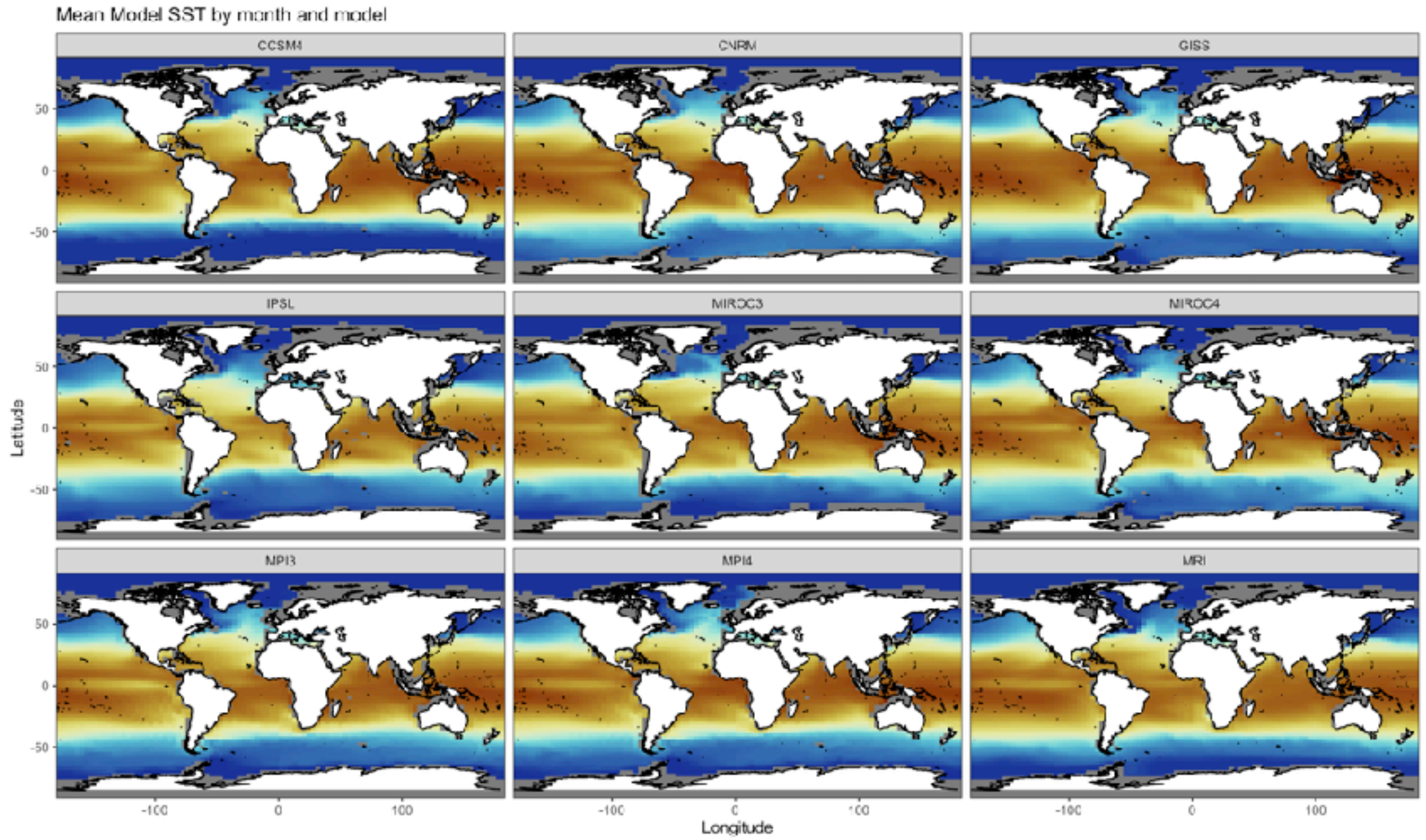
Model Runs




50
0
-50
-100
0
100
Longitude


Mean Model SST by month and model
CCSM4
CNRM
GISS
IPSL
MIROC3
MIROC4
MPI3
MPI4
MRI
Latitude
Longitude

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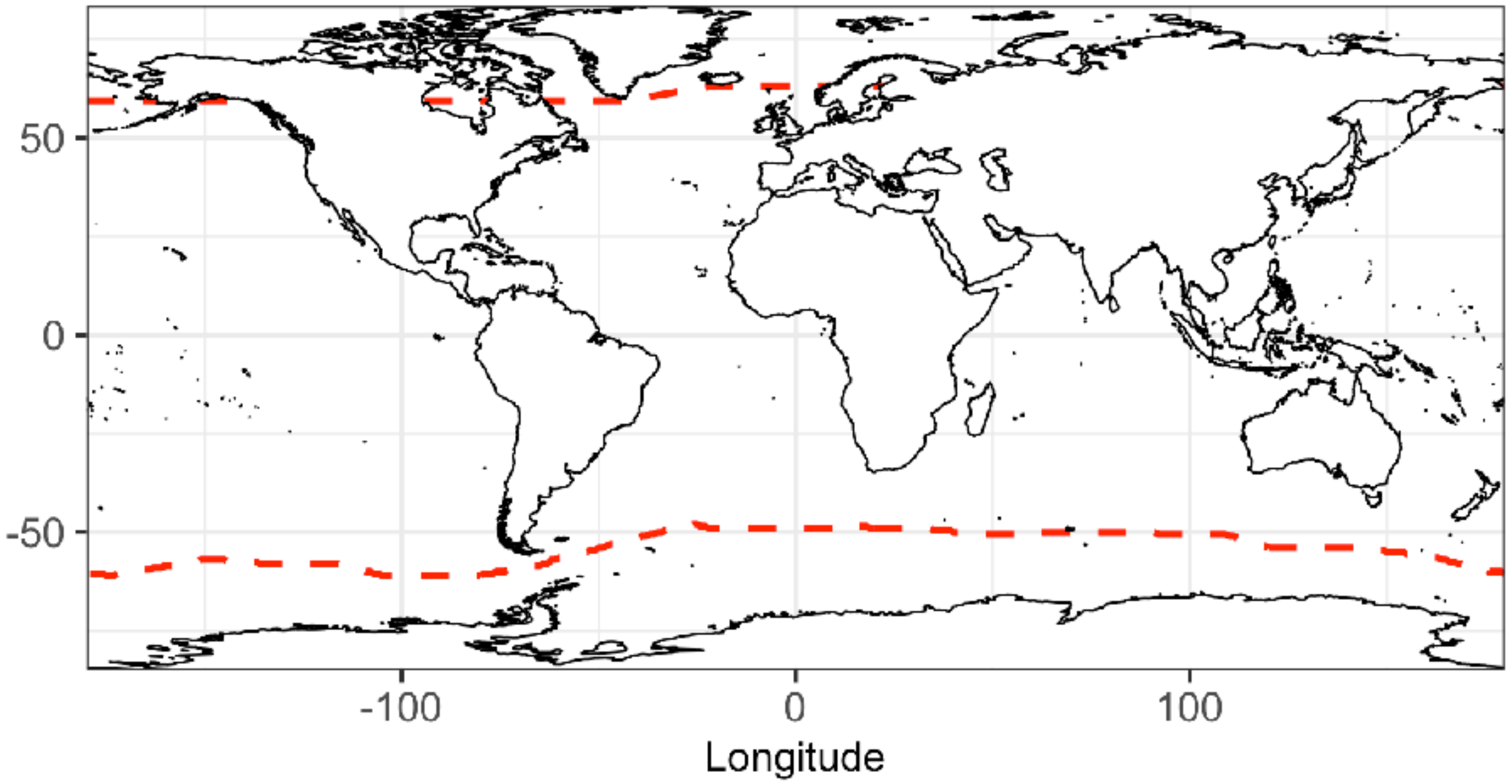
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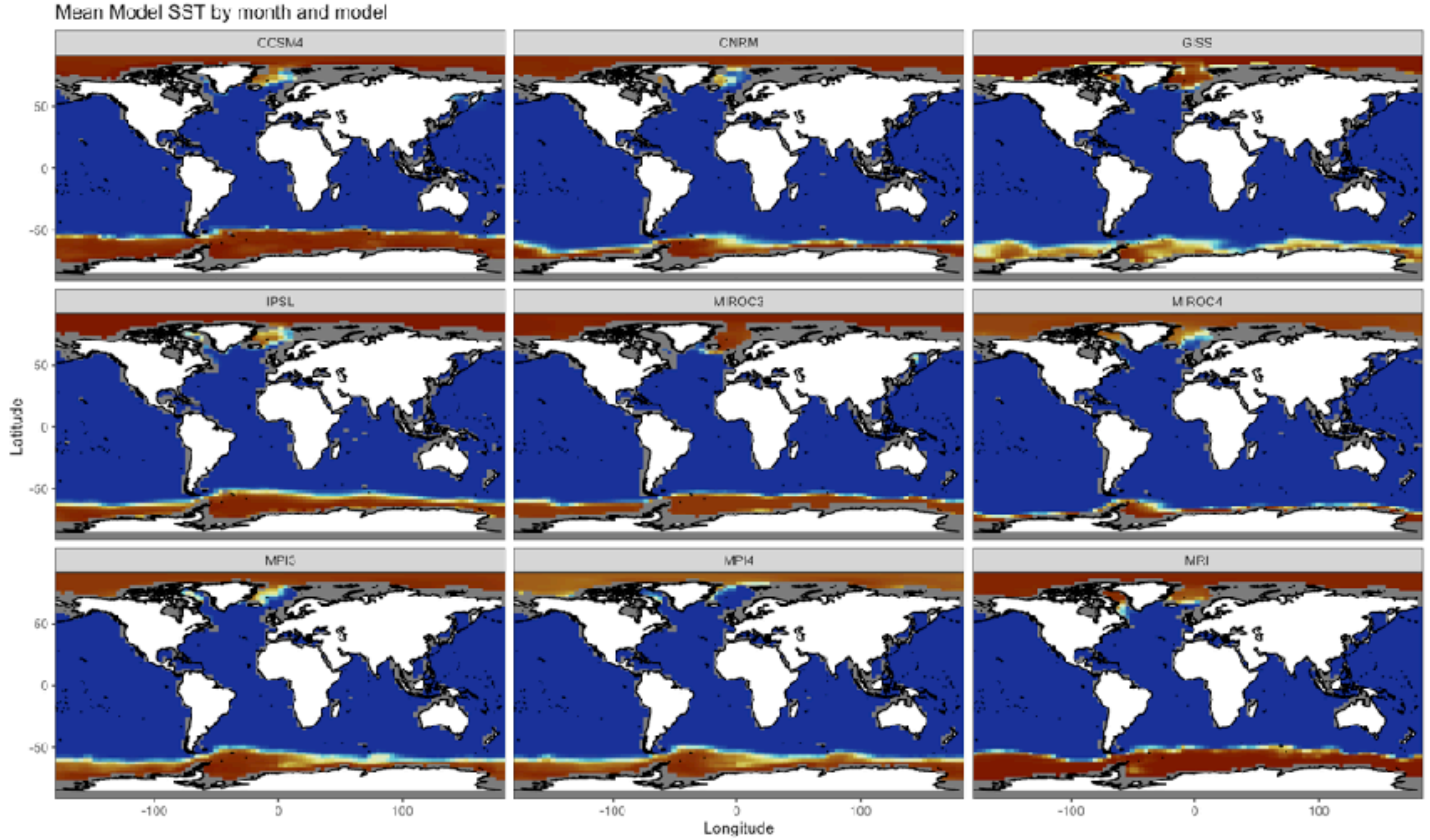
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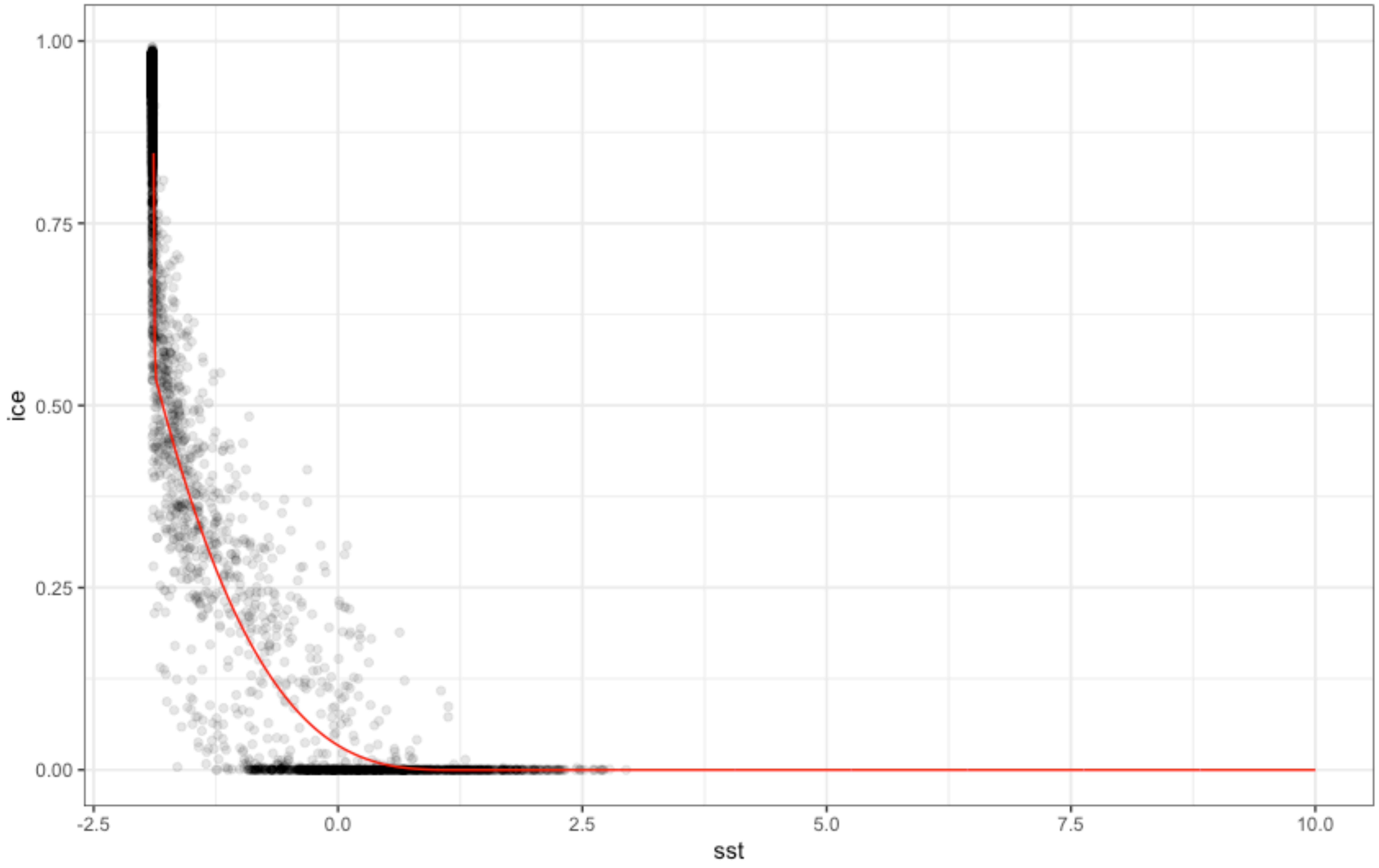
Joint behaviour of SST and SIC



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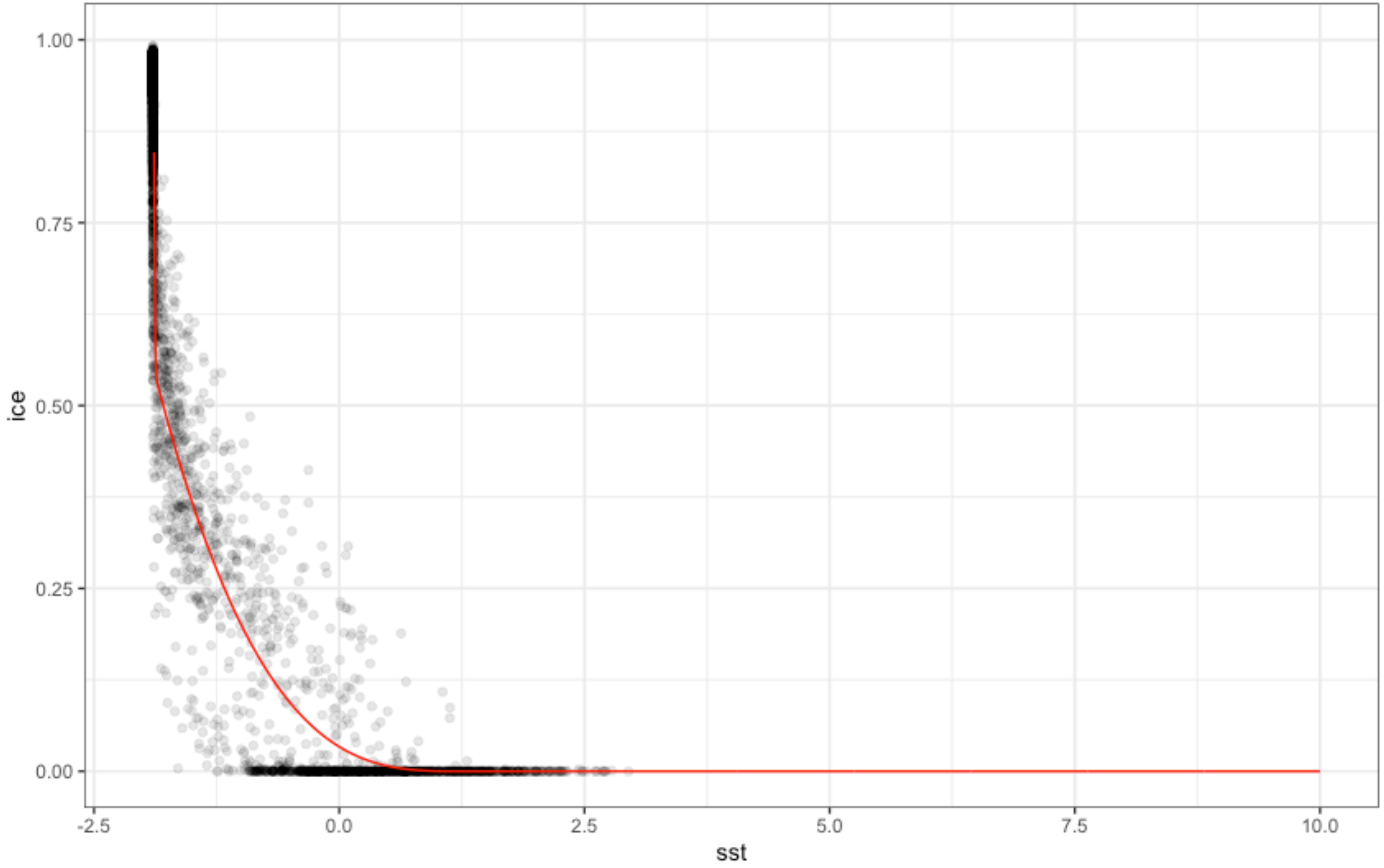

We have this at every grid cell in the model

$$
\begin{tikzcd}
\textbf{SIC:} & & Y_1 & & \cdots & & Y_m & \\
& & \Phi(X_1)\beta_1 \arrow[u] & & \cdots \arrow[u] & & \Phi(X_m)\beta_m \arrow[u] & \\
\textbf{SST:} & X_1 \arrow[ur] & & & \cdots \arrow[u] & & & X_m \arrow[ul] \\
& & & \beta_1 \arrow[uul] & & \beta_m \arrow[uur] & & \\
& & & & \mathrm{M}(\beta) \arrow[ul] \arrow[uu] \arrow[ur] & & &
\end{tikzcd}
$$

Adjusting Beliefs of $M(\beta)$

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_m \\ \beta_1 \\ \vdots \\ \beta_m \end{bmatrix} = \left[\begin{array}{ccc|c} \Phi_1 & \cdots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & \Phi_m & \\ \hline & 0 & & \mathbf{J}_{m\times 1} \otimes \mathbf{I} \end{array}\right] \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \\ \mathcal{M}(\beta) \end{bmatrix} + \begin{bmatrix} \mathcal{R}_1(Y) \\ \vdots \\ \mathcal{R}_m(Y) \\ \mathcal{R}_1(\beta) \\ \vdots \\ \mathcal{R}_m(\beta) \end{bmatrix}$$

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Adjusting Beliefs of $M(\beta)$

Following Hodges (1998), note that $0 = M(\beta) - \beta_i + R_i(\beta)$, and so with some manipulation

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_m \\ 0_{km\times 1} \end{bmatrix} = \left[\begin{array}{ccc|c} \Phi_1 & \cdots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & \Phi_m & \\ \hline & -\mathbf{I}_{km} & & \mathbf{J}_{m\times 1} \otimes \mathbf{I}_k \end{array}\right] \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \\ \mathcal{M}(\beta) \end{bmatrix} + \begin{bmatrix} \mathcal{R}_1(Y) \\ \vdots \\ \mathcal{R}_m(Y) \\ \mathcal{R}_1(\beta) \\ \vdots \\ \mathcal{R}_m(\beta) \end{bmatrix}$$

Now let’s make it fast

Define $\hat{\Phi}_i = (\Phi_i^\intercal \Phi_i)^{-1} \Phi_1^\intercal Y_1$ as the projection of Y_i onto the column space of Φ_i

$$
\begin{bmatrix} \hat{\Phi}_1 \\ \vdots \\ \hat{\Phi}_m \\ 0 \end{bmatrix}
=
\left[\begin{array}{ccc|c}
\mathbf{I}_k & \cdots & 0 & \\
\vdots & \ddots & \vdots & 0 \\
0 & \cdots & \mathbf{I}_k & \\
\hline
\multicolumn{3}{c|}{-\mathbf{I}_{km}} & \mathbf{J}_{m\times 1} \otimes \mathbf{I}_k
\end{array}\right]
\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \\ \mathcal{M}(\beta) \end{bmatrix}
+
\begin{bmatrix} \mathcal{R}_1(Y) \\ \vdots \\ \mathcal{R}_m(Y) \\ \mathcal{R}_1(\beta) \\ \vdots \\ \mathcal{R}_m(\beta) \end{bmatrix}
$$

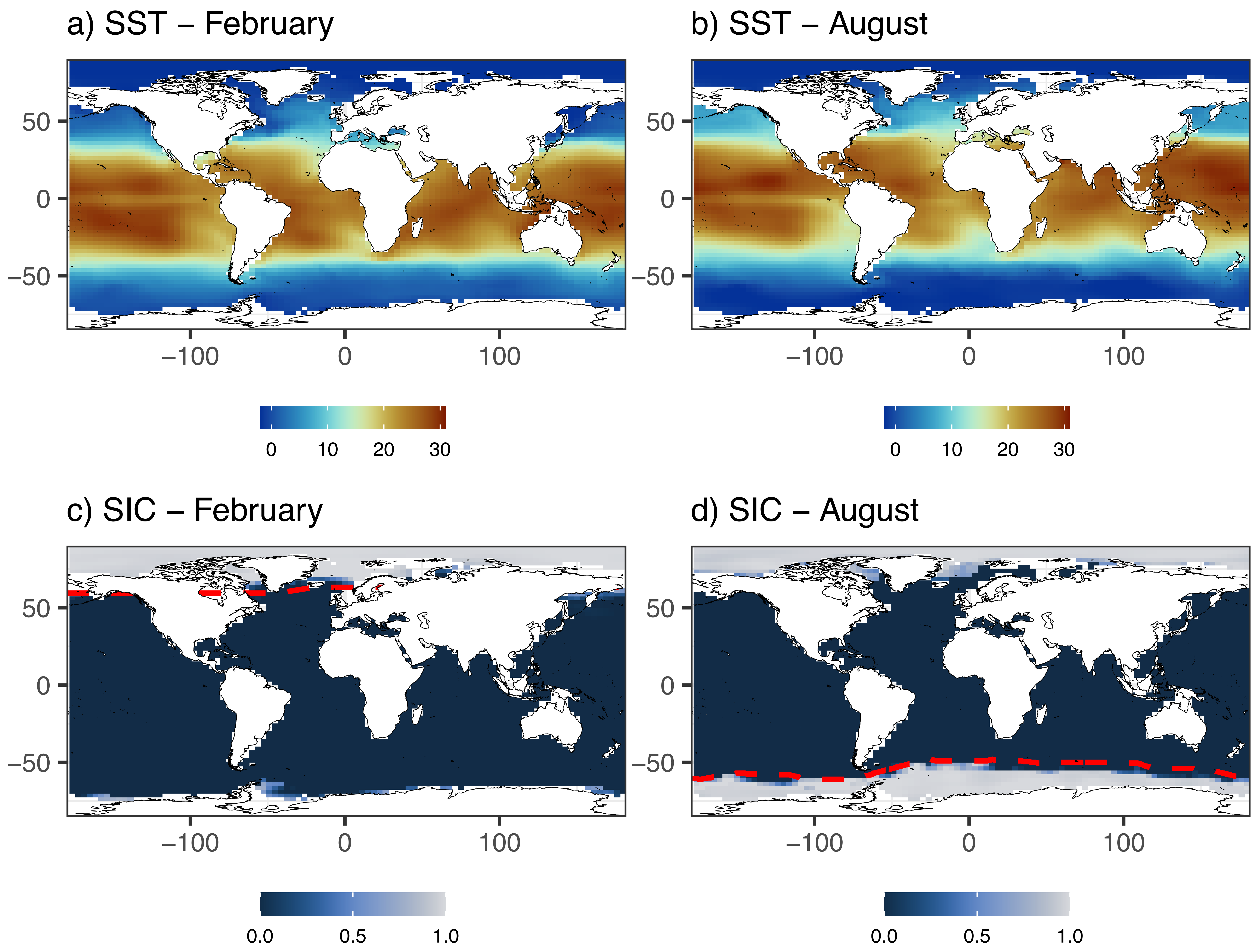
The statistical model

SST	SIC
$\mathbf{X}_i = \mathcal{M}(\mathbf{X}) + \mathcal{R}_i(\mathbf{X})$	$\mathbf{Y}_i = \Phi_{\mathbf{X}_i}\beta_i + \epsilon_i$ $\beta_i = \mathcal{M}(\beta) + \mathcal{R}_i(\beta)$
$\mathbf{T}_\mathbf{X} = \mathcal{M}(\mathbf{X}) + \mathbf{U}_\mathbf{X}$	$\mathbf{T}_\mathbf{Y} = \Phi_{\mathbf{T}_\mathbf{X}}\mathcal{M}(\beta) + \mathbf{U}_\mathbf{Y}$
$\mathbf{Z} = \mathbf{H}\mathbf{T}_\mathbf{X} + \mathbf{W}$	

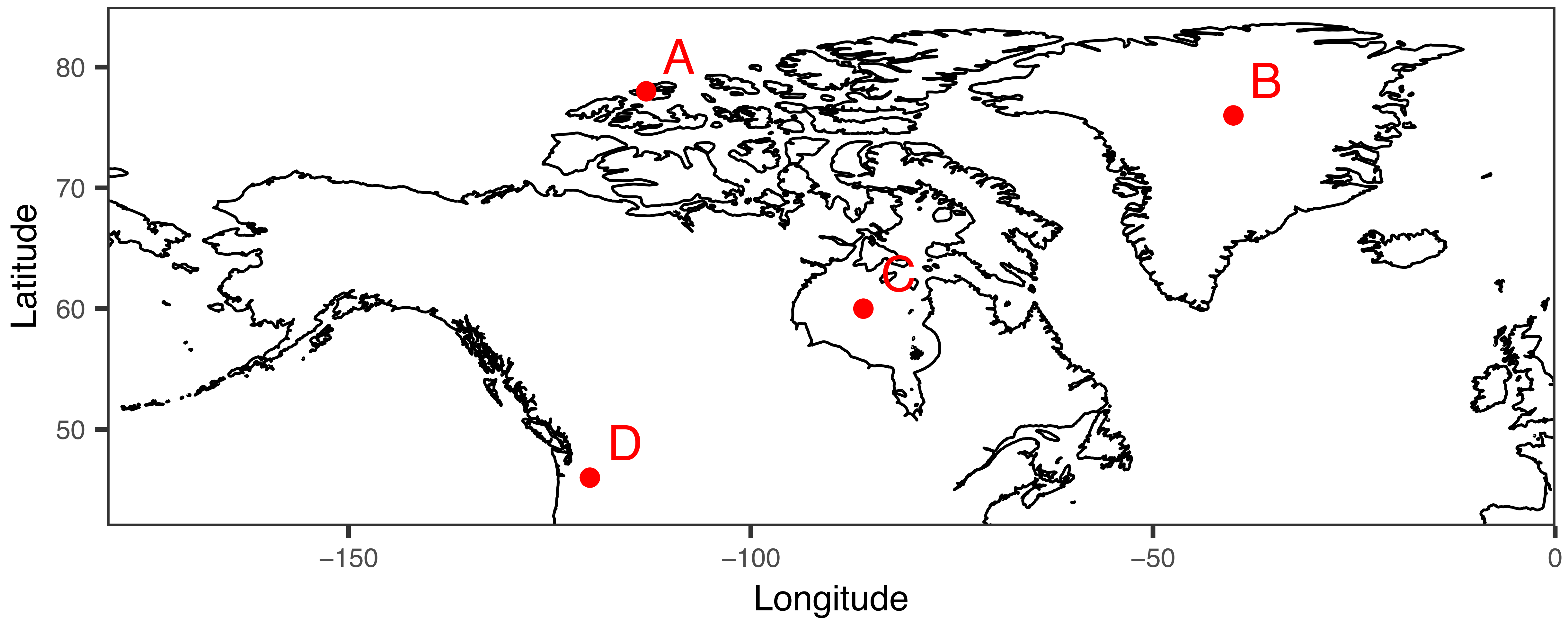
The coexchangeable model of Rougier et al. (2013)

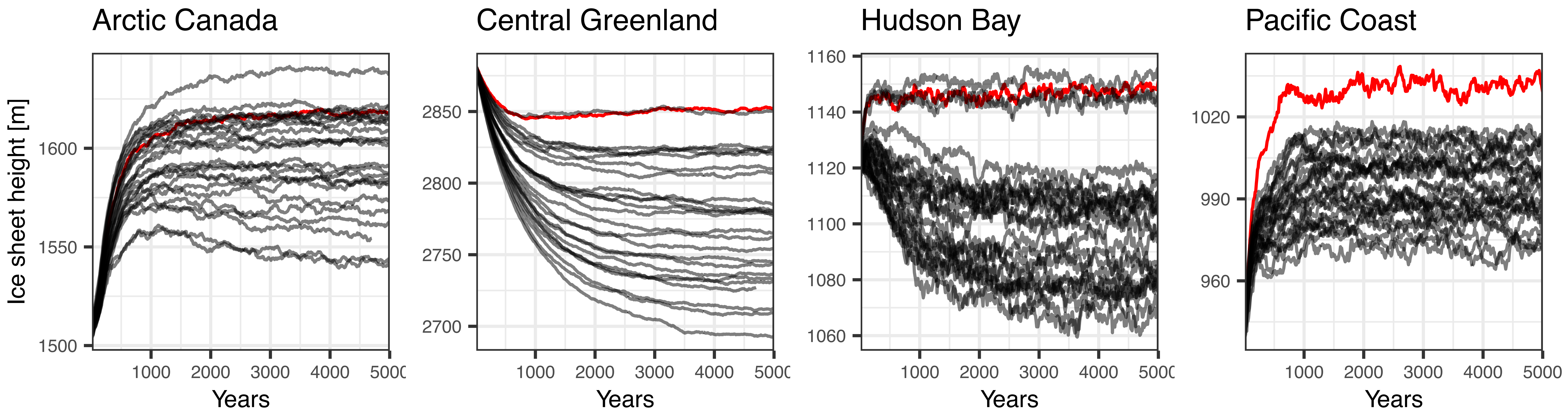
The coexchangeable process model of Astfalck et al. (2024)

Reconstructions of SST and SIC



Effects on the ice-sheet





Generalising Bayes Linear

A generalised Bayes inference

Property 1: An underlying geometry $\mathscr{G}$, establishing the space in which inference takes place

Property 2: A notion of closeness between objects in $\mathscr{G}$ to relate beliefs and data

Property 3: An optimisation, within solution space C, for the closest belief representation to the data generating process

Bayes Linear

Probabilistic Bayes

Bayes Linear

The product inner product

$$\langle X, Y \rangle = E[X^{\intercal} Y]$$

Probabilistic Bayes

The $\mathscr{L}_2$ inner product

$$\langle f(\theta), g(\theta) \rangle = \int f(\theta) g(\theta) \, \mathrm{d}\mu$$

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The belief structure, $\mathscr{B}$ ↔ Probability measure P

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Bayes Linear		Probabilistic Bayes
The product inner product $\langle \mathrm{X}, \mathrm{Y} \rangle = \mathrm{E}[\mathrm{X}^\intercal \mathrm{Y}]$	$\longleftrightarrow$	The $\mathscr{L}_2$ inner product $\langle f(\theta), g(\theta) \rangle = \int f(\theta) g(\theta)\, \mathrm{d}\mu$
The belief structure, $\mathscr{B}$	$\longleftrightarrow$	Probability measure P
Affine space of D $\mathrm{E}_\mathrm{D}[\mathrm{X}] = \mathrm{h} + \mathrm{HD}$	$\longleftrightarrow$	Posterior distributions Π

Property 3: An optimisation, within solution space C, for the closest belief representation to the data generating process

Bayes as optimisation

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Bissiri et al. (2016) recast probabilistic Bayes as the solution to

$$q^*(\theta) = \arg\min_{q \in \Pi} \left\{ \mathrm{E}_{q(\theta)} \left[\sum_{i=1}^{n} l(\theta, x_i) \right] + \mathrm{KLD}(q \| \pi) \right\}$$

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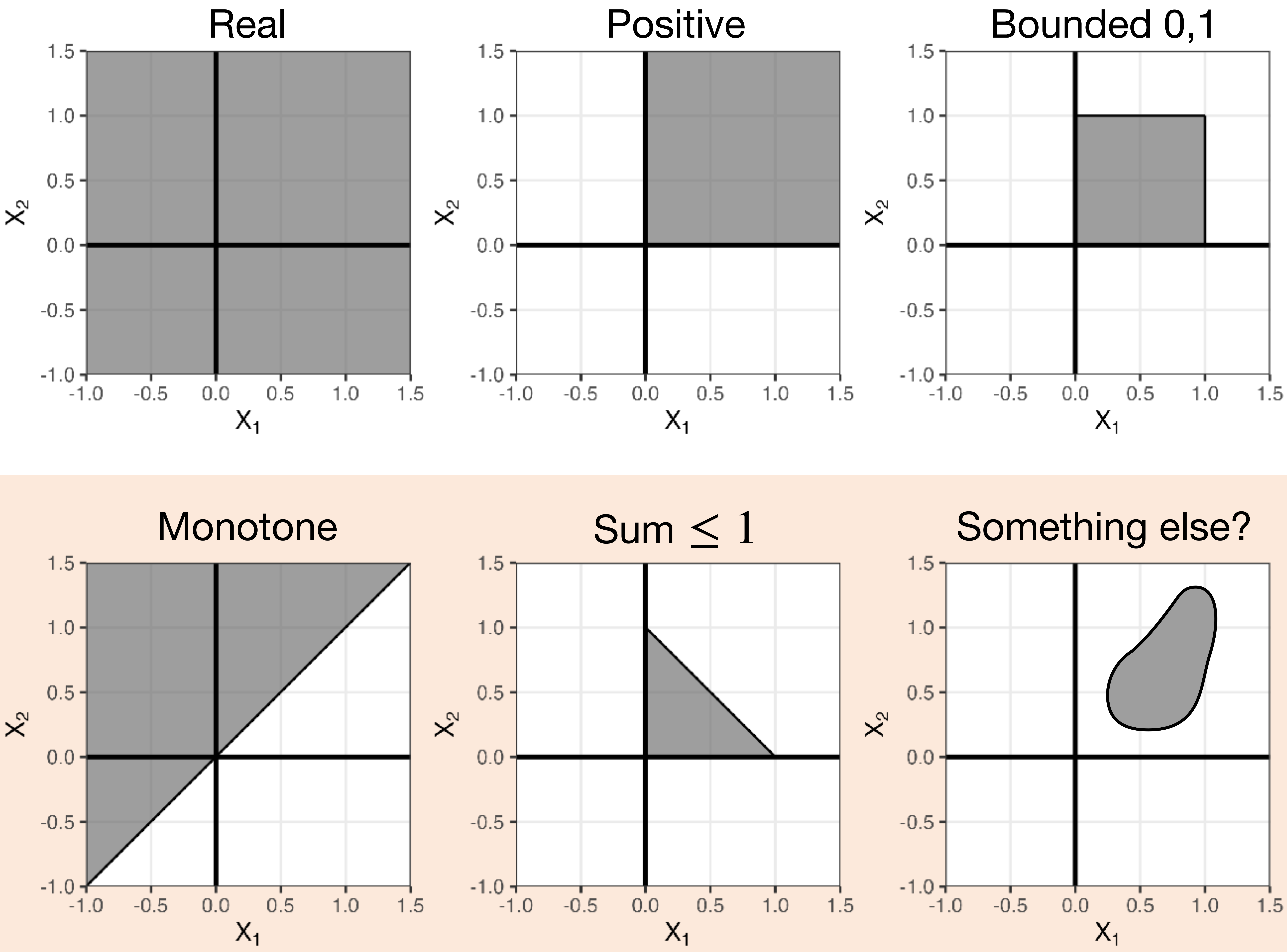
What do we achieve by playing with Π?



Real
Positive
Bounded 0,1
Monotone
Sum ≤ 1
Something else?
X_1
X_2
-1.0
-0.5
0.0
0.5
1.0
1.5

Inference with constrained solutions

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In a probabilistic Bayesian analysis we generally handle this in two ways:

1. Assign zero weight to regions in the prior (or equivalently, add a rejection step into the MCMC).

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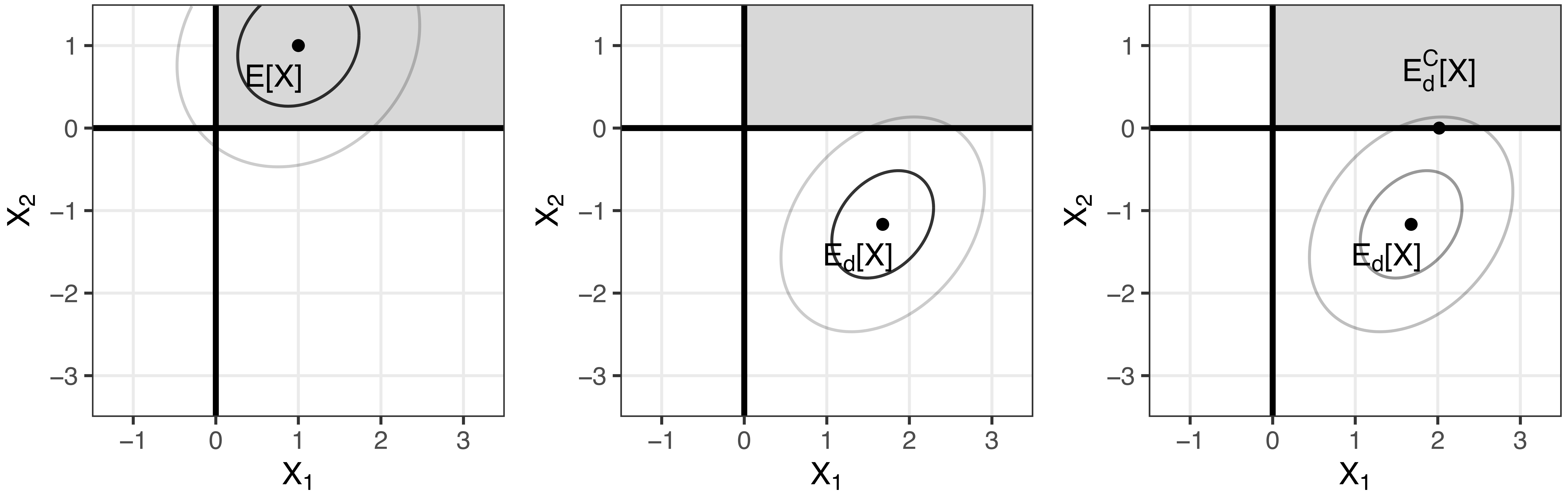
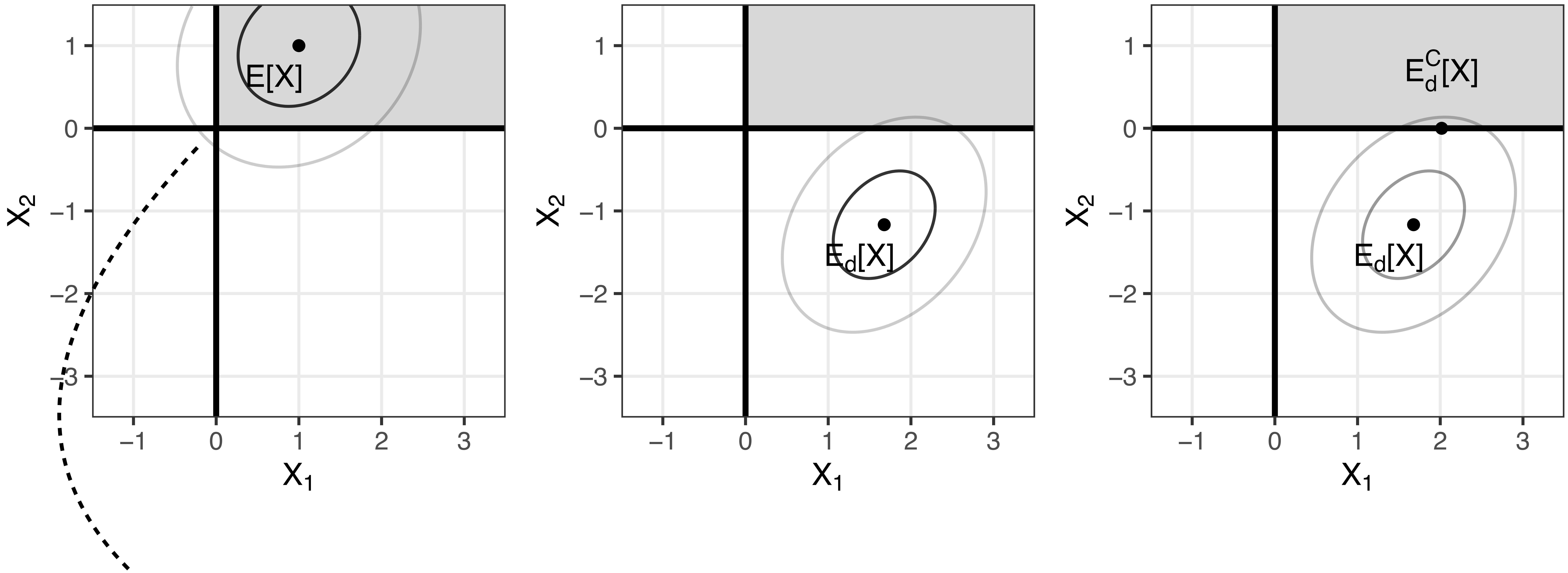
$$\mathrm{E_D[X]} = \underset{\mathrm{h+HD}}{\arg\min} \left\{ \langle \mathrm{X} - \mathrm{h} - \mathrm{HD}, \mathrm{h} + \mathrm{HD} \rangle \right\}$$

Constrain the solution to lie in some subset C and call this quantity $\mathrm{E}_{\mathrm{d}}^{C}[\mathrm{X}]$. Note, $\mathrm{E}_{\mathrm{d}}^{C}[\mathrm{X}]$ is not necessarily affine in D.

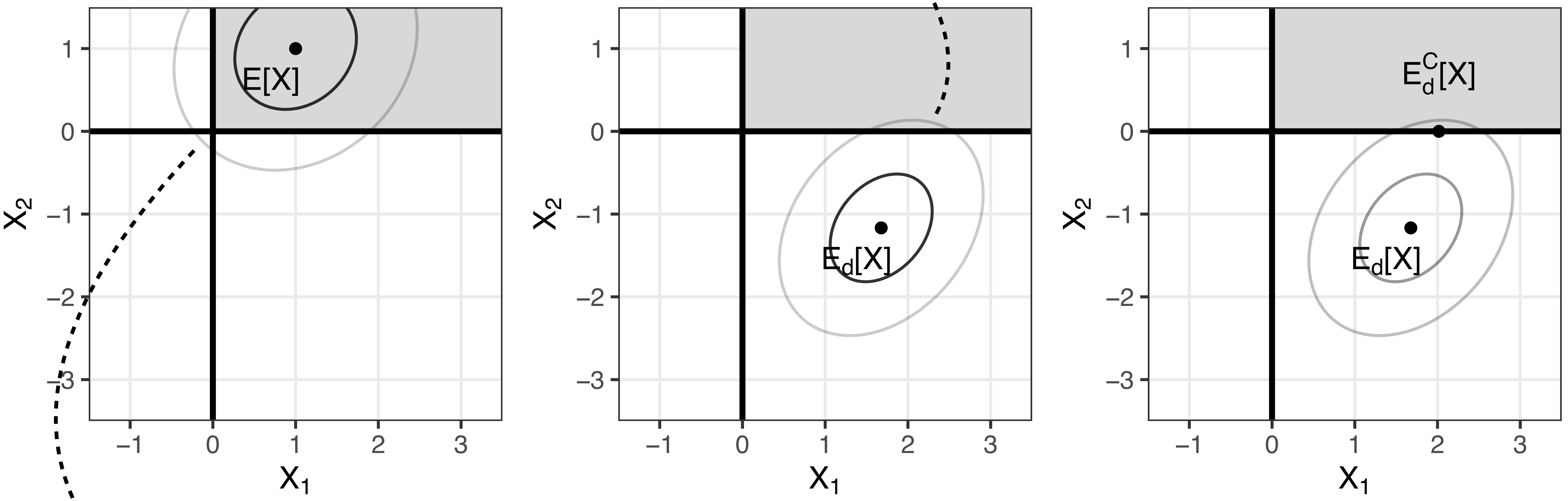


E[X]
$E_d[X]$
$E_d^C[X]$
X_1
X_2



$$\langle X, Y \rangle = E[X^\intercal Y]$$

$$\langle X, Y\rangle_D = \langle X - E_D[X], Y - E_D[Y]\rangle$$



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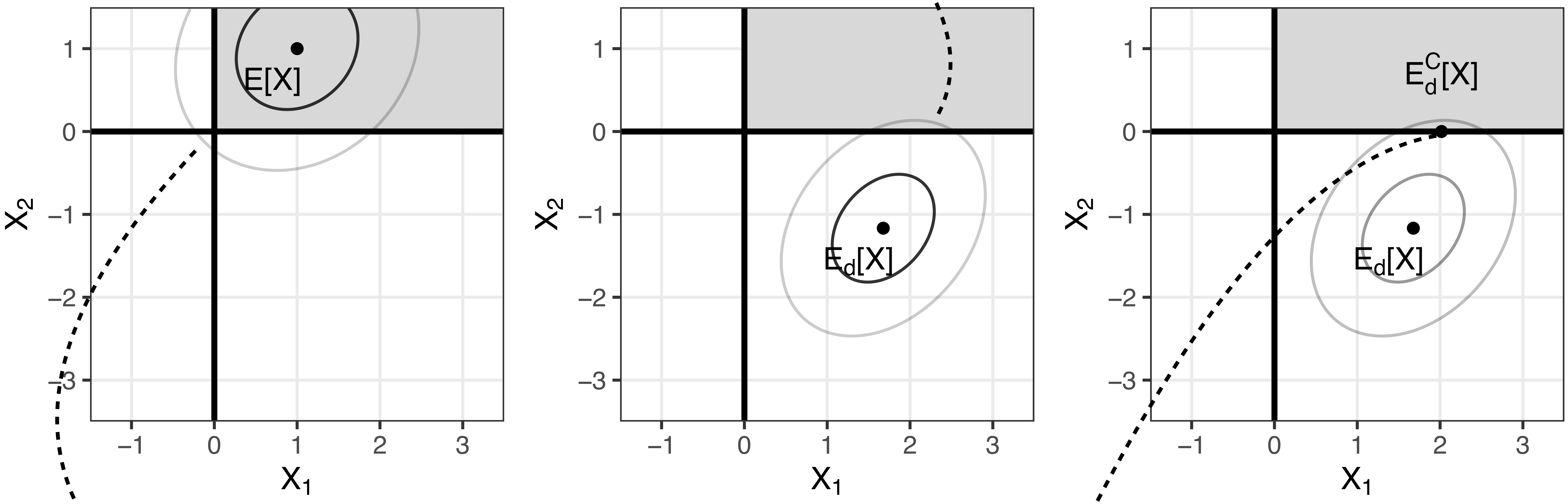

$$\langle X, Y\rangle = E[X^\intercal Y]$$

$$E_d^C[X] = \underset{q\in C}{\arg\min}\|E_d[X] - q\|$$

Generalised Adjusted Variance

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Adjusted variance is calculated from an outer product assuming affine $\mathrm{E}_{\mathrm{D}}[\mathrm{X}]$. If $\mathrm{E}_{\mathrm{d}}^{C}[\mathrm{X}] \neq \mathrm{E}_{\mathrm{d}}[\mathrm{X}]$, we break the affine assumption.

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Define L as a square-root decomposition $\mathrm{var}_{\mathrm{D}}[\mathrm{X}] = \mathrm{L}\mathrm{L}^{\intercal}$ (I like $\mathrm{L} = \mathrm{Q}\sqrt{\Lambda}$) and the *constraint discrepancy* $\mathrm{z} = \mathrm{L}^{-1}(\mathrm{E}^{C}_{\mathrm{d}}[\mathrm{X}] - \mathrm{E}_{\mathrm{d}}[\mathrm{X}])$.

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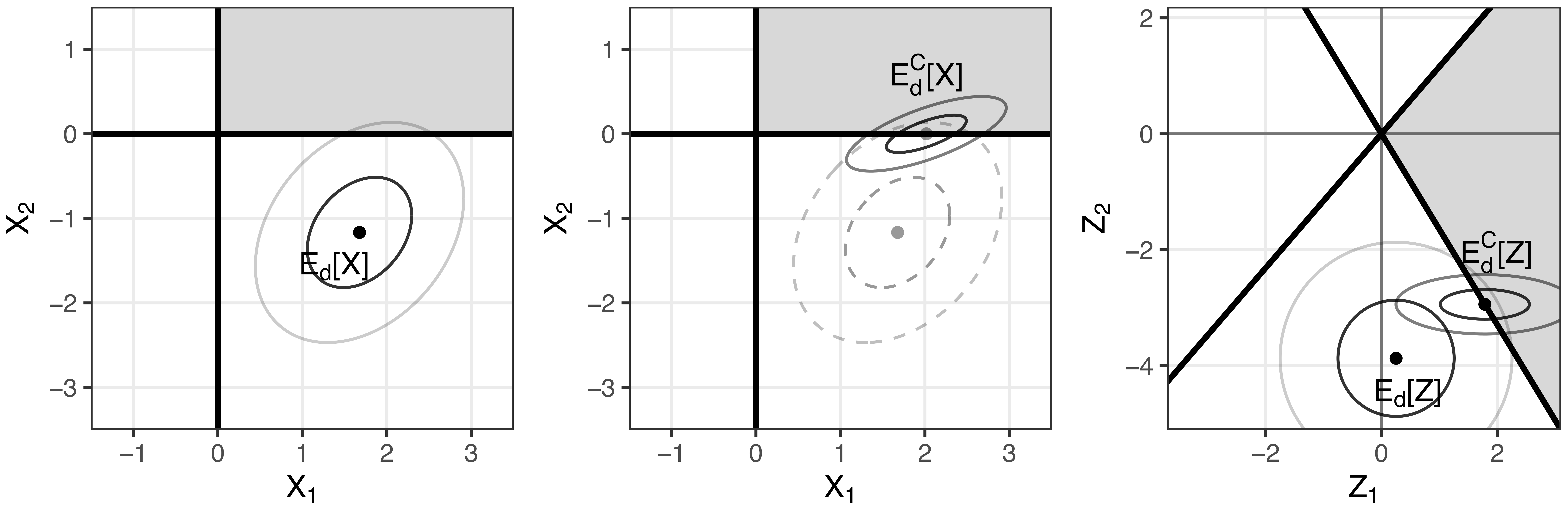
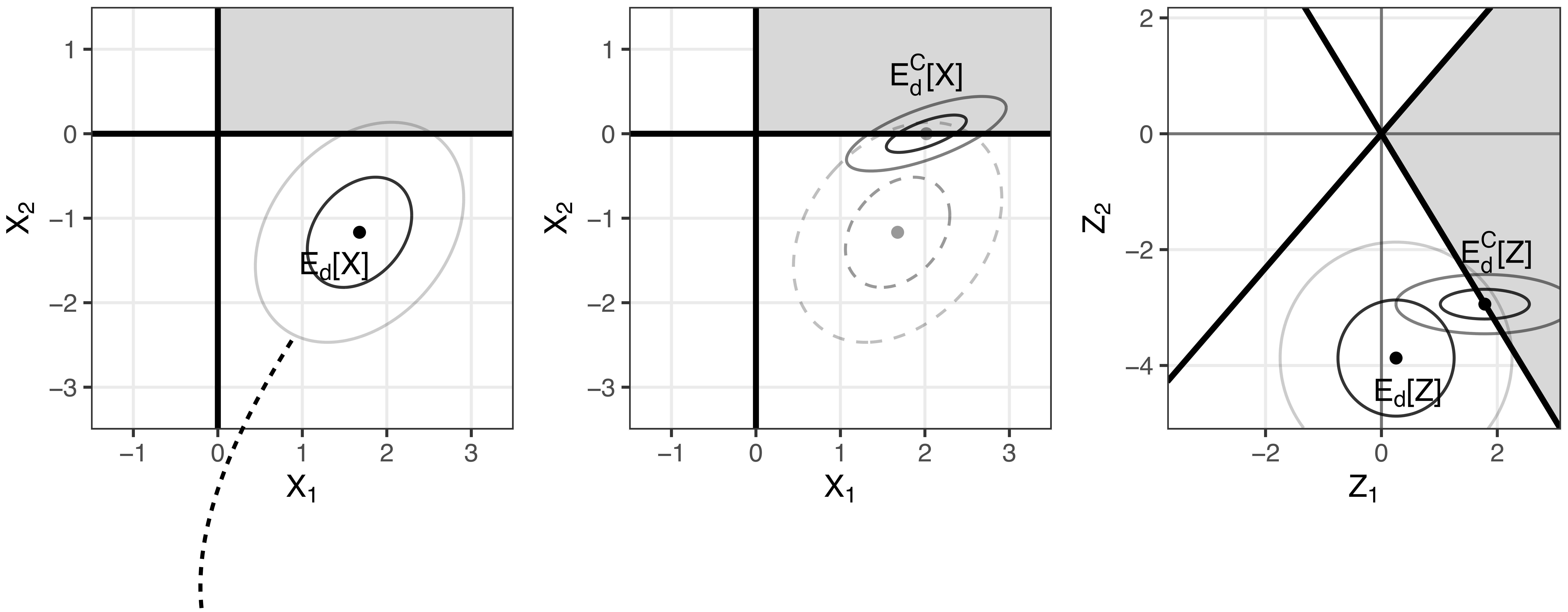
Define L as a square-root decomposition $\mathrm{var_D}[\mathrm{X}] = \mathrm{LL}^\intercal$ (I like $\mathrm{L} = \mathrm{Q}\sqrt{\Lambda}$) and the *constraint discrepancy* $\mathrm{z} = \mathrm{L}^{-1}(\mathrm{E}_\mathrm{d}^{C}[\mathrm{X}] - \mathrm{E_d}[\mathrm{X}])$.

The generalised adjusted variance is $\mathrm{var}_\mathrm{d}^{C}[\mathrm{X}] = \mathrm{LSL}^\intercal$, where

1. The limit $\lim_{|z_i|\to 0} \{\mathrm{S}_{ii}\} = 1$
2. The limit $\lim_{|z_i|\to \infty} \{\mathrm{S}_{ii}\} = 0$
3. $\mathrm{S}_{ii} = f(z_i)$ is non-increasing in z_i

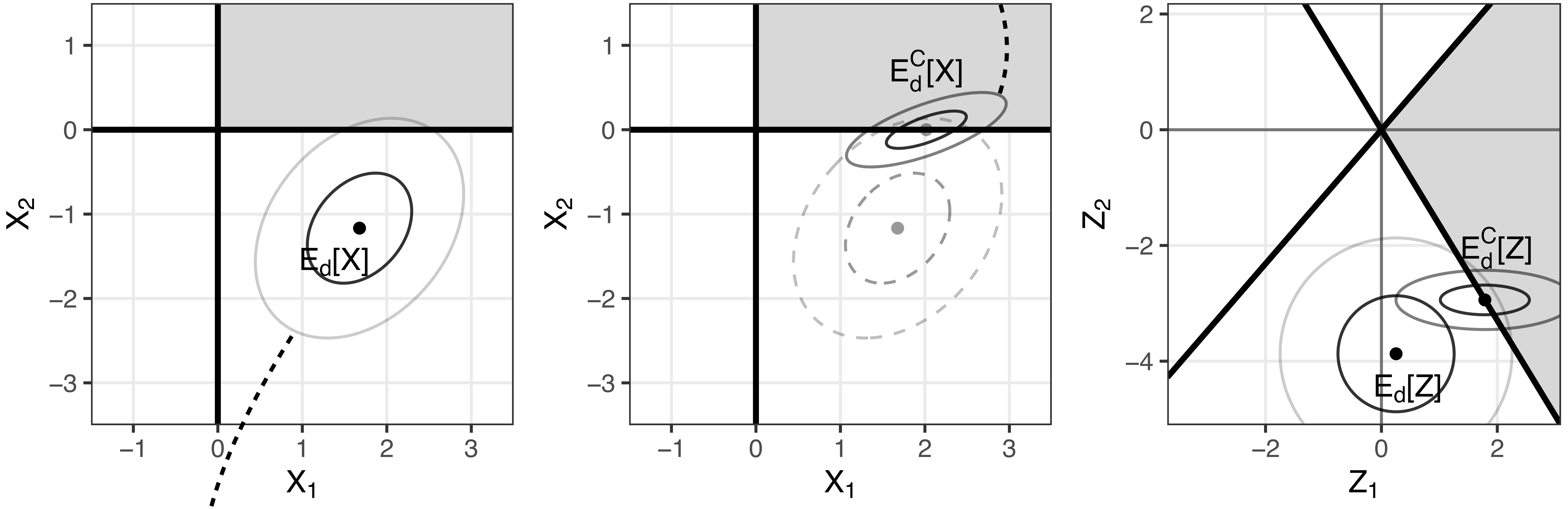


$E_d[X]$
$E_d^C[X]$
$E_d^C[Z]$
$E_d[Z]$
X_1
X_2
Z_1
Z_2



$\mathrm{var}_D[X] = LL^\intercal$, and $L = Q\sqrt{\Lambda}$

$$\mathrm{var}_{\mathrm{d}}^{C}[\mathrm{X}] = \mathrm{LSL}^{\intercal}$$



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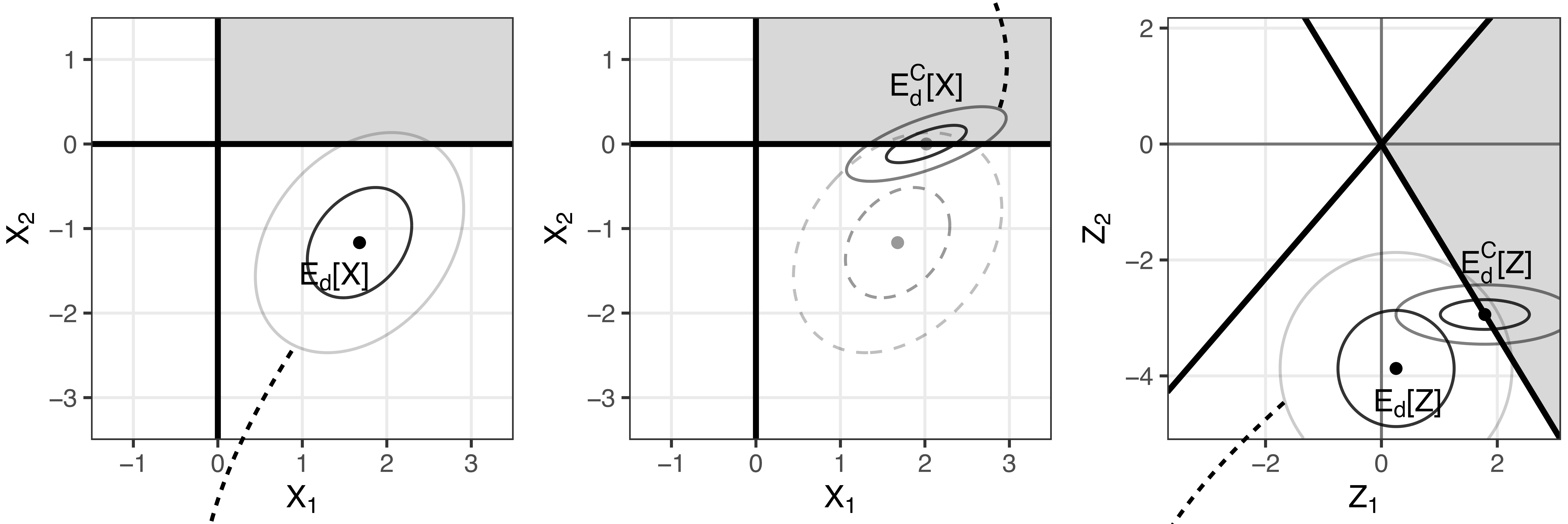

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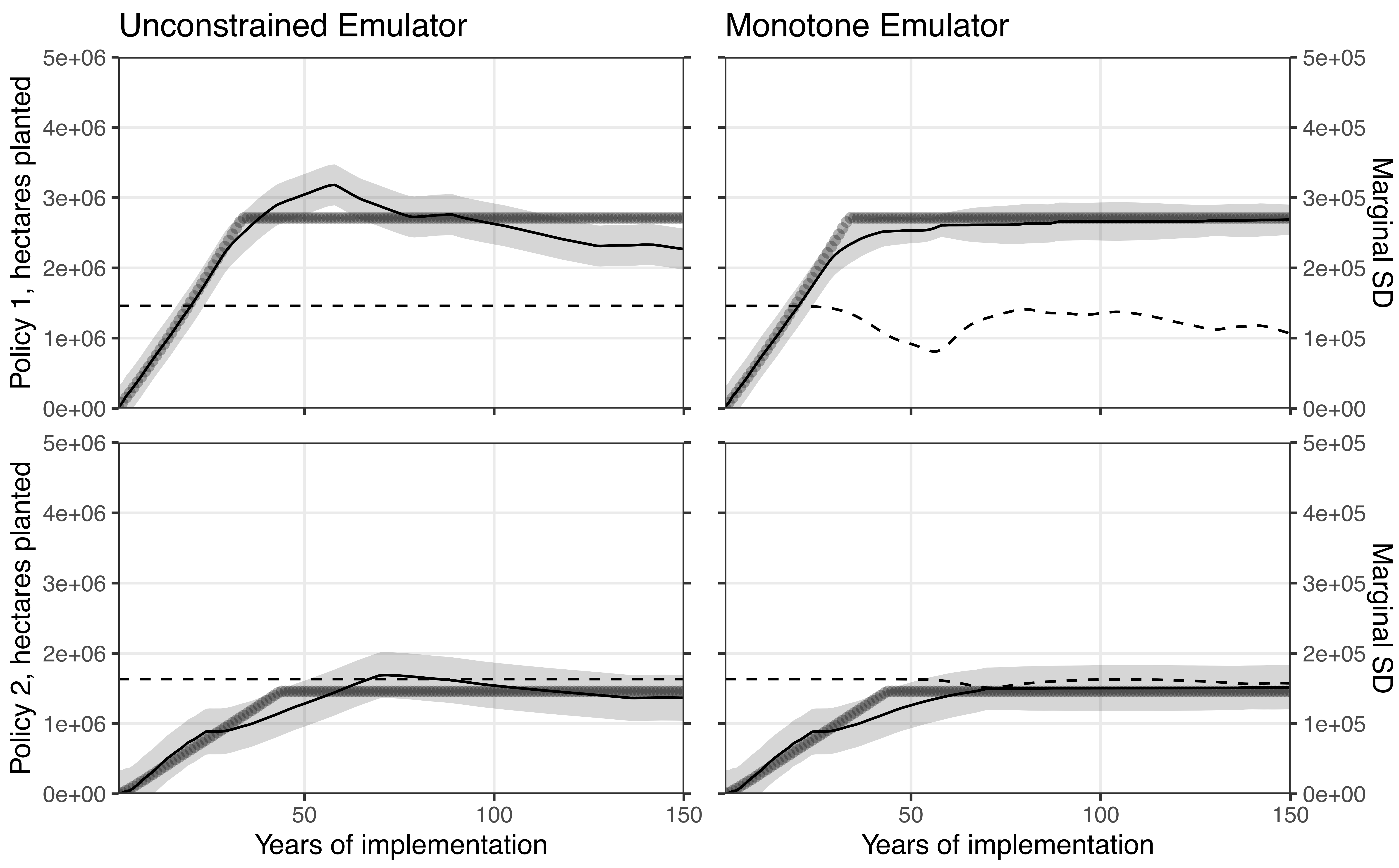
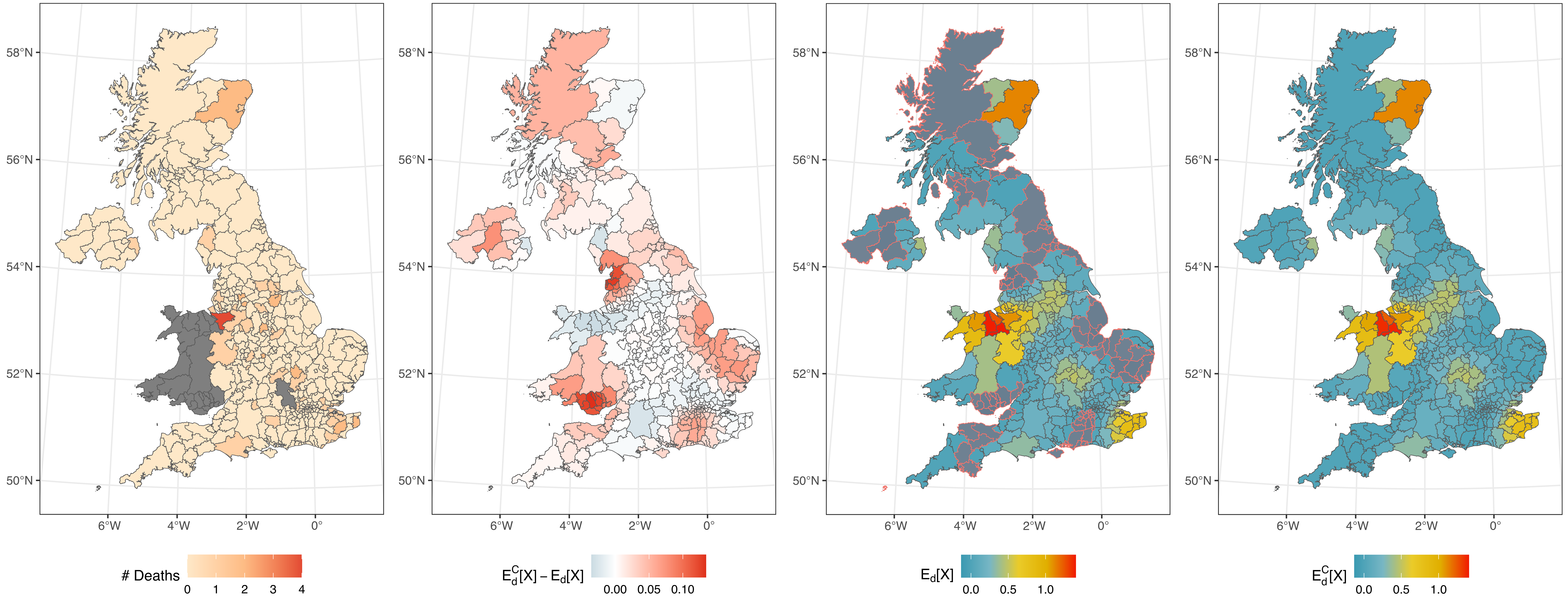
Rotate into $\mathscr{Z}$ via L^{-1}

An Afforestation Uptake Model



One day of COVID19 Deaths



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astfalckl.github.io/presentations