

Natural Deduction Inference Rules for Predicate Logic

Basic Inference Rules

1 $\neg\neg$ -elimination

$$\frac{(\neg(\neg\alpha))}{\alpha}$$

2 \perp -elimination

$$\frac{\perp}{\alpha}$$

3 \wedge -introduction

$$\frac{\alpha \quad \beta}{(\alpha \wedge \beta)}$$

4 \wedge -elimination

$$\frac{(\alpha \wedge \beta)}{\alpha} \quad \frac{(\alpha \wedge \beta)}{\beta}$$

5 \rightarrow -introduction

$$\frac{\begin{array}{|c|} \hline \alpha \\ \vdots \\ \beta \\ \hline \end{array}}{(\alpha \rightarrow \beta)}$$

7 \vee -introduction

$$\frac{\alpha}{(\alpha \vee \beta)} \quad \frac{\alpha}{(\beta \vee \alpha)}$$

9 \neg -introduction

$$\frac{\begin{array}{|c|} \hline \alpha \\ \vdots \\ \perp \\ \hline \end{array}}{(\neg\alpha)}$$

10 \neg -elimination
(\perp -introduction)

$$\frac{\alpha \quad (\neg\alpha)}{\perp}$$

Derived Inference Rules

A Modus Tollens (MT)

$$\frac{(\alpha \rightarrow \beta) \quad (\neg\beta)}{(\neg\alpha)}$$

B Law of Excluded Middle (LEM)

$$\frac{\emptyset}{(\alpha \vee (\neg\alpha))}$$

C Double-Negation Introduction

$$\frac{\alpha}{(\neg(\neg\alpha))}$$

D Proof by Contradiction (PBC)

$$\frac{\begin{array}{|c|} \hline \neg\alpha \\ \vdots \\ \perp \\ \hline \end{array}}{\alpha}$$

! Depending on the assignment you may not always be allowed to use all derived rules!

Subproofs

Assumption first line can be any formula

you must **never** refer to any **single** line on the inside from outside

can end at any time

y fresh

fresh variable must **not** occur outside

! All subproofs must be closed by the end of the proof

Soundness & Completeness

Soundness

"All formulae derived by ND are entailments"

$$\Sigma \vdash_{ND} \varphi \Rightarrow \Sigma \models \varphi$$

$$\Sigma \not\models \varphi \Rightarrow \Sigma \not\vdash_{ND} \varphi$$

Completeness

"All formulae that are entailments can be derived by ND"

$$\Sigma \models \varphi \Rightarrow \Sigma \vdash_{ND} \varphi$$

$$\Sigma \not\vdash_{ND} \varphi \Rightarrow \Sigma \not\models \varphi$$

Basic Inference Rules for Predicate Logic

1 \forall -introduction

$$\frac{\begin{array}{|c|} \hline y \text{ fresh} \\ \vdots \\ \alpha[y/x] \\ \hline \end{array}}{\forall x \alpha}$$

2 \forall -elimination

$$\frac{\forall x \alpha}{\alpha[t/x]}$$

3 \exists -introduction

$$\frac{\alpha[t/x]}{\exists x \alpha}$$

4 \exists -elimination

$$\frac{\exists x \alpha \quad \begin{array}{|c|} \hline \alpha[u/x] \text{ u fresh} \\ \vdots \\ \beta \\ \hline \end{array}}{\beta}$$

Formula $\alpha[y/x]$ Variable

Variable

"Substitute x with y in formula α "