Natural Deduction Inference Rules for Predicate Logic

Basic Inference Rules

1 ¬¬-elimination

$$\frac{(\neg(\neg\alpha))}{\alpha}$$

2 ⊥-elimination

$$\frac{\bot}{\alpha}$$

3 ∧-introduction

$$\frac{\alpha \quad \beta}{(\alpha \wedge \beta)}$$

4 ∧-elimination

$$\frac{(\alpha \wedge \beta)}{\alpha} \qquad \frac{(\alpha \wedge \beta)}{\beta}$$

5 →-introduction

$$\frac{\begin{array}{|c}\alpha \\ \vdots \\ \beta\end{array}}{(\alpha \rightarrow \beta)}$$

6 →-elimination

$$\frac{(\alpha \rightarrow \beta) \quad \alpha}{\beta}$$

7 ∨-introduction

$$\frac{\alpha}{(\alpha \vee \beta)} \qquad \frac{\alpha}{(\beta \vee \alpha)}$$

8 ∨-elimination

$$\frac{(\alpha_1 \vee \alpha_2) \quad \begin{array}{|c}\alpha_1 \\ \vdots \\ \beta\end{array} \quad \begin{array}{|c}\alpha_2 \\ \vdots \\ \beta\end{array}}{\beta}$$

9 ¬-introduction

$$\frac{\begin{array}{|c}\alpha \\ \vdots \\ \bot\end{array}}{(\neg\alpha)}$$

10 ¬-elimination (⊥-introduction)

$$\frac{\alpha \quad (\neg\alpha)}{\bot}$$

⚠ Depending on the assignment you may not always be allowed to use all derived rules!

Derived Inference Rules

A Modus Tollens (MT)

$$\frac{(\alpha \rightarrow \beta) \quad (\neg\beta)}{(\neg\alpha)}$$

B Law of Excluded Middle (LEM)

$$\frac{\varnothing}{(\alpha \vee (\neg\alpha))}$$

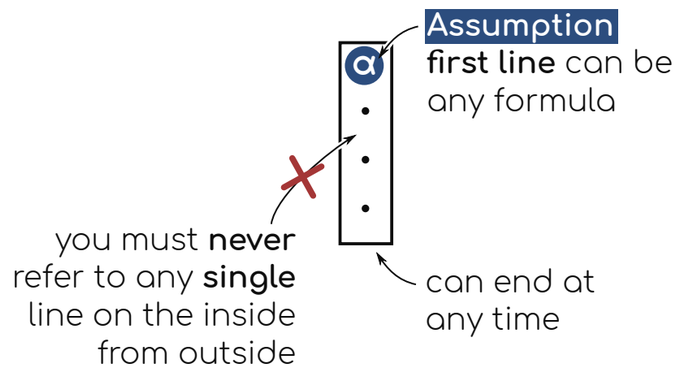
C Double-Negation Introduction

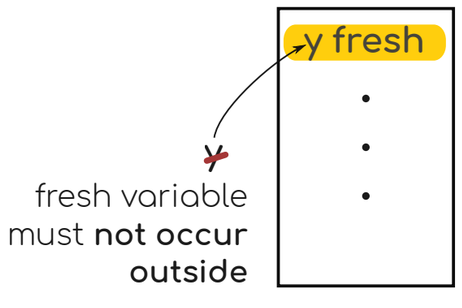
$$\frac{\alpha}{(\neg(\neg\alpha))}$$

D Proof by Contradiction (PBC)

$$\frac{\begin{array}{|c}\neg\alpha \\ \vdots \\ \bot\end{array}}{\alpha}$$

Subproofs





Fresh variable
Introduce new variable that has not been used before.

⚠ All subproofs must be closed by the end of the proof

Basic Inference Rules for Predicate Logic

1 ∀-introduction

$$\frac{\begin{array}{|l}y \text{ fresh} \\ \vdots \\ \alpha[y / x]\end{array}}{\forall x\, \alpha}$$

2 ∀-elimination

$$\frac{\forall x\, \alpha}{\alpha[t / x]}$$

3 ∃-introduction

$$\frac{\alpha[t / x]}{\exists x\, \alpha}$$

4 ∃-elimination

$$\frac{\exists x\, \alpha \quad \begin{array}{|l}\alpha\,[u / x] \quad u \text{ fresh} \\ \vdots \\ \beta\end{array}}{\beta}$$

Formula, Variable, Variable: $\alpha[y / x]$

"Substitute x with y in formula α"

Soundness & Completeness

Soundness
"All formulae derived by ND are entailments"

$$\Sigma \vdash_{ND} \varphi \quad \Rightarrow \quad \Sigma \models \varphi$$

$$\Sigma \not\models \varphi \quad \Rightarrow \quad \Sigma \not\vdash_{ND} \varphi$$

Completeness
"All formulae that are entailments can be derived by ND"

$$\Sigma \models \varphi \quad \Rightarrow \quad \Sigma \vdash_{ND} \varphi$$

$$\Sigma \not\vdash_{ND} \varphi \quad \Rightarrow \quad \Sigma \not\models \varphi$$