Natural Deduction for Propositional Logic

Basic Inference Rules

0 Reflexivity

$$\frac{\alpha}{\alpha}$$

1 ¬¬-elimination

$$\frac{\neg\neg\alpha}{\alpha}$$

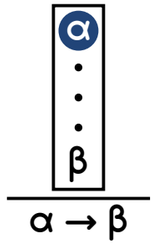
2 ∧-introduction

$$\frac{\alpha \quad \beta}{\alpha \wedge \beta}$$

3 ∧-elimination

$$\frac{\alpha \wedge \beta}{\alpha} \qquad \frac{\alpha \wedge \beta}{\beta}$$

4 →-introduction



$$\frac{\quad}{\alpha \to \beta}$$

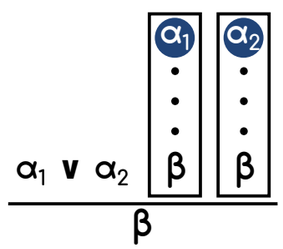
5 →-elimination

$$\frac{\alpha \to \beta \quad \alpha}{\beta}$$

5 ∨-introduction

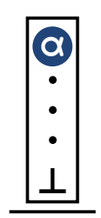
$$\frac{\alpha}{\alpha \vee \beta} \qquad \frac{\alpha}{\beta \vee \alpha}$$

6 ∨-elimination



$$\frac{\quad}{\beta}$$

7 ¬-introduction



$$\frac{\quad}{\neg\alpha}$$

8 ¬-elimination (⊥-introduction)

$$\frac{\alpha \quad \neg\alpha}{\bot}$$

Derived Inference Rules

⊥-elimination

$$\frac{\bot}{\alpha}$$

Modus Tollens (MT)

$$\frac{\alpha \to \beta \quad \neg\beta}{\neg\alpha}$$

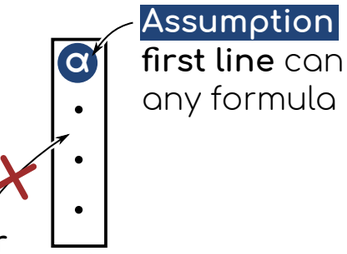
Law of Excluded Middle (LEM)

$$\frac{\varnothing}{\alpha \vee \neg\alpha}$$

Double-Negation Introduction

$$\frac{\alpha}{\neg\neg\alpha}$$

Subproofs



⚠ All subproofs must be closed by the end of the proof

Soundness & Completeness

Soundness
"All formulae derived by ND are entailments"

$$\Sigma \vdash_{ND} \varphi \quad \Rightarrow \quad \Sigma \models \varphi$$

$$\Sigma \nvdash_{ND} \varphi \quad \Leftarrow \quad \Sigma \not\models \varphi$$

Completeness
"All entailments can be derived by ND"

$$\Sigma \models \varphi \quad \Rightarrow \quad \Sigma \vdash_{ND} \varphi$$

$$\Sigma \not\models \varphi \quad \Leftarrow \quad \Sigma \nvdash_{ND} \varphi$$

https://github.com/astoeckel/cs245-f18/