Natural Deduction Inference Rules for Propositional Logic

Basic Inference Rules

1 ¬¬-elimination

$$\frac{(\neg(\neg\alpha))}{\alpha}$$

2 ⊥-elimination

$$\frac{\bot}{\alpha}$$

3 ∧-introduction

$$\frac{\alpha \quad \beta}{(\alpha \land \beta)}$$

4 ∧-elimination

$$\frac{(\alpha \land \beta)}{\alpha} \qquad \frac{(\alpha \land \beta)}{\beta}$$

5 →-introduction

$$\frac{\boxed{\begin{array}{c}\alpha\\ \vdots\\ \beta\end{array}}}{(\alpha \to \beta)}$$

6 →-elimination

$$\frac{(\alpha \to \beta) \quad \alpha}{\beta}$$

7 ∨-introduction

$$\frac{\alpha}{(\alpha \lor \beta)} \qquad \frac{\alpha}{(\beta \lor \alpha)}$$

8 ∨-elimination

$$\frac{(\alpha_1 \lor \alpha_2) \quad \boxed{\begin{array}{c}\alpha_1\\ \vdots\\ \beta\end{array}} \quad \boxed{\begin{array}{c}\alpha_2\\ \vdots\\ \beta\end{array}}}{\beta}$$

9 ¬-introduction

$$\frac{\boxed{\begin{array}{c}\alpha\\ \vdots\\ \bot\end{array}}}{(\neg\alpha)}$$

10 ¬-elimination (⊥-introduction)

$$\frac{\alpha \quad (\neg\alpha)}{\bot}$$

Derived Inference Rules

A Modus Tollens (MT)

$$\frac{(\alpha \to \beta) \quad (\neg\beta)}{(\neg\alpha)}$$

B Law of Excluded Middle (LEM)

$$\frac{\varnothing}{(\alpha \lor (\neg\alpha))}$$

C Double-Negation Introduction

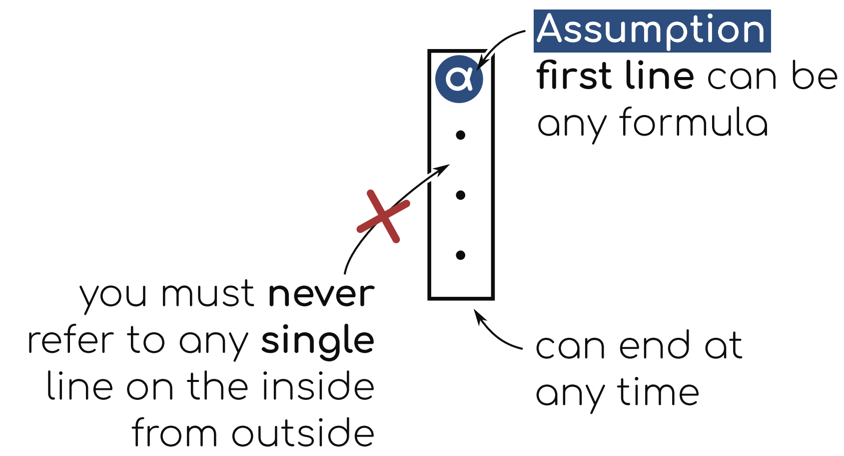
$$\frac{\alpha}{(\neg(\neg\alpha))}$$

D Proof by Contradiction (PBC)

$$\frac{\boxed{\begin{array}{c}\neg\alpha\\ \vdots\\ \bot\end{array}}}{\alpha}$$

⚠ Depending on the assignment you may not always be allowed to use all derived rules!

Subproofs



⚠ All subproofs must be closed by the end of the proof

Soundness & Completeness

Soundness
"All formulae derived by ND are entailments"

$$\Sigma \vdash_{ND} \varphi \quad \Rightarrow \quad \Sigma \models \varphi$$

$$\Sigma \not\models \varphi \quad \Rightarrow \quad \Sigma \nvdash_{ND} \varphi$$

Completeness
"All formulae that are entailments can be derived by ND"

$$\Sigma \models \varphi \quad \Rightarrow \quad \Sigma \vdash_{ND} \varphi$$

$$\Sigma \nvdash_{ND} \varphi \quad \Rightarrow \quad \Sigma \not\models \varphi$$