CSE 275
Homework 1
Written

Part I: 1.

$$\frac{p+q}{2} = \frac{1}{\sqrt{2}} + \frac{i}{2\sqrt{2}} + \frac{j}{2\sqrt{2}}$$

$$\left|\frac{p+q}{2}\right| = \sqrt{\frac{1}{2}+\frac{1}{8}+\frac{1}{8}} = \frac{\sqrt{3}}{2}$$

$$r_0 = \frac{2}{\sqrt{3}}\cdot\frac{p+q}{2} = \overset{q_0}{\frac{\sqrt{6}}{3}} + \overset{q_1}{\frac{\sqrt{6}}{6}}i + \overset{q_2}{\frac{\sqrt{6}}{6}}j \Rightarrow |r_0| = 1$$

$$M(r_0) = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \xrightarrow[\text{Eigen-decomposition}]{} r = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ with } \lambda_1 = 1$$

$\Rightarrow V = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is the rotation axis

$$tr(M) = \frac{5}{3} = 1 + 2\cos\theta \Rightarrow \theta = \arccos\left(\frac{1}{3}\right) \approx 70.5^\circ$$

2. $\omega_p = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\theta_p = \frac{\pi}{2}$, $\omega_q = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\theta_q = \frac{\pi}{2}$

$$\therefore \vec{W_p} = \omega_p\theta_p = \begin{bmatrix} \frac{\pi}{2} \\ 0 \\ 0 \end{bmatrix}, \quad \vec{W_q} = \omega_q\theta_q = \begin{bmatrix} 0 \\ \frac{\pi}{2} \\ 0 \end{bmatrix}$$

3. (a) $[\omega_p] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, $\theta_p = \frac{\pi}{2}$, $[\omega_q] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$, $\theta_q = \frac{\pi}{2}$

$$Rot(\omega_p, \theta_p) = e^{[\omega_p]\theta_p} \approx I + [\omega_p]\sin\theta_p + [\omega_p]^2(1-\cos\theta_p) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$Rot(\omega_q, \theta_q) = e^{[\omega_q]\theta_q} \approx I + [\omega_q]\sin\theta_q + [\omega_q]^2(1-\cos\theta_q) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

(b) $[\omega_p] + [\omega_q] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$

$$\exp\left(([\omega_p]+[\omega_q])\frac{\pi}{2}\right) \approx I + [\omega_p] + [\omega_q] + ([\omega_p]+[\omega_q])^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & -1 \end{bmatrix}$$

$$\exp\left([\omega_p]\frac{\pi}{2}\right)\exp\left([\omega_q]\frac{\pi}{2}\right) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \neq \exp\left(([\omega_p]+[\omega_q])\frac{\pi}{2}\right)$$

(c) i. Step 2: Solve the following optimization problem to get $\Delta\omega$:

$$\min_{\Delta\omega} \|R(I+[\Delta\omega])X - Y\|^2$$

$$\text{s.t.} \begin{cases} \|\Delta\omega\|^2 \leq \varepsilon \\ R(I+[\Delta\omega]) \in SO(3) \end{cases}$$

4. (a) $p = \frac{(1+i)}{\sqrt{2}}$, $-p = \frac{-1-i}{\sqrt{2}}$, $q = \frac{(1+j)}{\sqrt{2}}$, $-q = \frac{-1-j}{\sqrt{2}}$

$\theta_{-p} = 2\arccos(-\frac{\sqrt{2}}{2}) = \frac{-1}{2}\pi$, $\omega_{-p} = \frac{1}{\sin(\frac{-1}{4}\pi)} \cdot [-\frac{\sqrt{2}}{2}\ 0\ 0]^T = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$, $\overrightarrow{\omega_{-p}} = \omega_{-p}\theta_{-p} = \begin{bmatrix} \frac{\pi}{2} \\ 0 \\ 0 \end{bmatrix} = \overrightarrow{\omega_p}$

$\theta_{-q} = 2\arccos(-\frac{\sqrt{2}}{2}) = \frac{-1}{2}\pi$, $\omega_{-q} = \frac{1}{\sin(\frac{-\pi}{4})} \cdot [0\ -\frac{\sqrt{2}}{2}\ 0]^T = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, $\overrightarrow{\omega_{-q}} = \omega_{-q}\theta_{-q} = \begin{bmatrix} 0 \\ \frac{\pi}{2} \\ 0 \end{bmatrix} = \overrightarrow{\omega_q}$

Statement: Quaternion pair $(r, -r)$ represents the same rotation.

Proof: Suppose rotating vector $\vec{x}$ using quaternion r to get $R(\vec{x})$:

$$R_r(\vec{x}) = r\vec{x}r^{-1}$$

Now use $-r$, we get:

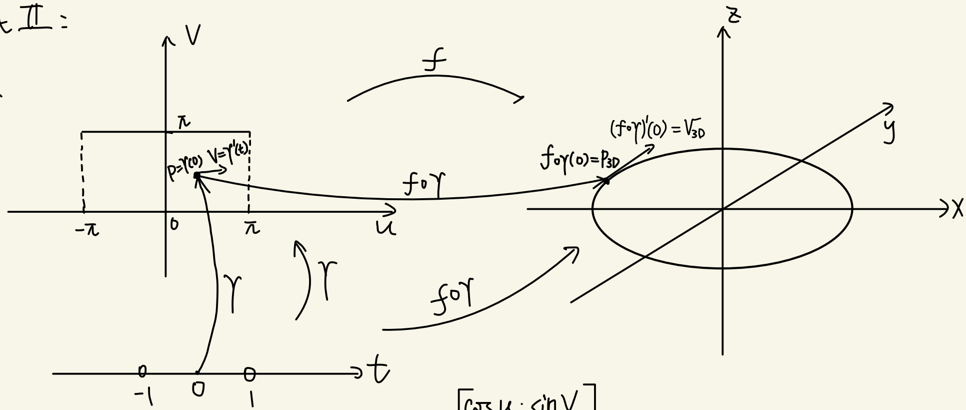
$$R_{-r}(\vec{x}) = (-r)\vec{x}(-r)^{-1} = (-1)^2 r\vec{x}r^{-1} = r\vec{x}r^{-1} = R_r(\vec{x})$$

$\Rightarrow R_r(\vec{x})$ and $R_{-r}(\vec{x})$ represent the same rotation, proved

(b) No. Since for such $(r, -r)$ having a large difference in domain, they yield the same rotation matrix at SO(3), and hence L2 distance learning probably gives the prediction in the middle $(\frac{r-r}{2} = 0)$, which is undesirable for both ground truths.

Part II:

1、



$$f(u,v) = \begin{bmatrix} \cos u \cdot \sin v \\ \sin u \cdot \sin v \\ \frac{1}{2}\cos v \end{bmatrix}$$

P: A point in the domain of f

V: Velocity of P in the domain

γ: A function mapping 1D input t to the domain

$f\circ\gamma$: A function mapping 1D input t to the 3D manifold

$(f\circ\gamma)'(0)$: Velocity of P_{3D}, projected from P to the 3D manifold, at $t=0$

3.(a) $$Df_p = \left.\begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{bmatrix}\right|_p = \begin{bmatrix} -\sin u \cdot \sin v & \cos u \cdot \cos v \\ \cos u \cdot \sin v & \sin u \cdot \cos v \\ 0 & -\frac{1}{2}\sin v \end{bmatrix}$$

(b) $Df_p = \left[\frac{\partial f}{\partial u} \ \frac{\partial f}{\partial v}\right]$ represents 2 3D vectors spanning the tangent plane at $f(u,v)$.

(d) $$Df_{p=(\frac{\pi}{4},\frac{\pi}{6})} = \begin{bmatrix} -\frac{\sqrt{2}}{4} & \frac{\sqrt{6}}{4} \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{6}}{4} \\ 0 & -\frac{1}{4} \end{bmatrix} = [f_{p,u} \ f_{p,v}], \quad N_{p=(u,v)} = \frac{\frac{\partial f}{\partial u}\times\frac{\partial f}{\partial v}}{\left|\frac{\partial f}{\partial u}\times\frac{\partial f}{\partial v}\right|} = \left[\frac{-\sin v\cos u}{\sqrt{1+3\cos^2 v}} \quad \frac{-\sin u \sin v}{\sqrt{1+3\cos^2 v}} \quad \frac{-2\cos v}{\sqrt{1+3\cos^2 v}}\right]^T$$

$$\therefore N_{p=(\frac{\pi}{4},\frac{\pi}{6})} = \frac{f_{p,u}\times f_{p,v}}{|f_{p,u}\times f_{p,v}|} = \frac{[-\frac{\sqrt{2}}{4} \ \frac{\sqrt{2}}{4} \ 0]^T \times [\frac{\sqrt{6}}{4} \ \frac{\sqrt{6}}{4} \ -\frac{1}{4}]^T}{|f_{p,u}\times f_{p,v}|} = \frac{[-\frac{1}{8\sqrt{2}} \ -\frac{1}{8\sqrt{2}} \ -\frac{\sqrt{3}}{4}]^T}{|[-\frac{1}{8\sqrt{2}} \ -\frac{1}{8\sqrt{2}} \ -\frac{\sqrt{3}}{4}]^T|} = \underline{\left[-\frac{1}{\sqrt{26}} \quad -\frac{1}{\sqrt{26}} \quad -2\sqrt{\frac{3}{13}}\right]^T}$$

4.(a) $p=(\frac{\pi}{4},\frac{\pi}{6})$, $v=(1,0)$,

$$\gamma(t) = \int_0^t \gamma'(t)\,dt = \gamma(0)+vt = \begin{bmatrix} \frac{\pi}{4}+t \\ \frac{\pi}{6} \end{bmatrix} \Rightarrow u=\frac{\pi}{4}+t, \ v=\frac{\pi}{6}, \ Df_t = \begin{bmatrix} -\sin(\frac{\pi}{4}+t)\cdot\frac{1}{2} & \cos(\frac{\pi}{4}+t)\cdot\frac{\sqrt{3}}{2} \\ \cos(\frac{\pi}{4}+t)\cdot\frac{1}{2} & \sin(\frac{\pi}{4}+t)\cdot\frac{\sqrt{3}}{2} \\ 0 & -\frac{1}{4} \end{bmatrix}$$

$$s(t) = \int_0^t \|g_v'(t)\|\,dt = \int_0^t \|Df_t[\gamma'(t)]\|\,dt$$
$$= \int_0^t \sqrt{\tfrac{1}{4}(\sin^2(\tfrac{\pi}{4}+t)+\cos^2(\tfrac{\pi}{4}+t))}\,dt$$
$$= \frac{t}{2}$$

(b) From (a) $\Rightarrow$ $t = 2s$, $u = \frac{\pi}{4} + 2s$, $v = \frac{\pi}{6}$

$$h_v(s) = \begin{bmatrix} \frac{1}{2}\cos(\frac{\pi}{4}+2s) \\ \frac{1}{2}\sin(\frac{\pi}{4}+2s) \\ \frac{\sqrt{3}}{4} \end{bmatrix}$$

(c) $N_v(s) = \dfrac{\frac{\partial h_v}{\partial s}}{\left\|\frac{\partial h_v}{\partial s}\right\|} = \begin{bmatrix} -\sin(\frac{\pi}{4}+2s) \\ \cos(\frac{\pi}{4}+2s) \\ 0 \end{bmatrix}$, $N_v(0) = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$ is different from $N_{p=(\frac{\pi}{4},\frac{\pi}{6})}$ at 3(d)

5. (a) $DN = \begin{bmatrix} \frac{\partial N}{\partial u} & \frac{\partial N}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\sin v \sin u}{\sqrt{1+3\cos^2 v}} & \frac{-4\cos u \cos v}{(3\cos^2 v+1)^{\frac{3}{2}}} \\ \frac{-\sin v \cos u}{\sqrt{1+3\cos^2 v}} & \frac{-4\sin u \cos v}{(3\cos^2 v+1)^{\frac{3}{2}}} \\ 0 & \frac{2\sin v}{(3\cos^2 v+1)^{\frac{3}{2}}} \end{bmatrix}$

$$DN_{p=(\frac{\pi}{4},\frac{\pi}{6})} = \begin{bmatrix} \frac{1}{\sqrt{26}} & -\frac{8}{13}\sqrt{\frac{6}{13}} \\ \frac{-1}{\sqrt{26}} & -\frac{8}{13}\sqrt{\frac{6}{13}} \\ 0 & \frac{8}{13}\sqrt{\frac{1}{13}} \end{bmatrix}$$

(b) Denote $S = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix}$; let $DN_p = Df_p \cdot S$, $Df_p = \begin{bmatrix} -\frac{\sqrt{2}}{4} & \frac{\sqrt{6}}{4} \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{6}}{4} \\ 0 & -\frac{1}{4} \end{bmatrix}$:

$$DN_p = Df_p S \Rightarrow \begin{cases} \frac{1}{\sqrt{26}} = -\frac{\sqrt{2}}{4}s_1 + \frac{\sqrt{6}}{4}s_3 \\ \frac{8}{-13}\sqrt{\frac{6}{13}} = -\frac{\sqrt{2}}{4}s_2 + \frac{\sqrt{6}}{4}s_4 \\ \frac{-1}{\sqrt{26}} = \frac{\sqrt{2}}{4}s_1 + \frac{\sqrt{6}}{4}s_3 \\ \frac{-8}{13}\sqrt{\frac{6}{13}} = \frac{\sqrt{2}}{4}s_2 + \frac{\sqrt{6}}{4}s_4 \\ 0 = -\frac{1}{4}s_3 \\ \frac{8}{13}\sqrt{\frac{1}{13}} = -\frac{1}{4}s_4 \end{cases} \Rightarrow \begin{cases} s_3 = 0 \\ s_2 = 0 \\ s_4 = \frac{-32}{13}\sqrt{\frac{1}{13}} \\ s_1 = -2\sqrt{\frac{1}{13}} \end{cases} \Rightarrow S = \begin{bmatrix} -2\sqrt{\frac{1}{13}} & 0 \\ 0 & -\frac{32}{13}\sqrt{\frac{1}{13}} \end{bmatrix}$$

is diagonal

$\Rightarrow$ Eigenvectors are:

$S_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $S_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(d) Orthogonal directions

Part III: 3、1、Denote the surface normal at p to be N_p:

$$M_pN_p = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_p(t_\theta) t_\theta t_\theta^T d\theta \cdot N_p$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_p(t_\theta)\, t_\theta (t_\theta^T N_p) d\theta$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_p(t_\theta)\cdot t_\theta \cdot 0 \cdot d\theta \quad (\text{since } t_\theta \perp N_p)$$
$$= 0$$

$\therefore N_p$ is an eigenvector of M_p with eigenvalue $\lambda_1 = 0$

2、$M_pT_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_p(t_\theta) t_\theta t_\theta^T d\theta \cdot T_1$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_p(t_\theta)\, t_\theta (t_\theta^T T_1)\, d\theta$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_p(t_\theta)\, t_\theta \cos\theta \, d\theta$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} (K_p^1\cos^2\theta + K_p^2\sin^2\theta)(\cos^2\theta\, T_1 + \sin\theta\cos\theta\, T_2)\, d\theta$$
$$= \frac{1}{2\pi}\left\{\left[\int_{-\pi}^{\pi}(K_p^1\cos^2\theta + K_p^2\sin^2\theta)\cos^2\theta\, d\theta\right]T_1 + \left[\int_{-\pi}^{\pi}(K_p^1\cos^2\theta + K_p^2\sin^2\theta)\sin\theta\cos\theta\, d\theta\right]T_2\right\} \quad ①$$

Denote $\varphi(\theta) = (K_p^1\cos^2\theta + K_p^2\sin^2\theta)\cos^2\theta$, $\varphi(-\theta) = \varphi(\theta) \Rightarrow \varphi(\theta)$ is even

Denote $h(\theta) = (K_p^1\cos^2\theta + K_p^2\sin^2\theta)\sin\theta\cos\theta$, $h(\theta) = -h(-\theta) \Rightarrow h(\theta)$ is odd

$\therefore ① \Rightarrow \; = \frac{1}{\pi}\cdot\int_0^{\pi}(K_p^1\cos^2\theta + K_p^2\sin^2\theta)\cos^2\theta\, d\theta\, T_1$

$= (\frac{3}{8}K_p^1 + \frac{1}{8}K_p^2)\, T_1$

$\Rightarrow T_1$ is an eigenvector with eigenvalue $\frac{3}{8}K_p^1 + \frac{1}{8}K_p^2$

$M_pT_2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_p(t_\theta) t_\theta t_\theta^T d\theta \cdot T_2$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_p(t_\theta)\, t_\theta (t_\theta^T T_2)\, d\theta$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_p(t_\theta)\, t_\theta \sin\theta \, d\theta$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} (K_p^1\cos^2\theta + K_p^2\sin^2\theta)(\sin\theta\cos\theta\, T_1 + \sin^2\theta\, T_2)\, d\theta$$
$$= \frac{1}{2\pi}\left\{\left[\int_{-\pi}^{\pi}(K_p^1\cos^2\theta + K_p^2\sin^2\theta)\sin\theta\cos\theta\, d\theta\right]T_1 + \left[\int_{-\pi}^{\pi}(K_p^1\cos^2\theta + K_p^2\sin^2\theta)\sin^2\theta\, d\theta\right]T_2\right\} \quad ①$$

Denote $a(\theta) = (K_p^1\cos^2\theta + K_p^2\sin^2\theta)\sin^2\theta$, $a(-\theta) = a(\theta) \Rightarrow a(\theta)$ is even

Denote $b(\theta) = (K_p^1\cos^2\theta + K_p^2\sin^2\theta)\sin\theta\cos\theta$, $b(\theta) = -b(-\theta) \Rightarrow b(\theta)$ is odd

$\therefore ① \Rightarrow \; = \frac{1}{\pi}\cdot\int_0^{\pi}(K_p^1\cos^2\theta + K_p^2\sin^2\theta)\sin^2\theta\, d\theta\, T_2$

$= (\frac{1}{8}K_p^1 + \frac{3}{8}K_p^2)\, T_2$

$\Rightarrow T_2$ is an eigenvector with eigenvalue $\frac{1}{8}K_p^1 + \frac{3}{8}K_p^2$