1 Rotation

1.1:

$$\frac{(p+q)}{2} = \frac{1}{\sqrt{2}} + \frac{i}{2\sqrt{2}} + \frac{j}{2\sqrt{2}}, |\frac{(p+q)}{2}| = \frac{\sqrt{3}}{2}$$

$$r_0 = \frac{2}{\sqrt{3}} \cdot \frac{p+q}{2} = \frac{\sqrt{6}}{3} + \frac{\sqrt{6}}{6} i + \frac{\sqrt{6}}{6} j, |r_0| = 1$$

$$M(r_0) = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{-2}{3} \\ \frac{-2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

By eigen-decomposition, we have eigen-vector $r = [1 \quad 1 \quad 0]^T$ for $\lambda_1 = 1$. Hence, r is the rotation axis.

$$tr(M) = \frac{3}{5} = 1 + 2\cos\theta \Rightarrow \theta = \arccos\frac{1}{3} \approx 70.5^\circ$$

1.2:

$$w_p = [1\ 0\ 0]^T, \theta_p = \frac{\pi}{2}, w_q = [0\ 1\ 0]^T, \theta_q = \frac{\pi}{2}$$

So the exponential coordinates for p ($\overrightarrow{w_p}$) and q ($\overrightarrow{w_q}$) are:

$$\overrightarrow{w_p} = w_p\theta_p = [\frac{\pi}{2}\ 0\ 0]^T, \overrightarrow{w_q} = w_q\theta_q = [0\ \frac{\pi}{2}\ 0]^T$$

1.3.a:

$$[w_p] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \theta_p = \frac{\pi}{2},$$

$$[w_q] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \theta_q = \frac{\pi}{2}$$

$$\mathrm{Rot}(w_p, \theta_p) = e^{[w_p]\theta_p} \approx I + [w_p]\sin\theta_p + [w_p]^2(1-\cos\theta_p)$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$
$$\mathrm{Rot}(w_q, \theta_q) = e^{[w_q]\theta_q} \approx I + [w_q]\sin\theta_q + [w_q]^2(1-\cos\theta_q)$$
$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

1.3.b:

$$[w_p] + [w_q] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix},$$
$$\exp(([w_p] + [w_q])\frac{\pi}{2}) \approx I + [w_p] + [w_q] + ([w_p] + [w_q])^2$$
$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & -1 \end{bmatrix},$$
$$\exp([w_p]\frac{\pi}{2})\exp([w_q]\frac{\pi}{2}) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \neq \exp(([w_p] + [w_q])\frac{\pi}{2})$$
$$\Rightarrow \exp(([w_p] + [w_q])) \neq \exp([w_p])\exp([w_q]),\ \text{proved}$$

1.3.c.i

The objective:

$$\|R(I + [\Delta w])X - Y\|^2$$
$$= \|R[\Delta w]X - (Y - RX)\|^2$$
$$= \sum_{i=1}^{n} \|R[\Delta w]X_i - (Y_i - RX_i)\|^2 \ (X = [X_1\ X_2 \ldots X_n], X_i \in R^{3\times1}, i \in \{1, 2, \ldots, n\})$$
$$= \sum_{i=1}^{n} \| - R[X_i]\Delta w - (Y_i - RX_i)\|^2$$
$$= \|C\Delta w - D\|^2, \text{where } C = \begin{bmatrix} -R[X_1] \\ -R[X_2] \\ \vdots \\ -R[X_n] \end{bmatrix}, D = \begin{bmatrix} Y_1 - RX_1 \\ Y_1 - RX_2 \\ \vdots \\ Y_n - RX_n \end{bmatrix}, [X_i] = \begin{bmatrix} 0 & -X_{i3} & X \\ X_{i3} & 0 & - \\ -X_{i2} & X_{i1} & \end{bmatrix}$$

So we write: Step 2: Solve the following optimization problem by least square:

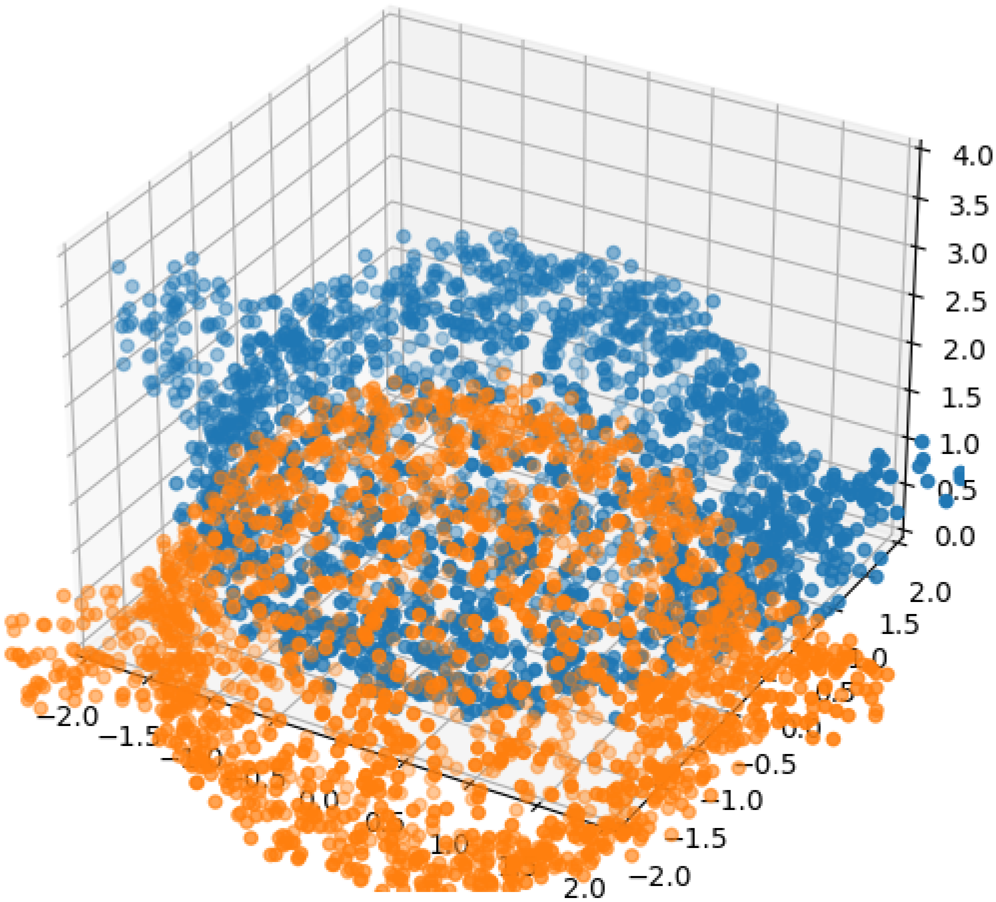
$$
\begin{aligned}
\min_{\Delta w} \quad & ||C\Delta w - D||^2 \\
\text{s. t.} \quad & ||\Delta w||^2 \leq \epsilon,\ \Delta w = [w_1\ w_2\ w_3]^T \\
& R^T R = I \\
& \det(R) = I
\end{aligned}
$$

1.3.c.ii

In [30]:
```python
# Note Matplotlib is only suitable for simple 3D visualization.
# For later problems, you should not use Matplotlib to do the plotting
from mpl_toolkits.mplot3d import Axes3D
import numpy as np
import matplotlib.pyplot as plt
def show_points(points):
    fig = plt.figure()
    # ax = fig.gca(projection = '3d')
    ax = fig.add_axes(Axes3D(fig))
    ax.set_xlim3d([-2, 2])
    ax.set_ylim3d([-2, 2])
    ax.set_zlim3d([0, 4])
    ax.scatter(points[0], points[2], points[1])

def compare_points(points1, points2):
    fig = plt.figure()
    # ax = fig.gca(projection = '3d')
    ax = fig.add_axes(Axes3D(fig))
    ax.set_xlim3d([-2, 2])
    ax.set_ylim3d([-2, 2])
    ax.set_zlim3d([0, 4])
    ax.scatter(points1[0], points1[2], points1[1]) # right->x, in->y, up->z
    ax.scatter(points2[0], points2[2], points2[1])
```

In [31]:
```python
npz = np.load('HW1_P1.npz')
X = npz['X']
Y = npz['Y']
print(X)
print(X.shape)
compare_points(X, Y)  # noisy teapotsand
```

```
[[-2.8002013  -0.36611453 -1.79778603 ...  0.93021646  0.03109836
  -0.60719447]
 [ 1.71286959  1.18759664  1.43200163 ...  0.1726376   2.61984994
   2.00916566]
 [ 0.00316965  1.80786287  0.12010375 ... -0.85181288 -0.11694988
   1.63876969]]
(3, 2000)
```



In [32]:
```
# copy-paste your hw0 solve module here
def hw0_solve(A, b, eps):
    x, _, _, _ = np.linalg.lstsq(A, b, rcond=None)
    # print(x)

    # case 1: unconstraint least square
    if x.T @ x < eps:
        return x

    # case 2: linear search over lambda s. t. xTx-epsilon goes to 0 (xTx goes to ep
    d, U = np.linalg.eigh(A.T@A) # SVD of A may be faster
    k = U.T@(A.T@b)
    def func(lam):
        return ((k / (d + 2 * lam))**2).sum() - eps
    def dfunc(lam):
        return -4 * ((k**2 / (d+2*lam)**3)).sum()

    # Newton, should converge in less than 10 iterations
    lam = 0
    while True:
        lam2 = lam - func(lam) / dfunc(lam)
        if abs(lam-lam2) < 1e-6:
            break
        lam = lam2
```

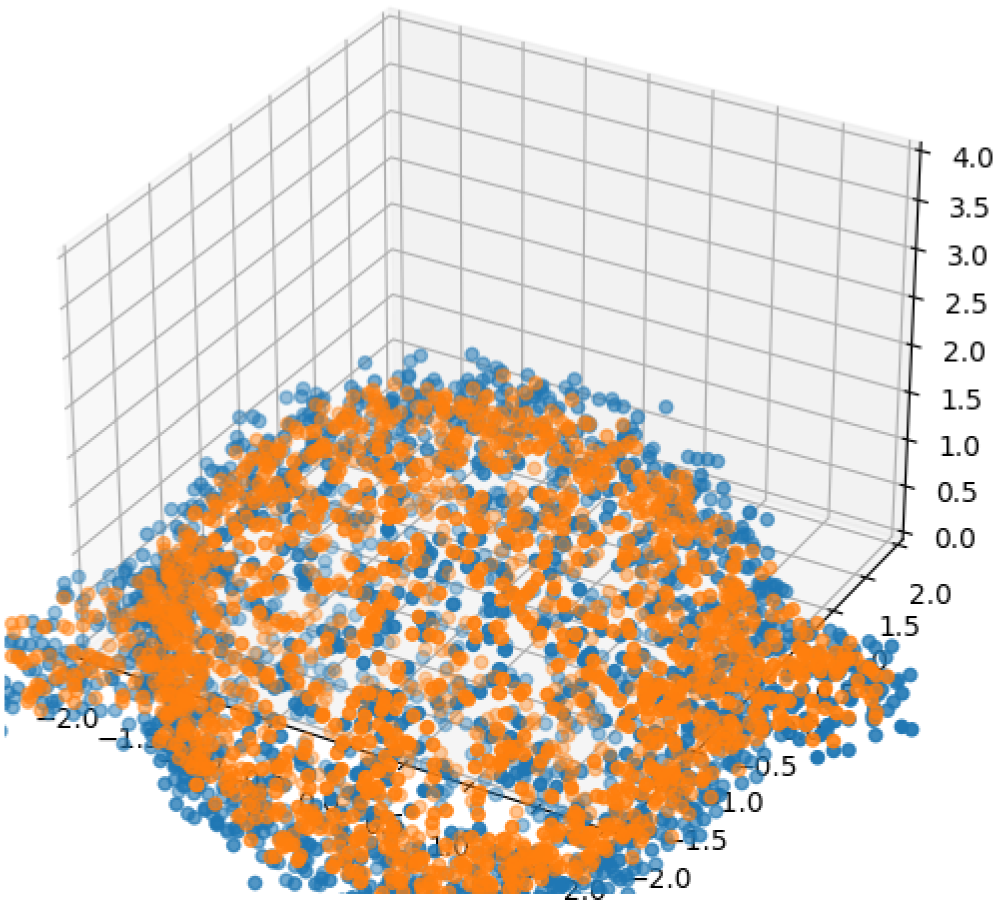
```python
    x = U@(np.diag(1/(d + 2 * lam))@(U.T@(A.T@b)))
    return x
```

In [33]:
```python
# Iterative solution to point-cloud alignment problem
# solve this problem here, and store your final results in R1
R1 = np.eye(3)
n = X.shape[1] # dataset size
# print(R1)
def skew(X):
    """
    Find skey-symmetric matrix of X
    """
    x1,x2,x3 = X[0],X[1],X[2]
    return np.array([[0,-x3,x2],[x3,0,-x1],[-x2,x1,0]])

for __ in range(100):
    # solve dw
    A = np.zeros((6000,3))
    for i in range(n):
        A[3*i:3*i+3, :] = - R1 @ skew(X[:,i])
    b = np.zeros((6000,1))
    for i in range(n):
        b[3*i:3*i+3, :] = np.expand_dims(Y[:,i] - R1 @ X[:,i], axis=1)
    dw = hw0_solve(A, b, 0.01)
    # update R1
    w1, w2, w3 = dw[0][0], dw[1][0], dw[2][0]
    wskew = np.array([[0, -w3, w2], [w3, 0, -w1], [-w2, w1, 0]])
    R1 = np.dot(R1, np.eye(3) + wskew)
    # print(R1)

print("Completed R1:")
print(R1)
```

```
Completed R1:
[[-1.03309053  0.23970278  0.22846469]
 [-0.17768836  0.26768134 -1.00887162]
 [-0.27443643 -0.96125508 -0.29015662]]
```

In [34]:
```python
# Testing code, you should see the points of the 2 teapots roughly overlap
compare_points(R1@X, Y)
R1.T@R1
```

Out[34]:
```
array([[ 1.17416456, -0.03139511,  0.02286958],
       [-0.03139511,  1.05312204,  0.06362204],
       [ 0.02286958,  0.06362204,  1.15420894]])
```



1.4.a:

$$p = \frac{1+i}{\sqrt{2}}, -p = \frac{-1-i}{\sqrt{2}}, q = \frac{1+j}{\sqrt{2}}, -q = \frac{-1-j}{\sqrt{2}}$$

$$\theta_{-p} = 2\arccos -\frac{\sqrt{2}}{2} = \frac{3\pi}{2}, w_{-p} = \frac{1}{\sin\frac{3\pi}{4}}[\frac{-1}{\sqrt{2}}\ 0\ 0]^T = [-1\ 0\ 0]^T$$

$$=> \overrightarrow{w_{-p}} = w_{-p}\theta_{-p} = [\frac{-3\pi}{2}\ 0\ 0]^T = \overrightarrow{w_p}$$

$$\theta_{-q} = 2\arccos -\frac{\sqrt{2}}{2} = \frac{3\pi}{2}, w_{-q} = \frac{1}{\sin\frac{3\pi}{4}}[0\ \frac{-1}{\sqrt{2}}\ 0]^T = [0\ -1\ 0]^T$$

$$=> \overrightarrow{w_{-q}} = w_{-q}\theta_{-q} = [0\ \frac{-3\pi}{2}\ 0]^T = \overrightarrow{w_q}$$

Statement: Quaternion pair $(r, -r)$ represents the same rotation.

Proof: Suppose rotating vector $\overrightarrow{x}$ using quaternion r to get $R_r(\overrightarrow{x})$:

$$R_r(\overrightarrow{x}) = r\overrightarrow{x}r^{-1}$$

Now use $-\mathrm{r}$, we get:

$$\mathrm{R}_{-\mathrm{r}}(\overrightarrow{\mathrm{x}}) = -\mathrm{r}\,\overrightarrow{\mathrm{x}}\,-\mathrm{r}^{-1} = (-1)^2\mathrm{r}\,\overrightarrow{\mathrm{x}}\,\mathrm{r}^{-1} = \mathrm{r}\,\overrightarrow{\mathrm{x}}\,\mathrm{r}^{-1} = \mathrm{R}_{\mathrm{r}}(\overrightarrow{\mathrm{x}})$$

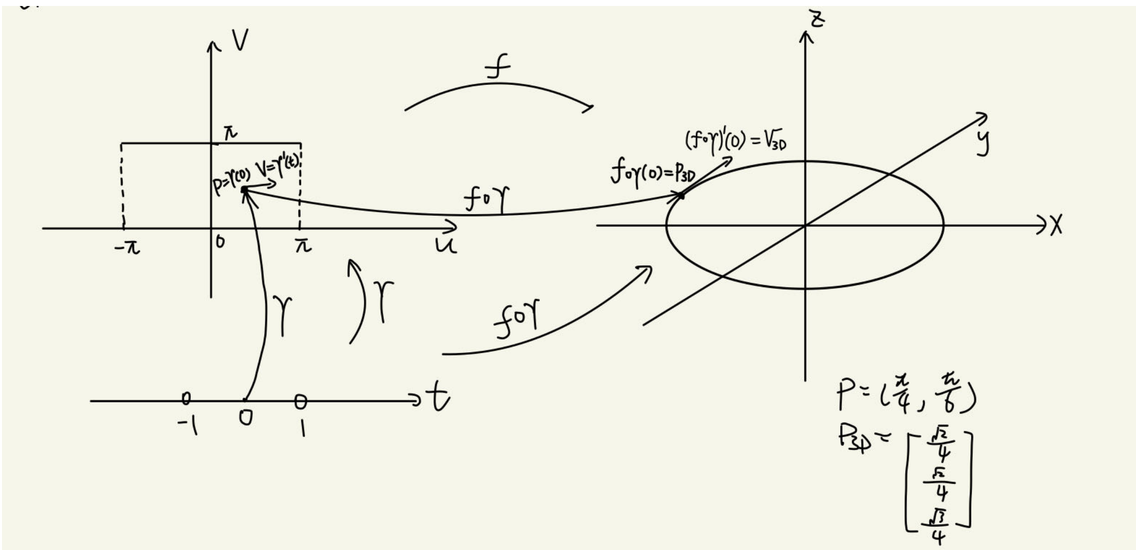
Hence we proved that $\mathrm{R}_{\mathrm{r}}(\overrightarrow{\mathrm{x}})$ has the same rotation with $\mathrm{R}_{-\mathrm{r}}(\overrightarrow{\mathrm{x}})$.

1.4.b:

No. Since for each $(\mathrm{r}, -\mathrm{r})$ having a large difference in domain, they will yield the same rotation matrix at $\mathrm{SO}(3)$, and hence the L2 difference learning will give the prediction in the middle ($\frac{\mathrm{r}-\mathrm{r}}{2} = 0$), which is undesirable for both ground truths (r and $-\mathrm{r}$).

2 Geometry

2.1:



P: A point in the domain of f

v: 2D Velocity of P in the domain

γ: A function mapping 1D input t to a point in the domain

$\mathrm{f} \circ \gamma$: A function mapping 1D input t to a point in the 3D manifold

$(\mathrm{f} \circ \gamma)'(0)$: Velocity of $\mathrm{P}_{3\mathrm{D}}$, the point projected from P to the 3D manifold, at t=0

2.2

In [35]:
```
a, b, c = 1, 1, 0.5
```

In [36]:
```
# These are some convenient functions to create open3d geometries and plot them
# The viewing direction is fine-tuned for this problem, you should not change them
import open3d
```

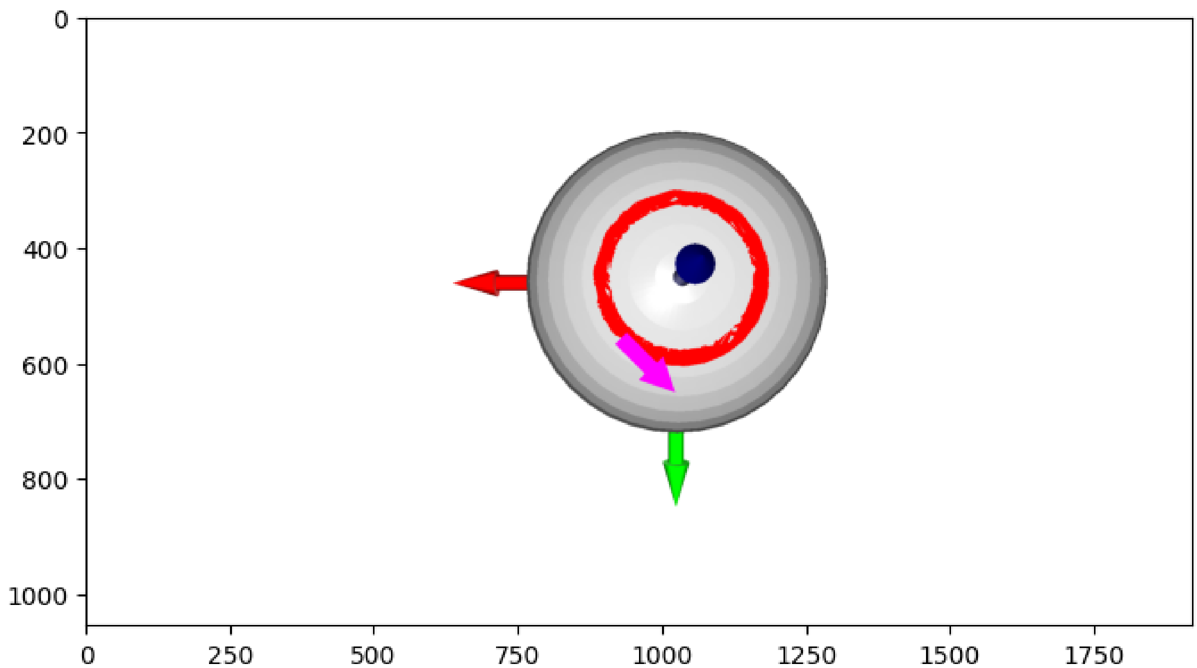
```python
import math
import numpy as np
import matplotlib.pyplot as plt

vis = open3d.visualization.Visualizer()
vis.create_window(visible = False)

def draw_geometries(geoms):
    for g in geoms:
        vis.add_geometry(g)
    view_ctl = vis.get_view_control()
    view_ctl.set_up((0, 1e-4, 1))
    view_ctl.set_front((0, 0.5, 2))
    view_ctl.set_lookat((0, 0, 0))
    # do not change this view point
    vis.update_renderer()
    img = vis.capture_screen_float_buffer(True)
    plt.figure(figsize=(8,6))
    plt.imshow(np.asarray(img)[::-1, ::-1])
    for g in geoms:
        vis.remove_geometry(g)

def create_arrow_from_vector(origin, vector):
    '''
    origin: origin of the arrow
    vector: direction of the arrow
    '''
    v = np.array(vector)
    v /= np.linalg.norm(v)
    z = np.array([0,0,1])
    angle = np.arccos(z@v)

    arrow = open3d.geometry.TriangleMesh.create_arrow(0.05, 0.1, 0.25, 0.2)
    arrow.paint_uniform_color([1,0,1])
    T = np.eye(4)
    T[:3, 3] = np.array(origin)
    T[:3,:3] = open3d.geometry.get_rotation_matrix_from_axis_angle(np.cross(z, v) *
    arrow.transform(T)
    return arrow

def create_ellipsoid(a,b,c):
    sphere = open3d.geometry.TriangleMesh.create_sphere()
    sphere.transform(np.diag([a,b,c,1]))
    sphere.compute_vertex_normals()
    return sphere


def create_lines(points):
    lines = []
    for p1, p2 in zip(points[:-1], points[1:]):
        height = np.linalg.norm(p2-p1)
        center = (p1+p2) / 2
        d = p2-p1
        d /= np.linalg.norm(d)
        axis = np.cross(np.array([0,0,1]), d)
        axis /= np.linalg.norm(axis)
```

```python
        angle = np.arccos(np.array([0,0,1]) @ d)
        R = open3d.geometry.get_rotation_matrix_from_axis_angle(axis * angle)

        T = np.eye(4)
        T[:3,:3]=R
        T[:3,3] = center
        cylinder = open3d.geometry.TriangleMesh.create_cylinder(0.02, height)
        cylinder.transform(T)
        cylinder.paint_uniform_color([1,0,0])
        lines.append(cylinder)
    return lines
```

In [37]:

```python
import math
# exapmle code to draw ellipsoid, curve, and arrows
arrow = create_arrow_from_vector([0.,0.,1.], [1.,1.,0.])
print(arrow)
ellipsoid = create_ellipsoid(a, b, c)
cf = open3d.geometry.TriangleMesh.create_coordinate_frame()
cf.scale(1.5, (0,0,0))

def get_3d_curve(p=np.array([math.pi/4, math.pi/6]), v=np.array([1, 0]), numPts=100
    """
    Get the points for the curve of p moving with v
    p, v: 2D numpy arrays
    numPts: int
    """
    xRight = math.pi
    incr = (xRight - p[0]) / numPts # increment in domain for each point
    pts = []
    for i in range(numPts):
        p[0] = p[0] + incr*i
        u0, v0 = p[0], p[1]
        pts.append([math.cos(u0)*math.sin(v0), math.sin(u0)*math.sin(v0), 0.5*math.
    return np.array(pts)


# 2.2: Draw the 3D curve
pts = get_3d_curve(p=np.array([math.pi/4, math.pi/6]), v=np.array([1, 0]), numPts=1
# print(pts)
curve = create_lines(pts)

# 2.3.c: Draw Dfp(v) on the ellipsoid
arrow = create_arrow_from_vector([math.sqrt(2)/4, math.sqrt(2)/4, math.sqrt(3)/4],

draw_geometries([ellipsoid, cf, arrow] + curve)
```

```
TriangleMesh with 124 points and 240 triangles.
```



2.3.a:

$$Df_p = [\frac{\partial f}{\partial u} \; \frac{\partial f}{\partial v}]|_p = \begin{bmatrix} -\sin u \sin v & \cos u \cos v \\ \cos u \sin v & \sin u \cos v \\ 0 & -\frac{1}{2}\sin v \end{bmatrix}$$

2.3.b:

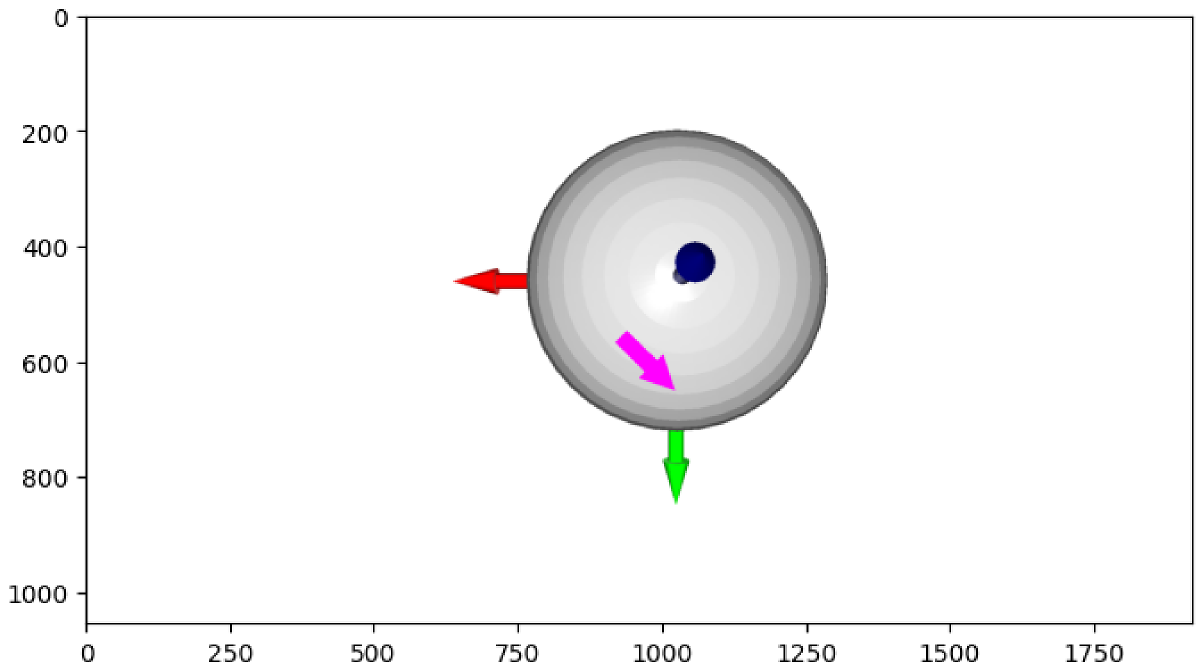
$Df_p = [\frac{\partial f}{\partial u} \; \frac{\partial f}{\partial v}]$ represents 2 3D vecetors spanning the tangent plane at $f(u, v)$

2.3.c:

In [38]:
```
# Run the function definitions at 2.2 first before running the codes here!
import math
# exapmle code to draw ellipsoid, curve, and arrows
arrow = create_arrow_from_vector([0.,0.,1.], [1.,1.,0.])
# curve = create_lines(np.array([[1,1,1], [-1,1,1], [-1,-1,1], [1,-1,1], [1,1,1]],
print(arrow)
ellipsoid = create_ellipsoid(a, b, c)
cf = open3d.geometry.TriangleMesh.create_coordinate_frame()
cf.scale(1.5, (0,0,0))

def get_3d_curve(p=np.array([math.pi/4, math.pi/6]), v=np.array([1, 0]), numPts=100
    """
    Get the points for the curve of p moving with v
    p, v: 2D numpy arrays
    numPts: int
    """
    xRight = math.pi
    incr = (xRight - p[0]) / numPts # increment in domain for each point
```

```
    pts = []
    for i in range(numPts):
        p[0] = p[0] + incr*i
        u0, v0 = p[0], p[1]
        pts.append([math.cos(u0)*math.sin(v0), math.sin(u0)*math.sin(v0), 0.5*math.
    return np.array(pts)

# 2.3.c: Draw Dfp(v) on the ellipsoid
arrow = create_arrow_from_vector([math.sqrt(2)/4, math.sqrt(2)/4, math.sqrt(3)/4],

draw_geometries([ellipsoid, cf, arrow])
```

```
TriangleMesh with 124 points and 240 triangles.
```



2.3.d:

$$Df_{p=(\frac{\pi}{4},\frac{\pi}{6})} = \begin{bmatrix} -\frac{\sqrt(2)}{4} & \frac{\sqrt{6}}{4} \\ \frac{\sqrt(2)}{4} & \frac{\sqrt{6}}{4} \\ 0 & -\frac{1}{4} \end{bmatrix} = [\frac{\partial f}{\partial u}|_{(\frac{\pi}{4},\frac{\pi}{6})} \quad \frac{\partial f}{\partial v}|_{(\frac{\pi}{4},\frac{\pi}{6})}]$$

$$N_{p=(\frac{\pi}{4},\frac{\pi}{6})} = \frac{\frac{\partial f}{\partial u}|_{(\frac{\pi}{4},\frac{\pi}{6})} \times \frac{\partial f}{\partial v}|_{(\frac{\pi}{4},\frac{\pi}{6})}}{||\frac{\partial f}{\partial u}|_{(\frac{\pi}{4},\frac{\pi}{6})} \times \frac{\partial f}{\partial v}|_{(\frac{\pi}{4},\frac{\pi}{6})}||} = [\frac{-1}{\sqrt{26}} \quad \frac{-1}{\sqrt{26}} \quad \frac{-2\sqrt{3}}{\sqrt{13}}]^T$$

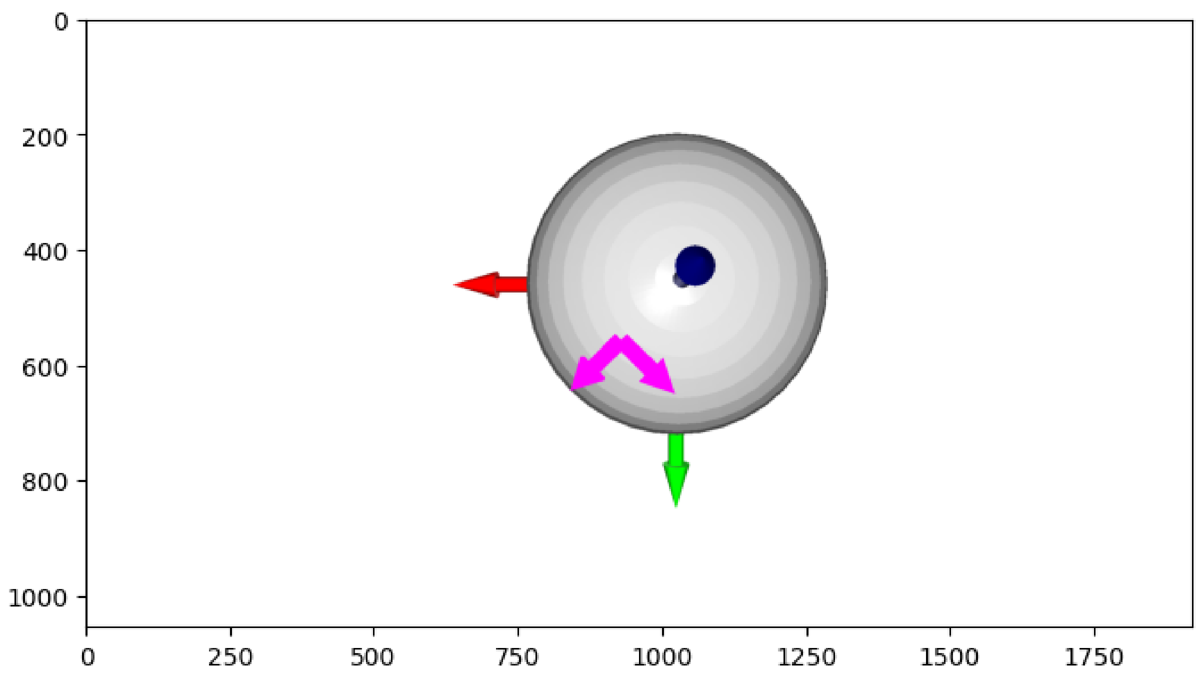
2.3.e:

In [39]:

```
# Run the function definitions at 2.2 first before running the codes here!
import math
# exapmle code to draw ellipsoid, curve, and arrows
arrow = create_arrow_from_vector([0.,0.,1.], [1.,1.,0.])
# curve = create_lines(np.array([[1,1,1], [-1,1,1], [-1,-1,1], [1,-1,1], [1,1,1]],
```

```
print(arrow)
ellipsoid = create_ellipsoid(a, b, c)
cf = open3d.geometry.TriangleMesh.create_coordinate_frame()
cf.scale(1.5, (0,0,0))

def get_3d_curve(p=np.array([math.pi/4, math.pi/6]), v=np.array([1, 0]), numPts=100
    """
    Get the points for the curve of p moving with v
    p, v: 2D numpy arrays
    numPts: int
    """
    xRight = math.pi
    incr = (xRight - p[0]) / numPts # increment in domain for each point
    pts = []
    for i in range(numPts):
        p[0] = p[0] + incr*i
        u0, v0 = p[0], p[1]
        pts.append([math.cos(u0)*math.sin(v0), math.sin(u0)*math.sin(v0), 0.5*math.
    return np.array(pts)

# 2.3.e: Draw orthonomal basis
arrowOrtho1 = create_arrow_from_vector([math.sqrt(2)/4, math.sqrt(2)/4, math.sqrt(3
arrowOrtho2 = create_arrow_from_vector([math.sqrt(2)/4, math.sqrt(2)/4, math.sqrt(3


draw_geometries([ellipsoid, cf, arrowOrtho1, arrowOrtho2])
```

```
TriangleMesh with 124 points and 240 triangles.
```



2.4.a:

$$
\begin{aligned}
\gamma(t) &= \int_0^t \gamma^{'}(t)dt \\
&= \gamma(0) + vt \\
&= [\frac{\pi}{4} + t \quad \frac{\pi}{6}]^T
\end{aligned}
$$

$$
\begin{aligned}
=> Df_{u=\frac{\pi}{4}+t, v=\frac{\pi}{6}} &= \begin{bmatrix} -\frac{\sin(\frac{\pi}{4}+t)}{2} & \frac{\sqrt{3}}{2}\cos(\frac{\pi}{4}+t) \\ \frac{\cos(\frac{\pi}{4}+t)}{2} & \frac{\sqrt{3}}{2}\sin(\frac{\pi}{4}+t) \\ 0 & -\frac{1}{4} \end{bmatrix} \\
&= Df_t
\end{aligned}
$$

$$
\begin{aligned}
S(t) &= \int_0^t ||g_v^{'}(t)||dt \\
&= \int_0^t ||Df_t \cdot \gamma^{'}(t)||dt \\
&= \int_0^t \sqrt{\frac{1}{4}(\sin^2(\frac{\pi}{4}+t) + \cos^2(\frac{\pi}{4}+t))}dt \\
&= \frac{t}{2}, \ t \in (-1, 1)
\end{aligned}
$$

2.4.b:

From (a) => $t = 2s, u = \frac{\pi}{4} + 2s, v = \frac{\pi}{6}$, and we have
$h_v(s) = [\frac{1}{2}\cos(\frac{\pi}{4} + 2s) \quad \frac{1}{2}\sin(\frac{\pi}{4} + 2s) \quad \frac{\sqrt{3}}{4}]^T$

2.4.c:

$$
\begin{aligned}
T_v(s) &= \frac{\partial h_v}{\partial s} = [-\sin(\frac{\pi}{4} + 2s) \quad \cos(\frac{\pi}{4} + 2s) \quad 0]^T, \\
N_v(s) &= \frac{\frac{\partial T_v}{\partial s}}{||\frac{\partial T_v}{\partial s}||} = [\cos(\frac{\pi}{4} + 2s) \quad \sin(\frac{\pi}{4} + 2s) \quad 0]^T,
\end{aligned}
$$

So the curve normal is given as:

$$
N_v(0) = [\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \quad 0]^T
$$

Which is different from the normal at 3(d) given below when $s = 0$:

$$
N_{p=(\frac{\pi}{4}, \frac{\pi}{6})} = [\frac{-1}{\sqrt{26}} \quad \frac{-1}{\sqrt{26}} \quad \frac{-2\sqrt{3}}{\sqrt{13}}]^T
$$

2.5.a:

$$DN = [\frac{\partial N}{\partial u} \quad \frac{\partial N}{\partial v}]$$

$$= \begin{bmatrix} \frac{\sin(u)\sin(v)}{(3\cos^2(v)+1)^{\frac{1}{2}}} & \frac{-4\cos(u)\cos(v)}{(3\cos^2(v)+1)^{\frac{3}{2}}} \\ \frac{-\sin(v)\cos(u)}{(3\cos^2(v)+1)^{\frac{1}{2}}} & \frac{-4\sin(u)\cos(v)}{(3\cos^2(v)+1)^{\frac{3}{2}}} \\ 0 & \frac{2\sin(v)}{(3\cos^2(v)+1)^{\frac{3}{2}}} \end{bmatrix}$$

$$DN_{p=(\frac{\pi}{4},\frac{\pi}{6})} = \begin{bmatrix} \frac{1}{\sqrt{26}} & -\frac{8\sqrt{6}}{13\sqrt{13}} \\ \frac{-1}{\sqrt{26}} & -\frac{8\sqrt{6}}{13\sqrt{13}} \\ 0 & \frac{8}{13\sqrt{13}} \end{bmatrix}$$

2.5.b:

Denote $S = [\begin{matrix} s_1 & s_2 \\ s_3 & s_4 \end{matrix}]$: Let $DN_{p=(\frac{\pi}{4},\frac{\pi}{6})} = Df_{p=(\frac{\pi}{4},\frac{\pi}{6})} \cdot S$, where $Df_{p=(\frac{\pi}{4},\frac{\pi}{6})}$ is mentioned at 2.3.d:

From $DN_p = Df_p S$ we have:

$$\begin{aligned} \frac{1}{\sqrt{26}} &= -\frac{\sqrt{2}}{4}s_1 + \frac{\sqrt{6}}{4}s_3 \\ \frac{-8\sqrt{6}}{13\sqrt{13}} &= -\frac{\sqrt{2}}{4}s_2 + \frac{\sqrt{6}}{4}s_4 \\ \frac{1}{\sqrt{26}} &= \frac{\sqrt{2}}{4}s_1 + \frac{\sqrt{6}}{4}s_3 \\ \frac{-8\sqrt{6}}{13\sqrt{13}} &= \frac{\sqrt{2}}{4}s_2 + \frac{\sqrt{6}}{4}s_4 \\ 0 &= \frac{-1}{4}s_3 \\ \frac{8}{13\sqrt{13}} &= \frac{-1}{4}s_4 \end{aligned}$$

Solving the above 6 equations yield: $s_1 = \frac{-2}{\sqrt{13}}, s_2 = 0, s_3 = 0, s_4 = \frac{-32}{13\sqrt{13}}$

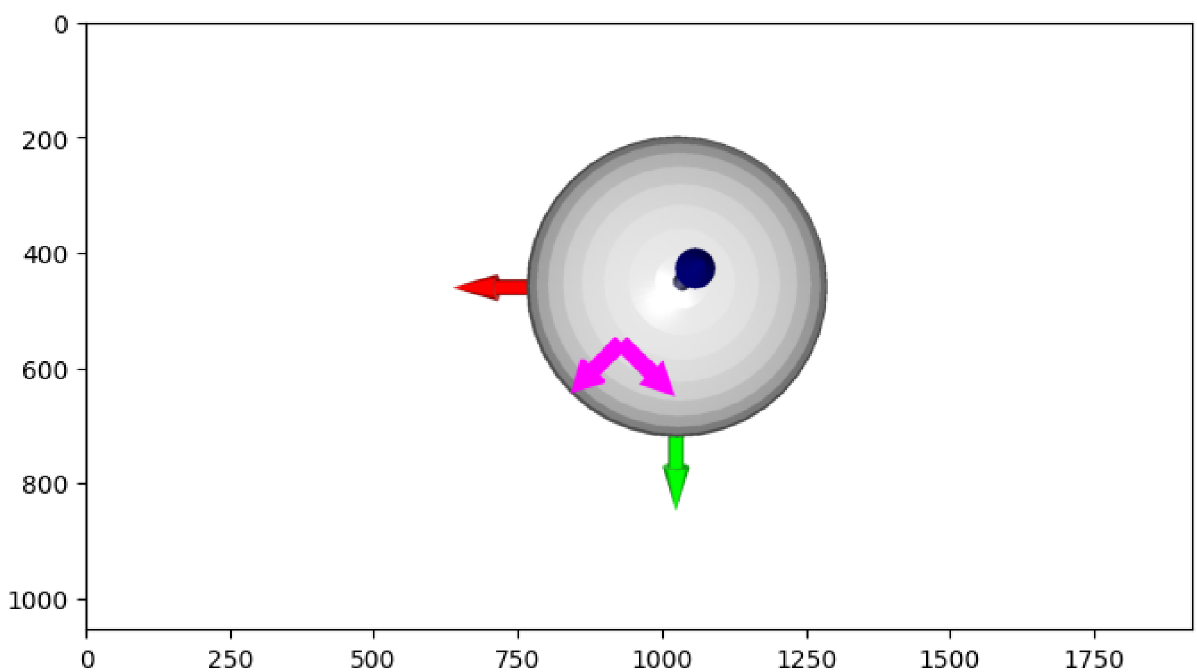
Hence $S = \begin{bmatrix} \frac{-2}{\sqrt{13}} & 0 \\ 0 & \frac{-32}{13\sqrt{13}} \end{bmatrix}$ is diagonal with eigenvectors $s_1 = [1 \quad 0]^T$ and $s_2 = [0 \quad 1]^T$

2.5.c:

In [40]:

```python
# Run the function definitions at 2.2 first before running the codes here!
import math
# exapmle code to draw ellipsoid, curve, and arrows
arrow = create_arrow_from_vector([0.,0.,1.], [1.,1.,0.])
# curve = create_lines(np.array([[1,1,1], [-1,1,1], [-1,-1,1], [1,-1,1], [1,1,1]],
print(arrow)
ellipsoid = create_ellipsoid(a, b, c)
cf = open3d.geometry.TriangleMesh.create_coordinate_frame()
cf.scale(1.5, (0,0,0))

def get_3d_curve(p=np.array([math.pi/4, math.pi/6]), v=np.array([1, 0]), numPts=100
    """
    Get the points for the curve of p moving with v
    p, v: 2D numpy arrays
    numPts: int
    """
    xRight = math.pi
    incr = (xRight - p[0]) / numPts # increment in domain for each point
    pts = []
    for i in range(numPts):
        p[0] = p[0] + incr*i
        u0, v0 = p[0], p[1]
        pts.append([math.cos(u0)*math.sin(v0), math.sin(u0)*math.sin(v0), 0.5*math.
    return np.array(pts)

# 2.5.c: Draw principle curvature directions at 3D
arrowOrtho1 = create_arrow_from_vector([math.sqrt(2)/4, math.sqrt(2)/4, math.sqrt(3
arrowOrtho2 = create_arrow_from_vector([math.sqrt(2)/4, math.sqrt(2)/4, math.sqrt(3

draw_geometries([ellipsoid, cf, arrowOrtho1, arrowOrtho2])
```

```
TriangleMesh with 124 points and 240 triangles.
```



2.5.d: Orthogonal directions.

3 Mesh

3.1:

Denote the surface normal at P to be $\mathrm{N_p}$:

$$\begin{aligned}
\mathrm{M_p N_p} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \kappa_p(t_\theta)t_\theta t_\theta^T\, d\theta \cdot N_p \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \kappa_p(t_\theta)t_\theta(t_\theta^T \cdot N_p)d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \kappa_p(t_\theta)t_\theta \cdot (0)d\theta \quad (\text{since } t_\theta \perp N_p) \\
&= 0
\end{aligned}$$

Hence $\mathrm{N_p}$ is an eigenvector of $\mathrm{M_p}$ with eigenvalue $\lambda_1 = 0$.

3.2:

$$\begin{aligned}
M_p T_1 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \kappa_p(t_\theta)t_\theta t_\theta^T\, d\theta \cdot T_1 \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \kappa_p(t_\theta)t_\theta(t_\theta^T \cdot T_1)d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \kappa_p(t_\theta)t_\theta(\cos\theta)d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\kappa_p^1(\cos\theta)^2 + \kappa_p^2(\sin\theta)^2)(T_1(\cos\theta)^2 + T_2(\sin\theta)\cos\theta)d\theta \\
&= \frac{1}{2\pi}\{[\int_{-\pi}^{\pi} ((\kappa_p^1(\cos\theta)^2 + \kappa_p^2(\sin\theta)^2)(\cos\theta)^2)d\theta]T_1 + [\int_{-\pi}^{\pi} ((\kappa_p^1(\cos\theta)^2 + \kappa_p^2(\sin\theta)^2
\end{aligned}$$

Let $\psi(\theta) = (\kappa_p^1(\cos\theta)^2 + \kappa_p^2(\sin\theta)^2)(\cos\theta)^2$, $\psi(-\theta) = \psi(\theta) \Rightarrow \psi(\theta)$ is even.

Let $h(\theta) = (\kappa_p^1(\cos\theta)^2 + \kappa_p^2(\sin\theta)^2)\sin\theta\cos\theta$, $h(-\theta) = -h(\theta) \Rightarrow h(\theta)$ is odd.

Hence we can further simplify the above equation (1) to:

$$\begin{aligned}
&= \frac{1}{\pi}\int_{0}^{\pi} (\kappa_p^1(\cos\theta)^2 + \kappa_p^2(\sin\theta)^2)(\cos\theta)^2 d\theta \cdot T_1 \\
&= (\frac{3}{8}\kappa_p^1 + \frac{1}{8}\kappa_p^2) \cdot T_1
\end{aligned}$$

Hence T_1 is an eigenvector with eigenvalue $\frac{3}{8}\kappa_p^1 + \frac{1}{8}\kappa_p^2$.

$$
\begin{aligned}
M_p T_2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \kappa_p(t_\theta) t_\theta t_\theta^T d\theta \cdot T_2 \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \kappa_p(t_\theta) t_\theta (t_\theta^T \cdot T_2) d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \kappa_p(t_\theta) t_\theta (\sin\theta) d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\kappa_p^1(\cos\theta)^2 + \kappa_p^2(\sin\theta)^2)(T_1 \sin\theta\cos\theta + T_2(\sin\theta)^2) d\theta \\
&= \frac{1}{2\pi}\{[\int_{-\pi}^{\pi} ((\kappa_p^1(\cos\theta)^2 + \kappa_p^2(\sin\theta)^2)\sin\theta\cos\theta) d\theta]T_1 + [\int_{-\pi}^{\pi} ((\kappa_p^1(\cos\theta)^2 + \kappa_p^2(\sin\theta
\end{aligned}
$$

Let $a(\theta) = (\kappa_p^1(\cos\theta)^2 + \kappa_p^2(\sin\theta)^2)(\sin\theta)^2$, $a(-\theta) = a(\theta) \Rightarrow a(\theta)$ is even.

Let $b(\theta) = (\kappa_p^1(\cos\theta)^2 + \kappa_p^2(\sin\theta)^2)\sin\theta\cos\theta$, $b(-\theta) = -b(\theta) \Rightarrow b(\theta)$ is odd.

Hence we can further simplify the above equation (2) to:

$$
\begin{aligned}
&= \frac{1}{\pi}\int_{0}^{\pi} (\kappa_p^1(\cos\theta)^2 + \kappa_p^2(\sin\theta)^2)(\sin\theta)^2 d\theta \cdot T_2 \\
&= (\frac{1}{8}\kappa_p^1 + \frac{3}{8}\kappa_p^2) \cdot T_2
\end{aligned}
$$

Hence T_2 is an eigenvector with eigenvalue $\frac{1}{8}\kappa_p^1 + \frac{3}{8}\kappa_p^2$.

3.3:

In [41]:

```
# You may want to restart your notebook here, to reinitialize Open3D

import open3d
import numpy as np
import matplotlib.pyplot as plt

vis = open3d.visualization.Visualizer()
vis.create_window(visible = False)

# Make sure you call this function to draw the points for proper viewing direction
def draw_geometries(geoms):
    for g in geoms:
        vis.add_geometry(g)
    view_ctl = vis.get_view_control()
    view_ctl.set_up((0, 1, 0))
    view_ctl.set_front((0, 2, 1))
    view_ctl.set_lookat((0, 0, 0))
    view_ctl.set_zoom(1)
    # do not change this view point
    vis.update_renderer()
    img = vis.capture_screen_float_buffer(True)
    plt.figure(figsize=(8,6))
    plt.imshow(np.asarray(img))
```

```python
    for g in geoms:
        vis.remove_geometry(g)
```

In [42]:
```python
# R's method definition

def Rusinkiewicz(norms, verts, faces):
    # fetch vertex normals, coordinates, and faces
    pcurs, pdirs = np.zeros(shape=(len(faces), 2)), np.zeros(shape=(len(faces), 4))
    # Rusinkiewicz's algorithm for computing the curvatures
    for i in range(len(faces)):
        # Solve S
        face = faces[i]
        p2, p0, p1 = verts[face[0]], verts[face[1]], verts[face[2]]
        n2, n0, n1 = norms[face[0]], norms[face[1]], norms[face[2]]
        e1, e2, e0 = p0-p2, p1-p0, p2-p1
        ksaiu = e2 / np.linalg.norm(e2)
        ksaiv = np.cross(n0, ksaiu)
        ksaiv /= np.linalg.norm(ksaiv)
        # print(ksaiu, ksaiv, ksaiu.shape, ksaiv.shape, ksaiu@ksaiv)

        Df = np.vstack((ksaiu, ksaiv)).T
        A = np.vstack((
                    np.concatenate(((Df.T@e0).T, np.array([0,0]))),  # 6*4 matrix
                np.concatenate((np.array([0,0]), (Df.T@e0).T)),
                np.concatenate(((Df.T@e1).T, np.array([0,0]))),
                np.concatenate((np.array([0,0]), (Df.T@e1).T)),
                np.concatenate(((Df.T@e2).T, np.array([0,0]))),
                np.concatenate((np.array([0,0]), (Df.T@e2).T))
                ))
        b = np.concatenate((Df.T@(n2-n1),
                    Df.T@(n0-n2),
                    Df.T@(n1-n0))).T # 6*1 matrix
        # print(A, A.shape)
        # print(b, b.shape)
        s, _, _, _ = np.linalg.lstsq(A, b, rcond=None) # S = [[s1,s2],[s3,s4]] flat
        S = np.vstack((s[0:2], s[2:])) # 4*4 S matrix recovered
        # Eigen-decompose S to get principal directions and curvatures
        fpcurs, fpdirs = np.linalg.eig(S)
        if (fpcurs[0] < fpcurs[1]): # Align to Kmax, Kmin
            fpcurs[0], fpcurs[1] = np.copy(fpcurs[1]), np.copy(fpcurs[0])
            fpdirs[:,0], fpdirs[:,1] = np.copy(fpdirs[:,1]), np.copy(fpdirs[:,0])
        # update the principal curvature set
        pcurs[i] = fpcurs
        pdirs[i] = np.concatenate((fpdirs[:,0].T, fpdirs[:,1].T))
    return pcurs, pdirs
```

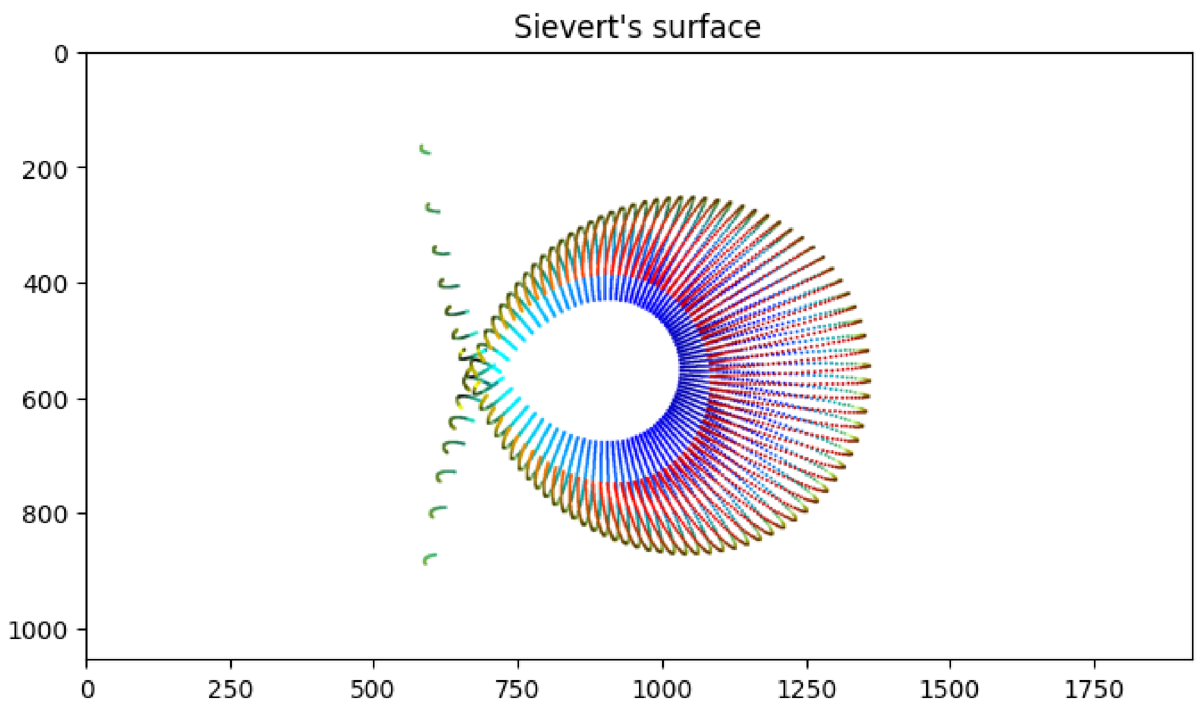
In [43]:
```python
# Principal curvature computations for sievert.obj
import open3d
import trimesh
import warnings
warnings.filterwarnings("ignore")

mesh = trimesh.load('sievert.obj')
print(mesh)
pcd = open3d.geometry.PointCloud()
```

```python
pcd.points = open3d.utility.Vector3dVector(mesh.vertices)
pcd.estimate_normals(open3d.geometry.KDTreeSearchParamKNN(knn=50))

# fetch vertex normals, coordinates, and faces
verts = np.asarray(pcd.points)
norms = np.asarray(pcd.normals)
faces = mesh.faces

# Apply R's method to get principal curvatures
pcursSie, pdirsSie = Rusinkiewicz(norms, verts, faces)

print("Principal curvatures ((Kmax, Kmin) corresponding to max and min curvatures):
print(pcursSie)
print(pcursSie.shape)
print("Principal directions in 2D ((x1, y1, x2, y2) where x1, y1 corresponding to K
print(pdirsSie)
print(pdirsSie.shape)

draw_geometries([pcd])
plt.title("Sievert's surface")
```

```
<trimesh.Trimesh(vertices.shape=(10201, 3), faces.shape=(20000, 3), name=`sievert.ob
j`)>
Principal curvatures ((Kmax, Kmin) corresponding to max and min curvatures):
[[ 1.16586428e-13 -1.16125901e-02]
 [-4.67985026e-14 -1.16334043e-02]
 [-3.63944985e-14 -1.17763662e-02]
 ...
 [ 6.96977198e-14 -4.73133049e-02]
 [ 1.34822709e-13 -5.76065985e+00]
 [ 1.33336051e-13 -3.55466334e-02]]
(20000, 2)
Principal directions in 2D ((x1, y1, x2, y2) where x1, y1 corresponding to Kmax, and
x2, y2 corresponding to Kmin):
[[ 0.03422705 -0.99941408 -0.11111641  0.9938074 ]
 [-0.03454718 -0.99940307 -0.11175038 -0.99373631]
 [-0.03419781 -0.99941508 -0.11259544 -0.99364091]
 ...
 [-0.03495435 -0.99938891 -0.10076161 -0.9949106 ]
 [-0.02862704 -0.99959016 -0.88922524  0.45746963]
 [-0.02848466 -0.99959423 -0.09610212  0.99537148]]
(20000, 4)
```

Out[43]:

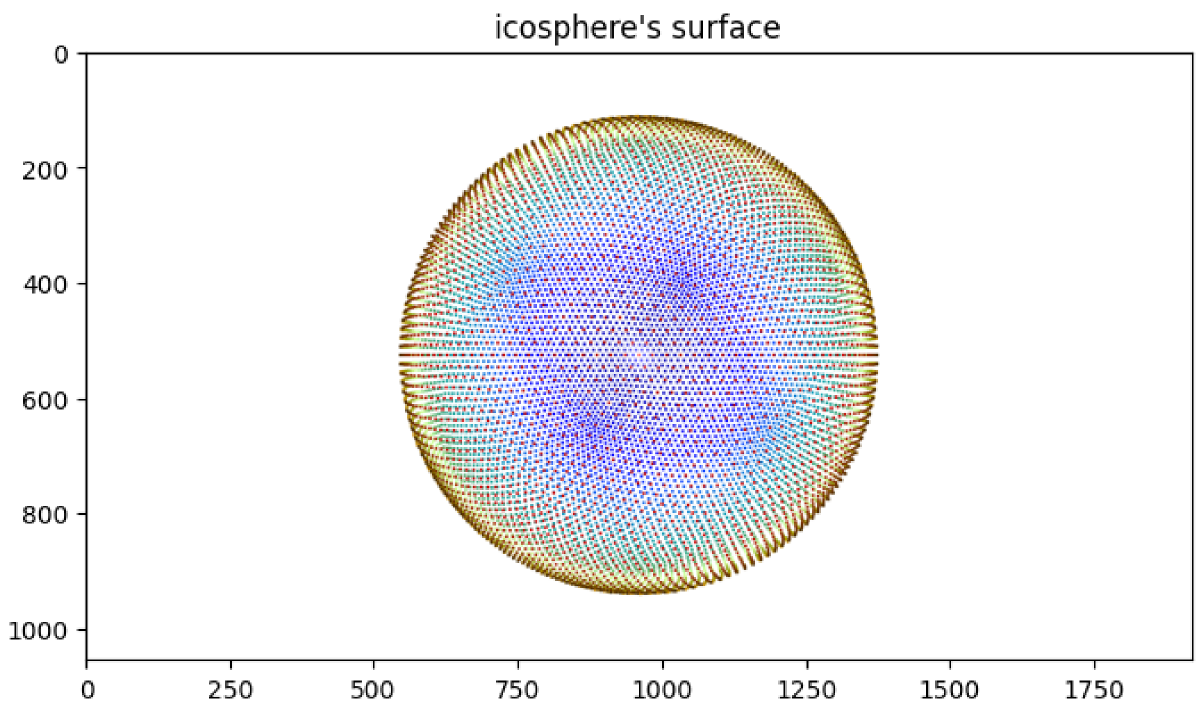
```
Text(0.5, 1.0, "Sievert's surface")
```



In [44]:
```
# Principal curvature computations for icosphere.obj
import open3d
import trimesh
import warnings
warnings.filterwarnings("ignore")

mesh = trimesh.load('icosphere.obj')
print(mesh)
pcd = open3d.geometry.PointCloud()
pcd.points = open3d.utility.Vector3dVector(mesh.vertices)
pcd.estimate_normals(open3d.geometry.KDTreeSearchParamKNN(knn=50))

# fetch vertex normals, coordinates, and faces
verts = np.asarray(pcd.points)
norms = np.asarray(pcd.normals)
faces = mesh.faces

# Apply R's method to get principal curvatures
pcursSph, pdirsSph = Rusinkiewicz(norms, verts, faces)

print("Principal curvatures ((Kmax, Kmin) corresponding to max and min curvatures):
print(pcursSph)
print(pcursSph.shape)
print("Principal directions in 2D ((x1, y1, x2, y2) where x1, y1 corresponding to K
print(pdirsSph)
print(pdirsSph.shape)

draw_geometries([pcd])
plt.title("icosphere's surface")
```

```
<trimesh.Trimesh(vertices.shape=(61432, 3), faces.shape=(20480, 3), name=`icosphere.
obj`)>
Principal curvatures ((Kmax, Kmin) corresponding to max and min curvatures):
[[1.197281   1.08617268]
 [1.2637722  0.89339637]
 [1.56127018 1.22152556]
 ...
 [1.12145781 0.8692251 ]
 [1.31803086 0.73781594]
 [1.21171006 1.21171006]]
(20480, 2)
Principal directions in 2D ((x1, y1, x2, y2) where x1, y1 corresponding to Kmax, and
x2, y2 corresponding to Kmin):
[[ 0.99999782  0.00208966  0.95085472  0.30963738]
 [ 0.99977965 -0.02099193  0.0565818   0.99839797]
 [ 0.65196255 -0.75825117 -0.98895485 -0.1482171 ]
 ...
 [ 0.15666677 -0.98765152 -0.59521006 -0.80357015]
 [ 0.99864915 -0.05196024  0.00569803  0.99998377]
 [-0.35240925  0.9340836  -0.35240925  0.9340836 ]]
(20480, 4)
```

Out[44]:
```
Text(0.5, 1.0, "icosphere's surface")
```



3.4:

In [45]:
```
# Function defs of Gaussian and Mean Curvature computations for sievert.obj
import open3d
import trimesh
import warnings
warnings.filterwarnings("ignore")

def curvature_to_Gaussian(curv):
    "Convert to Gaussian curvature"
```

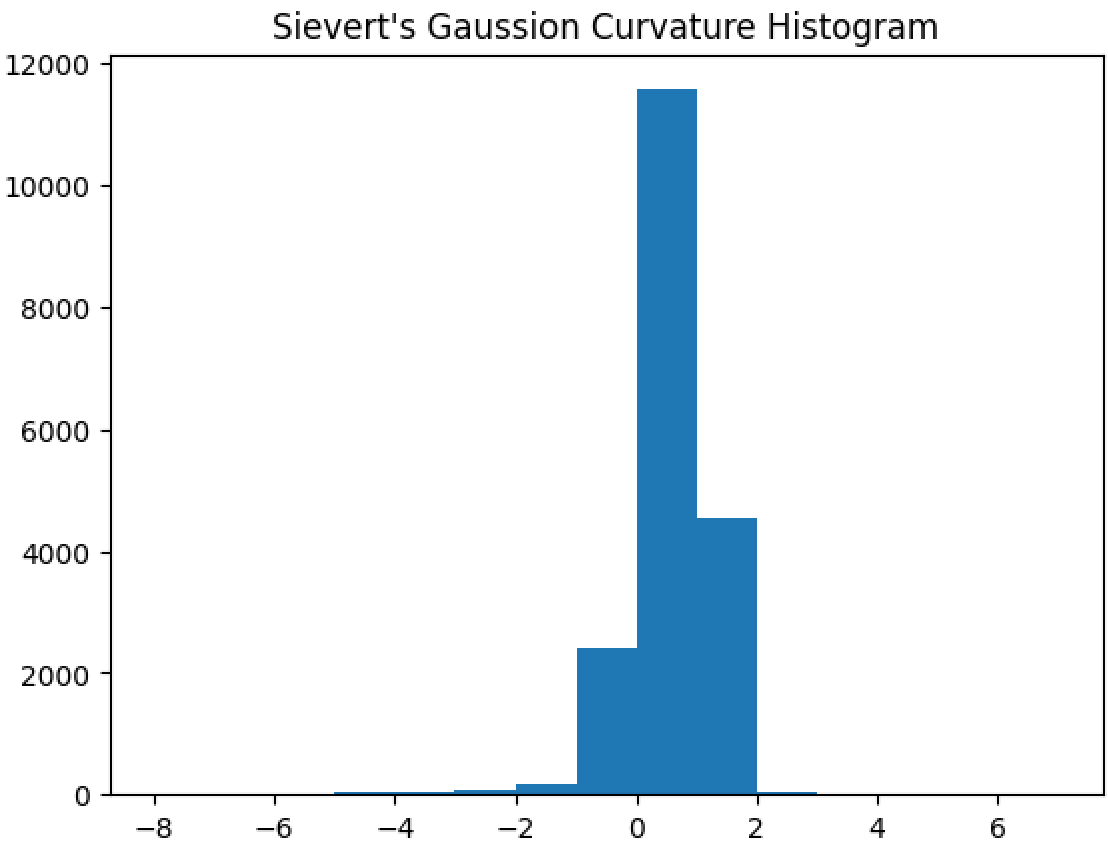
```
    return curv[:,0]*curv[:,1]

def curvature_to_Mean(curv):
    "Convert to Mean curvature"
    return (curv[:,0]+curv[:,1])/2
```

In [46]:
```
# Gaussian and Mean curvature computations for icosphere.obj and sievert.obj (run t

pGauSph = curvature_to_Gaussian(pcursSph)
pGauSie = curvature_to_Gaussian(pcursSie)
pMeanSph = curvature_to_Mean(pcursSph)
pMeanSie = curvature_to_Mean(pcursSie)
print("Gaussian curvatures for icosphere.obj:")
print(pGauSph)
print("Gaussian curvatures for sievert.obj:")
print(pGauSie)
print("Mean curvatures for icosphere.obj:")
print(pMeanSph)
print("Mean curvatures for sievert.obj:")
print(pMeanSie)
```

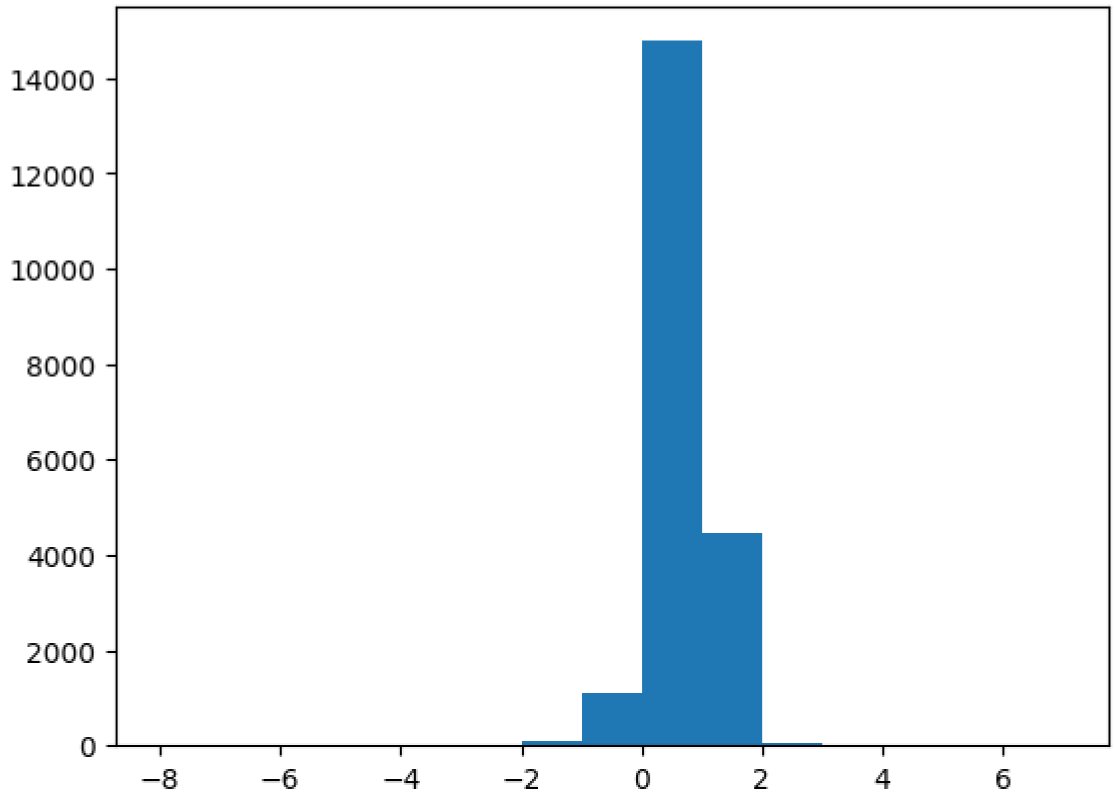
```
Gaussian curvatures for icosphere.obj:
[1.30045391 1.1290495  1.90713144 ... 0.97479927 0.97246418 1.46824127]
Gaussian curvatures for sievert.obj:
[-1.35387040e-15  5.44425903e-16  4.28594941e-16 ... -3.29762947e-15
 -7.76667765e-13 -4.73964771e-15]
Mean curvatures for icosphere.obj:
[1.14172684 1.07858429 1.39139787 ... 0.99534145 1.0279234  1.21171006]
Mean curvatures for sievert.obj:
[-0.0058063  -0.0058167  -0.00588818 ... -0.02365665 -2.88032993
 -0.01777332]
```

In [47]:
```
# Comparing Mean and Gaussian curvatures of icosphere.obj and sievert.obj (Run the
from matplotlib import pyplot as plt
import numpy as np
fig = plt.figure()

# Gaussian plots:
# For Sievert's surface
plt.hist(pGauSie, bins = range(-8, 8, 1))
plt.title("Sievert's Gaussion Curvature Histogram")
plt.show()

# For Sphere's surface
plt.hist(pGauSph, bins = range(-8, 8, 1))
plt.title("Sphere's Gaussion Curvature Histogram")
plt.show()
# We see that the Gaussian Curvature histograms of the two objects are roughly the
# which verifies that they're isometric.

# Mean plots:
# For Sievert's surface
# print(len(pMeanSie))
plt.hist(pMeanSie, bins = range(-8, 8, 1))
plt.title("Sievert's Mean Curvature Histogram")
plt.show()
```

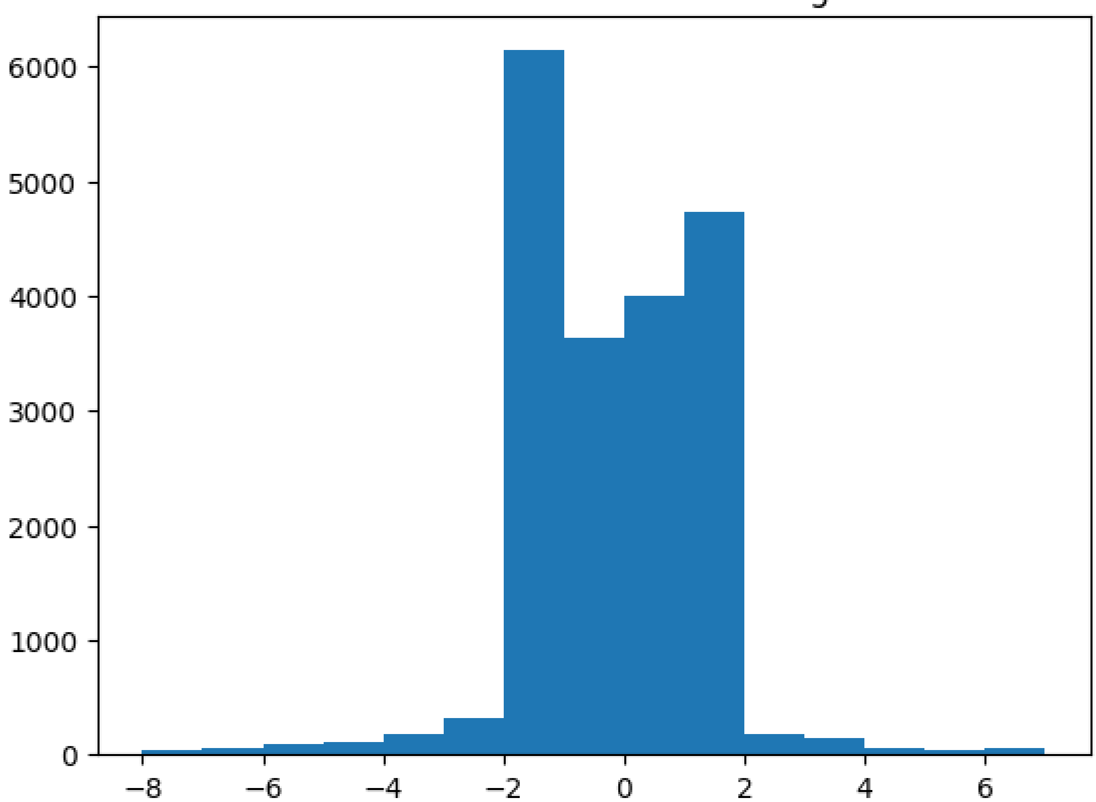
```python
# For Sphere's surface
# print(len(pMeanSph))
plt.hist(pMeanSph, bins = range(-8, 8, 1))
plt.title("Sphere's Mean Curvature Histogram")
plt.show()
```

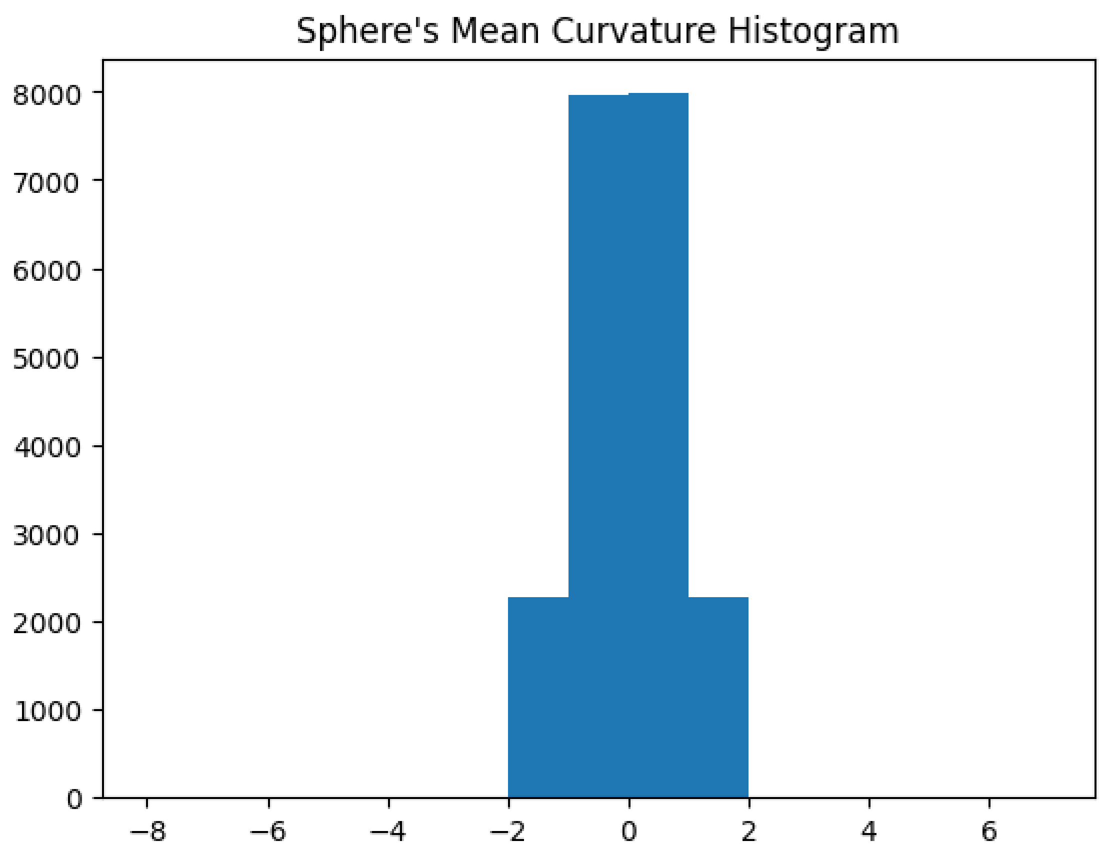


Sphere's Gaussion Curvature Histogram



Sievert's Mean Curvature Histogram





4 Point Cloud

4.1:

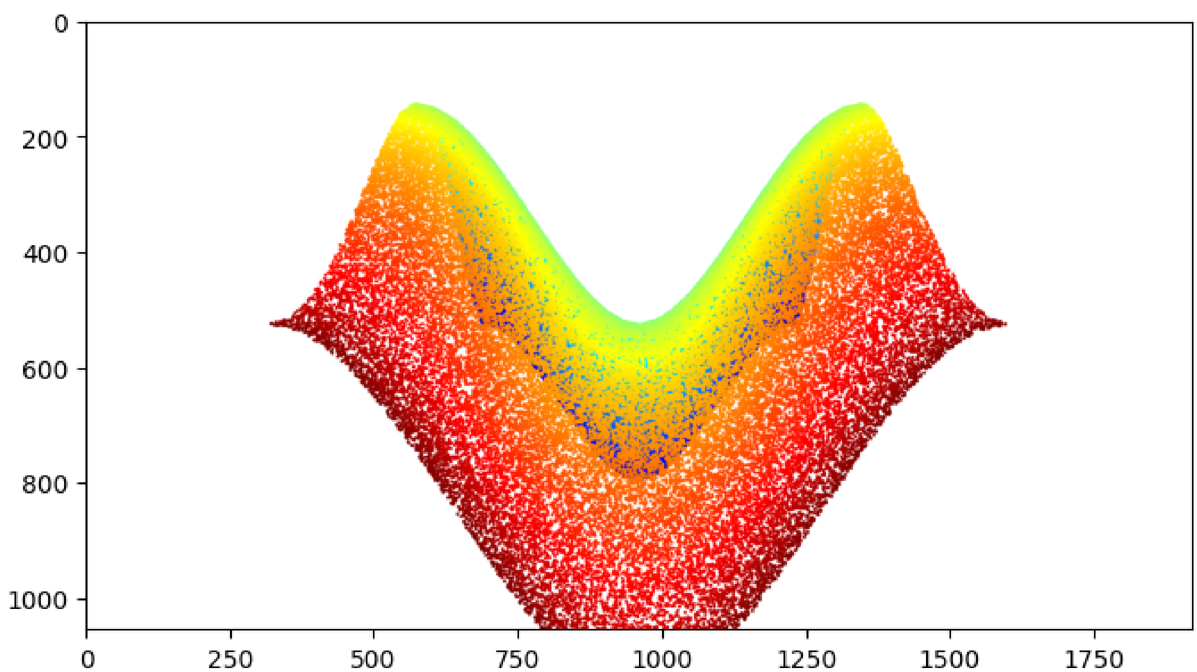
In [48]:

```
# These are some convenient functions to create open3d geometries and plot them
# The viewing direction is fine-tuned for this problem, you should not change them
import open3d
import math
import numpy as np
import matplotlib.pyplot as plt

vis = open3d.visualization.Visualizer()
vis.create_window(visible = False)

def draw_geometries(geoms):
    for g in geoms:
        vis.add_geometry(g)
    view_ctl = vis.get_view_control()
    view_ctl.set_up((0, 1e-4, 1))
    view_ctl.set_front((0, 0.5, 2))
    view_ctl.set_lookat((0, 0, 0))
    # do not change this view point
    vis.update_renderer()
    img = vis.capture_screen_float_buffer(True)
    plt.figure(figsize=(8,6))
    plt.imshow(np.asarray(img)[::-1, ::-1])
```

```python
    for g in geoms:
        vis.remove_geometry(g)
```

In [49]:
```python
# Sample 100K points from saddle.obj
import trimesh
import trimesh.sample

Nsamples = 100000
mesh = trimesh.load_mesh("saddle.obj")
pts, ptinds = trimesh.sample.sample_surface(mesh, Nsamples, seed=1)

print("Sampled 100K points:")
print(pts)
print(pts.shape)
print("Sampled 100K points' indices:")
print(ptinds)
print(ptinds.shape)


pcd = open3d.geometry.PointCloud()
pcd.points = open3d.utility.Vector3dVector(pts)
draw_geometries([pcd])
```

```
Sampled 100K points:
[[ 0.4666489   0.54707479 -0.97690362]
 [-0.8447135  -0.44934622 -0.50869453]
 [ 0.44529958 -0.39383187  0.12025959]
 ...
 [ 0.32206893 -0.043264    0.28268161]
 [-0.80940607 -0.88490067 -0.08279984]
 [-0.58706528 -0.05346066 -0.54113807]]
(100000, 3)
Sampled 100K points' indices:
[ 696 1271  187 ...  886  590  521]
(100000,)
```

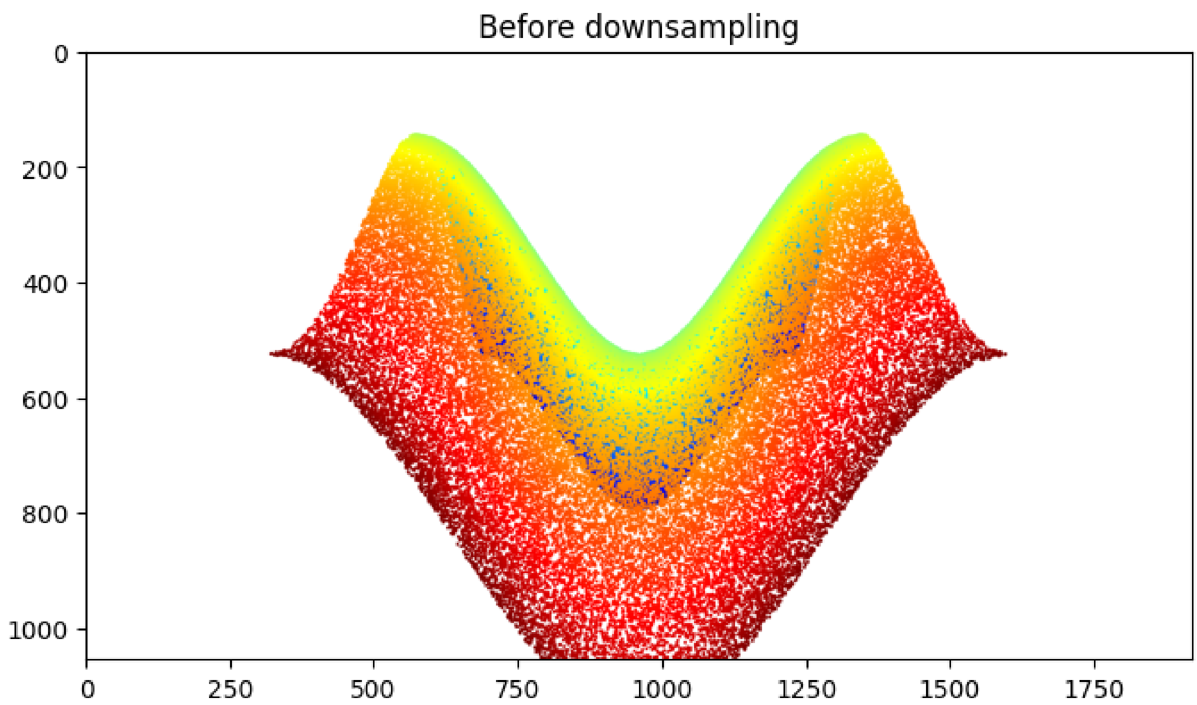


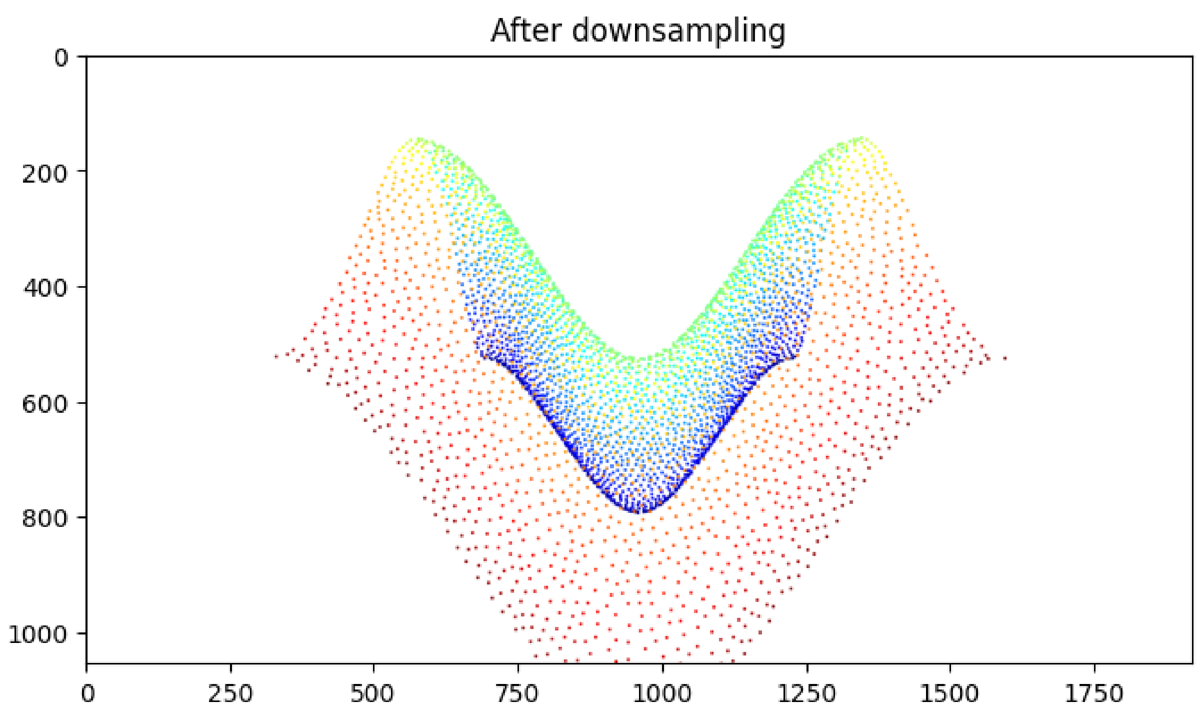
4.2

In [50]:
```python
# Sample 4K points from the 100K ones using Iterative Farthest Sampling
import trimesh
import trimesh.sample
from tqdm import tqdm

Nsamples = 100000
Ndnsamps = 4000
mesh = trimesh.load_mesh("saddle.obj")
pts, ptinds = trimesh.sample.sample_surface(mesh, Nsamples, seed=1)

# union points -> compute distance -> take min for set distance
def fps_downsample(points, number_of_points_to_sample):
    selected_points = np.zeros((number_of_points_to_sample, 3))
    dist = np.ones(points.shape[0]) * np.inf # distance to the selected set
    for i in tqdm(range(number_of_points_to_sample)):
        # pick the point with max dist
        idx = np.argmax(dist)
        selected_points[i] = points[idx]
        dist_ = ((points - selected_points[i]) ** 2).sum(-1)
        dist = np.minimum(dist, dist_)
    return selected_points

samp_pts = fps_downsample(pts, Ndnsamps)

print("Sampled 4K points:")
print(samp_pts)
print(samp_pts.shape)

pcd = open3d.geometry.PointCloud()
pcd.points = open3d.utility.Vector3dVector(pts)
draw_geometries([pcd])
plt.title("Before downsampling")

pcd = open3d.geometry.PointCloud()
pcd.points = open3d.utility.Vector3dVector(samp_pts)
draw_geometries([pcd])
plt.title("After downsampling")
```

```
  0%|          | 0/4000 [00:00<?, ?it/s]100%|██████████| 4000/4000 [00:13<00:00, 30
2.15it/s]
Sampled 4K points:
[[ 4.66648899e-01  5.47074793e-01 -9.76903621e-01]
 [-9.94262176e-01  4.21190122e-04  9.96173964e-01]
 [ 9.96549144e-01 -5.86438268e-02  8.53244279e-01]
 ...
 [ 7.79742173e-01 -8.43722450e-01  1.10400391e-01]
 [-4.11577570e-02  3.87159903e-01  4.26708900e-01]
 [-4.62089705e-01  5.15203100e-01 -8.38720485e-01]]
(4000, 3)
```

Out[50]:
```
Text(0.5, 1.0, 'After downsampling')
```





4.3:

In [51]:
```
# Estimate normals for the 4K points (Run the previous cell to get sampled points f
import trimesh
import trimesh.sample
from tqdm import tqdm
from sklearn.decomposition import PCA

Nsamples = 100000
mesh = trimesh.load_mesh("saddle.obj")
pts, ptinds = trimesh.sample.sample_surface(mesh, Nsamples, seed=1)
Ndnsamps = 4000
```

```
samp_pts = samp_pts

def find_near_k_pts(pt, pts, k):
    """
    Find 50 nearest points (not necessarily ordered) to pt in pts, and return the 5
    """
    dist_ = ((pts - pt) ** 2).sum(-1)
    idx = np.argpartition(dist_, k)
    return pts[idx[:k]]

def esti_norm_PCA(samp_pts_near_50):
    """
    Estimate a normal given 50 points, using PCA, and return 1*3 normal vector esti
    """
    M = np.zeros((3,3))
    xbar = np.mean(samp_pts_near_50, axis=0)
    for xi in samp_pts_near_50:
        M += np.outer(xi-xbar, xi-xbar)
    # PCA on M to get w, corresponding to the 3rd principal component
    pca = PCA(3)
    pca.fit(M)
    # print(pca.components_)
    # print(pca.singular_values_)
    norm = pca.components_[:,-1] # Fetch the third principal component to be the no
    if (norm@np.array([0,1,0]) < 0): # Align to Y axis
        norm = -norm
    return norm


def esti_norms(samp_pts):
    Ndnsamps = samp_pts.shape[0]
    norms = np.zeros((Ndnsamps, 3))
    for i in tqdm(range(Ndnsamps)):
        samp_pts_near_50 = find_near_k_pts(samp_pts[i], np.asarray(mesh.vertices),
        norm = esti_norm_PCA(samp_pts_near_50)
        norms[i] = norm
    return norms

norms = esti_norms(samp_pts)
print("Estimated normals:")
print(norms.shape)
print(norms)
```

```
  0%|          | 0/4000 [00:00<?, ?it/s]100%|██████████| 4000/4000 [00:02<00:00, 147
0.77it/s]
Estimated normals:
(4000, 3)
[[-0.02396244  0.91208134  0.40930847]
 [ 0.42111041  0.68200905  0.59793785]
 [-0.31124628  0.7920378  -0.52516843]
 ...
 [-0.39081875  0.9024415  -0.1812734 ]
 [ 0.39850363  0.27458315 -0.87509939]
 [ 0.51612205  0.67592656  0.52606208]]
```

4.4: I use the Poisson Surface Reconstruction algorithm to first reconstruct the Mesh, which contains the essential face information for Rusinkiewicz's method curvature estimation, from the Point Cloud. Then I estimate the curvatures based on the Mesh (the face, vertex, and normal information inside) generated.

In [52]:

```
# Function definition of sample 4K points from the 100K ones using Iterative Farthe
import trimesh
import trimesh.sample
from tqdm import tqdm

Nsamples = 100000
Ndnsamps = 4000
mesh = trimesh.load_mesh("saddle.obj")
pts, ptinds = trimesh.sample.sample_surface(mesh, Nsamples, seed=1)

samp_pts = samp_pts
norms = norms

# union points -> compute distance -> take min for set distance
def fps_downsample(points, number_of_points_to_sample):
    selected_points = np.zeros((number_of_points_to_sample, 3))
    dist = np.ones(points.shape[0]) * np.inf # distance to the selected set
    for i in tqdm(range(number_of_points_to_sample)):
        # pick the point with max dist
        idx = np.argmax(dist)
        selected_points[i] = points[idx]
        dist_ = ((points - selected_points[i]) ** 2).sum(-1)
        dist = np.minimum(dist, dist_)
    return selected_points
```

In [53]:

```
# Function definition of R's method.
import warnings
warnings.filterwarnings("ignore")

samp_pts = samp_pts
norms = norms

def Rusinkiewicz(norms, verts, faces):
    # fetch vertex normals, coordinates, and faces
    pcurs, pdirs = np.zeros(shape=(len(faces), 2)), np.zeros(shape=(len(faces), 4))

    # Rusinkiewicz's algorithm for computing the curvatures
    for i in range(len(faces)):
        # Solve S
        face = faces[i]
        p2, p0, p1 = verts[face[0]], verts[face[1]], verts[face[2]]
        n2, n0, n1 = norms[face[0]], norms[face[1]], norms[face[2]]
        e1, e2, e0 = p0-p2, p1-p0, p2-p1
        ksaiu = e2 / np.linalg.norm(e2)
        ksaiv = np.cross(n0, ksaiu)
        ksaiv /= np.linalg.norm(ksaiv)
        # print(ksaiu, ksaiv, ksaiu.shape, ksaiv.shape, ksaiu@ksaiv)

        Df = np.vstack((ksaiu, ksaiv)).T
```

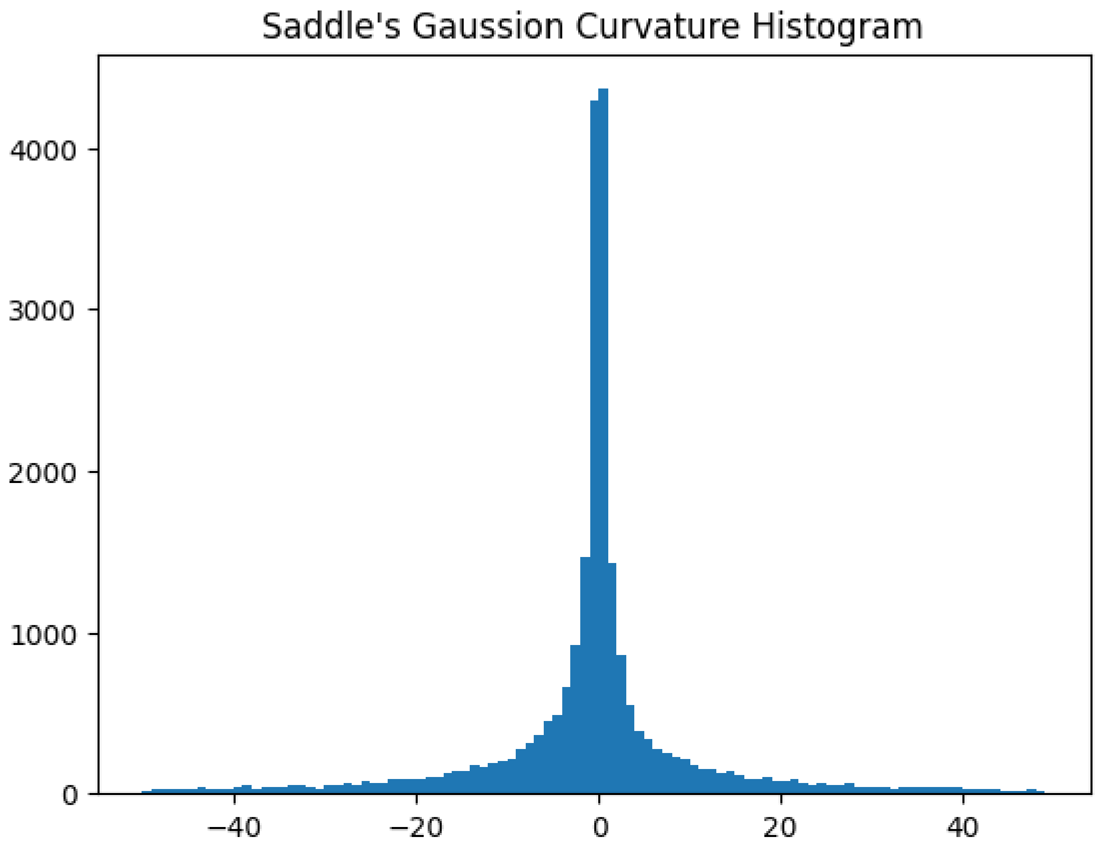
```
        # print(Df, Df.shape)
        # print(np.concatenate(((Df.T@e0).T, np.array([0,0]))))
        A = np.vstack((
                    np.concatenate(((Df.T@e0).T, np.array([0,0]))),  # 6*4 matrix
                np.concatenate((np.array([0,0]), (Df.T@e0).T)),
                np.concatenate(((Df.T@e1).T, np.array([0,0]))),
                np.concatenate((np.array([0,0]), (Df.T@e1).T)),
                np.concatenate(((Df.T@e2).T, np.array([0,0]))),
                np.concatenate((np.array([0,0]), (Df.T@e2).T))
                ))
        b = np.concatenate((Df.T@(n2-n1),
                    Df.T@(n0-n2),
                    Df.T@(n1-n0))).T # 6*1 matrix
        # print(A, A.shape)
        # print(b, b.shape)
        s, _, _, _ = np.linalg.lstsq(A, b, rcond=None) # S = [[s1,s2],[s3,s4]] flat
        S = np.vstack((s[0:2], s[2:])) # 4*4 S matrix recovered
        # Eigen-decompose S to get principal directions and curvatures
        fpcurs, fpdirs = np.linalg.eig(S)
        if (fpcurs[0] < fpcurs[1]): # Align to Kmax, Kmin
            fpcurs[0], fpcurs[1] = np.copy(fpcurs[1]), np.copy(fpcurs[0])
            fpdirs[:,0], fpdirs[:,1] = np.copy(fpdirs[:,1]), np.copy(fpdirs[:,0])
        # update the principal curvature set
        pcurs[i] = fpcurs
        # print(fpdirs[:,0].T, fpdirs[:,0].T.shape)
        pdirs[i] = np.concatenate((fpdirs[:,0].T, fpdirs[:,1].T))
    return pcurs, pdirs

def curvature_to_Gaussian(curv):
    "Convert to Gaussian curvature"
    # print(curv)
    return curv[:,0]*curv[:,1]

def curvature_to_Mean(curv):
    "Convert to Mean curvature"
    # print(curv)
    return (curv[:,0]+curv[:,1])/2
```

In [54]:

```
# R's method on Point Cloud (Run the previous cells to get sampled points and estim
import trimesh
import open3d as o3d
import trimesh.sample
from tqdm import tqdm
from sklearn.decomposition import PCA

# fetch sampled 4k points and the normals
Nsamples = 100000
Ndnsamps = 4000
mesh = trimesh.load_mesh("saddle.obj")
pts, ptinds = trimesh.sample.sample_surface(mesh, Nsamples, seed=1)
samp_pts = fps_downsample(pts, Ndnsamps)
print("Down-sampling 4K points finished")
norms = esti_norms(samp_pts)
print("Normal estimations for the 4K points finished")

# Init the sampled Point Cloud
```

```python
pcd = open3d.geometry.PointCloud()
pcd.points = open3d.utility.Vector3dVector(samp_pts)
# pcd.estimate_normals(open3d.geometry.KDTreeSearchParamKNN(knn=50))
pcd.normals = open3d.utility.Vector3dVector(norms)
# print(len(samp_pts))

# Reconstruct the Mesh using the Poisson Surface Reconstruction method
print("Poisson Surface Reconstruction started!")
mesh, densities = o3d.geometry.TriangleMesh.create_from_point_cloud_poisson(pcd,dep
print("Poisson Surface Reconstruction finished!")

# Re-construct the point cloud since some vertex interpolations are done
re_pcd = open3d.geometry.PointCloud()
re_pcd.points = open3d.utility.Vector3dVector(mesh.vertices)
re_pcd.estimate_normals(open3d.geometry.KDTreeSearchParamKNN(knn=50))
re_pcd.orient_normals_to_align_with_direction(np.array([0.0, 1.0, 0.0]))

# Compute curvatures using the mesh, by R's method
# print(mesh)
faces = np.asarray(mesh.triangles)
verts = np.asarray(re_pcd.points)
norms = np.asarray(re_pcd.normals)

# faces = mesh.faces
print("Rusinkiewicz's curvature estimation started!")
pcurs, pdirs = Rusinkiewicz(norms, verts, faces)
print("Rusinkiewicz's curvature estimation finished!")
pGauSad = curvature_to_Gaussian(pcurs)
print("Generating Gaussian histogram...")
plt.hist(pGauSad, bins = range(-50, 50, 1))
plt.title("Saddle's Gaussion Curvature Histogram")
plt.show()
```

```
  1%|          | 22/4000 [00:00<00:18, 217.01it/s]100%|██████████| 4000/4000 [00:14<
00:00, 276.92it/s]
Down-sampling 4K points finished
100%|██████████| 4000/4000 [00:02<00:00, 1451.31it/s]
Normal estimations for the 4K points finished
Poisson Surface Reconstruction started!
Poisson Surface Reconstruction finished!
Rusinkiewicz's curvature estimation started!
Rusinkiewicz's curvature estimation finished!
Generating Gaussian histogram...
```



5 Feedbacks

5.1: A week. Working at least 6 hours on this per day.

5.2: At very least 36 hours.

5.3: Hopefully we can have more time for the assignments (since they're indeed a lot), and hopefully the grades will be given more leniently eventually. I can learn a lot though the assignments are tough.